TUTTE POWER SERIES AND TUTTE EVALUATIONS ON THE MODULI SPACE OF METRIC GRAPHS

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ABSTRACT. We define a way to assign a formal power series to a metric graph with arbitrary (positive) real edge lengths. When all edge lengths are positive integers, this recovers the usual Tutte polynomial. We prove that for positive inputs, evaluation of the Tutte power series defines a continuous function of the moduli space of metric graphs, which also extends to the compactification by tropical curves. We study how Tutte power series evaluations are related to various structures on a metric graph.

1. INTRODUCTION

Given a graph G, the Tutte polynomial $T(G; x, y)$ is a two-variable polynomial introduced by Tutte in [13]. Many important graph invariants arise as evaluations of the Tutte polynomial at specific (real) parameters x, y. For a comprehensive modern overview of the Tutte polynomial see [2] [6].

The following characterization of the Tutte polynomial was initially introduced by Crapo [5], using the *rank generating function* of G (see also: [6, Definition 3]). Given a connected graph G, the Tutte polynomial of G is

eq:tutte-graph

$$T(G; x, y) = \sum_{A \subset E(G)} (x-1)^{h_0(G \setminus A) - 1} (y-1)^{h_1(G \setminus A)} \tag{1}$$

where $G \setminus A$ denotes the graph with edges in A deleted, and h_0 and h_1 denote the zeroth and first Betti numbers of a topological space. In graph theoretic terms,

$$h_0(G) = \#(\text{connected components of } G), \quad \text{and}$$
$$h_1(G) = \#E(G) - \#V(G) + h_0(G).$$

The purpose of this paper is to explain that this definition of the Tutte polynomial may be extended meaningfully to a metric graph. As a consequence, evaluation of the Tutte polynomial (for certain real inputs) extends to a continuous function on the moduli space of metric graphs.

1.1. **Statement of results.** Suppose Γ is a metric graph with combinatorial model $\Gamma = (G, \ell)$, where $\ell : E(G) \to \mathbb{R}_{>0}$ is a function assigning a length to each edge of G. Given real parameters x, y with $x > 0$, let

eq:tutte-metric-graph

$$T(\Gamma; x, y) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_x \right) (x-1)^{h_0(G \setminus A) - 1} (y-1)^{h_1(G \setminus A)} \tag{2}$$

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where the notation $[\alpha]_x$ for real $\alpha, x > 0$ means

$$[\alpha]_x = \frac{x^\alpha - 1}{x - 1} \quad \text{if } x \neq 1, \qquad [\alpha]_1 = \alpha.$$

For a fixed metric graph Γ, the expression (eq:tutte-metric-graph 2) defines a function $\mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ by associating $(x, y) \mapsto T(\Gamma; x, y)$. This function is generally not a polynomial in x; moreover, it does not admit a formal power series expansion in x if some edge length $\ell(e_i)$ is non-integral.

We can recover a power series expression for $T(\Gamma; x, y)$ by a simple change of variables. Let $T^+(\Gamma; u, w) := T(\Gamma; 1 + u, 1 + w)$, so that

eq:tutte-power-series

$$T^+(\Gamma; u, w) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_{1+u} \right) u^{h_0(G \backslash A) - 1} w^{h_1(G \backslash A)}. \tag{3}$$

Note that

$$[\alpha]_{1+u} = \frac{(1 + u)^\alpha - 1}{u} = \sum_{k \geq 1} \binom{\alpha}{k} u^{k-1} \qquad \text{for } |u| < 1$$

has a power series expansion in u

$$[\alpha]_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \in \mathbb{R}[[u]]. \tag{4}$$

♢ use different notation to distinguish $[a]_{1+u}$ as a power series in variable u vs $[a]_{1+u}$ as the real-valued expression $1/u((1 + u)^a - 1)$ ♢

Our first main result is that the expression (eq:tutte-power-series 3) does not depend on which model (G, ℓ) we choose for the metric graph Γ.

