TUTTE POWER SERIES ON METRIC GRAPHS

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ABSTRACT. We define a Tutte power series for a metric graph with arbitrary positive real edge lengths, which recovers the usual Tutte polynomial when all edge lengths are positive integers. We prove that evaluation of the Tutte power series defines a continuous function of the moduli space of metric graphs, under certain constraints on the input parameters. The evaluations give rise to a large family of functions generalizing the function that sends a metric graph to the volume of its graph Jacobian.

1. INTRODUCTION

Given a graph G, the Tutte polynomial $T(G;x,y)$ is a two-variable polynomial introduced by Tutte in [16]. Many important graph invariants arise as evaluations of the Tutte polynomial at specific integer parameters x, y. For a comprehensive modern overview of the Tutte polynomial see [2, 6].

The following characterization of the Tutte polynomial was initially introduced by Crapo [5], using the *rank generating function* of G (see also [6, Definition 3]). Given a connected graph G, the Tutte polynomial of G is

eq:tutte-graph

$$T(G;x,y) = \sum_{A \subset E(G)} (x-1)^{h_0(G\setminus A)-1} (y-1)^{h_1(G\setminus A)} \tag{1}$$

where $G \setminus A$ denotes the graph with edges in A deleted, and h_0 and h_1 denote the zeroth and first Betti numbers of a topological space. In graph theoretic terms,

$$\begin{aligned} h_0(G) &= \#(\text{connected components of } G), \quad \text{and} \\ h_1(G) &= \#E(G) - \#V(G) + h_0(G). \end{aligned}$$

The purpose of this paper is to explain that this definition of the Tutte polynomial may be extended meaningfully to a metric graph. As a consequence, evaluation of the Tutte polynomial (for certain real parameters) extends to a continuous function on the moduli space of metric graphs.

1.1. Statement of results.

Suppose Γ is a metric graph with combinatorial model $\Gamma = (G,\ell)$, where $\ell : E(G) \to \mathbb{R}_{>0}$ assigns a positive length to each edge of G.

We define the *Tutte power series* $T^+(\Gamma;u,w)$ of Γ as

eq:tutte-power-series

$$T^+(\Gamma;u,w) = \sum_{A \subset E(G)} \left(\prod_{e_i \in A} [\![\ell(e_i)]\!]_{1+u} \right) u^{h_0(G\setminus A)-1} w^{h_1(G\setminus A)} \tag{2}$$

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where $[\![\alpha]\!]_{1+u}$ is the formal power series

$$[\![\alpha]\!]_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \quad \in \mathbb{R}[[u]] \tag{3}$$

and $\binom{\alpha}{k} = \frac{1}{k!}\alpha(\alpha-1)\cdots(\alpha-k+1)$. Note this power series converges for $|u| < 1$, and in this range satisfies

$$[\![\alpha]\!]_1 = \alpha \qquad \text{and} \qquad [\![\alpha]\!]_{1+u} = \frac{(1+u)^\alpha - 1}{u} \qquad \text{if } u \neq 0.$$

Our first main result is that the expression (2) eq:tutte-power-series does not depend on the chosen model (G, ℓ) for the metric graph Γ.

thm:intro-tutte-series **Theorem 1** (Tutte power series)**.** *Given a metric graph $\Gamma = (G, \ell)$, the expression $T^+(\Gamma; u, w)$ is a well-defined power series in $\mathbb{R}[\![u]\!][w]$; in particular, $T^+(\Gamma; u, w)$ does not depend on the choice of model (G, ℓ) for Γ.*

To prove this result, we show that $T^+(\Gamma; u, w)$ satisfies a familiar deletion-contraction identity.

thm:deletion-contraction **Theorem 2** (Deletion-contraction relation)**.** *Given a metric graph $\Gamma = (G, \ell)$ and an edge $e \in E(G)$, which is not a loop, the Tutte power series satisfies*

$$T^+(\Gamma; u, w) = [\![\ell(e)]\!]_{1+u}\, T^+(\Gamma \backslash e; u, w) + T^+(\Gamma / e; u, w). \tag{4}$$

1.2. **Evaluating Tutte power series.** Instead of considering a fixed metric graph Γ, we can vary Γ over a natural moduli space of metric graphs. As Γ varies, the value of $T^+(\Gamma; u, w)$ also varies continuously under certain mild assumptions on the parameters (u, w).

Let $\mathcal{M}_g^{\mathrm{graph}}$ denote the moduli space of metric graphs of genus g.

thm:tutte-eval-moduli **Theorem 3** (Continuity of Tutte evaluation)**.** *Let u and w be fixed (nonnegative?) real numbers, with $u > -1$. The Tutte evaluation at (u, w),*

$$\mathrm{ev}^+(u, w) : \Gamma \mapsto T^+(\Gamma; u, w),$$

defines a continuous function $\mathrm{ev}^+(u, w) : \mathcal{M}_g^{\mathrm{graph}} \to \mathbb{R}$.

