SMA2172 - CALCULUS PROF. OKELLO

TOPICS

1. Functions & Mappings.
   - Def.
   - Domain
   - Range
   - Codomain.
   - Composition / composite
   - Inverse

2. Limits, Continuity and differentiability of a function

3. Differentiation by first principle and for the rule 'x^n' integral and fractional n.

4. Other techniques of differentiation. i.e. Sums
   - Products
   - Quotients
   - Chain rule

   } Applied to algebraic, trigonometric functions, logarithmic function, exponential, inverse trig function, all of a single variable.

5. Implicit and Parametric differentiation.

6. Application of differentiation to:
   - Rates of change
   - Small changes
   - Stationary points
   - Equations of tangents and normal lines
   - Kinematics
   - Cost, revenue and profit.

7. Introduction to Integration and its application to area and volume.

References:

1. Calculus and analytic Geometry by: Finney and Thomas. Larson and Hostlers Penny
2. Any book in Calculus of a single variable.

FUNCTIONS AND MAPPING

Function - A function is a rule that assigns / associates each element in the independent set say x to a unique element in the dependent set say y.

Examples of functions : $y = x + 5$ linear function.

$y(x) = x^2 - 2x + 5$ Quadratic function

$y(x) = x^3 - 1$ Cubic function.

$y(x) = \sin(2x + 5)$ Trigonometric function

$y(x) = \log(3x + 1)$ logarithmic function (log base 10)

$y(x) = \ln(5x + 1)$ Natural log function (log base e)

$y(x) = \tan^{-1}(2x + 1)$ Inverse of trigonometric function

$y(x) = e^{2x+1}$ Exponential function

Domain - Consists of all the elements in the independent set for which the function is defined. (not undefined) (x-values)

Range - All the images of elements in the domain.

Co-domain - All elements in the dependent set.

Examples

1. State the domain and range of :

a) $f(x) = 2x^2 + 7$

x		y
1	$2(1)^2 + 7$	9
2	$2(2^2) + 7$	15
3	$2(3^2) + 7$	25
		30
		40

Domain $\{1, 2, 3\}$

Range $\{9, 15, 25\}$

Codomain $\{9, 15, 25, 30, 40\}$

b) $f(x) = \dfrac{4}{x^2 - 5x + 6}$

N/B: $\dfrac{4}{0}$ = Infinity, undefined, very large value $\dfrac{0}{4} = 0$

- To identify the domain find values of x for which the denominator is 0 and exclude them in the domain statement.

$x^2 - 5x + 6 = 0$

$x = \dfrac{5 \pm \sqrt{25 - 24}}{2}$

$\dfrac{5+1}{2}$ or $\dfrac{5-1}{2}$

3 or 2

Therefore : Domain include all real values of x except $x = 2$ and $x = 3$

b) $y(x) = 2x^2 - 5x + 1$
- The domain is from $-\infty$ to ∞. (∞ Infinity)

d) $f(x) = \sqrt{x-1}$ $\sqrt{-ve}$ is undefined.

$\sqrt{x-1}$ is not defined for $x - 1 < 0$
$= x < 1$

Domain $[1 \ldots +\infty]$ open.
Shows that it is closed.

Range $[0, \ldots \infty)$

NOTE : $y(x)$ is read as y of x meaning that y depends on x
$f(x)$ is read as f or x meaning that f depends on x.

- y and f are dependent variables.
- x is the only independent variable.

d) State the range and domain of :
1. $f(x) = 6 - x^2$
2. $f(x) = \dfrac{6}{x^{2/3} + x^{1/3} - 12 \neq 0}$

3. $f(x) = \dfrac{6 + 3x}{1 - 2x}$ $1 - 2x = 0$ $\frac{1}{2} = \frac{2x}{2}$ $x = \frac{1}{2}$
4. $f(x) = \sqrt{x - 5}$

EVALUATION OF FUNCTIONS.

- This involves replacing x in the function by the suggested value and retaining the rule of the function.

Eg. Given $f(x) = 2x + 1$ Find :
(i) $f(0)$ $\rightarrow 2(0) + 1 = 1$
(ii) $f(1)$ $\rightarrow 2(1) + 1 = 3$
(iii) $f(0+1) \rightarrow 2(0+1)+1 = 2 + 1 = 3$.
(iv) $f(x+2) \rightarrow 2(x+2) + 1 = 2x + 4 + 1 = 2x + 5$
(v) $\dfrac{f(x+h) - f(x)}{h}$ $h \neq 0$

$\dfrac{\cancel{2x}+1+2h - \cancel{2x}+1}{h} = \dfrac{2\cancel{h}}{\cancel{h}}$ $\dfrac{\cancel{2x} + 2h + \cancel{1} - (\cancel{2x} + \cancel{1})}{h}$ $= 2$.