thm:intro-tutte-series

Theorem 1 (Tutte power series)**.** *Given a metric graph $\Gamma = (G, \ell)$, the expression $T^+(\Gamma; u, w)$ is a well-defined power series in $\mathbb{R}[[u]][w]$; in particular, $T^+(\Gamma; u, w)$ does not depend on the choice of model (G, ℓ) for Γ.*

thm:deletion-contraction

Theorem 2 (Deletion-contraction relation)**.** *Given a metric graph $\Gamma = (G, \ell)$ and an edge $e \in E(G)$, which is neither a bridge nor a loop, the Tutte power series satisfies*

$$T^+(\Gamma; u, w) = [\ell(e)]_{1+u} T^+(\Gamma \backslash e; u, w) + T^+(\Gamma / e; u, w).$$

Our next result concerns the convergence of the Tutte powers series of a fixed metric graph Γ.

thm:tutte-convergence

Theorem 3 (Tutte convergence)**.** *Given a metric graph Γ, the Tutte power series $T^+(\Gamma; u, w)$ converges when $|u| < 1$. If some edge length $\ell(e)$ is not an integer* ♢ in a minimal model ♢ *, then the radius of convergence in u is equal to 1.*

Instead of fixing a metric graph Γ and varying the parameters u, w, we can instead fix a choice of (u, w) and vary the metric graph. As Γ varies, the value of $T^+(\Gamma; u, w)$ also varies continuously.

Let $\mathcal{M}_g$ denote the moduli space of metric graphs of genus g.

thm:tutte-eval-moduli

Theorem 4 (Continuity of Tutte evaluation)**.** *Let u and w be fixed (nonnegative?) real numbers, with $u > -1$. The Tutte evaluation at (u, w),*

$$\mathrm{ev}^+(u, w) : \Gamma \mapsto T^+(\Gamma; u, w),$$

defines a continuous function $\mathrm{ev}^+(u, w) : \mathcal{M}_g \to \mathbb{R}$.

Namely,

(1) *(Continuity on cells) for each combinatorial graph G, the Tutte evaluation* $\mathrm{ev}(u, w)$ *restricts to a continuous function*

$$\mathrm{ev}(u, w) : \mathbb{R}_{>0}^{E(G)} \to \mathbb{R},$$

where a point in the domain $\in \mathbb{R}_{>0}^{E(G)}$ *represents a choice of (positive, real) edge lengths* $\ell : E(G) \to \mathbb{R}_{>0}$.

(2) *(Continuity between cells) If $u > -1$, then as the length of a non-loop edge $e \in E(G)$ approaches zero in the metric graph $\Gamma = (G, \ell)$ while other edge lengths are fixed, the value of* $\mathrm{ev}(u, w)$ *at (G, ℓ) approaches the value of* $\mathrm{ev}(u, w)$ *at the contraction* $\Gamma/e = (G/e, \ell\big|_{E\setminus e})$.

Example 5 ($x = 1$, $y = 1$)**.** The Tutte evaluation $\mathrm{ev}(1, 1)$ on a graph G gives the number of spanning trees. On a metric graph, $\mathrm{ev}(1, 1)$ gives the volume of the Jacobian of $\Gamma = (G, \ell)$, which can be expressed as a weighted sum of spanning trees of G ♢ [cite a reference] ♢ . The function $\mathrm{ev}(1, 1)$ is continuous on $\mathcal{M}_g$, and extends continuously to $\overline{\mathcal{M}}_g$ (where it has value zero on the boundary).

Example 6 ($x = 0$, $y = 2$)**.** The Tutte evaluation $\mathrm{ev}(0, 2)$ on $\Gamma = (G, \ell)$ gives the number of totally cyclic orientations of Γ. This number does not depend on the edge lengths of Γ; i.e. $\mathrm{ev}(0, 2)$ is constant on metric graphs of a fixed combinatorial model G. However, $\mathrm{ev}(0, 2)$ is not continuous as some edge length approaches 0. Namely, the value of $T(G; 0, 2)$ generally differs from the value of $T(G/e; 0, 2)$ on the contraction.

