THE MÖBIUS FUNCTION OF THE POSET OF TRIANGULAR NUMBERS UNDER DIVISIBILITY

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Abstract. This paper experimentally investigates the Möbius function ($\mu_{\mathcal{T}}(i)$) defined on the partially ordered set of triangular numbers ($\mathcal{T}(i)$) under the divisibility relation. We call $\mu_{\mathcal{T}}$ the "triangular Möbius function." We are motivated by the goal of better understanding the classical Möbius function. The partial sums of the classical Möbius function have been widely studied, as knowing its precise asymptotic behavior would answer the Riemann hypothesis.

We make conjectures on the asymptotic behavior of the triangular Möbius function on the basis of experimental data. We first study the growth of partial sums of $\mu_{\mathcal{T}}(i)$ and then analyze the partial sums of $|\mu_{\mathcal{T}}(i)|$. To generate values of $\mu_{\mathcal{T}}$, we create zeta and Möbius matrices for the underlying partial order in the Python programming language. Using Python libraries we also create visualizations for demonstrating the previously mentioned patterns. We conclude the paper with divisibility patterns in Appendix C, with proofs which may be helpful for future work on our conjectures.

Contents

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1. Introduction

The classical Möbius function is an important and well-studied function for understanding the distribution of prime numbers, and plays a central role in multiplicative number theory. It has connections to many theorems and unresolved conjectures. Notably, the Riemann hypothesis is equivalent to a certain asymptotic bound on the partial sums of the Möbius function.

In this paper, we introduce and study a variant on the classical Möbius function. The classical Möbius function arises from the positive integers under division, which forms the structure of a partial order. We consider the partial order on the positive integers which records divisibility among the triangular numbers. In more detail: let $\mathcal{T}(i) = \frac{1}{2}i(i+1)$ denote the i-th triangular number, and let $\leq_{\mathcal{T}}$ be the relation on $\mathbb{N}$ defined by

$$i \leq_{\mathcal{T}} j \qquad \Leftrightarrow \qquad \mathcal{T}(i) \text{ divides } \mathcal{T}(j).$$

Equivalently, we have $i \leq_{\mathcal{T}} j$ if and only if $i(i+1)$ divides $j(j+1)$.

The partially ordered set $(\mathbb{N}, \leq_{\mathcal{T}})$ has a Möbius function, which we denote $\mu_{\mathcal{T}}$. Based on experimental data, we make the follow conjectures.

Conjecture 1 (Growth of partial sums of $\mu_{\mathcal{T}}$)**.** *There is a positive constant C such that*

$$\sum_{i=1}^{n} \mu_{\mathcal{T}}(i) \leq -Cn \quad \textit{for all sufficiently large } n.$$

Conjecture 2 (Partial sums of $|\mu_{\mathcal{T}}|$)**.** *As $n \to \infty$,*

$$\sum_{i=1}^{n} |\mu_{\mathcal{T}}(i)| = \frac{1}{2}n + o(n).$$

The little-o asymptotic notation here means that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\mu_{\mathcal{T}}(i)| = \frac{1}{2}$. Conjecture 1 is notably different from the corresponding behavior of the classical Möbius function μ, for which $\sum_{i=1}^{n} \mu(i)$ changes sign infinitely often, and it is conjectured that $\sum_{i=1}^{n} \mu(i) = O(n^{1/2+\epsilon})$ for any $\epsilon > 0$. Regarding Conjecture 2, the corresponding asymptotic for the classical Möbius function μ is

$$\sum_{i=1}^{n} |\mu(i)| = \frac{1}{\zeta(2)}n + o(n),$$

where the leading constant is $1/\zeta(2) = 6/\pi^2 \approx 0.608$. See Figures 1 and 2 for plots illustrating the data supporting Conjectures 1 and 2.

The next conjecture concerns the size of the triangular Möbius function values. This is also in notable contrast with the classical Möbius function, for which $|\mu(i)| \leq 1$.

Conjecture 3. *For any integer $M \geq 0$, there exists n such that $\mu_{\mathcal{T}}(n) \geq M$.*

These conjectures are made on the basis of experimental data. We write code in Python to compute the Möbius function $\mu_{\mathcal{T}}$ of the poset $(\mathbb{N}, \leq_{\mathcal{T}})$, and examine plots of the relevant Möbius function sums. The Python code is included in an appendix. The data on the Möbius function $\mu_{\mathcal{T}}$ and its partial sums were submitted to OEIS as entries A350682 and A351167 [9, 10].

This investigation of the Möbius function $\mu_{\mathcal{T}}$ was partially motivated by [6], which defines a "floor quotient" relation on integers, and explores its asymptotic properties motivated by a better understanding the interplay between addition and multiplication at the heart of the Riemann hypothesis.

Acknowledgements. Figures were created in Python [13] using Matplotlib [4] and Plotly [11].

