TUTTE POWER SERIES AND TUTTE EVALUATIONS ON THE MODULI SPACE OF METRIC GRAPHS

HARRY RICHMAN

Abstract. We define a formal power series assigned to a metric graph with arbitrary (positive) real edge lengths. When all edge lengths are positive integers, this recovers the usual Tutte polynomial. We prove that for positive inputs, evaluation of the Tutte power series defines a continuous function of the moduli space of metric graphs, which also extends to the compactification by tropical curves. We study how this metric Tutte polynomial is related to various structures on a metric graph.

1. Introduction

Given a graph G, the Tutte polynomial $T(G; x, y)$ is a two-variable polynomial introduced by Tutte in [cite reference]. Many important graph invariants arise as evaluations of the Tutte polynomial at specific (real) parameters x, y.

Given a connected graph G, the Tutte polynomial of G is

eq:tutte-graph

$$T(G; x, y) = \sum_{A \subset E(G)} (x-1)^{h_0(G \setminus A) - 1} (y-1)^{h_1(G \setminus A)} \tag{1}$$

where $G \setminus A$ denotes the graph with edges in A deleted, and h_0 and h_1 denote the zeroth and first Betti numbers of a topological space. In graph theoretic terms,

$$\begin{aligned} h_0(G) &= \#(\text{connected components of } G), \quad \text{and} \\ h_1(G) &= \#E(G) - \#V(G) + h_0(G). \end{aligned}$$

1.1. **Statement of results.** Suppose Γ is a metric graph with combinatorial model $\Gamma = (G, \ell)$, where $\ell : E(G) \to \mathbb{R}_{>0}$ is a function assigning a positive, real length to each edge of G. Given real parameters x, y with $x > 0$, let

eq:tutte-metric-graph

$$T(\Gamma; x, y) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_x \right) (x-1)^{h_0(G \setminus A) - 1} (y-1)^{h_1(G \setminus A)} \tag{2}$$

where the notation $[\alpha]_x$ means

$$[\alpha]_x = \frac{x^\alpha - 1}{x - 1} \quad \text{if } x \neq 1, \qquad [\alpha]_1 = \alpha.$$

(If $x < 0$, then the expression $[\alpha]_x$ can be considered a complex-valued function, by taking the principal branch of the complex logarithm.) For fixed Γ, the expression (2) eq:tutte-metric-graph defines a function $(x, y) \mapsto T(\Gamma; x, y)$ from $\mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$. This function is generally not a polynomial in x; moreover, it is generally not even a formal power

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series in x since $[\ell(e_i)]_x$ is not analytic at $x = 0$ if the edge length $\ell(e_i)$ is non-integral.

We can recover a power series expression for $T(\Gamma; x, y)$ by a simple linear change of variables. Let $T^+(\Gamma; u, w) = T(\Gamma; 1 + u, 1 + w)$, so that

$$T^+(\Gamma; u, w) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_{1+u} \right) u^{h_0(G \setminus A) - 1} w^{h_1(G \setminus A)}. \tag{3}$$

Note that

$$[\alpha]_{1+u} = \frac{(1+u)^\alpha - 1}{u} = \sum_{k \geq 1} \binom{\alpha}{k} u^{k-1}.$$

so as a power series in u we have

$$[\alpha]_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \in \mathbb{R}[[u]]. \tag{4}$$

Theorem 1 (Tutte power series)**.** *Given a metric graph $\Gamma = (G, \ell)$, the expression $T^+(\Gamma; u, w)$ is a well-defined power series in $\mathbb{R}[[u]][w]$; in particular, $T^+(\Gamma; u, w)$ does not depend on the choice of model (G, ℓ).*

The power series $T^+(\Gamma; u, w)$ converges when $|u| < 1$; if some edge length $\ell(e)$ is not an integer, then this is the radius of convergence in u.