♢ Maybe remove this theorem ♢

Theorem 7 (Continuity of Tutte coefficient)**.** *For fixed indices $i, j \geq 0$, let* $\mathrm{coeff}(i, j; \Gamma)$ *denote the coefficient of $u^i w^j$ in the power series expansion of $T^+(\Gamma; u, w)$. Then the function* $\mathrm{coeff}(i, j)$ *defines a continuous(?) function* $\mathcal{M}_g \to \mathbb{R}$. ♢ extra assumptions needed? ♢

1.2. **Previous work.** Several authors have investigated the behavior of the Tutte polynomial under the operation of subdividing an edge into multiple edges.

Read and Whitehead RW2 [8]

Brylawski Bry [3]

Tra1,Tra2,Tra3 [10, 11, 12]

RW1,RW2 [7, 8]

Multivariate Tutte polynomial Sok-potts [9] also known as the *Potts-model partition function.* Sokal Sok-potts [9] asks:

> Let me conclude by observing that numerous specific evaluations of the Tutte polynomial have been given combinatorial interpretations, as counting some set of objects associated to the graphs G. It would be an interesting project to seek to extend these counting problems to "counting with weights," i.e., to obtain suitably defined univariate or multivariate generating polynomials for the objects in question as specializations of $Z_G(q, v)$ or $Z_G(q, \mathbf{v})$, respectively.

Zeros of Tutte polynomials?

1.3. **Notation.** Γ a compact metric graph

G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G
(G, ℓ) a combinatorial model for a metric graph, where
$\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on edges of G
$T(G; x, y)$ the Tutte polynomial of G
$T^+(G; u, w) = T(G; 1 + u, 1 + w)$ "additive" centered Tutte polynomial
$T^+(\Gamma; u, w)$ the Tutte power series of Γ

2. Background

2.1. ***q*-analogs.** For a positive integer ℓ, the *q-analog* $[\ell]_q$ is defined as the polynomial

$$[\ell]_q = \frac{q^\ell - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{\ell-1} \in \mathbb{Z}[q].$$

When ℓ is not an integer, $[\ell]_q$ does not admit a Laurent expansion in the variable q. However, we can obtain a well-defined power series under a change of variable. Namely, note that

$$[\alpha]_{1+u} = \frac{(1 + u)^\alpha - 1}{u} = \sum_{k\geq 0} \binom{\alpha}{k+1} u^k$$

so we have

$$[\alpha]_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \in \mathbb{R}[[u]]. \tag{5}$$

Here α can be any real number, and

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1)\cdots(\alpha - k + 1)}{k!} \tag{6}$$

$[[\alpha]]_q$ $[\alpha]_q$ $[\![\alpha]\!]_q$

Proposition 8.

(1) For positive integers n, m we have

$$[n + m]_q = [n]_q + [m]_q + (q - 1)[n]_q[m]_q.$$

(2) For positive real numbers α, β, we have

$$[\alpha + \beta]_{1+u} = [\alpha]_{1+u} + [\beta]_{1+u} + u[\alpha]_{1+u}[\beta]_{1+u}.$$

Proof. Observe that

$$\begin{aligned}
[\alpha + \beta]_{1+u} &= \frac{(1 + u)^{\alpha+\beta} - 1}{u} \\
&= \frac{(1 + u)^{\alpha+\beta} - (1 + u)^\alpha}{u} + \frac{(1 + u)^\alpha - 1}{u} \\
&= (1 + u)^\alpha[\beta]_{1+u} + [\alpha]_{1+u} \\
&= \left(u\frac{(1 + u)^\alpha - 1}{u} + 1\right)[\beta]_{1+u} + [\alpha]_{1+u} \\
&= u[\alpha]_{1+u}[\beta]_{1+u} + [\beta]_{1+u} + [\alpha]_{1+u}.
\end{aligned}$$

∎

Note that the q-analog satisfies the following properties

(1) If $q_0 > 0$ is fixed and $q_0 \neq 1$, the map

$$\ell \mapsto [\ell]_{q_0} = \frac{q_0^\ell - 1}{q_0 - 1}$$

defines a continuous function from $\mathbb{R}$ to $\mathbb{R}$, which sends $1 \mapsto 1$ and $0 \mapsto 0$.