2. Given $f(x) = 3x^2 - 2x + 4$

Find: $f(0)$. $\quad 3(0^2) - 2(0) + 4 = 4$

$f(-1) \quad 3(-1^2) - 2(-1) + 4 = 3 + 2 + 4 = 9$

$f(x+2) \quad 3(x+2)^2 - 2(x+2) + 4$

$$3(x^2 + 4x + 4) - 2x \cancel{+4} \cancel{-4}$$
$$3x^2 + 12x + 12 - 2x$$
$$= 3x^2 + 10x + 12$$

$$\frac{f(x+h) - f(x)}{h} =$$

$$\frac{3(x+h)^2 - 2(x+h) + 4 - \cancel{3(x+h}\,(3x^2 - 2x + 4)}{h}$$

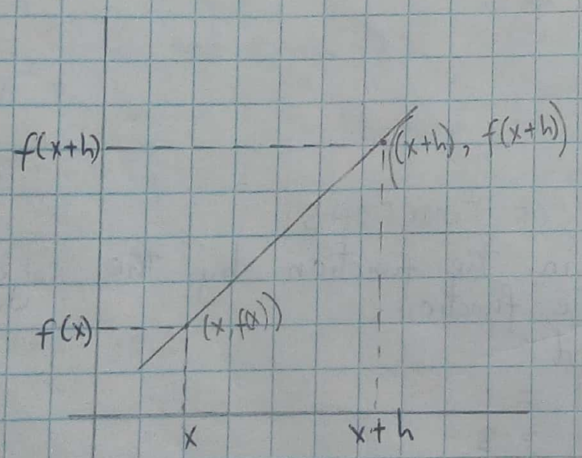
$$\frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{2x} - 2h + \cancel{4} - \cancel{3x^2} + \cancel{2x} - \cancel{4}}{h}$$

$$\frac{3h^2 - 2h + 6xh}{h}$$

$$\frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} = 6x + 3h - 2.$$

NOTE: $\left(\frac{f(x+h) - f(x)}{h} \quad h \neq 0\right)$ is called difference - quotient.

- It is very important in First Principle of differentiation.



$$\text{Gradient} = \frac{\Delta V}{\Delta H}.$$

$$G = \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$$

$$= \frac{f(x+h) - f(x)}{h}$$

EXERCISE

1. Given $f(x) = x^3 + 2x + 1$ Find:

a) $f(-a)$
b) $f(-9)$
c) $f(0)$
d) $f(x+2)$
e) $\frac{f(x+h) - f(x)}{h} \quad h \neq 0$

2. Given $g(x) = \frac{1}{\sqrt{x} + 1}$ Find

a) $g(1)$

b) $g(0)$

c) $g(x+2)$

d) $\frac{g(x+h) - g(x)}{h}$ $h \neq 0$

3. Given $p(x) = \frac{6 - 2x}{1 + 3x}$ Find

$p(0)$

$p(-1)$

$p(-x + 2)$

$\frac{p(x+h) - p(x)}{h}$

4. Given $f(x) = x^2 + 5x + 1$ Find:

$f(x + \Delta x)$

$f(x + h)$

$\frac{f(x + \Delta x) - f(x)}{(\Delta x)}$ $\Delta x \neq 0$

$\frac{f(x+h) - f(x)}{h}$ $h \neq 0$

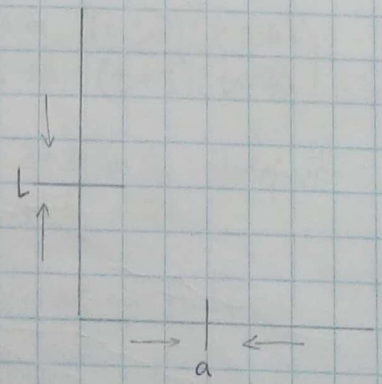
COMPOSITE FUNCTIONS → Later (Done at the back)

The Concept of Limits.

- If x approaches a value 'a' from either RHS or LHS of "a", then $f(x)$ approaches value L.

