TUTTE POWER SERIES AND TUTTE EVALUATIONS ON THE MODULI SPACE OF METRIC GRAPHS

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ABSTRACT. We define a way to assign a formal power series to a metric graph with arbitrary (positive) real edge lengths. When all edge lengths are positive integers, this recovers the usual Tutte polynomial. We prove that for positive inputs, evaluation of the Tutte power series defines a continuous function of the moduli space of metric graphs, which also extends to the compactification by tropical curves. We study how this metric Tutte polynomial is related to various structures on a metric graph.

1. INTRODUCTION

Given a graph G, the Tutte polynomial $T(G; x, y)$ is a two-variable polynomial introduced by Tutte in [8]. Many important graph invariants arise as evaluations of the Tutte polynomial at specific (real) parameters x, y. For a comprehensive modern overview of the Tutte polynomial see [2].

Given a connected graph G, the Tutte polynomial of G is

$$T(G; x, y) = \sum_{A \subset E(G)} (x-1)^{h_0(G \backslash A)-1} (y-1)^{h_1(G \backslash A)} \tag{1}$$

where $G \setminus A$ denotes the graph with edges in A deleted, and h_0 and h_1 denote the zeroth and first Betti numbers of a topological space. In graph theoretic terms,

$$h_0(G) = \#(\text{connected components of } G), \quad \text{and}$$
$$h_1(G) = \#E(G) - \#V(G) + h_0(G).$$

$\diamondsuit$ attribute to Crapo [1]. $\diamondsuit$ The purpose of this paper is to explain that this definition of the Tutte polynomial may be extended meaninfully to metric graphs.

1.1. **Statement of results.** Suppose Γ is a metric graph with combinatorial model $\Gamma = (G, \ell)$, where $\ell : E(G) \to \mathbb{R}_{>0}$ is a function assigning a positive, real length to each edge of G. Given real parameters x, y with $x > 0$, let

$$T(\Gamma; x, y) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_x \right) (x-1)^{h_0(G \backslash A)-1} (y-1)^{h_1(G \backslash A)} \tag{2}$$

where the notation $[\alpha]_x$ means

$$[\alpha]_x = \frac{x^\alpha - 1}{x - 1} \quad \text{if } x \neq 1, \qquad [\alpha]_1 = \alpha.$$

(If $x < 0$, then the expression $[\alpha]_x$ can be considered a complex-valued function, by taking the principal branch of the complex logarithm.)

For fixed metric graph Γ, the expression (2) defines a function $\mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ by associating $(x, y) \mapsto T(\Gamma; x, y)$. This function is generally not a polynomial in x; moreover, it does not even admit a formal power series expansion in x if some edge length $\ell(e_i)$ is non-integral.

We can recover a power series expression for $T(\Gamma; x, y)$ by a simple change of variables. Let $T^+(\Gamma; u, w) = T(\Gamma; 1 + u, 1 + w)$, so that

$$\boxed{\texttt{eq:tutte-power-series}} \qquad (3) \qquad T^+(\Gamma; u, w) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_{1+u} \right) u^{h_0(G \backslash A) - 1} w^{h_1(G \backslash A)}.$$

Note that

$$[\alpha]_{1+u} = \frac{(1+u)^\alpha - 1}{u} = \sum_{k \geq 1} \binom{\alpha}{k} u^{k-1} \qquad \text{for } |u| < 1$$

so as a power series in u we have

$$(4) \qquad [\alpha]_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \in \mathbb{R}[[u]].$$

◇ use different notation to distinguish $[a]_{1+u}$ as a power series vs $[a]_{1+u}$ as the expression $1/u((1+u)^a - 1)$ ◇

Our first main result is that the expression (3) does not depend on which model (G, ℓ) we choose for the metric graph Γ.

Theorem 1 (Tutte power series). *Given a metric graph $\Gamma = (G, \ell)$, the expression $T^+(\Gamma; u, w)$ is a well-defined power series in $\mathbb{R}[[u]][w]$; in particular, $T^+(\Gamma; u, w)$ does not depend on the choice of model (G, ℓ) for Γ.*

Our next result concerns the convergence of the Tutte powers series of a fixed metric graph Γ.