If $q_0 = 1$, we use the convention that $[\ell]_1 = \ell$.

If $q_0 = 0$, we have $[\ell]_0 = 1$ for any $\ell > 0$.

(2) If $\ell_0 \geq 0$ is fixed and $q > 0$, the map

$$q \mapsto [\ell_0]_q = \frac{q^{\ell_0} - 1}{q - 1}$$

defines a continuous function from $\mathbb{R}_{>0} \setminus \{1\}$ to $\mathbb{R}$, which satsifies

$$\lim_{q \to 0^+} [\ell_0]_q = \lim_{q \to 0^+} \frac{q^{\ell_0} - 1}{q - 1} = \begin{cases} 1 & \text{if } \ell_0 > 0 \\ 0 & \text{if } \ell_0 = 0 \\ -\infty & \text{if } \ell_0 < 0. \end{cases}$$

and has a continuous extension to $\mathbb{R}_{>0} \to \mathbb{R}$ that sends $1 \mapsto \ell_0$.

(3) In particular, for $\ell, q > 0$ we have

$$[\ell]_0 = 1 \qquad \text{and} \qquad [0]_q = 0.$$
$$\lim_{\ell \to 0} [\ell]_0 = 1 \qquad \text{and} \qquad \lim_{q \to 0} [0]_q = 0.$$

Considering $[\alpha]_{1+u}$ as a power series in u and α:

$$\begin{aligned} [\alpha]_{1+u} &= \sum_{k \geq 0} \binom{\alpha}{k+1} u^k = \sum_{k \geq 0} \frac{1}{(k+1)!} \alpha^{\underline{k+1}} u^k \\ &= \sum_{k \geq 0} \left(\sum_{j \geq 0} \frac{s(k+1, j)}{(k+1)!} \alpha^j \right) u^k \\ &= (\alpha) + (-\frac{1}{2}\alpha + \frac{1}{2}\alpha^2) u + (\frac{1}{3}\alpha - \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha^3) u^2 + \cdots \end{aligned}$$

where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ denotes the falling factorial and $s(k, j)$ denotes the Stirling number of the first kind.

$$\begin{aligned} [\alpha]_{1+u} &= \frac{(1+u)^\alpha - 1}{u} = \frac{\exp(\alpha \log(1+u)) - 1}{u} \\ &= \frac{1}{u} \left(-1 + \sum_{j \geq 0} \frac{\log(1+u)^j}{j!} \alpha^j \right) \\ &= \sum_{j \geq 1} \frac{\log(1+u)^j}{j!\, u} \alpha^j. \end{aligned}$$

$$T(\Gamma; 1+u, 1+w) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_{1+u} \right) u^{h_0(G \setminus A) - 1} w^{h_1(G \setminus A)}$$

For a fixed metric graph $\Gamma = (G, \ell)$, the Tutte power series is an element of $\mathbb{R}[[u]][w]$.

This function satisfies the deletion-contraction relation

$$T(\Gamma; x, y) = [\ell(e)]_x \cdot T(\Gamma \setminus e; x, y) + T(\Gamma / e; x, y).$$

$$T^+(\Gamma; u, w) = [\ell(e)]_{1+u} \cdot T^+(\Gamma \backslash e; u, w) + T^+(\Gamma / e; u, w).$$

2.2. Graph theory.

$$h_0(G|A) - 1 = rk(G) - rk(A), \quad \text{and}$$
$$h_1(G|A) = \#(A) - rk(A).$$

2.3. Tutte polynomial.

3. Metric graphs

A metric graph is a compact, connected metric space which comes from assigning edge lengths to a finite, connected graph. If the metric graph Γ comes from a combinatorial graph G by assigning edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, we say (G, ℓ) is a *combinatorial model* for Γ and we write $\Gamma = (G, \ell)$.