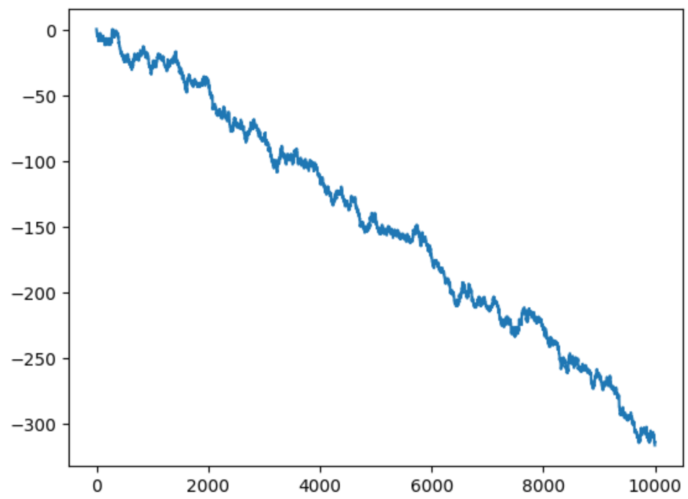

FIGURE 1. Partial sums of $\mu_{\mathcal{T}}$ from 1 to n for $n \leq 10,000$.

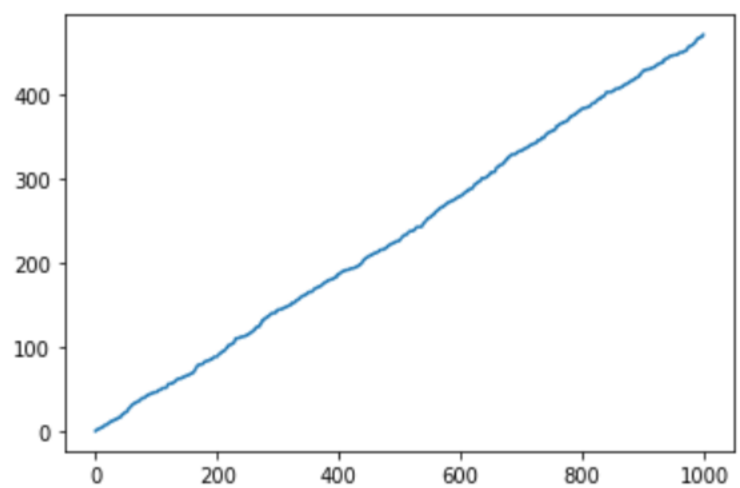

FIGURE 2. Partial sums of $|\mu_{\mathcal{T}}|$ from 1 to n, for $n \leq 1000$.

2. Background

Further information is included in this section to clarify terms and diagrams and other information. For further background on number theory, see Burton [2]. For further background on the combinatorics of posets, see [1].

2.1. **The classical Möbius and Mertens functions.** The classical Möbius function [8] is defined, for a positive integer n, by the following rules:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ where } p_i \text{ are distinct primes}, \\ 0 & \text{if } p^2 \mid n \ \text{ for some prime } p. \end{cases}$$

The behavior of the Möbius function is fundamentally linked to the structure of prime numbers and prime factorization. In broad terms, the Möbius function captures the multiplicative structure of the integer n. Studying the Möbius function under an additive perspective, i.e. what happens when $n \to \infty$, reveals the interplay between multiplication and addition on the positive integers.

The Mertens function $\mathcal{M}(n)$ is defined by taking partial sums of the Möbius function,

$$\mathcal{M}(n) = \sum_{k=1}^{n} \mu(k).$$

Many properties of this function were studied by Mertens [7]. Significantly, the Riemann hypothesis is equivalent to the asymptotic bound

$$\mathcal{M}(n) = O(n^{1/2+\epsilon}) \qquad \text{for any } \epsilon > 0.$$

Whether this bounds holds is currently open, and is a subject of active research. Kotnik and van de Lune [5] investigate the asymptotics of the Mertens function by numerical experiment.

2.2. **Posets and Hasse diagrams.** The classical definition of the Möbius function depends on the prime factorization of a positive integer, which in turn depends on the relation of integer divisibility. Integer divisibility defines a *partial order* relation on the positive integers.

The classical Möbius function is uniquely characterized by the following equations, coming from the integer divisibility relation.

(M.1) For $n = 1$, we have $\mu(1) = 1$.
(M.2) For any $n \geq 2$, we have $\sum_{d \mid n} \mu(d) = 0$.

These relations can be adapted to form the definition of the "Möbius function" for an arbitrary partially ordered set.

Before discussing this generalized Möbius function, let us first recall some definitions. A partially ordered set, also known as a *poset*, is a tool for ordering combinatorial objects. For more background, see Bona [1, Chapter 16]. A poset $(S, \preccurlyeq)$ consists of an underlying set S and a binary relation "$\preccurlyeq$" on S satisfying the following axioms:

- (Reflexivity) $x \preccurlyeq x$ for all $x \in S$;
- (Antisymmetry) if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$;
- (Transitivity) if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$.

If these hold, then we call $\preccurlyeq$ a *partial order relation*.

The *Hasse diagram* of a poset is a nice way to represent a poset graphically. It is a directed graph whose vertices are the elements of the poset, and whose edges correspond to covering relations in the poset. A *covering relation* is a pair (x, y) of distinct elements such that $x \preccurlyeq y$, and there is no element $z \notin \{x, y\}$ such that $x \preccurlyeq z \preccurlyeq y$. Rather than indicating the directions of edges with arrows, it is typical to display Hasse diagrams so that all edges implicitly point "upwards," i.e. if $x \preccurlyeq y$ is a covering relation, then we draw the x-vertex lower and the y-vertex higher in the diagram.