Theorem 2 (Tutte convergence). *Given a metric graph Γ, the Tutte power series $T^+(\Gamma; u, w)$ converges when $|u| < 1$. If some edge length $\ell(e)$ is not an integer, then the radius of convergence in u is equal to 1.*

Instead of fixing a metric graph Γ and varying the parameters u, w, we can instead fix a choice of (u, w) and vary the metric graph. As Γ varies, the value of $T^+(\Gamma; u, w)$ also varies continuously.

Theorem 3 (Continuity of Tutte evaluation). *Consider the Tutte evaluation function $\mathrm{ev}(u, w) : \mathcal{M}_g \to \mathbb{R}$ on the moduli space $\mathcal{M}_g$ of metric graphs (of genus g), defined by*

$$\mathrm{ev}(u, w) : \Gamma \mapsto T^+(\Gamma; u, w),$$

where u and w are fixed (nonnegative?) real numbers. If $u > -1$, then the function $\mathrm{ev}(u, w)$ is continuous on $\mathcal{M}_g$.

Namely, for each combinatorial graph G, the Tutte evaluation $\mathrm{ev}(u, w)$ restricts to a continuous function

$$\mathrm{ev}(u, w) : \mathbb{R}_{>0}^{E(G)} \to \mathbb{R},$$

where a point in the domain $\in \mathbb{R}_{>0}^{E(G)}$ represents a choice of (positive, real) edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$. If $u > -1$, then as the length of a non-loop edge $e \in E(G)$ approaches zero in the metric graph $\Gamma = (G, \ell)$ while other edge lengths are fixed,

the value of $\mathrm{ev}(u, w)$ *at* (G, ℓ) *approaches the value of* $\mathrm{ev}(u, w)$ *at the contraction* $\Gamma/e = (G/e, \ell\big|_{E \setminus e})$.

Example 4 ($x = 1$, $y = 1$)**.** The Tutte evaluation $\mathrm{ev}(1, 1)$ on a graph G gives the number of spanning trees. On a metric graph, $\mathrm{ev}(1, 1)$ gives the volume of the Jacobian of $\Gamma = (G, \ell)$, which can be expressed as a weighted sum of spanning trees of G [cite a reference]. The function $\mathrm{ev}(1, 1)$ is continuous on $\mathcal{M}_g$, and extends continuously to $\overline{\mathcal{M}}_g$ (where it has value zero on the boundary).

Example 5 ($x = 0$, $y = 2$)**.** The Tutte evaluation $\mathrm{ev}(0, 2)$ on $\Gamma = (G, \ell)$ gives the number of totally cyclic orientations of Γ. This number does not depend on the edge lengths of Γ; i.e. $\mathrm{ev}(0, 2)$ is constant on metric graphs of a fixed combinatorial model G. However, $\mathrm{ev}(0, 2)$ is not continuous as some edge length approaches 0. Namely, the value of $T(G; 0, 2)$ generally differs from the value of $T(G/e; 0, 2)$ on the contraction.

Theorem 6 (Continuity of Tutte coefficient)**.** *For fixed indices* $i, j \geq 0$, *let* $\mathrm{coeff}(i, j; \Gamma)$ *denote the coefficient of* $u^i w^j$ *in the power series expansion of* $T^+(\Gamma; u, w)$. *Then the function* $\mathrm{coeff}(i, j)$ *defines a continuous(?) function* $\mathcal{M}_g \to \mathbb{R}$. $\diamondsuit$ extra assumptions needed? $\diamondsuit$

Theorem 7 (Deletion-contraction relation)**.** *Given a metric graph* $\Gamma = (G, \ell)$ *and an edge* $e \in E(G)$, *the Tutte power series satisfies*

$$T^+(\Gamma; u, w) = [\ell(e)]_{1+u} T^+(\Gamma \setminus e; u, w) + T^+(\Gamma/e; u, w).$$

1.2. **Previous work.** Read and Whitehead [4]

Several authors have investigated the behavior of the Tutte polynomial under the operation of subdividing an edge into multiple edges. [5, 6, 7]
[3, 4]

Zeros of Tutte polynomials?