Namely,

(i) (Continuity on cells) for each combinatorial graph G, the Tutte evaluation $\mathrm{ev}^+(u, w)$ restricts to a continuous function

$$\mathrm{ev}^+(u, w) : \mathbb{R}_{>0}^{E(G)} \to \mathbb{R},$$

where a point in the positive orthant $\mathbb{R}_{>0}^{E(G)}$ represents a choice of (positive, real) edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$.

(ii) (Continuity between cells) As the length of a non-loop edge $e \in E(G)$ approaches zero in the metric graph $\Gamma = (G, \ell)$ while other edge lengths are fixed, the value of $\mathrm{ev}^+(u, w)$ at (G, ℓ) approaches the value of $\mathrm{ev}^+(u, w)$ at the contraction Γ / e.

Example 4 $(u = 0, w = 0)$**.** On a metric graph, the Tutte evaluation $\mathrm{ev}^+(0, 0)$ gives the volume of the Jacobian of $\Gamma = (G, \ell)$, which can be expressed as a weighted sum of spanning trees of G ♢ [cite a reference] ♢ . The function $\mathrm{ev}^+(0, 0)$ is continuous on $\mathcal{M}_g^{\mathrm{graph}}$, and extends continuously to $\overline{\mathcal{M}_g^{\mathrm{graph}}}$ (where it has value zero on the boundary).

Example 5 ($u = -1$, $w = 1$)**.** The Tutte evaluation $\mathrm{ev}^+(-1,1)$ on $\Gamma = (G,\ell)$ gives the number of totally cyclic orientations of Γ. This number does not depend on the edge lengths of Γ; i.e. $\mathrm{ev}^+(-1,1)$ is constant on metric graphs of a fixed combinatorial model G. However, $\mathrm{ev}^+(-1,1)$ is not continuous as some edge length approaches 0. Namely, the value of $T(G;-1,1)$ generally differs from the value of $T(G/e;-1,1)$ after contracting an edge.

1.3. **Tutte power series coefficients.** ♢ Maybe remove this theorem ♢

Theorem 6 (Continuity of Tutte coefficient)**.** *For fixed indices $i, j \geq 0$, let $\mathrm{coeff}(i,j;\Gamma)$ denote the coefficient of $u^i w^j$ in the power series expansion of $T^+(\Gamma;u,w)$. Then the function $\mathrm{coeff}(i,j)$ defines a locally-polynomial function $\mathcal{M}_g^{\mathrm{graph}} \to \mathbb{R}$.* ♢ extra assumptions needed? ♢

1.4. **Previous work.**

Multivariate Tutte polynomial [12] also known as the *Potts-model partition function.* Sokal [12] asks:

> Let me conclude by observing that numerous specific evaluations of the Tutte polynomial have been given combinatorial interpretations, as counting some set of objects associated to the graphs G. It would be an interesting project to seek to extend these counting problems to "counting with weights," i.e., to obtain suitably defined univariate or multivariate generating polynomials for the objects in question as specializations of $Z_G(q,v)$ or $Z_G(q,\mathbf{v})$, respectively.

Several authors have investigated the behavior of the Tutte polynomial under the operation of subdividing an edge into multiple edges.

Previous work on *chain polynomials*:

The identities presented in this paper are essentially in the work of Read and Whitehead [11], but with the restriction that edge lengths are positive integers. In their work, edge lengths are called "chain lengths."

The family of graphs obtained by varying chain lengths of a fixed graph is called a "homeomorphism class" of graphs.

Read and Whitehead [11] [10, 11]

The essential contribution of this work is to enforce a dependence of the edge weights $(\alpha(e))$ on the polynomial parameter u, namely $\alpha(e) = [\![\ell(e)]\!]_{1+u}$.

Traldi [13] considers the weighted Tutte polynomial

$$\widetilde{T}^+(G;u,w) = \sum_{A \subset E} \left(\prod_{e \in A} c(e) \right) u^{h_0(G|A)} w^{h_1(G|A)}.$$

The same weighted polynomial was previously studied by Fortuin–Kasteleyn [7], but in harder-to-understand notation for a modern reader.

Brylawski [3]

Traldi [13, 14, 15]

Application: Zeros of Tutte polynomials?

Jackson and Sokal [8] study zero-free regions of the Tutte polynomial.

Ok and Perrett [9] study the density of the real zeros of the Tutte polynomial.