Theorem 2 (Continuity of Tutte evaluation)**.** *Consider the Tutte evaluation function $\mathrm{ev}(x, y) : \mathcal{M}_g \to \mathbb{R}$ on the moduli space $\mathcal{M}_g$ of metric graphs (of genus g), defined by*

$$\mathrm{ev}(x, y) : \Gamma \mapsto T(\Gamma; x, y),$$

where x and y are fixed (nonnegative?) real numbers. If $x > 0$, then the function $\mathrm{ev}(x, y)$ is continuous on $\mathcal{M}_g$.

Namely, for each combinatorial graph G, the Tutte evaluation $\mathrm{ev}(x, y)$ restricts to a continuous function

$$\mathrm{ev}(x, y) : \mathbb{R}_{>0}^{E(G)} \to \mathbb{R},$$

where a point in the domain $\in \mathbb{R}_{>0}^{E(G)}$ represents a choice of (positive, real) edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$. If $x > 0$, then as the length of a non-loop edge $e \in E(G)$ approaches zero in the metric graph $\Gamma = (G, \ell)$ while other edge lengths are fixed, the value of $\mathrm{ev}(x, y)$ at (G, ℓ) approaches the value of $\mathrm{ev}(x, y)$ at the contraction $\Gamma/e = (G/e, \ell\big|_{E \setminus e})$.

Example 3 ($x = 1$, $y = 1$)**.** The Tutte evaluation $\mathrm{ev}(1, 1)$ on a graph G gives the number of spanning trees. On a metric graph, $\mathrm{ev}(1, 1)$ gives the volume of the Jacobian of $\Gamma = (G, \ell)$, which can be expressed as a weighted sum of spanning trees of G [cite a reference]. The function $\mathrm{ev}(1, 1)$ is continuous on $\mathcal{M}_g$, and extends continuously to $\overline{\mathcal{M}}_g$ (where it has value zero on the boundary).

Example 4 ($x = 0$, $y = 2$)**.** The Tutte evaluation $\mathrm{ev}(0, 2)$ on $\Gamma = (G, \ell)$ gives the number of totally cyclic orientations of Γ. This number does not depend on the edge lengths of Γ; i.e. $\mathrm{ev}(0, 2)$ is constant on metric graphs of a fixed combinatorial model G. However, $\mathrm{ev}(0, 2)$ is not continuous as some edge length approaches 0. Namely, the value of $T(G; 0, 2)$ generally differs from the value of $T(G/e; 0, 2)$ on the contraction.

Theorem 5 (Continuity of Tutte coefficient)**.** *For fixed indices $i, j \geq 0$, let* $\operatorname{coeff}(i, j; \Gamma)$ *denote the coefficient of $u^i w^j$ in the power series expansion of $T^+(\Gamma; u, w)$. Then the function* $\operatorname{coeff}(i, j)$ *defines a continuous(?) function $\mathcal{M}_g \to \mathbb{R}$.*

1.2. **Previous work.** Read and Whitehead [RW2] [11]

1.3. **Notation.** Γ a compact metric graph

G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G

(G, ℓ) a combinatorial model for a metric graph, where $\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on edges of G

D a divisor on a metric graph

$T(G; x, y)$ the Tutte polynomial of G

$T^+(G; u, w) = T(G; 1 + u, 1 + w)$ "additive" centered Tutte polynomial

$T^\times(\Gamma; u, w)$ the Tutte polynomial (power series) of Γ

2. Background

For a positive integer ℓ, the *q-analog* $[\ell]_q$ is defined as the polynomial

$$[\ell]_q = \frac{q^\ell - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{\ell-1}.$$

When ℓ is not a positive integer, the above expression is no longer a polynomial. When ℓ is not an integer, the above expression is not even has a valid Laurent expansion in the variable q. However, we can obtain a well-defined power series under a simple change of variable. Namely, note that

$$[\alpha]_{1+u} = \frac{(1+u)^\alpha - 1}{u} = \sum_{k \geq 1} \binom{\alpha}{k} u^{k-1}.$$

so we have

$$[\alpha]_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \in \mathbb{R}[[u]]. \tag{5}$$