We say the limits of $f(x)$ as x tends to a is L, mathematically written as

$$\lim_{x \to a} f(x) = L$$



Properties of Limits

If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$ Then

1. $\lim \left(f(x) \pm g(x)\right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L_1 \pm L_2$

2. $\lim \left(f(x)\, g(x)\right) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$

③. $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$ Provided $g(a) \neq 0$
The denominator is not zero at $x = a$

④. $\lim_{x\to a} \sqrt[n]{f(x)} = \left(\lim_{x\to a} f(x)\right)^{1/n} = (L_1)^{1/n} = \sqrt[n]{L_1}$

⑤ If $f(x) = c$ (Where c is a constant then $\lim_{x\to a} f(x) = \lim_{x\to a} c = c$

Techniques of Evaluating Limits

1. By Direct Substitution (DS).

- Substitute the limiting value in the problem, conclude if the result is finite. Otherwise look for alternative methods.

Example: Evaluate

① $\lim_{x\to a}\left(x^3+6\right) = 5$

By DS in place of x put -1

$(-1)^3 + 6 = 5$.

$-1 + 6 = 5$

$= 5$.

② $\lim_{x\to 2}\left(\frac{6x-1}{1+3x}\right) = 11/7$

By D/s in place of x put 2.

$\frac{6(2)-1}{1+3(2)} = \frac{12-1}{1+6} = \frac{11}{7}$

③. $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)$

By DS $0/0$ undefined, → Look for alternative method.

2. By Factorization

-Factorize either the numerator or the denominator in order to eliminate the bracket that contributes to ~~either~~ getting a zero in either the numerator or denominator.

Recall: $(a+b)(a-b) = a^2 - b^2$

$ax^2 + bx + c \rightarrow$ Sum $= b$
Prod $= ac$

Eg. $\lim_{x \to 1} \left(\frac{x^2-1}{x-1}\right)$

By D/S $0/0$.

By Factorization $\lim_{x\to 1} \left(\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}\right)$

$\lim_{x\to 1} (x+1)$

$1+1$

$=2$.

② $\lim_{x\to 1} \left(\frac{x^2-x-2}{x^2+3x-4}\right)$ $1+3-4$ $\frac{-2}{0}$

By D/s $0/0$ $-2/0$

By Factorization $\lim_{x\to 1} \left(\frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}(x+4)}\right)$

$\lim_{x\to 1} \left(\frac{x+2}{x+4}\right)$ $\frac{1+2}{1+4} = 3/5$

③ $\lim_{t\to 3} \left(\frac{t^2+6t+9}{t^2-9}\right)$

By D/s $36/0$

By Factorization. $\lim_{t\to 3} \left(\frac{\cancel{(t+3)}(t+3)}{\cancel{(t+3)}(t-3)}\right)$

$\frac{3+3}{3-3} = \frac{6}{0}$ Undefined.

N/B. Factorization fails.

3. L'Hospital's rule.

- If on direct substitution of the limiting value you get a zero in the denominator, L'Hospital's rule require that we differentiate the numerator and denominator separately.
- ✓ Subject the result of the derivatives to the limit condition.
- ✓ Conclude if the result is finite, Otherwise repeat the procedure.

Procedure → Redifferentiate the current numerator and denominator again.

Recall:

x^n	nx^{n-1}
x^2	$2x^{2-1} = 2x$
x^3	$3x^{3-1} = 3x^2$
t^2	$2t$
6	0
9	0
$6x^3$	$6(3)x^{3-1} = 18x^2$

Evaluate:

① $\lim_{x\to 1} \left(\frac{x^2-1}{x-1}\right)$

a) D/s $= 0/0$

b) ~~Factorization~~. L'Hospital's rule

Diff $x^2-1 \Rightarrow 2x$
$x-1 \Rightarrow 1$

$\lim_{x\to 1} \left(\frac{2x}{1}\right) = \frac{2}{1} = 2.$

② $\lim_{x\to 2} \left(\frac{x^4-16}{x^2-4}\right)$

D/s $= 0/0$

L'Hospital's rule

$N = x^4-16 \to 4x^3$

$D = x^2-4 \to 2x$

$$\lim_{x\to 2}\left(\frac{4x^3}{2x}\right) = \frac{4(2^3)}{2(2)} = \frac{\cancel{4}\times 8}{\cancel{4}} = 8$$

2. $$\lim_{t\to 3}\left(\frac{t^2+6t+9}{t^2-9}\right)$$

By D/S = 36/0

Factorization 6/0 Undefined

Hospital's rule →

N t^2+6t+9 $\frac{dy}{dt} \to 2t+6$

D t^2-9 $\frac{dy}{dt} \to 2t$

$$\lim_{t\to 3}\left(\frac{2t+6}{2t}\right) = \frac{2(3)+6}{2(3)} = \frac{12}{6} = 2$$

$$\therefore \lim_{t\to 3}\left(\frac{t^2+6t+9}{t^2-9}\right) = 2$$

4. By Rationalization.

Rationalize either the numerator or the denominator by multiplying each by an appropriate conjugate.