1.5. **Notation.**

Γ a compact metric graph

G a finite graph, loops and parallel edges allowed, possibly disconnected

$E(G)$ edge set of G

$V(G)$ vertex set of G

(G, ℓ) a combinatorial model for a metric graph, where
$\ell : E(G) \to \mathbb{R}_{>0}$ is a length function on edges of G
$T(G; x, y)$ the Tutte polynomial of G
$T^+(G; u, w) = T(G; 1 + u, 1 + w)$ "additive" centered Tutte polynomial
$T^+(\Gamma; u, w)$ the Tutte power series of Γ

2. Background

2.1. **q-analogs.** For a positive integer ℓ, the *q-analog* $[\ell]_q$ is defined as the polynomial

$$[\ell]_q = \frac{q^\ell - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{\ell-1} \in \mathbb{Z}[q].$$

When ℓ is not an integer, $[\ell]_q$ does not admit a Laurent expansion in the variable q. However, we can obtain a well-defined power series under a change of variable. Namely, note that

$$[\alpha]_{1+u} = \frac{(1+u)^\alpha - 1}{u} = \sum_{k \geq 0} \binom{\alpha}{k+1} u^k$$

so we have

$$\llbracket \alpha \rrbracket_{1+u} = \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots \in \mathbb{R}[[u]]. \tag{5}$$

Here α can be any real number, and $\binom{\alpha}{k}$ is the real number

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!} \tag{6}$$

♢ TODO: decide on notation $\llbracket \alpha \rrbracket_q$ $[\alpha]_q$ $\llbracket \alpha \rrbracket_q$ $[\alpha]_q$ ♢

Fact 7.

(1) For a positive integer n, the expression $[n]_q$ is a polynomial in $\mathbb{Z}[q]$. As $q \to 1$, we have $[n]_q \to n$. As $q \to 0$, we have $[n]_q \to 1$.
(2) For a real number α, the expression $\llbracket \alpha \rrbracket_{1+u}$ is a power series in $\mathbb{R}[[u]]$. As $u \to 0$, we have $\llbracket \alpha \rrbracket_{1+u} \to \alpha$. ♢ If α is positive, then as $u \to -1$, we have $\llbracket \alpha \rrbracket_{1+u} \to 1$. ♢

For positive integers n, m we have

$$[n + m]_q = [n]_q + [m]_q + (q - 1)[n]_q[m]_q.$$

The analogous identity holds for power series $\llbracket \alpha \rrbracket_{1+u}$.

Proposition 8. *For* ♢ positive ? ♢ *real numbers α, β, we have*

$$\llbracket \alpha + \beta \rrbracket_{1+u} = \llbracket \alpha \rrbracket_{1+u} + \llbracket \beta \rrbracket_{1+u} + u \llbracket \alpha \rrbracket_{1+u} \llbracket \beta \rrbracket_{1+u}.$$

as elements of $\mathbb{R}[[u]]$.

Proof. Observe that on the open disc $|u| < 1$,

$$\begin{aligned} \llbracket \alpha + \beta \rrbracket_{1+u} &= \frac{(1+u)^{\alpha+\beta} - 1}{u} \\ &= \frac{(1+u)^{\alpha+\beta} - (1+u)^\alpha}{u} + \frac{(1+u)^\alpha - 1}{u} \\ &= (1+u)^\alpha \llbracket \beta \rrbracket_{1+u} + \llbracket \alpha \rrbracket_{1+u} \end{aligned}$$

and $(1 + u)^\alpha = u \llbracket \alpha \rrbracket_{1+u} + 1$. ∎

Note that the q-analog satisfies the following properties

(1) (Varying the edge length) If $q_0 > 0$ is fixed and $q_0 \neq 1$, the map
$$\ell \mapsto [\ell]_{q_0} = \frac{q_0^\ell - 1}{q_0 - 1}$$
defines a continuous function from $\mathbb{R}$ to $\mathbb{R}$, which sends $1 \mapsto 1$ and $0 \mapsto 0$.
If $q_0 = 1$, we use the convention that $[\ell]_1 = \ell$.
If $q_0 = 0$, we have $[\ell]_0 = 1$ for any $\ell > 0$.

(2) (Varying the formal q-parameter) If $\ell_0 \geq 0$ is fixed and $q > 0$, the map
$$q \mapsto [\ell_0]_q = \frac{q^{\ell_0} - 1}{q - 1}$$
defines a continuous function from $\mathbb{R}_{>0} \setminus \{1\}$ to $\mathbb{R}$, which satisfies
$$\lim_{q \to 0^+} [\ell_0]_q = \lim_{q \to 0^+} \frac{q^{\ell_0} - 1}{q - 1} = \begin{cases} 1 & \text{if } \ell_0 > 0 \\ 0 & \text{if } \ell_0 = 0 \\ -\infty & \text{if } \ell_0 < 0. \end{cases}$$
and has a continuous extension to $\mathbb{R}_{>0} \to \mathbb{R}$ that sends $1 \mapsto \ell_0$.

(3) In particular, for $\ell, q > 0$ we have
$$[\ell]_0 = 1 \qquad \text{and} \qquad [0]_q = 0.$$
$$\lim_{\ell \to 0} [\ell]_0 = 1 \qquad \text{and} \qquad \lim_{q \to 0} [0]_q = 0.$$

Considering $[\alpha]_{1+u}$ as a power series in u and α:
$$\begin{aligned} [\![\alpha]\!]_{1+u} &= \sum_{k \geq 0} \binom{\alpha}{k+1} u^k = \sum_{k \geq 0} \frac{1}{(k+1)!} \alpha^{\underline{k+1}} u^k \\ &= \sum_{k \geq 0} \left(\sum_{j \geq 0} \frac{s(k+1, j)}{(k+1)!} \alpha^j \right) u^k \\ &= (\alpha) + (-\frac{1}{2}\alpha + \frac{1}{2}\alpha^2)u + (\frac{1}{3}\alpha - \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha^3)u^2 + \cdots \end{aligned}$$

where $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1)$ denotes the falling factorial and $s(k, j)$ denotes the Stirling number of the first kind.