Note that

$$[\alpha + \beta]_q = [\alpha]_q + q^\alpha [\beta]_q = [\alpha]_q + [\beta]_q + (q - 1)[\alpha]_q [\beta]_q.$$

$$\begin{aligned} [\alpha + \beta]_{1+u} &= [\alpha]_{1+u} + (1 + u)^\alpha [\beta]_{1+u} \\ &= [\alpha]_{1+u} + [\beta]_{1+u} + u[\alpha]_{1+u}[\beta]_{1+u}. \end{aligned}$$

$$T(\Gamma; 1 + u, 1 + w) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\ell(e_i)]_{1+u} \right) u^{h_0(G \backslash A) - 1} w^{h_1(G \backslash A)}$$

For a fixed metric graph $\Gamma = (G, \ell)$, the Tutte power series is an element of $\mathbb{R}[[u]][w]$.

This function satisfies the deletion-contraction relation

$$T(\Gamma; x, y) = [\ell(e)]_x \cdot T(\Gamma \backslash e; x, y) + T(\Gamma/e; x, y).$$

$$\begin{aligned} T(\Gamma; 1 + u, 1 + w) = [\ell(e)]_{1+u} &\cdot T(\Gamma \backslash e; 1 + u, 1 + w) \\ &+ T(\Gamma/e; 1 + u, 1 + w). \end{aligned}$$

Note that the q-analog satisfies the following properties

(1) If $q_0 > 0$ is fixed and $q_0 \neq 1$, the map
$$\ell \mapsto [\ell]_{q_0} = \frac{q_0^\ell - 1}{q_0 - 1}$$
defines a continuous function from $\mathbb{R}$ to $\mathbb{R}$, which sends $1 \mapsto 1$ and $0 \mapsto 0$.

(2) As q_0 approaches 0 from the right, we have
$$\lim_{q \to 0^+} \frac{q_0^\ell - 1}{q_0 - 1} = \begin{cases} 1 & \text{if } \ell > 0, \\ 0 & \text{if } \ell = 0, \\ -\infty & \text{if } \ell < 0. \end{cases}$$

(3) If $\ell_0 \neq 0$ is fixed and $q > 0$, the map
$$q \mapsto [\ell_0]_q = \frac{q^{\ell_0} - 1}{q - 1}$$
defines a continuous function from $\mathbb{R}_{>0}$ to $\mathbb{R}$, which sends $1 \mapsto \ell_0$ and
$$\lim_{q \to 0^+} [\ell_0]_q = \lim_{q \to 0^+} \frac{q^{\ell_0} - 1}{q - 1} = \begin{cases} 1 & \text{if } \ell_0 > 0 \\ -\infty & \text{if } \ell_0 < 0. \end{cases}$$

3. Metric graphs

A metric graph is a compact, connected metric space which comes from assigning edge lengths to a finite, connected graph. If the metric graph Γ comes from a combinatorial graph G by assigning edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, we say (G, ℓ) is a *combinatorial model* for Γ and we write $\Gamma = (G, \ell)$.

3.1. **Tropical curves.** Here we use "tropical curve" to refer to a metric graph which possibly has contracted loops, which we think of as "infinitesimally small" loops attached to a vertex. We record the number of

4. Tutte polynomial

The polynomial $T_G(x, y) = T(G; x, y)$ can also be defined inductively by the deletion-contraction relation:
$$T(G; x, y) = T(G \backslash e; x, y) + T(G/e; x, y).$$
along with the bases cases for a loop edge and bridge edge.
$$T(G; x, y) = x^i y^j \qquad \text{if } G \text{ consists of } i \text{ bridges and } j \text{ loops.}$$
$$T(G; x, y) = \begin{cases} x & \text{if } G \text{ is a bridge} \\ y & \text{if } G \text{ is a loop.} \end{cases}$$