Recall:

① $\sqrt{x} - 2$ $C = \sqrt{x} + 2$

$\sqrt{x+9} - 3$ $C = \sqrt{x+9} + 3$

$\sqrt{y} + \sqrt{x}$ $C = \sqrt{y} - \sqrt{x}$

② $(a-b)(a+b) = a^2 - b^2$

$(\sqrt{x+9} - 3)(\sqrt{x+9} + 3) = ((x+9)-3) - (x+9+3)$

$(x+9) + 3\cancel{\sqrt{x+9}} - 3\cancel{\sqrt{x+9}} - 9 \neq (x+9-3) - (x+12)$

$x+9-3-\cancel{x}-12$

$x+\cancel{9}-9$ $9-3-12$

$= x$ $= -6$

$$\left(\sqrt{x+1} - \sqrt{1-x}\right)\left(\sqrt{x+1} + \sqrt{1-x}\right) = 1+x-(1-x) = 2x$$

$(x+1)-(1-x) = x+\cancel{1}-\cancel{1}+x = 2x$

Evaluate:

1. $$\lim_{x\to 2}\left(\frac{x-4}{\sqrt{x}-2}\right)$$

D/S → 0/0

Rationalization × both N and D by $\sqrt{x}+2$

$$\frac{x-4\,(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{\cancel{x-4}\,(\sqrt{x}+2)}{\cancel{x-4}}$$

$$\lim_{x\to 2}\left(\sqrt{x}+2\right) = \sqrt{2}+2$$

$$\lim_{x\to 2}(\sqrt{x}+2) = 4.$$

2. $$\lim_{x\to 0}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{x}\right)$$

D/S ⇒ 0/0

Rationalize × both N & D by $\sqrt{x+1} + \sqrt{1-x}$

$$\lim_{x\to 0} \frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{x\,(\sqrt{1+x} + \sqrt{1-x})}$$

$$\frac{2\cancel{x}}{\cancel{x}(\sqrt{1+x})+(\sqrt{1-x})} = \frac{2}{(\sqrt{1+x}) + (\sqrt{1-x})}$$

$$= \frac{2}{2} = 1$$

$$\lim_{x\to 0}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{x}\right) = 1$$

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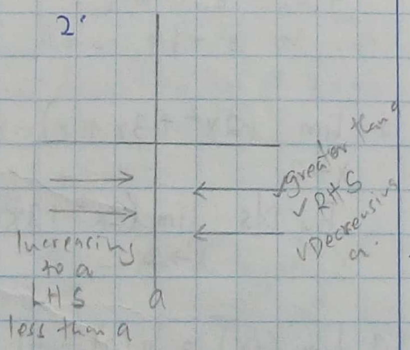
- If x approaches a from the LHS of "a" or from values less than "a" or ~~and the limit~~ increases to "a" $f(x)$ approachs L and the limit we obtain a left hand limit of $f(x)$ written as $\lim\limits_{x \to a} f(x)$

↳ Indicate the direction of approach.

2. If x approaches a from the right HS of "a" or from values greater than "a" or decreases to "a". $f(x)$ approaches L, and we obtain a RHLimit of $f(x)$ written as

$\lim\limits_{x \to a^{+}} f(x)$ ↘ Indicates the direction of approach.

"a" can be negative or positive.



3. $f(x)$ is said to have a limit if and only if the LHLimit = RHLimit. Otherwise, it will possess one-sided limits.

4. Meaning of absolutely stated functions such af as

$f(x) = |x-3|$

$f(x) = |x+5|$

$f(x) = |x|$

- First identify the reference by equating the middle term to Z.

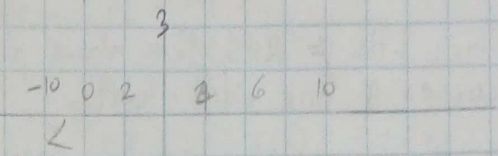
$x - 3 = 0 \qquad x = 3$

$x + 5 = 0 \qquad x = -5$

$x = 0 \qquad x = 0$

} Reference

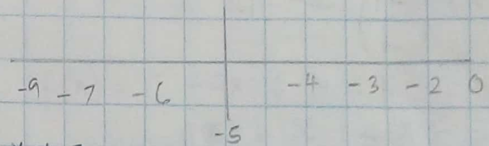
- Investigate the behavior of the function to the left and the right of the Reference point.