As a power series in α:
$$\begin{aligned} [\![\alpha]\!]_{1+u} &= \frac{(1+u)^\alpha - 1}{u} = \frac{\exp(\alpha \log(1+u)) - 1}{u} \\ &= \frac{1}{u} \left(-1 + \sum_{j \geq 0} \frac{\log(1+u)^j}{j!} \alpha^j \right) \\ &= \sum_{j \geq 1} \frac{\log(1+u)^j}{j!\, u} \alpha^j. \end{aligned}$$

2.2. **Graph theory.**

$$h_0(G|A) - 1 = rk(G) - rk(A), \quad \text{and}$$
$$h_1(G|A) = \#(A) - rk(A).$$

2.3. **Tutte polynomial.**

3. Metric graphs

A metric graph is a compact, connected metric space which comes from assigning edge lengths to a finite, connected graph. If the metric graph Γ comes from a combinatorial graph G by assigning edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, we say (G, ℓ) is a *combinatorial model* for Γ and we write $\Gamma = (G, \ell)$.

3.1. **Deletion and contraction.** Suppose $\Gamma = (G, \ell)$, and $e \in E(G)$ is a non-loop edge.

The *edge deletion* $\Gamma \setminus e$ is the metric graph obtained from Γ by removing the points in the interior of e.

The *edge contraction* Γ / e is the metric graph obtained from Γ by removing the points in the interior of e, and then joining the endpoints of e.

3.2. **Moduli spaces of metric graphs.** See Melody Chan [4].

Abramovich–Caporaso–Payne [1].

A graph G is *stable* if it is connected, and every vertex has degree at least three. A metric graph is *semistable* if every point has valence at least two.

The moduli space of genus g metric graphs is a polyhedral fan

♢ is it natural here to restrict to stable graphs, or have infinitely many graphs per genus? ♢

3.3. **Tropical curves.** ♢ consider deleting ♢ Here we use "tropical curve" to refer to a metric graph which possibly has contracted loops, which we think of as "infinitesimally small" loops attached to a vertex. We record the number of

4. Tutte power series

Let $T^+(\Gamma; u, w)$ be the power series in $\mathbb{R}[\![u]\!][w]$ defined by

$$T^+(\Gamma; u, w) = \sum_{A \subset E(G)} \left(\prod_{e \in A} [\![\ell(e)]\!]_{1+u} \right) u^{h_0(G \setminus A) - 1} w^{h_1(G \setminus A)}. \tag{7}$$

Proposition 9.

(1) *Let $\Gamma = \Gamma_1 \vee \Gamma_2$ denote the wedge sum of two metric graphs. Then*

$$T^+(\Gamma_1 \vee \Gamma_2; u, w) = T^+(\Gamma_1; u, w)\, T^+(\Gamma_2; u, w).$$

(2) *Let $\Gamma = \Gamma_1 \bigsqcup \Gamma_2$ denote the (disconnected) disjoint union of two metric graphs. Then*

$$T^+(\Gamma_1 \sqcup \Gamma_2; u, w) = u\, T^+(\Gamma; u, w)\, T^+(\Gamma; u, w).$$

4.1. **Deletion-contraction.** The Tutte polynomial $T(G; x, y)$ can be characterized inductively by the deletion-contraction relation:

$$T(G; x, y) = T(G \setminus e; x, y) + T(G/e; x, y).$$

if e is not a loop or bridge, along with the base cases

$$T(G; x, y) = x^i y^j \qquad \text{if } G \text{ consists of } i \text{ bridges and } j \text{ loops.}$$

The Tutte power series $T^+(\Gamma; u, w)$ satisfies a similar deletion-contraction relation.

Theorem 10. *For a metric graph Γ,*

$$T^+(\Gamma; u, w) = [\![\ell(e)]\!]_{1+u} T^+(\Gamma \backslash e; u, w) + T^+(\Gamma/e; u, w). \tag{8}$$

Proof. Observe that

$$T^+(\Gamma; u, w) = \sum_{A \subset E} (\cdots) = \sum_{\substack{A \subset E \\ e_0 \in A}} (\cdots) + \sum_{\substack{A \subset E \\ e_0 \notin A}} (\cdots).$$