3.1. Moduli spaces of metric graphs.

See Melody Chan [4]. [1]

♢ is it natural here to restrict to stable graphs, or have infinitely many graphs per genus? ♢

3.2. Tropical curves.

Here we use "tropical curve" to refer to a metric graph which possibly has contracted loops, which we think of as "infinitesimally small" loops attached to a vertex. We record the number of

4. Tutte power series

The polynomial $T(G; x, y)$ can also be defined inductively by the deletion-contraction relation:

$$T(G; x, y) = T(G \backslash e; x, y) + T(G/e; x, y).$$

along with the base cases for a loop edge and bridge edge.

$$T(G; x, y) = x^i y^j \qquad \text{if } G \text{ consists of } i \text{ bridges and } j \text{ loops.}$$

$$T(G; x, y) = \begin{cases} x & \text{if } G \text{ is a bridge} \\ y & \text{if } G \text{ is a loop.} \end{cases}$$

The Tutte power series $T^+(\Gamma; u, w)$ satsifies a similar deletion-contraction relation:

$$T^+(\Gamma; u, w) = [\ell(e)]_{1+u} T^+(\Gamma \backslash e; u, w) + T^+(\Gamma/e; u, w). \tag{7}$$

Check compatibility:

$$\begin{aligned} T(\text{bridge}; x, y) &= (x-1)^{\tilde{h}_0(\text{bridge})}(y-1)^{h_1(\text{bridge})} + (x-1)^{\tilde{h}_0(\text{two pts.})}(y-1)^{h_1(\text{two pts.})} \\ &= 1 + (x - 1) = x; \end{aligned}$$

$$\begin{aligned} T(\text{loop}; x, y) &= (x-1)^{\tilde{h}_0(\text{loop})}(y-1)^{h_1(\text{loop})} + (x-1)^{\tilde{h}_0(\text{one pt.})}(y-1)^{h_1(\text{one pt.})} \\ &= (y - 1) + 1 = y. \end{aligned}$$

Example 9 (Tutte power series of a line)**.** Suppose Γ is a segment of length α, then

$$T^+(\Gamma; u, w) = [\alpha]_{1+u} u + 1 = (1 + u)^\alpha$$

The power series expansion is

$$T^+(\Gamma; u, w) = \sum_{k=0}^{\infty} \binom{\alpha}{k} u^k = 1 + \alpha u + \binom{\alpha}{2} u^2 + \cdots.$$

If G is a line graph consisting of n edges, then

$$T(G;x,y)=x^n \qquad \text{and} \qquad T^+(G;u,w)=(1+u)^n=1+nu+\binom{n}{2}u^2+\cdots+u^n.$$

Example 10 (Tutte power series of a circle)**.** If Γ is a circle of length λ, then

$$T^+(\Gamma;u,w)=[\lambda]_{1+u}+w=\frac{(1+u)^\lambda-1}{u}+w$$

which has power series expansion

$$T^+(\Gamma;u,w)=\sum_{k=0}^{\infty}\binom{\lambda}{k+1}u^k+w$$

Suppose G is a cycle graph consisting of n edges. Then

$$T(G;x,y)=x+x^2+\cdots+x^{n-2}+x^{n-1}+y=\frac{x^n-1}{x-1}+y-1.$$

and

$$T^+(G;u,w)=n+\binom{n}{2}u+\binom{n}{3}u^2+\cdots+nu^{n-2}+u^{n-1}+w=\frac{(1+u)^n-1}{u}+w$$