$2-3=-1$ | RP $4-3=1$
$0-3=-3$ | $6-3=3$
$-10-3=-13$ | $10-3=7$

$-(x-3)$ of $x<3$ $\qquad +(x-3)$ of $x>3$

$$f(x) = |x-3| = \begin{cases} -(x-3) & \text{of } x < 3 \\ +(x-3) & \text{of } x > 3 \end{cases}$$



$x+5$ | $x+5$
$-9+5=-4$ | $-4+5=-1$
$-7+5=-2$ | $-3+5=-2$
$-6+5=-1$ | $0+5=5$

$-(x+5)$ of $x<-5$ $\qquad +(x+5)$ of $x>-5$

$$f(x) = |x+5| = \begin{cases} -(x+5) & \text{of } x < -5 \\ +(x+5) & \text{of } x > -5 \end{cases}$$

<u>Evaluate</u>

$$\lim_{x \to 7} \left(\frac{|x-7|}{(x-7)} \right)$$

Solution.

(i) Separate $|x-7| = \begin{cases} -(x-7) & \text{of } x < 7 \\ x-7=0 & \text{of } x = 7 \\ +(x-7) & \text{of } x > 7 \end{cases}$

LHS $<$ 7	RHS $>$
$\lim_{x\to 7}\left(\frac{-(x-7)}{(x-7)}\right)$	$\lim_{x\to 7}\left(\frac{+(x-7)}{(x-7)}\right)$
$\lim_{x\to 7}(-1)$	$\lim_{x\to 7}(+1)$
Limit of a constant is the constant $\therefore \lim_{x\to 7} = -1$	$\lim_{x\to 7}(+1) = +1$

Since LHS $\neq$ RHS, the above problem posseses a one sided limits.

Solution II

LHS

$$\lim_{x\to 7}\left(\frac{-(x-7)}{(x-7)}\right)$$

D/s 0/0

$$\lim_{x\to 7}\left(\frac{-x+7}{x-7}\right) \quad \text{Diff } \frac{N' = -1}{D' = 1}$$

$$\lim_{x\to 7}\left(\frac{-1}{1}\right) = -1$$

RHS

$$\lim_{x\to 7}\left(\frac{+(x-7)}{(x-7)}\right) \quad \frac{N' = 1}{D' = 1}$$

Since LHS $\neq$ RHS, it has one sided limits.

Exercise: Evaluate

1. $\lim_{x\to 6}\left(\frac{x-6}{|x-6|}\right)$
2. $\lim_{x\to 9}\left(\frac{|x+9|}{x+9}\right)$
3. $\lim_{x\to 1}\left(2 + \frac{1}{|x-1|}\right) = 2$

CONTINUITY OF A FUNCTION

- A function $f(x)$ is said to be continuous at a point $x = a$ if the following 3 conditions are satisfied:

1. $f(x)$ must be defined at $x = a$ ($f(a)$ must exist. D/S must give defined.)
2. $\lim_{x\to a} f(x)$ must exist (LHL = RHL at $x = a$)
3. $\lim_{x\to a} f(x) = f(a)$
   (ii) = (i)

- Otherwise, $f(x)$ will be discontinuous at $x = a$.

- All polynomials of any degree are continuous in the interval $(-\infty, +\infty)$

For example:

1. Investigate whether $y(x)$ is continuous at $x = 2$
   $y(x) = -2x^2 + 3x + 1$.

Soln

1. $y(2) = -2(2)^2 + 3(2) + 1$
   $y(2) = -8 + 6 + 1$
   $= -1$

2. $\lim_{x\to 2}(-2x^2 + 3x + 1)$
   $-8 + 6 + 1$
   By D/s $\lim_{x\to 2}(-2x^2 + 3x + 1) = -1$

3. $\lim_{x\to 2} f(x) = y(2) = -1$

$y(x)$ is continuous at $x = 2$

2. Investigate whether $f(x) = \frac{x-1}{x^2+x-2}$ is Continuous at $x = 1$

Soln

Test $f(1) = \frac{1-1}{1+1-2} = \frac{0}{0}$ Undefined hence $f(x)$ is not continuous at $x=1$

But by rewriting $f(x) = \frac{\cancel{x-1}}{\cancel{(x-1)}(x+2)}$

$$\frac{1}{x+2} = \frac{1}{3}$$

Test (i) $\frac{1}{x+2} = \frac{1}{3}$

2) $\lim_{x \to 1} \left(\frac{1}{x+2}\right) = \frac{1}{3}$

3) Since $\lim_{x \to 1} (f(x)) = f(1) = \frac{1}{3}$

- $f(x)$ becomes continuous at $x = 1$, hence $f(x)$ has a removable point of discontinuity.