In Figure 3 is an example of a Hasse diagram of the integers $\{1, 2, \ldots, 20\}$ under the divisibility relation. With the divisibility relation, the covering relations are those of the form n divides pn, where p is a prime number.

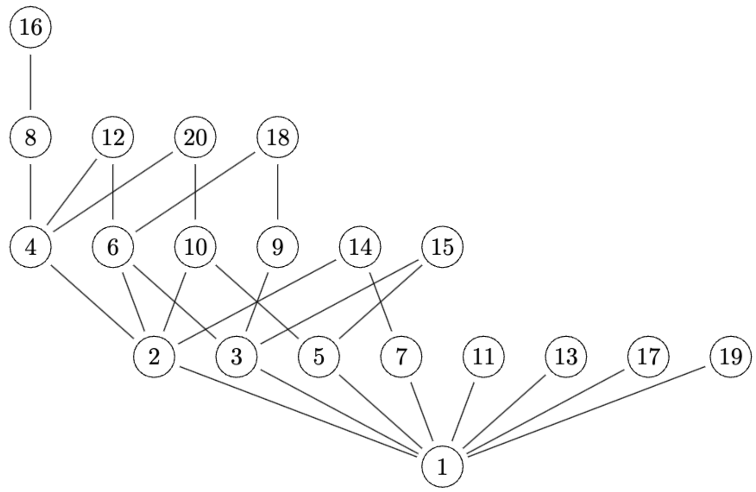

FIGURE 3. Hasse Diagram for Integer Divisibility

2.3. **Generalized Möbius functions.** We now return to the generalized Möbius functions on a poset. This idea was introduced by Rota [12]. For more details on the Möbius function, see Bona [1, Chapter 16.2]. In order to define the Möbius function for $(S, \preccurlyeq)$ we need one additional technical definition: we say a poset $(S, \preccurlyeq)$ is *locally finite* if for any $x, y \in S$, the set $\{z : x \preccurlyeq z \preccurlyeq y\}$ of elements between x and y is finite.

The *Möbius function* of a locally finite poset $(S, \preccurlyeq)$ is defined as follows: $\mu_S : S \times S \to \mathbb{Z}$ satisfies

(GM.0) $\mu_S(x, y) = 0$ if $x \not\preccurlyeq y$;
(GM.1) $\mu_S(x, x) = 1$ for all $x \in S$;
(GM.2) if $x \preccurlyeq y$ and $x \neq y$, then $\displaystyle\sum_{z:\, x \preccurlyeq z \preccurlyeq y} \mu_s(x, z) = 0$.

Now suppose there is an element $1 \in S$, such that $1 \preccurlyeq x$ for all $x \in S$. The Möbius function of $(S, \preccurlyeq)$ is defined by $\mu_S(1, 1) = 1$ and

$$\sum_{z \preccurlyeq y} \mu_P(1, z) = 0 \qquad \text{if } y \neq 1.$$

2.4. **Matrix computation: Zeta matrix and Möbius matrix.** We now describe how the generalized Möbius function, defined in the previous section, has an equivalent description in terms of matrices. This matrix version has the advantage of being straightforward to implement in a programming language, using a well-supported linear algebra library. For the rest of the paper, we assume for convenience that our poset is defined on the underlying set $S = \mathbb{N}$.

The *zeta matrix* of a poset $P = (\mathbb{N}, \preccurlyeq_P)$ on the natural numbers is the $\{0, 1\}$-valued matrix whose entries are

$$Z_{i,j} = \begin{cases} 1 & \text{if } j \preccurlyeq_P i, \\ 0 & \text{otherwise.} \end{cases}$$

Examples of these matrices using the triangular numbers and integers (for comparison) are shown in Section 3.2. Creating the zeta matrix for a poset is equivalent to figuring out the poset relation between all pairs of elements in the poset; to find a submatrix of the zeta matrix, we just need to figure out the poset relation between pairs of corresponding elements. The i-th row of the zeta matrix records which elements are larger than i in the poset; the j-th column of the zeta matrix records which elements are smaller than j in the poset.

We may compute the Möbius function of a poset by finding the matrix inverse of the zeta matrix. Namely, if P is a locally finite poset, let

$$M = Z^{-1} \tag{1}$$

where Z denotes the zeta matrix of P from above. Then the entries of the matrix M are exactly the Möbius function values:

$$M_{i,j} = \mu_P(j, i). \tag{2}$$

This is because the properties defining the Möbius function, (GM.0 - GM.2) above, are equivalent to the matrix relation

$$[\mu_P(i, j)]_{i,j\in\mathbb{N}}\, Z = I,$$

where I denotes the infinite $\mathbb{N} \times \mathbb{N}$ identity matrix.

Note: In this paper we focus on studying the first row of the Möbius matrix.

3. Poset of triangular numbers

In this section we define the poset which is the main focus of this paper. As before, let $\mathcal{T}(n) = \frac{1}{2}n(n + 1)$ denote the n-th triangular number. By abuse of notation, let $\mathcal{T}$ also denote the set of *triangular numbers*, i.e.,

$$\mathcal{T} = \left\{\tfrac{1}{2}n(n + 1) : n = 1, 2, \ldots\right\} = \{1, 3, 6, 10, \ldots\}.$$

Consider the poset $(\mathbb{N}, \leq_\mathcal{T})$ define by $i \leq j$ if and only if $\mathcal{T}(i)$ divides $\mathcal{T}(j)$.