Example 11 (Tutte power series of theta graph)**.** Suppose G is the graph with two vertices connected by three edges. Suppose Γ is the metric graph which assigns lengths a,b,c to the edges of G. Then

$$\begin{aligned}T^+(G;u,w)=&([a]_{1+u}[b]_{1+u}+[a]_{1+u}[c]_{1+u}+[b]_{1+u}[c]_{1+u})\\&+([a]_{1+u}[b]_{1+u}[c]_{1+u})u+([a]_{1+u}+[b]_{1+u}+[c]_{1+u})w+w^2.\end{aligned}$$

Using deletion-contraction, we also deduce

$$\begin{aligned}T^+(G;u,w)&=[a]_{1+u}T^+(\text{circle of length } b+c)\\&\quad+T^+(\text{circle of length } b)T^+(\text{circle of length } c)\\&=[a]_{1+u}([b+c]_{1+u}+w)+([b]_{1+u}+w)([c]_{1+u}+w)\\&=([a]_{1+u}[b+c]_{1+u}+[b]_{1+u}[c]_{1+u})+([a]_{1+u}+[b]_{1+u}+[c]_{1+u})w+w^2.\end{aligned}$$

Note that at $w=0$, we have

$$\begin{aligned}T^+(G;u,0)&=[a]_{1+u}[b+c]_{1+u}+[b]_{1+u}[c]_{1+u}\\&=\sum_{k\geq0}\binom{a}{k+1}u^k\sum_{k\geq0}\binom{b+c}{k+1}u^k+\sum_{k\geq0}\binom{b}{k+1}u^k\sum_{k\geq0}\binom{c}{k+1}u^k\\&=\left(a+\binom{a}{2}u+\cdots\right)\left(b+c+\binom{b+c}{2}u+\cdots\right)+\left(b+\binom{b}{2}u+\cdots\right)\left(c+\binom{c}{2}u+\cdots\right)\\&=(ab+ac+bc)+\left(a\binom{b+c}{2}+\binom{a}{2}(b+c)+b\binom{c}{2}+\binom{b}{2}c\right)u+()u^2+\cdots\\&=(ab+ac+bc)+\frac{1}{2}\left(a(b+c)(a+b+c-2)+bc(b+c-2)\right)u+\cdots\\&=(ab+ac+bc)+\frac{1}{2}\left(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2+2abc-2ab-2ac-2bc\right)u+\cdots\\&=(ab+ac+bc)+\left(abc+\frac{1}{2}(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)-ab-ac-bc\right)u+\cdots\end{aligned}$$

$$
\begin{aligned}
T^+(G;u,0) &= [a]_{1+u}[b+c]_{1+u} + [b]_{1+u}[c]_{1+u} \\
&= [a]_{1+u}\left([b]_{1+u} + [c]_{1+u} + u[b]_{1+u}[c]_{1+u}\right) + [b]_{1+u}[c]_{1+u} \\
&= [a][b] + [a][c] + [b][c] + u[a][b][c].
\end{aligned}
$$

Example 12 (Tutte power series of K_4)**.** Suppose $G = K_4$, the complete graph on four vertices. Suppose Γ is the metric graph assigning edge lengths a, b, c, d, e, f to G, as shown in Figure ♢ [fill in] ♢ .

Then we have

$$
\begin{aligned}
T^+(\Gamma;u,w) = {} & ([a][b][d] + [a][b][e] + [a][b][f] + [a][c][d] + [a][c][e] + [a][c][f] + [a][d][e] + [a][d][f] \\
& \quad + [b][c][d] + [b][c][e] + [b][c][f] + [b][d][e] + [b][e][f] + [c][d][f] + [c][e][f] + [d][e][f]) \\
& + ([a][b][c][d] + [a][b][c][e] + [a][b][c][f] + [a][b][d][e] + [a][b][d][f] + [a][b][e][f] \\
& \quad + [a][c][d][e] + [a][c][d][f] + [a][c][e][f] + [a][d][e][f] + [b][c][d][e] \\
& \quad + [b][c][d][f] + [b][c][e][f] + [b][d][e][f] + [c][d][e][f])u \\
& + ([a][b][c][d][e] + [a][b][c][d][f] + [a][b][c][e][f] + [a][b][d][e][f] \\
& \quad + [a][c][d][e][f] + [b][c][d][e][f])u^2 \\
& + [a][b][c][d][e][f]u^3 \\
& + ([a][b] + [a][c] + [a][d] + [a][e] + [a][f] + [b][c] + [b][d] + [b][e] \\
& \quad + [b][f] + [c][d] + [c][e] + [c][f] + [d][e] + [d][f] + [e][f])w \\
& + ([a][b][c] + [a][e][f] + [b][d][f] + [c][d][e])uw \\
& + ([a] + [b] + [c] + [d] + [e] + [f])w^2 + w^3
\end{aligned}
$$