1.3. **Notation.** Γ a compact metric graph

G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G

(G, ℓ) a combinatorial model for a metric graph, where $\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on edges of G

D a divisor on a metric graph

$T(G; x, y)$ the Tutte polynomial of G

$T^+(G; u, w) = T(G; 1 + u, 1 + w)$ "additive" centered Tutte polynomial

$T^+(\Gamma; u, w)$ the Tutte power series of Γ

2. Background

For a positive integer ℓ, the *q-analog* $[\ell]_q$ is defined as the polynomial

$$[\ell]_q = \frac{q^\ell - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{\ell-1}.$$

When ℓ is not an integer, $[\ell]_q$ does not admit a Laurent expansion in the variable q. However, we can obtain a well-defined power series under a change of variable.

Namely, note that

$$[\alpha]_{1+u} = \frac{(1+u)^\alpha - 1}{u} = \sum_{k \geq 0} \binom{\alpha}{k+1} u^k$$

so we have

(5) $$[\alpha]_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \in \mathbb{R}[[u]].$$

Note that for positive integers n, m we have

$$[n+m]_q = [n]_q + q^n [m]_q = [n]_q + [m]_q + (q-1)[n]_q[m]_q.$$

For positive real numbers α, β, we have

$$[\alpha + \beta]_{1+u} = [\alpha]_{1+u} + (1+u)^\alpha [\beta]_{1+u}$$
$$= [\alpha]_{1+u} + [\beta]_{1+u} + u[\alpha]_{1+u}[\beta]_{1+u}.$$

Note that the q-analog satisfies the following properties
  (1) If $q_0 > 0$ is fixed and $q_0 \neq 1$, the map

$$\ell \mapsto [\ell]_{q_0} = \frac{q_0^\ell - 1}{q_0 - 1}$$

  defines a continuous function from $\mathbb{R}$ to $\mathbb{R}$, which sends $1 \mapsto 1$ and $0 \mapsto 0$.
      If $q_0 = 1$, we use the convention that $[\ell]_1 = \ell$.
      If $q_0 = 0$, we have $[\ell]_0 = 1$ for any $\ell > 0$.
  (2) If $\ell_0 \geq 0$ is fixed and $q > 0$, the map

$$q \mapsto [\ell_0]_q = \frac{q^{\ell_0} - 1}{q - 1}$$

  defines a continuous function from $\mathbb{R}_{>0} \setminus \{1\}$ to $\mathbb{R}$, which satsifies

$$\lim_{q \to 0^+} [\ell_0]_q = \lim_{q \to 0^+} \frac{q^{\ell_0} - 1}{q - 1} = \begin{cases} 1 & \text{if } \ell_0 > 0 \\ 0 & \text{if } \ell_0 = 0 \\ -\infty & \text{if } \ell_0 < 0. \end{cases}$$

  and has a continuous extension to $\mathbb{R}_{>0} \to \mathbb{R}$ that sends $1 \mapsto \ell_0$.
  (3) In particular, for $\ell, q > 0$ we have

$$[\ell]_0 = 1 \qquad \text{and} \qquad [0]_q = 0.$$

$$\lim_{\ell \to 0} [\ell]_0 = 1 \qquad \text{and} \qquad \lim_{q \to 0} [0]_q = 0.$$

Considering $[\alpha]_{1+u}$ as a power series in u and α:

$$[\alpha]_{1+u} = \sum_{k \geq 0} \binom{\alpha}{k+1} u^k = \sum_{k \geq 0} \frac{1}{(k+1)!} (\alpha)_{k+1} u^k$$

$$= \sum_{k \geq 0} \left(\sum_{j \geq 0} \frac{s(k+1, j)}{(k+1)!} \alpha^j \right) u^k$$

$$= (\alpha) + \left(-\frac{1}{2}\alpha + \frac{1}{2}\alpha^2\right)u + \left(\frac{1}{3}\alpha - \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha^3\right)u^2 + \cdots$$

where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ denotes the falling factorial and $s(k,j)$ denotes the Stirling number of the first kind.