Check compatibility:
$$\begin{aligned} T(\text{bridge}; x, y) &= (x-1)^{\tilde{h}_0(\text{bridge})}(y-1)^{h_1(\text{bridge})} + (x-1)^{\tilde{h}_0(\text{two pts.})}(y-1)^{h_1(\text{two pts.})} \\ &= 1 + (x - 1) = x; \end{aligned}$$

$$\begin{aligned} T(\text{loop}; x, y) &= (x-1)^{\tilde{h}_0(\text{loop})}(y-1)^{h_1(\text{loop})} + (x-1)^{\tilde{h}_0(\text{one pt.})}(y-1)^{h_1(\text{one pt.})} \\ &= (y - 1) + 1 = y. \end{aligned}$$

4.1. **Deleting bridges and contracting loops.**

5. Specializations of the Tutte polynomial

5.1. **Constants.** For a graph $G = (V, E)$,

- $T_G(2,2)$ counts the number of subsets of edges; $T_G(2,2) = 2^{\#E}$.
- $T_G(1,1)$ counts the number of spanning trees.
- $T_G(1,2)$ counts the number of spanning subset of edges.
- $T_G(2,1)$ counts the number of independent subsets of edges.
- $T_G(0,2)$ counts the number of totally cyclic orientations.
- $T_G(2,0)$ counts the number of acyclic orientations.

For a metric graph $\Gamma = (G, \ell)$,

$$T_\Gamma(2,2) = \sum_{A \subset E(G)} \prod_{e_i \in A} [\ell(e_i)]_2 = \sum_{A \subset E(G)} \prod_{e_i \in A} (2^{\ell(e_i)} - 1).$$
$$= \prod_{e_i \in E(G)} (1 + (2^{\ell(e_i)} - 1)) = 2^{\sum_i \ell(e_i)}$$

Example 6. Suppose Γ is the theta graph with edge lengths a, b and c,

$$\mathrm{eval}(\Gamma; 2, 2) = 2^{a+b+c}.$$

$T(\Gamma; 1, 1) = ab + ac + bc.$
$T(\Gamma; 1, 2) = 1 + (a + b + c) + (ab + ac + bc).$
$T(\Gamma; 2, 1) = (2^{a+b} - 2 \cdot 2^a + 3) + (2^{a+b+c} - 2^{a+b} + 2^a - 1) = 2^{a+b+c} - 2^a - 2^b - 2^c + 2.$
$T(\Gamma; 0, 2) = 1 + 3 + 3 - 1 = 6.$
$T(\Gamma; 2, 0) = 2^{a+b+c} - 2(2^a + 2^b + 2^c) + 6.$

Example 7. For the theta graph, we have

$$\begin{aligned} T^+(\Gamma; u, w) = w^2 &+ ([a]_{1+u} + [b]_{1+u} + [c]_{1+u})w \\ &+ ([a]_{1+u}[b]_{1+u} + [a]_{1+u}[c]_{1+u} + [b]_{1+u}[c]_{1+u}) \\ &+ ([a]_{1+u}[b]_{1+u}[c]_{1+u})u \end{aligned}$$

$$\mathrm{coeff}(0, 0; \Gamma) = ab + ac + bc.$$
$$\mathrm{coeff}(0, 1; \Gamma) = a + b + c.$$
$$\mathrm{coeff}(k, 1; \Gamma) = \binom{a}{k+1} + \binom{b}{k+1} + \binom{c}{k+1}.$$
$$\mathrm{coeff}(0, 2; \Gamma) = 1.$$

5.2. **Chromatic polynomial.** At $y = 0$ we obtain the chromatic polynomial of a graph:

$$\chi(G; \lambda) = (-1)^{\#V} (-\lambda)^{h_0(G)} T(G; 1 - \lambda, 0)$$

5.3. **Flow polynomial.** At $x = 0$ we obtain the flow polynomial of a graph:

$$F(G; \lambda) = (-1)^{h_1(G)} T(G; 0, 1 - \lambda)$$

5.4. **Reliability polynomial.** The reliability polynomial of a graph satisfies

$$R(G; p) = (1 - p)^{\#V - h_0(G)} p^{h_1(G)} T(G; 1, \frac{1}{p})$$

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