The first sum reduces to

$$\begin{aligned} &\sum_{\substack{A \subset E \\ e_0 \in A}} \left(\prod_{e \in A} [\![\ell(e)]\!]_{1+u} \right) u^{h_0(G \backslash A) - 1} w^{h_1(G \backslash A)} \\ &\qquad = [\![\ell(e_0)]\!]_{1+u} \sum_{A' \subset E \backslash e_0} \left(\prod_{e \in A'} [\![\ell(e)]\!]_{1+u} \right) u^{h_0(G \backslash e_0 \backslash A) - 1} w^{h_1(G \backslash e_0 \backslash A)} \\ &\qquad = [\![\ell(e_0)]\!]_{1+u} T^+(G \setminus e_0; u, w), \end{aligned}$$

while the second sum reduces to

$$\begin{aligned} &\sum_{\substack{A \subset E \\ e_0 \notin A}} \left(\prod_{e \in A} [\![\ell(e)]\!]_{1+u} \right) u^{h_0(G \backslash A) - 1} w^{h_1(G \backslash A)} \\ &\qquad = \sum_{A \subset E \backslash e_0} \left(\prod_{e \in A} [\![\ell(e)]\!]_{1+u} \right) u^{h_0(G \backslash A / e_0) - 1} w^{h_1(G \backslash A / e_0)} \\ &\qquad = T^+(G/e_0; u, w). \end{aligned}$$

In the second equality, we use the fact that the quotient map $(G \setminus A) \to (G \setminus A)/e_0$ is a homotopy equivalence, so it preserves homology groups. The third line uses the fact that deletion and contraction commute, $(G \setminus A)/e_0 = (G/e_0) \setminus A$. $\square$

Example 11 (Tutte power series of a line)**.** Suppose Γ is a segment of length α, then

$$T^+(\Gamma; u, w) = 1 + \alpha u + \binom{\alpha}{2} u^2 + \cdots.$$

If G is a line graph consisting of n edges, then

$$T^+(G; u, w) = (1+u)^n = 1 + nu + \binom{n}{2} u^2 + \cdots + u^n.$$

Example 12 (Tutte power series of a circle)**.** If Γ is a circle of length α, then

$$T^+(\Gamma; u, w) = w + \alpha + \binom{\alpha}{2} u + \binom{\alpha}{3} u^2 + \cdots.$$

Suppose G is a cycle graph consisting of n edges. Then

$$T^+(G; u, w) = n + \binom{n}{2} u + \binom{n}{3} u^2 + \cdots + nu^{n-2} + u^{n-1} + w = \frac{(1+u)^n - 1}{u} + w$$

Example 13 (Tutte power series of theta graph)**.** Suppose G is the graph with two vertices connected by three edges. Suppose Γ is the metric graph which assigns lengths a, b, c to the edges of G. Then

$$\begin{aligned} T^+(G; u, w) &= (\llbracket a \rrbracket_{1+u} \llbracket b \rrbracket_{1+u} + \llbracket a \rrbracket_{1+u} \llbracket c \rrbracket_{1+u} + \llbracket b \rrbracket_{1+u} \llbracket c \rrbracket_{1+u}) \\ &\quad + (\llbracket a \rrbracket_{1+u} \llbracket b \rrbracket_{1+u} \llbracket c \rrbracket_{1+u}) u + (\llbracket a \rrbracket_{1+u} + \llbracket b \rrbracket_{1+u} + \llbracket c \rrbracket_{1+u}) w + w^2. \end{aligned}$$

4.2. **Deleting bridges and contracting loops.** In this section we describe how the definition of Tutte power series $T^+(\Gamma; u, w)$ may be extended to a more general concept of metric graphs.

Definition 14. A *genus-weighted metric graph* $\Gamma = (G, \ell, \mathfrak{g})$ consists of a graph $G = (V, E)$, a length function $\ell : E \to \mathbb{R}_{>0}$ on edges, and a genus function $\mathfrak{g} : V \to \mathbb{Z}_{\geq 0}$ on vertices.

- If $\Gamma = \bigcup_{i=1}^k \Gamma_i$ is a disjoint union of k connected metric graphs Γ_i, then
$$T^+\left(\bigcup_{i=1}^k \Gamma_i; u, w \right) = u^{k-1}\, T^+\left(\vee_{i=1}^k \Gamma_i; u, w \right).$$
- If $\Gamma^{\mathfrak{g}} = (G, \ell, \mathfrak{g})$ is a genus-weighted metric graph, with underlying metric graph $\Gamma^0 = (G, \ell)$, then
$$T^+(\Gamma^{\mathfrak{g}}; u, w) = w^{\sum \mathfrak{g}(v)}\, T^+(\Gamma^0; u, w).$$

5. Proofs

Proof of Theorem thm:intro-tutte-series *1.* It suffices to show that the Tutte power series is invariant under an edge subdivision. Suppose $G = (V, E)$ contains the edge e_0, which we subdivide into $e_1 \cup e_2$ to obtain the graph G'.