3.1. **Hasse diagram.** Here we show the Hasse diagram of the first 20 elements of $(\mathbb{N}, \leq_\mathcal{T})$, which shows the divisibility relations among the first 20 triangular numbers, in Figure 4. Recall that a line is drawn between two numbers if they are "minimally related" to one other. For example, $\mathcal{T}(4)$ divides $\mathcal{T}(19)$, so there is a line from 4 to 19.

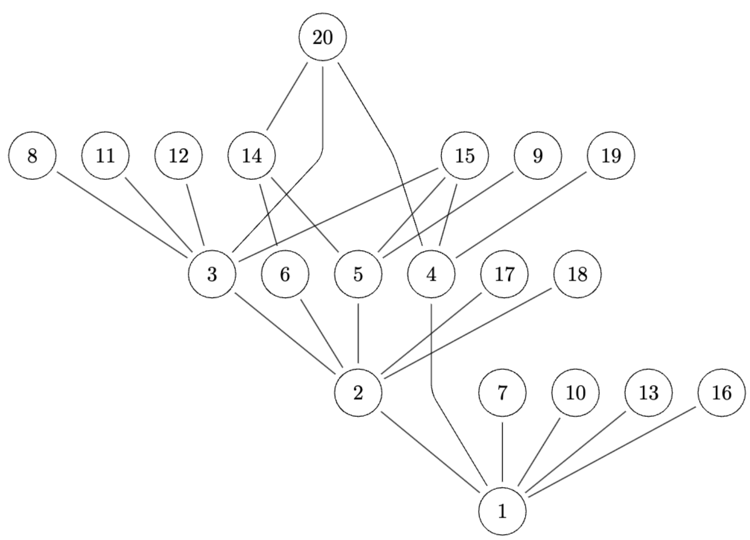

Figure 4. Hasse diagram for $\leq_\mathcal{T}$, encoding triangular number divisibility.

It's important to note that in the Hasse diagram, relations implied by transitivity are not shown. For example, the relations $5 \leq_\mathcal{T} 14$ and $14 \leq_\mathcal{T} 20$ are shown by edges in the Hasse diagram, since $\mathcal{T}(5)$ divides $\mathcal{T}(14)$ and $\mathcal{T}(14)$ divides $\mathcal{T}(20)$. But even though $5 \leq_{\mathcal{T}} 20$, we don't have this edge in the Hasse diagram because this relation is already implied by the upwards-path of edges from 5 to 20.

From the Hasse diagram, we can visibly see that the poset $(\mathbb{N}, \leq_{\mathcal{T}})$ is more disordered than the usual divisibility poset, shown earlier in Figure 3. We note some patterns in Section C.

3.2. **Triangular numbers: zeta matrix and Möbius matrix.** Let $\mu_{\mathcal{T}}$ denote Möbius function for the triangular numbers under the divisibility relation, in the sense defined in Section 2.3. Namely, $\mu_{\mathcal{T}} : \mathbb{N} \to \mathbb{Z}$ is the unique function that satisfies

$$\mu_{\mathcal{T}}(1,1) = 1, \qquad \mu_{\mathcal{T}}(n) = -\sum_{\substack{d \leq_{\mathcal{T}} n \\ d \neq n}} \mu_{\mathcal{T}}(1,d) \quad \text{for all } n \geq 2. \tag{3}$$

As mentioned earlier in Section 2.4, the values of the Möbius function $\mu_{\mathcal{T}}$ can be computed by inverting the zeta matrix. Each row of the zeta matrix records which elements are greater than a given element, in the poset relation, and each column records which elements are smaller. For every relation $i \leq_{\mathcal{T}} j$, a 1 is inserted in the corresponding i-th row and j-th column of the zeta matrix.

The following shows the initial part of the zeta matrix:

$$Z = \begin{array}{c} 1 \\ 3 \\ 6 \\ 10 \\ 15 \\ 21 \\ 28 \\ 36 \\ 45 \\ 55 \end{array} \begin{bmatrix} 1 \\ 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Note that the first column is filled with 1's for every row.

The following shows the initial part of the Möbius matrix, for the poset $(\mathbb{N}, \leq_{\mathcal{T}})$:

$$M = \begin{array}{c} 1 \\ 3 \\ 6 \\ 10 \\ 15 \\ 21 \\ 28 \\ 36 \\ 45 \\ 55 \end{array} \begin{bmatrix} 1 \\ -1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

This Möbius matrix M is the inverse of the zeta matrix Z in (4).

4. Results: data on Möbius values

In this section, we report some empirical observations concerning the values of the Möbius function $\mu_{\mathcal{T}}$. The values are available on the Online Encyclopedia of Integer Sequences (OEIS) as sequence A350682 [9].

4.1. **Möbius values with $m = 1$.** In Figure 5, we show the Möbius values of the partial order $(\mathbb{N}, \leq_{\mathcal{T}})$. The values are highly erratic, rapidly switching between positive and negative values.