Compare to the Example in Read–Whitehead RW2 [8, p. 272].

4.1. **Deleting bridges and contracting loops.**

In this section we describe how the definition of Tutte power series $T^+(\Gamma;u,w)$ may be extended to a more general concept of metric graphs.

Definition 13. A *genus-weighted metric graph* $\Gamma = (G, \ell, wt)$ consists of a graph $G = (V,E)$, a length function $\ell : E \to \mathbb{R}_{>0}$, and a genus function $wt : V \to \mathbb{Z}_{\geq 0}$.

- If $\Gamma = \bigcup_{i=1}^k \Gamma_i$ is a disjoint union of k connected metric graphs Γ_i, then

$$
\begin{aligned}
T^+\Big(\bigcup_{i=1}^k \Gamma_i; u, w\Big) &= u^{k-1}\, T^+\Big(\bigvee_{i=1}^k \Gamma_i; u, w\Big) \\
&= u^{k-1} \prod_{i=1}^k T^+(\Gamma_i; u, w).
\end{aligned}
$$

- If $\Gamma^{wt} = (G, \ell, wt)$ is a genus-weighted metric graph, with underlying metric graph $\Gamma^0 = (G, \ell)$, then

$$
T^+(\Gamma^{wt}; u, w) = w^{wt(G)}\, T^+(\Gamma^0; u, w).
$$

4.2. **Proofs.**

Proof of Theorem thm:intro-tutte-series *1.* It sufficies to show that the Tutte power series is invariant under an edge subdivision. Suppose $G = (V,E)$ contains the edge e_0, which we subdivide into $e_1 \cup e_2$ to obtain the graph G'.

By the deletion-contraction relation ♢ cite ♢ we have

$$T^+(G;u,w) = [a+b]_{1+u}T^+(G\backslash e_0;u,w) + T^+(G/e_0;u,w)$$

and

$$\begin{aligned} T^+(G';u,w) &= [a]_{1+u}T^+(G\backslash e_1;u,w) + T^+(G/e_1;u,w) \\ &= [a]_{1+u}\left(T^+(G\backslash\{e_1,e_2\};u,w) + T^+(G\backslash e_1/e_2;u,w)\right) \\ &\quad + \left([b]_{1+u}T^+(G/e_1\backslash e_2;u,w) + T^+(G/\{e_1,e_2\};u,w)\right) \\ &= ([a]_{1+u} + [b]_{1+u} + u[a]_{1+u}[b]_{1+u})T^+(G\backslash e_0;u,w) + T^+(G/e;u,w). \end{aligned}$$

Therefore. ...

$$[a]_{1+u} + [b]_{1+u} + u[a]_{1+u}[b]_{1+u}$$

Then

$$\begin{aligned} T^+(G';u,w) &= \sum_{A\subset E(G')}\left(\prod_{e\in A}[\ell(e)]_{1+u}\right)u^{h_0(G\backslash A)-1}w^{h_1(G\backslash A)} \\ &= \sum_{A\subset E(G')\backslash\{e_1,e_2\}}\ \sum_{B\subset\{e_1,e_2\}} \end{aligned}$$