By the deletion-contraction relation, Theorem thm:deletion-contraction 2, we have

$$T^+(G; u, w) = \llbracket a + b \rrbracket_{1+u} T^+(G \setminus e_0; u, w) + T^+(G/e_0; u, w)$$

and

$$\begin{aligned} T^+(G'; u, w) &= \llbracket a \rrbracket_{1+u} T^+(G \setminus e_1; u, w) + T^+(G/e_1; u, w) \\ &= \llbracket a \rrbracket_{1+u} \big(\llbracket b \rrbracket_{1+u} T^+(G \setminus \{e_1, e_2\}; u, w) + T^+(G \setminus e_1/e_2; u, w) \big) \\ &\quad + \big(\llbracket b \rrbracket_{1+u} T^+(G/e_1 \setminus e_2; u, w) + T^+(G/\{e_1, e_2\}; u, w) \big) \\ &= (\llbracket a \rrbracket_{1+u} + \llbracket b \rrbracket_{1+u} + u \llbracket a \rrbracket_{1+u} \llbracket b \rrbracket_{1+u}) T^+(G \setminus e_0; u, w) + T^+(G/e_0; u, w). \end{aligned}$$

Therefore. ...

$$\llbracket a \rrbracket_{1+u} + \llbracket b \rrbracket_{1+u} + u \llbracket a \rrbracket_{1+u} \llbracket b \rrbracket_{1+u}$$

□

6. Specializations of the Tutte polynomial

6.1. **Constants.** For a graph $G = (V, E)$, the Tutte polynomial has the following specializations to graph invariants when evaluated at particular integer points.

- $T^+(G; 1, 1) =$ the number of edge subsets; $T_G(2, 2) = 2^{\#E}$.
- $T^+(G; 0, 0) =$ the number of spanning trees.
- $T^+(G; 0, 1) =$ the number of connected spanning subgraphs.
- $T^+(G; 1, 0) =$ the number of acyclic spanning subgraphs.
- $T^+(G; -1, 1) =$ the number of totally cyclic orientations.

- $T^+(G;1,-1) =$ the number of acyclic orientations.
- $T^+(G;-k,-1) = (\pm 1/k)\cdot$ the number of k-colorings.

For a metric graph $\Gamma = (G,\ell)$,

$$T^+(\Gamma;1,1) = \sum_{A\subset E(G)} \prod_{e_i\in A} [\ell(e_i)]_2 = \sum_{A\subset E(G)} \prod_{e_i\in A} (2^{\ell(e_i)}-1).$$

$$= \prod_{e_i\in E(G)} (1+(2^{\ell(e_i)}-1)) = 2^{\sum_i \ell(e_i)}$$

- $T^+(\Gamma;1,1) = 2^{\mathrm{vol}(\Gamma)}$
- $T^+(\Gamma;0,0) = \mathrm{vol}(\mathrm{Jac}(\Gamma))$
- $T^+(\Gamma;0,1) = \sum_{k=0}^{g} \mathrm{vol}(\mathrm{Eff}^k(\Gamma))$?

Example 15. Suppose Γ is the circle graph with edge length a. Then

- the number of spanning trees is $T^+(\Gamma;0,0) = a$
- the number of connected spanning subgraphs is $T^+(\Gamma;0,1) = a+1$;
- the number of acyclic spanning subgraphs is $T^+(\Gamma;1,0) = 2^a - 1$;
- the number of totally acyclic orientations is $T^+(\Gamma;1,-1) = 2^a - 2$;
- the number of totally cyclic orientations is $T^+(\Gamma;-1,1) = 2$;

Example 16. Suppose Γ is the theta graph with edge lengths a, b and c,

$$\mathrm{ev}^+(\Gamma;1,1) = 2^{a+b+c}.$$

Number of spanning trees:
$T^+(\Gamma;0,0) = ab+ac+bc$.
Number of connected spanning subgraphs:
$T^+(\Gamma;0,1) = 1+(a+b+c)+(ab+ac+bc)$.
Number of acyclic spanning subgraphs:
$T^+(\Gamma;1,0) = 2^{a+b+c} - 2^a - 2^b - 2^c + 2$.
Number of totally cyclic orientations:
$T^+(\Gamma;-1,1) = 1+3+3-1 = 6$.
Number of totally acyclic orientations:
$T^+(\Gamma;1,-1) = 2^{a+b+c} - 2(2^a+2^b+2^c) + 6$.

Example 17. For the theta graph, we have

$$\begin{aligned} T^+(\Gamma;u,w) = w^2 &+ ([\![a]\!]_{1+u} + [\![b]\!]_{1+u} + [\![c]\!]_{1+u})w \\ &+ ([\![a]\!]_{1+u}[\![b]\!]_{1+u} + [\![a]\!]_{1+u}[\![c]\!]_{1+u} + [\![b]\!]_{1+u}[\![c]\!]_{1+u}) \\ &+ ([\![a]\!]_{1+u}[\![b]\!]_{1+u}[\![c]\!]_{1+u})u \end{aligned}$$