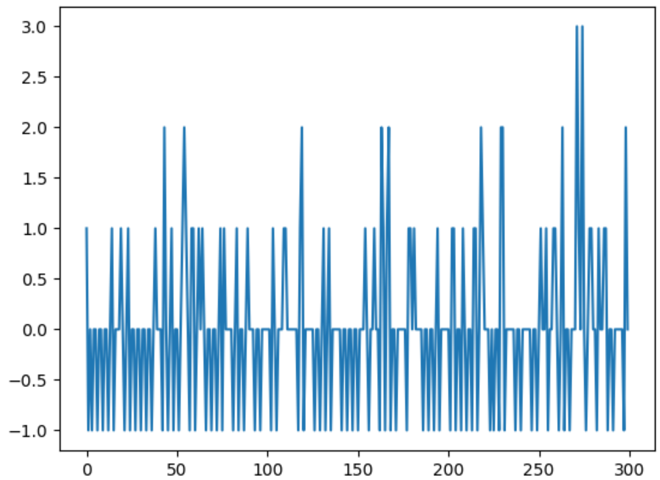

FIGURE 5. The Möbius values $\mu_{\mathcal{T}}(n)$ for $n \leq 300$.

4.2. **Data: Partial sums of Möbius values.** In this section we show data on the partial sums of the Möbius values $\mu_{\mathcal{T}}(n)$

$$\sum_{i=1}^{n} \mu_{\mathcal{T}}(i).$$

This sequence of partial sums is available at the OEIS entry A351167 [10]. Figure 6 shows a graph of these partial sums, for up to $n = 1,000$.

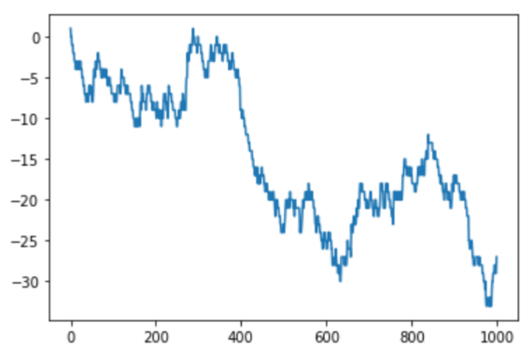

FIGURE 6. Partial sums of Möbius function values, for $n \leq 1000$.

Unlike Figure 5, Figure 6 shows a slight downward trend. The downward trend becomes much more apparent when the range is expanded to $n \leq 10,000$, as shown in Figure 1. This leads us to make the following conjecture.

Conjecture 1 (Growth of partial sums of $\mu_{\mathcal{T}}$)**.** *There is a positive constant C such that*

$$\sum_{i=1}^{n} \mu_{\mathcal{T}}(i) \leq -Cn \quad \textit{for all sufficiently large } n.$$

In other words, this conjecture states that the average value of the Möbius function $\mu_{\mathcal{T}}$ is eventually bounded above by $-C$, i.e.

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_{\mathcal{T}}(i) \leq -C.$$

4.3. **Data: Partial sums of Möbius value absolute values.** In Figure 7 we show the partial sums of the absolute values $|\mu_{\mathcal{T}}(n)|$. In this figure, the trend is even smoother than in Figure 6.

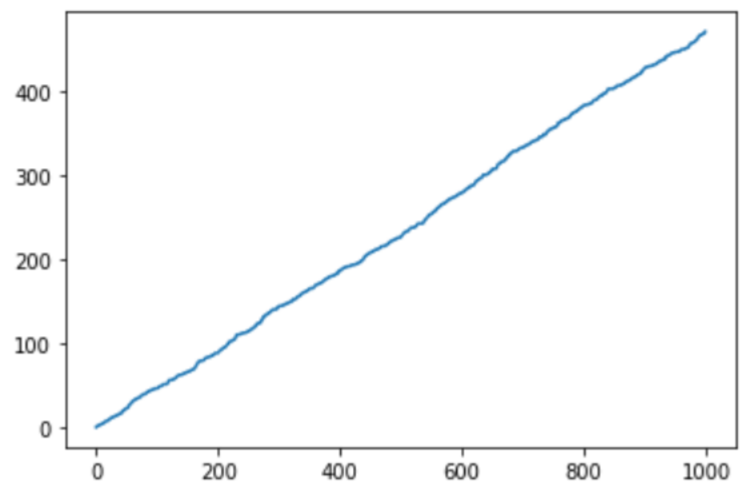

FIGURE 7. Partial sums of Möbius function absolute values $|\mu_{\mathcal{T}}(n)|$.

This data in Figure 7 leads us to make the following conjecture.

Conjecture 2 (Partial sums of $|\mu_{\mathcal{T}}|$)**.** *As $n \to \infty$,*

$$\sum_{i=1}^{n} |\mu_{\mathcal{T}}(i)| = \frac{1}{2}n + o(n).$$

4.3.1. *Average magnitude of Möbius values.* The conjecture can be rephrased in terms of the average magnitute of the the Möbius function values. Namely, that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\mu_{\mathcal{T}}(i)| = \frac{1}{2}.$$

4.4. **Möbius values with large magnitude.** We observe empirically that the values of the Möbius function $\mu_{\mathcal{T}}$ seem to achieve arbitrarily large magnitude. This is in contrast with the classical Möbius function μ, which only has values in $\{-1, 0, 1\}$.

This data leads us to make the following conjecture.