Note that

$$G'\setminus(A\cup\{e_1\}) \simeq G'\setminus(A\cup\{e_2\}) \simeq G\setminus(A\cup\{e_0\})$$

while

$$G'\setminus(A\cup\{e_1,e_2\}) \simeq G\setminus(A\cup\{e_0\})$$

□

5. Specializations of the Tutte polynomial

5.1. Constants. For a graph $G=(V,E)$,

- $T^+(G;1,1)$ = the number of subsets of edges; $T_G(2,2) = 2^{\#E}$.
- $T^+(G;0,0)$ = the number of spanning trees.
- $T^+(G;0,1)$ = the number of spanning subsets of edges.
- $T^+(G;1,0)$ = the number of acyclic subsets of edges.
- $T^+(G;-1,1)$ = the number of totally cyclic orientations.
- $T^+(G;1,-1)$ = the number of acyclic orientations.

For a metric graph $\Gamma=(G,\ell)$,

$$\begin{aligned} T^+(\Gamma;1,1) &= \sum_{A\subset E(G)}\prod_{e_i\in A}[\ell(e_i)]_2 = \sum_{A\subset E(G)}\prod_{e_i\in A}(2^{\ell(e_i)}-1). \\ &= \prod_{e_i\in E(G)}(1+(2^{\ell(e_i)}-1)) = 2^{\sum_i \ell(e_i)} \end{aligned}$$

- $T^+(\Gamma;1,1) = 2^{\mathrm{vol}(\Gamma)}$
- $T^+(\Gamma;0,0) = \mathrm{vol}(\mathrm{Jac}(\Gamma))$
- $T^+(\Gamma;0,1) = \sum_{k=0}^{g}\mathrm{vol}(\mathrm{Eff}^k(\Gamma))$?

Example 14. Suppose Γ is the theta graph with edge lengths a, b and c,

$$\mathrm{eval}(\Gamma;2,2) = 2^{a+b+c}.$$

$T(\Gamma;1,1) = ab+ac+bc.$
$T(\Gamma;1,2) = 1+(a+b+c)+(ab+ac+bc).$

$T(\Gamma;2,1) = (2^{a+b} - 2\cdot 2^a + 3) + (2^{a+b+c} - 2^{a+b} + 2^a - 1) = 2^{a+b+c} - 2^a - 2^b - 2^c + 2.$
$T(\Gamma;0,2) = 1 + 3 + 3 - 1 = 6.$
$T(\Gamma;2,0) = 2^{a+b+c} - 2(2^a + 2^b + 2^c) + 6.$

Example 15. For the theta graph, we have

$$\begin{aligned} T^+(\Gamma;u,w) = w^2 &+ ([a]_{1+u} + [b]_{1+u} + [c]_{1+u})w \\ &+ ([a]_{1+u}[b]_{1+u} + [a]_{1+u}[c]_{1+u} + [b]_{1+u}[c]_{1+u}) \\ &+ ([a]_{1+u}[b]_{1+u}[c]_{1+u})u \end{aligned}$$

$$\mathrm{coeff}(0,0;\Gamma) = ab + ac + bc.$$
$$\mathrm{coeff}(0,1;\Gamma) = a + b + c.$$
$$\mathrm{coeff}(0,k;\Gamma) =$$
$$\mathrm{coeff}(k,1;\Gamma) = \binom{a}{k+1} + \binom{b}{k+1} + \binom{c}{k+1}.$$
$$\mathrm{coeff}(0,2;\Gamma) = 1.$$

5.2. **Chromatic polynomial.** At $y = 0$ we obtain the chromatic polynomial of a graph:

$$\chi(G;\lambda) = (-1)^{\#V}(-\lambda)^{h_0(G)}T(G;1-\lambda,0)$$

5.3. **Flow polynomial.** At $x = 0$ we obtain the flow polynomial of a graph:

$$F(G;\lambda) = (-1)^{h_1(G)}T(G;0,1-\lambda)$$

5.4. **Reliability polynomial.** The reliability polynomial of a graph satisfies

$$R(G;p) = (1-p)^{\#V - h_0(G)}p^{h_1(G)}T(G;1,\frac{1}{p})$$

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