The Tutte constant coefficient is

$$\mathrm{coeff}(0,0;\Gamma) = ab+ac+bc.$$

The Tutte coefficient of w is

$$\mathrm{coeff}(0,1;\Gamma) = a+b+c.$$

The Tutte coefficient of u^0w^k is

$$\mathrm{coeff}(0,2;\Gamma) = 1.$$

$$\mathrm{coeff}(0,k;\Gamma) =$$

The Tutte coefficient of u is

$$\mathrm{coeff}(1,0;\Gamma) = a\binom{b}{2} + \binom{a}{2}b + \cdots + abc = abc + \frac{1}{2}\sum a^2b - \sum ab$$

$$\mathrm{coeff}(2,0;\Gamma) = a\binom{b}{3} + \binom{a}{2}\binom{b}{2} + \binom{a}{3}b + \cdots + ab\binom{c}{2} + a\binom{b}{2}c + \binom{a}{2}bc$$

$$= \frac{1}{6}\sum ab^3 - \frac{1}{2}\sum ab^2 + \frac{2}{3}\sum ab + \frac{1}{4}\sum a^2b^2 - \frac{1}{4}\sum ab^2 + \frac{1}{4}ab + \frac{1}{2}abc(a+b+c) - \frac{3}{2}abc$$

$$\mathrm{coeff}(k,1;\Gamma) = \binom{a}{k+1} + \binom{b}{k+1} + \binom{c}{k+1}.$$

Note that at $w = 0$, we have

$$\begin{aligned}
T^+(G;u,0) &= [\![a]\!]_{1+u}[b+c]_{1+u} + [\![b]\!]_{1+u}[\![c]\!]_{1+u} \\
&= \sum_{k\geq 0}\binom{a}{k+1}u^k \sum_{k\geq 0}\binom{b+c}{k+1}u^k + \sum_{k\geq 0}\binom{b}{k+1}u^k \sum_{k\geq 0}\binom{c}{k+1}u^k \\
&= \left(a + \binom{a}{2}u + \cdots\right)\left(b + c + \binom{b+c}{2}u + \cdots\right) + \left(b + \binom{b}{2}u + \cdots\right)\left(c + \binom{c}{2}u + \cdots\right) \\
&= (ab+ac+bc) + \left(a\binom{b+c}{2} + \binom{a}{2}(b+c) + b\binom{c}{2} + \binom{b}{2}c\right)u + ()u^2 + \cdots \\
&= (ab+ac+bc) + \frac{1}{2}\left(a(b+c)(a+b+c-2) + bc(b+c-2)\right)u + \cdots \\
&= (ab+ac+bc) + \frac{1}{2}\left(a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 + 2abc - 2ab - 2ac - 2bc\right)u + \cdots \\
&= (ab+ac+bc) + \left(abc + \frac{1}{2}(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) - ab - ac - bc\right)u + \cdots
\end{aligned}$$

$$\begin{aligned}
T^+(G;u,0) &= [\![a]\!]_{1+u}[b+c]_{1+u} + [\![b]\!]_{1+u}[\![c]\!]_{1+u} \\
&= [\![a]\!]_{1+u}\left([\![b]\!]_{1+u} + [\![c]\!]_{1+u} + u[\![b]\!]_{1+u}[\![c]\!]_{1+u}\right) + [\![b]\!]_{1+u}[\![c]\!]_{1+u} \\
&= [a][b] + [a][c] + [b][c] + u[a][b][c].
\end{aligned}$$

6.2. **Chromatic polynomial.** At $y = 0$ we obtain the chromatic polynomial of a (connected) graph:

$$\chi(G;\lambda) = (-1)^{\#V}(-\lambda)T(G;1-\lambda,0)$$

For a metric graph,

$$\chi(\Gamma;\lambda) = (-\lambda)\,T^+(\Gamma;-\lambda,-1) = \sum_{A\subset E}(-\lambda)^{h_0(\Gamma\backslash A)}(-1)^{h_1(\Gamma\backslash A)}\prod_{e\in A}[\![\ell(e)]\!]_{1-\lambda}.$$

6.3. **Flow polynomial.** At $x = 0$ we obtain the flow polynomial of a graph:

$$F(G;\lambda) = (-1)^{h_1(G)}T(G;0,1-\lambda)$$

For a metric graph,

$$\begin{aligned}
F(\Gamma;\lambda) = (-1)^{h_1(\Gamma)}T^+(\Gamma;-1,-\lambda) &= \sum_{A\subset E}(-1)^{h_0(\Gamma\backslash A)-1}(-\lambda)^{h_1(\Gamma\backslash A)}\prod_{e\in A}[\![\ell(e)]\!]_0 \\
&= \sum_{A\subset E}(-1)^{\chi(\Gamma\backslash A)}\lambda^{h_1(\Gamma\backslash A)}
\end{aligned}$$

Conclusion: (positive) edge lengths don't change the flow polynomial.