$$[\alpha]_{1+u} = \frac{(1+u)^\alpha - 1}{u} = \frac{\exp(\alpha \log(1+u)) - 1}{u}$$

$$= \frac{1}{u}\left(-1 + \sum_{j\geq 0} \frac{\log(1+u)^j}{j!}\alpha^j\right)$$

$$= \sum_{j\geq 1} \frac{\log(1+u)^j}{j!\,u}\alpha^j.$$

$$T(\Gamma; 1+u, 1+w) = \sum_{A\subset E(G)} \left(\prod_{e_i\in A} [\ell(e_i)]_{1+u}\right) u^{h_0(G\backslash A)-1}w^{h_1(G\backslash A)}$$

For a fixed metric graph $\Gamma = (G, \ell)$, the Tutte power series is an element of $\mathbb{R}[[u]][w]$.
This function satisfies the deletion-contraction relation

$$T(\Gamma; x, y) = [\ell(e)]_x \cdot T(\Gamma\backslash e; x, y) + T(\Gamma/e; x, y).$$

$$T^+(\Gamma; u, w) = [\ell(e)]_{1+u} \cdot T^+(\Gamma\backslash e; u, w) + T^+(\Gamma/e; u, w).$$

3. Metric graphs

A metric graph is a compact, connected metric space which comes from assigning edge lengths to a finite, connected graph. If the metric graph Γ comes from a combinatorial graph G by assigning edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, we say (G, ℓ) is a *combinatorial model* for Γ and we write $\Gamma = (G, \ell)$.

3.1. **Tropical curves.** Here we use "tropical curve" to refer to a metric graph which possibly has contracted loops, which we think of as "infinitesimally small" loops attached to a vertex. We record the number of

4. Tutte power series

The polynomial $T_G(x, y) = T(G; x, y)$ can also be defined inductively by the deletion-contraction relation:

$$T(G; x, y) = T(G\backslash e; x, y) + T(G/e; x, y).$$

along with the base cases for a loop edge and bridge edge.

$$T(G; x, y) = x^i y^j \qquad \text{if } G \text{ consists of } i \text{ bridges and } j \text{ loops.}$$

$$T(G; x, y) = \begin{cases} x & \text{if } G \text{ is a bridge} \\ y & \text{if } G \text{ is a loop.} \end{cases}$$

The Tutte power series $T^+(\Gamma; u, w)$ satsifies a similar deletion-contraction relation:

$$(6) \qquad T^+(\Gamma; u, w) = [\ell(e)]_{1+u}T^+(\Gamma\backslash e; u, w) + T^+(\Gamma/e; u, w).$$

Check compatibility:

$$T(\text{bridge}; x, y) = (x-1)^{\tilde{h}_0(\text{bridge})}(y-1)^{h_1(\text{bridge})} + (x-1)^{\tilde{h}_0(\text{two pts.})}(y-1)^{h_1(\text{two pts.})}$$

$$= 1 + (x-1) = x;$$

$$T(\text{loop}; x, y) = (x - 1)^{\tilde{h}_0(\text{loop})}(y - 1)^{h_1(\text{loop})} + (x - 1)^{\tilde{h}_0(\text{one pt.})}(y - 1)^{h_1(\text{one pt.})}$$
$$= (y - 1) + 1 = y.$$

Example 8 (Tutte power series of a line). Suppose G is a line graph consisting of n edges. Then

$$T(G; x, y) = x^n \qquad \text{and} \qquad T^+(G; u, w) = (1 + u)^n = 1 + nu + \binom{n}{2}u^2 + \cdots + u^n.$$

If Γ is a segment of length α, then

$$T^+(\Gamma; u, w) = (1 + u)^\alpha = [\alpha]_{1+u}u + 1 = \sum_{k=0}^{\infty} \binom{\alpha}{k}u^k = 1 + \alpha u + \binom{\alpha}{2}u^2 + \cdots .$$

Example 9 (Tutte power series of a circle). Suppose G is a cycle graph consisting of n edges. Then

$$T(G; x, y) = x + x^2 + \cdots + x^{n-2} + x^{n-1} + y = \frac{x^n - 1}{x - 1} + y - 1.$$

and

$$T^+(G; u, w) = n + \binom{n}{2}u + \binom{n}{3}u^2 + \cdots + nu^{n-2} + u^{n-1} + w = \frac{(1 + u)^n - 1}{u} + w$$