Conjecture 4. *For any positive integer M, there is a positive integer n such that $|\mu_{\mathcal{T}}(n)| \geq M$.*

4.5. **Two-variable Möbius values.** In Appendix A we show a heatmap illustrating the values of the two-variable Möbius function for $(\mathbb{N}, \leq_{\mathcal{T}})$. This should allow further explorations for patterns in the Möbius values in future work.

M	first n such that $\lvert\mu_{\mathcal{T}}(n)\rvert = M$
1	1
2	44
3	272
4	1274
5	2639
6	6720
7	3024
8	2079

TABLE 1. Inputs of the Möbius function $\mu_{\mathcal{T}}$ with increasing magnitude.

5. Further questions

- If Conjecture 1 holds, what are bounds on the constant C? Can the exact value of C be computed?
- Other statistics to measure on $\mu_{\mathcal{T}}$?
- Similar to Conjecture 5, is there an asymptotic relation of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mu_{\mathcal{T}}(i)}{\mathcal{T}(i)} \approx D, \tag{6}$$

where D is some constant?

Conjecture 5. *There is a positive constant E such that*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mu_{\mathcal{T}}(i)}{i} = -E. \tag{7}$$

This constant $-E$, can be seen to be around approximately -0.239:

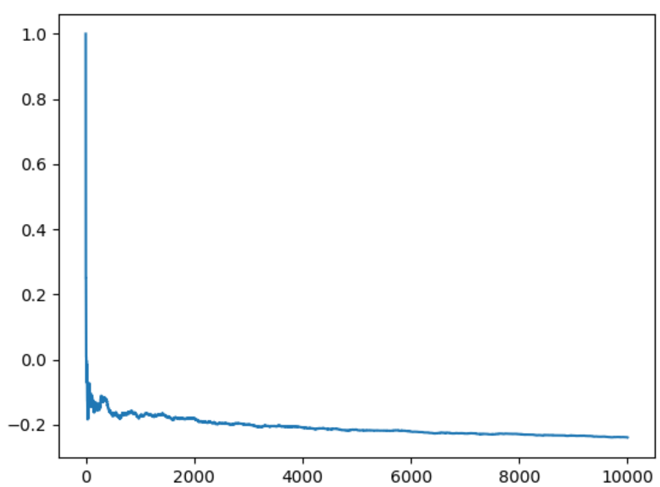

FIGURE 8. Partial sums of $\mu_{\mathcal{T}}(n)/n$ up to 10,000.

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Appendix A. Möbius matrices

In this appendix, we visualize the values of the two-variable Möbius function of $(\mathbb{N}, \leq_{\mathcal{T}})$. A heatmap showing the values of the Möbius matrix is given in Figure 9. Positive values are indicated by blue and negative values are indicated by red; zero values are light gray.

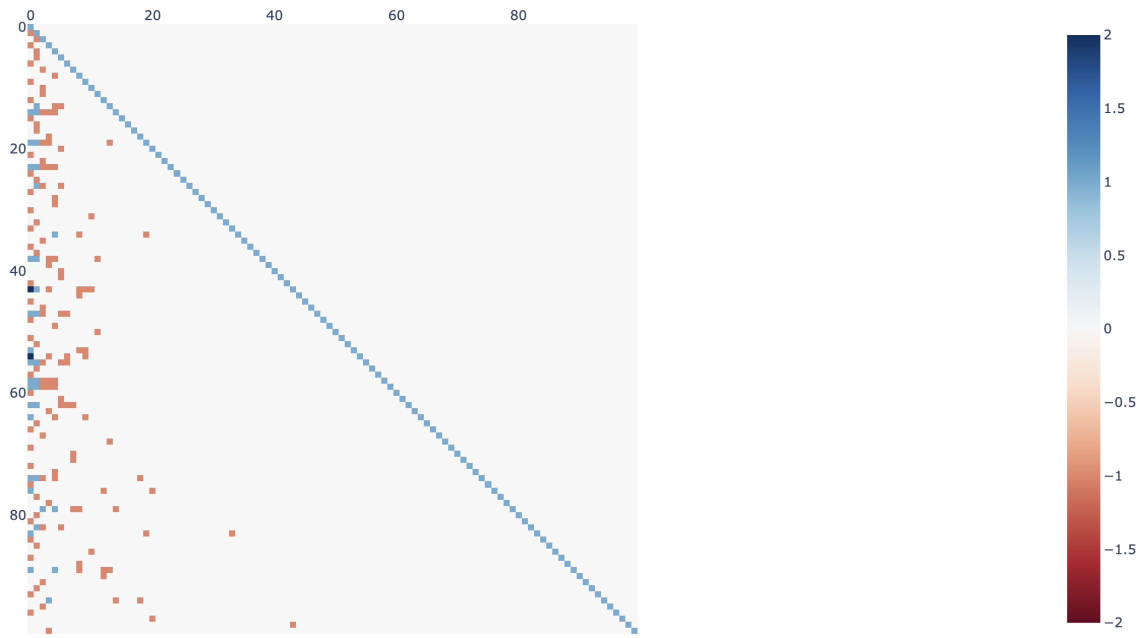

Figure 9. Two-variable Möbius function for $(\mathbb{N}, \leq_{\mathcal{T}})$ for $1 \leq m, n \leq 100$.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Figure 10. Möbius matrix of 20 Integers

For comparison, Figure 11 shows the values of the classical Möbius function.