6.4. **Reliability polynomial.** The reliability polynomial of a graph satisfies

$$R(G;p) = (1-p)^{\#V - h_0(G)} p^{h_1(G)} T\left(G; 1, \frac{1}{p}\right)$$

For a metric graph,

$$\begin{aligned} R(\Gamma;p) &= (1-p)^{\infty} p^{h_1(\Gamma)} T^+\left(\Gamma; 0, \frac{1-p}{p}\right) \\ &= (1-p)^{\infty} p^{h_1(\Gamma)} \sum_{\substack{A\subset E \\ \Gamma\setminus A \text{ connected}}} \left(\frac{1-p}{p}\right)^{h_1(\Gamma\setminus A)} \prod_{e\in A} \llbracket \ell(e) \rrbracket_1 \\ &= (1-p)^{\infty} \sum_{\substack{A\subset E \\ \Gamma\setminus A \text{ connected}}} p^{\#A}(1-p)^{h_1(\Gamma\setminus A) - \#A} \prod_{e\in A} \ell(e) \end{aligned}$$

6.5. **Potts model polynomial.** Following Sokal [12, Section 2.5]

The (modified) Potts model polynomial, or cluster-generating function, $\widetilde{Z}(G;q,v)$ is

$$\widetilde{Z}(G;q,v) = \sum_{A\subset E} q^{h_0(G|A) - |V|} v^{|A|} = \sum_{A\subset E} q^{h_1(G|A)} (v/q)^{|A|}$$

$$\widetilde{Z}(G;q,v) = (q/v)^{h_0(G)}(v/q)^{|V|} T(G; 1+\frac{q}{v}, 1+v) = (v/q)^{|V|-h_0(G)} T^+(G; \frac{q}{v}, v).$$

$$\begin{aligned} Z(\Gamma;q,v) &= (v/q)^{\infty} T^+(\Gamma; q/v, v) \\ &= (v/q)^{\infty} \sum_{A\subset E} (q/v)^{h_0(\Gamma\setminus A)-1} v^{h_1(\Gamma\setminus A)} \prod_{e\in A} \llbracket \ell(e) \rrbracket_{1+q/v} \end{aligned}$$

7. Miscellaneous

7.1. **Tutte power series as real function.** Given real parameters x, y with $x > 0$, let

eq:tutte-metric-graph

$$T(\Gamma;x,y) = \sum_{A\subset E(G)} \left(\prod_{e\in A} \llbracket \ell(e) \rrbracket_x \right) (x-1)^{h_0(G\setminus A)-1} (y-1)^{h_1(G\setminus A)} \tag{9}$$

where the notation $[\alpha]_x$ for real $\alpha, x > 0$ means

$$[\alpha]_x = \frac{x^{\alpha}-1}{x-1} \quad \text{if } x \neq 1, \qquad [\alpha]_1 = \alpha.$$

For a fixed metric graph Γ, the expression (9) defines a function $\mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ by associating $(x,y) \mapsto T(\Gamma;x,y)$. This function is generally not a polynomial in x; moreover, it does not admit a formal power series expansion in x if any edge length $\ell(e_i)$ is non-integral.

It is straightforward to verify that the Tutte power series $T^+(\Gamma;u,w)$ converges to a real value when $|u| < 1$. For a generic choice of edge lengths, the radius of convergence in u is equal to 1.

8. Further questions

Observation: for a fixed combinatorial graph G, the (i,j)-coefficient of the Tutte power series $T^+(\Gamma;u,w)$ is a polynomial in the edge lengths.

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Appendix: extra material

[probably to delete]

Example 18 (Tutte power series of K_4)**.** Suppose $G = K_4$, the complete graph on four vertices. Suppose Γ is the metric graph assigning edge lengths a, b, c, d, e, f to G, as shown in Figure ♢ [fill in] ♢ .

Then we have

$$\begin{aligned}
T^+(\Gamma; u, w) = {} & ([a][b][d] + [a][b][e] + [a][b][f] + [a][c][d] + [a][c][e] + [a][c][f] + [a][d][e] + [a][d][f] \\
& \quad + [b][c][d] + [b][c][e] + [b][c][f] + [b][d][e] + [b][e][f] + [c][d][f] + [c][e][f] + [d][e][f]) \\
& + ([a][b][c][d] + [a][b][c][e] + [a][b][c][f] + [a][b][d][e] + [a][b][d][f] + [a][b][e][f] \\
& \quad + [a][c][d][e] + [a][c][d][f] + [a][c][e][f] + [a][d][e][f] + [b][c][d][e] \\
& \quad + [b][c][d][f] + [b][c][e][f] + [b][d][e][f] + [c][d][e][f])u \\
& + ([a][b][c][d][e] + [a][b][c][d][f] + [a][b][c][e][f] + [a][b][d][e][f] \\
& \quad + [a][c][d][e][f] + [b][c][d][e][f])u^2 \\
& + [a][b][c][d][e][f]u^3 \\
& + ([a][b] + [a][c] + [a][d] + [a][e] + [a][f] + [b][c] + [b][d] + [b][e] \\
& \quad + [b][f] + [c][d] + [c][e] + [c][f] + [d][e] + [d][f] + [e][f])w \\
& + ([a][b][c] + [a][e][f] + [b][d][f] + [c][d][e])uw \\
& + ([a] + [b] + [c] + [d] + [e] + [f])w^2 + w^3
\end{aligned}$$

Compare to the Example in Read–Whitehead RW2 [11, p. 272].

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