If Γ is a circle of length λ, then

$$T^+(\Gamma; u, w) = [\lambda]_{1+u} + w = \frac{(1 + u)^\lambda - 1}{u} + w = \sum_{k=0}^{\infty} \binom{\lambda}{k+1}u^k + w$$

Example 10 (Tutte power series of theta graph). Suppose G is the graph with two vertices connected by three edges. Suppose Γ is the metric graph which assigns lengths a, b, c to the edges of G. Then

$$T^+(G; u, w) = [a]_{1+u}T^+(\text{circle of length } b + c)$$
$$+ T^+(\text{circle of length } b)T^+(\text{circle of length } c)$$
$$= [a]_{1+u}([b + c]_{1+u} + w) + ([b]_{1+u} + w)([c]_{1+u} + w)$$
$$= ([a]_{1+u}[b + c]_{1+u} + [b]_{1+u}[c]_{1+u}) + ([a]_{1+u} + [b]_{1+u} + [c]_{1+u})w + w^2.$$

Note that at $w = 0$, we have

$$T^+(G; u, 0) = [a]_{1+u}[b + c]_{1+u} + [b]_{1+u}[c]_{1+u}$$

$$= \sum_{k \geq 0}\binom{a}{k+1}u^k \sum_{k \geq 0}\binom{b + c}{k+1}u^k + \sum_{k \geq 0}\binom{b}{k+1}u^k \sum_{k \geq 0}\binom{c}{k+1}u^k$$

$$= \left(a + \binom{a}{2}u + \cdots\right)\left(b + c + \binom{b + c}{2}u + \cdots\right) + \left(b + \binom{b}{2}u + \cdots\right)\left(c + \binom{c}{2}u + \cdots\right)$$

$$= (ab + ac + bc) + \left(a\binom{b + c}{2} + \binom{a}{2}(b + c) + b\binom{c}{2} + \binom{b}{2}c\right)u + ()u^2 + \cdots$$

$$= (ab + ac + bc) + \frac{1}{2}\left(a(b + c)(a + b + c - 2) + bc(b + c - 2)\right)u + \cdots$$

$$= (ab + ac + bc) + \frac{1}{2}\left(a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 + 2abc - 2ab - 2ac - 2bc\right)u + \cdots$$

$$T^+(G; u, 0) = [a]_{1+u}[b+c]_{1+u} + [b]_{1+u}[c]_{1+u}$$
$$= [a]_{1+u}\left([b]_{1+u} + [c]_{1+u} + u[b]_{1+u}[c]_{1+u}\right) + [b]_{1+u}[c]_{1+u}$$
$$= [a][b] + [a][c] + [b][c] + u[a][b][c].$$

Example 11 (Tutte power series of K_4). Suppose $G = K_4$, the complete graph on four vertices. Suppose Γ is the metric graph assigning edge lengths a, b, c, d, e, f to G, as shown in Figure [fill in].

Then we have

$$T^+(\Gamma; u, w) = ([a][b][d] + [a][b][e] + [a][b][f] + [a][c][d] + [a][c][e] + [a][c][f] + [a][d][e] + [a][d][f]$$
$$+ [b][c][d] + [b][c][e] + [b][c][f] + [b][d][e] + [b][e][f] + [c][d][f] + [c][e][f] + [d][e][f])$$
$$+ ([a][b][c][d] + [a][b][c][e] + [a][b][c][f] + [a][b][d][e] + [a][b][d][f] + [a][b][e][f]$$
$$+ [a][c][d][e] + [a][c][d][f] + [a][c][e][f] + [a][d][e][f] + [b][c][d][e]$$
$$+ [b][c][d][f] + [b][c][e][f] + [b][d][e][f] + [c][d][e][f])u$$
$$+ ([a][b][c][d][e] + [a][b][c][d][f] + [a][b][c][e][f] + [a][b][d][e][f]$$
$$+ [a][c][d][e][f] + [b][c][d][e][f])u^2$$
$$+ [a][b][c][d][e][f]u^3$$
$$+ ([a][b] + [a][c] + [a][d] + [a][e] + [a][f] + [b][c] + [b][d] + [b][e]$$
$$+ [b][f] + [c][d] + [c][e] + [c][f] + [d][e] + [d][f] + [e][f])w$$
$$+ ([a][b][c] + [a][e][f] + [b][d][f] + [c][d][e])uw$$
$$+ ([a] + [b] + [c] + [d] + [e] + [f])w^2 + w^3$$

Compare to the Example in Read–Whitehead [4, p. 272].