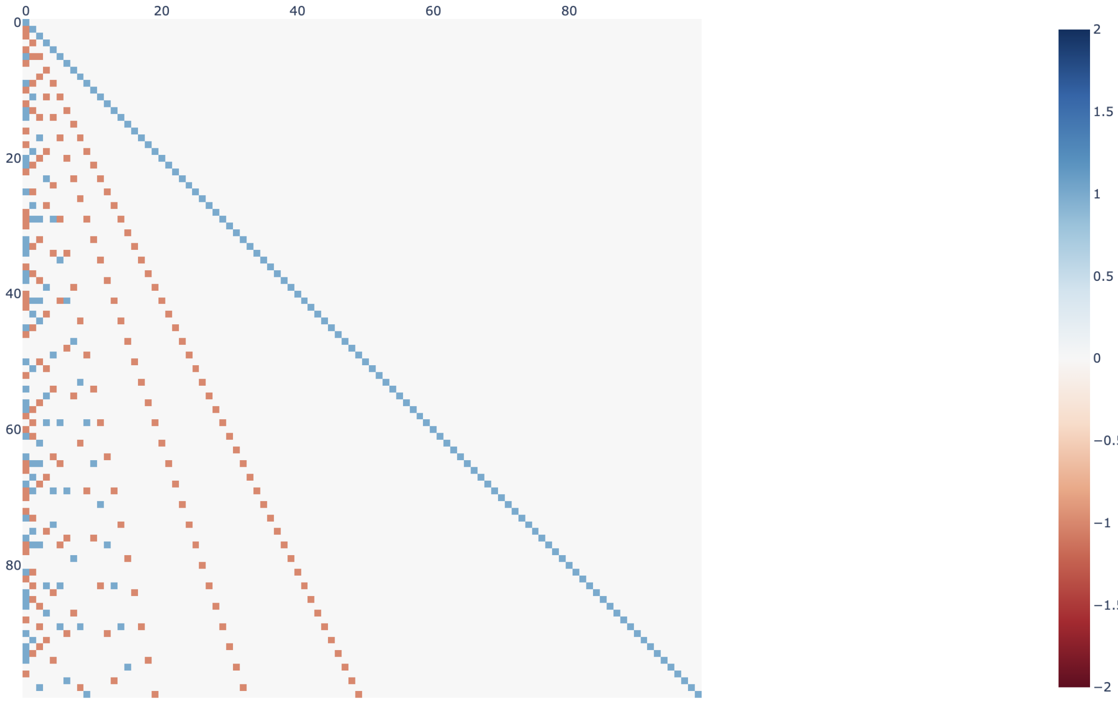

FIGURE 11. Classical two-variable Möbius function for $1 \leq m, n \leq 100$.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

FIGURE 12. Möbius matrix of 20 Triangular Numbers

APPENDIX B. CODE

In this section, we include the code in the Python programming language [13] which we wrote to generate values of the Möbius function $\mu_{\mathcal{T}}$. In addition to Python, we made use of the Matplotlib [4], Numpy [3], and Plotly [11] libraries.

The first is for generating the zeta matrix. Note: It was slightly faster and easier to generate matrices by dividing column by the rows, so at the end we transposed the matrix to get the desired result.

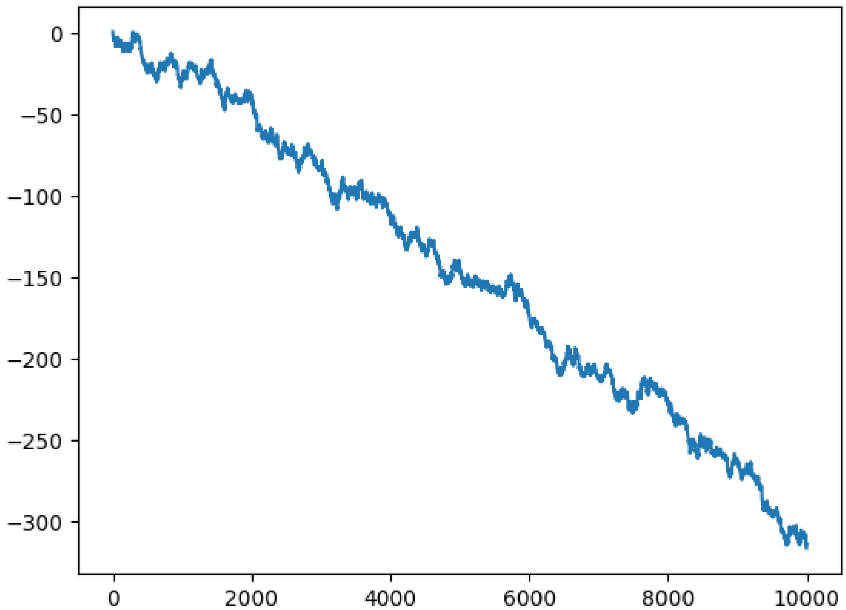

Figure 13. Sums of Möbius values $\mu_{\mathcal{T}}(n)$ up to 10,000.

```
from sympy import *
import matplotlib.pyplot as plt
import numpy as np

def triangular_numbers(n):
    # returns a list of triangular numbers
    return [(x * (x + 1) // 2) for x in range(1,n+1)]

def zeta_matrix(n):
    # returns zeta matrix for partial order on first n triangular
        numbers
    lst = triangular_numbers(n)
    zeta_array = [[0 if a % b != 0 else 1 for a in lst] for b in lst]
    Z = Matrix(zeta_array).T
    return Z

# to create zeta matrix for first 100 triangular numbers:
zeta_matrix(100)
```

The second is for generating the Möbius matrix.

```
def mobius_matrix(a):
    Z = Zeta_Matrix(a)
    M = Z ** -1
```

```
    return M

def plot_mobius_values(n):
    M = mobius_matrix(n)
    a = M[0, :n].tolist()
    plt.plot(a)
    plt.legend()
    plt.show()
```

The third is for generating the sum of the Möbius values:

```
number = int(input("Number: "))

M = mobius_matrix(triangular_numbers(number))
N = M[0, :].tolist()

def sum_function(lst):
    sum_list = [sum(lst[:i+1]) for i in range(len(lst))]
    return sum_list

S = sum_function(N[0])
plt.plot(S)
plt.show()
```

The fourth is for generating the absolute Möbius value sums.

```
number = int(input("Number: "))

M = mobius_matrix(triangular_numbers(number))
N = M[0, :].tolist()

S = sum_function(abs_value(N[0]))
plt.plot(S)
plt.show()
#slope approaching 0.5
```

The slope was found using the equation

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{8}$$

where the change in y was the difference between the last Möbius value and the first one, and the change in x was just the number input minus 0.

The fifth is for creating heatmaps for visualizing the Möbius matrix, which are generated using the Plotly package [11].

```
import plotly.express as px
from mobius_matrix import mobius_matrix, triangular_numbers

def plot_mobius_values():
    M = mobius_matrix(triangular_numbers())
    M = M.transpose()
    fig = px.imshow(M, color_continuous_scale='RdBu',
        color_continuous_midpoint=0.0)
```

```
    fig.update_layout(
    xaxis={'side': 'top'})
    fig.show()

plot_mobius_values()
```

The sixth is for the partial sums:

```
number = int(input("Number: "))

M = mobius_matrix(triangular_numbers(number))
N = M[0, :].tolist()

def partial_sums(lst):
    result = [sum(lst[:i+1]) for i in range(len(lst))]
    return result

n_divided = [N[i]/(i+1) for i in range(len(N))]

S = partial_sums(n_divided)

print(min(S))

plt.plot(S)
plt.show()
```

Appendix C. Triangular number divisibility patterns

In this section, we investigate some properties of when one triangular number divides another. The results of this section are independent of the rest of the paper.

Let $\mathcal{T}(n) = \frac{1}{2}n(n+1)$ denote the n-th triangular number. Recall that the partial order $(\mathbb{N}, \leq_{\mathcal{T}})$ records, for each pair of positive integers i, j, whether or not $\mathcal{T}(i) \mid \mathcal{T}(j)$ holds. The following statements may be useful for studying asymptotics of Möbius values $\mu_{\mathcal{T}}(m,n)$ as $n \to \infty$, for fixed $m \geq 2$.

Proposition 6. *For any n, $\mathcal{T}(n)$ divides $\mathcal{T}(n(n+1))$.*

Proof. We can calculate directly

$$\mathcal{T}(n(n+1)) = \frac{n(n+1)(n(n+1)+1)}{2} = \mathcal{T}(n)(n(n+1)+1).$$

Therefore the ratio $\mathcal{T}(n(n+1))/\mathcal{T}(n)$ simplifies to

$$\frac{\mathcal{T}(n(n+1))}{\mathcal{T}(n)} = n(n+1)+1$$

which is an integer for any n. □

Proposition 7. *For any n, $\mathcal{T}(n)$ divides $\mathcal{T}(\frac{1}{2}n(n+1))$ if and only if $n \equiv 1$ or 2 mod 4.*

Proof. We have

$$\mathcal{T}\left(\frac{n(n+1)}{2}\right) = \frac{1}{2}\left(\frac{n(n+1)}{2}\right) \cdot \left(\frac{n(n+1)}{2} + 1\right) = \mathcal{T}(n) \cdot \frac{n(n+1)+2}{4}.$$

So, for the proposition it suffices to prove that $\frac{1}{4}(n(n+1)+2)$ is an integer exactly when $n \equiv 1$ or 2 modulo 4. So, we have to find all n such that $n(n+1)+2 = n^2+n+2$ is congruent to 0 modulo 4, i.e.

$$n(n+1) \equiv 2 \pmod{4} \tag{9}$$

So, in the set of residues modulo 4, what values of n will satisfy equation (9)?

$$\begin{aligned}
n = 0 \quad &\Rightarrow \quad n(n+1) = 0 \not\equiv 2 \pmod{4} \\
n = 1 \quad &\Rightarrow \quad n(n+1) = 2 \equiv 2 \pmod{4} \\
n = 2 \quad &\Rightarrow \quad n(n+1) = 6 \equiv 2 \pmod{4} \\
n = 3 \quad &\Rightarrow \quad n(n+1) = 12 \not\equiv 2 \pmod{4}
\end{aligned}$$

Therefore, $n(n+1) \equiv 2 \mod 4$ if and only if $n \equiv 1$ or $2 \mod 4$, as desired. □