4.1. **Deleting bridges and contracting loops.**

5. SPECIALIZATIONS OF THE TUTTE POLYNOMIAL

5.1. **Constants.** For a graph $G = (V, E)$,

- $T^+(G; 1, 1) =$ the number of subsets of edges; $T_G(2, 2) = 2^{\#E}$.
- $T^+(G; 0, 0) =$ the number of spanning trees.
- $T^+(G; 0, 1) =$ the number of spanning subsets of edges.
- $T^+(G; 1, 0) =$ the number of independent subsets of edges.
- $T^+(G; -1, 1) =$ the number of totally cyclic orientations.
- $T^+(G; 1, -1) =$ the number of acyclic orientations.

For a metric graph $\Gamma = (G, \ell)$,

$$T_\Gamma(2, 2) = \sum_{A \subset E(G)} \prod_{e_i \in A} [\ell(e_i)]_2 = \sum_{A \subset E(G)} \prod_{e_i \in A} (2^{\ell(e_i)} - 1).$$

$$= \prod_{e_i \in E(G)} (1 + (2^{\ell(e_i)} - 1)) = 2^{\sum_i \ell(e_i)}$$

- $T^+(\Gamma; 1, 1) = 2^{\mathrm{vol}(\Gamma)}$
- $T^+(\Gamma; 0, 0) = \mathrm{vol}(\mathrm{Jac}(\Gamma))$
- $T^+(\Gamma; 0, 1) = \sum_{k=0}^{g} \mathrm{vol}(\mathrm{Eff}^k(\Gamma))$?

Example 12. Suppose Γ is the theta graph with edge lengths a, b and c,

$$\mathrm{eval}(\Gamma; 2, 2) = 2^{a+b+c}.$$

$T(\Gamma; 1, 1) = ab + ac + bc.$
$T(\Gamma; 1, 2) = 1 + (a + b + c) + (ab + ac + bc).$
$T(\Gamma; 2, 1) = (2^{a+b} - 2 \cdot 2^a + 3) + (2^{a+b+c} - 2^{a+b} + 2^a - 1) = 2^{a+b+c} - 2^a - 2^b - 2^c + 2.$
$T(\Gamma; 0, 2) = 1 + 3 + 3 - 1 = 6.$
$T(\Gamma; 2, 0) = 2^{a+b+c} - 2(2^a + 2^b + 2^c) + 6.$

Example 13. For the theta graph, we have

$$\begin{aligned}
T^+(\Gamma; u, w) = {}& w^2 + ([a]_{1+u} + [b]_{1+u} + [c]_{1+u})w \\
& + ([a]_{1+u}[b]_{1+u} + [a]_{1+u}[c]_{1+u} + [b]_{1+u}[c]_{1+u}) \\
& + ([a]_{1+u}[b]_{1+u}[c]_{1+u})u
\end{aligned}$$

$$\mathrm{coeff}(0, 0; \Gamma) = ab + ac + bc.$$
$$\mathrm{coeff}(0, 1; \Gamma) = a + b + c.$$
$$\mathrm{coeff}(k, 1; \Gamma) = \binom{a}{k+1} + \binom{b}{k+1} + \binom{c}{k+1}.$$
$$\mathrm{coeff}(0, 2; \Gamma) = 1.$$

5.2. Chromatic polynomial.
At $y = 0$ we obtain the chromatic polynomial of a graph:

$$\chi(G; \lambda) = (-1)^{\#V}(-\lambda)^{h_0(G)}T(G; 1 - \lambda, 0)$$

5.3. Flow polynomial.
At $x = 0$ we obtain the flow polynomial of a graph:

$$F(G; \lambda) = (-1)^{h_1(G)}T(G; 0, 1 - \lambda)$$

5.4. Reliability polynomial.
The reliability polynomial of a graph satisfies

$$R(G; p) = (1 - p)^{\#V - h_0(G)}p^{h_1(G)}T(G; 1, \frac{1}{p})$$

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