Identify points where $f(x) = \frac{1}{x^2+5x+6}$ is

- ✓ Not continuous
- ✓ Not defined
- ✓ Not Discontinuous

where $x^2 + 5x + 6 = 0$

$$x = \frac{-5 \pm \sqrt{25-24}}{2}$$

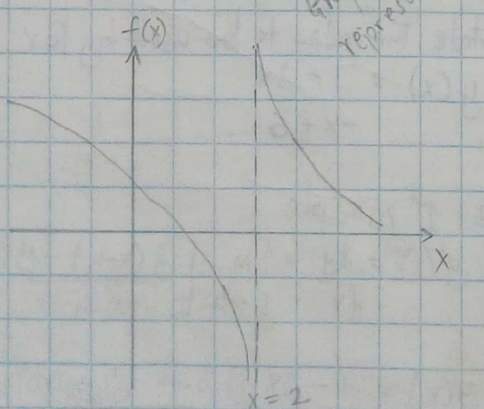
$$= \frac{-5 \pm 1}{2} \quad x = -3 \text{ or } -2$$

$f(x)$ is not continuous, not defined and has points of discontinuity at $x = -3$ and at $x = -2$.

$f(x) = \frac{4}{x-2}$

$x - 2 = 0 \rightarrow x = 2$

Graphical representation.



x

f(x)

Find the values of the constants in the equations below if $f(x)$ is continuous everywhere in the real number line.

$$f(x) = \begin{cases} ax & \text{if } x < -1 \\ 7+2x & \text{if } x > -1 \end{cases}$$

N/B → In such a question make use of the 2^{nd} condition of continuity in particular, LHL = RHL at any pt $x = a$.

Reference pt = −1

<	>
$f(x) = ax$	$f(x) = 7 + 2x$
LHL	RHL
$\lim_{x \to -1} (ax)$	$\lim_{x \to -1} = 7 + 2(-1)$
$-1 \times a$	$= 7 - 2$
$= -a$	$= 5$

LHL = RHL at $x = -1$

$-a = 5$

$a = -5$

$$f(x) = \begin{cases} -2x & \text{if } x < 1 \\ ax + b & \text{if } 1 \leq x \leq 3 \\ 16 & \text{if } x \geq 3 \end{cases}$$

Find a and b.

(1)

<	>
$f(x) = -2x$	$f(x) = ax + b$
LHL	RHL
$\lim_{x \to 1} (-2x)$	$\lim_{x \to 1} (ax + b)$
$-2(1)$	$a(1) + b$
$= -2$	$= a + b$

RHL = LHL at $x = 1$

$-2 = a + b$ (i)

Where $f(x) = ax + b$ if $1 \leq x$

Rewritten it is $x \geq 1$

R.P → 3

< 3	> 3
$f(x) = ax + b$	$f(x) = 16$
LHL	RHL
$\lim_{x \to 3} (ax+b)$	$\lim_{x\to 3} (16) = 16$ constant
$a(3) + b$	
$= 3a + b$	

RHL = LHL at $x = 3$

$3a + b = 16$ (ii)

$$a + b = -2$$
$$3a + b = 16 \; -$$
$$-2a = -18$$
$$a = 9$$

$$b = 16 - 27$$
$$= -11$$

$\therefore a = 9$ and $b = -11$.

Quiz.

3. $f(x) = \begin{cases} -2x & \text{if } x < 1 \\ b - ax^2 & \text{if } 1 \le x < 4 \\ -16x & \text{if } x \ge 4 \end{cases}$

Find a and b.

4. $f(x) = \begin{cases} 6 & \text{if } x < -2 \\ ax^2 + b & \text{if } -2 \le x < 3 \\ -16x & \text{if } x \ge 3 \end{cases}$

Find a and b.

DERIVATIVE OF FUNCTIONS

- The derivative of a fxn $f(x)$ denoted by $f'(x)$ or $\frac{df}{dx}$ is given by:

$$\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \quad \text{—— (1)}$$

Equation (1) gives the rate of change of f with respect to x and it is called First principle of differentiation. or differentiation by the definition. or differentiation of first kind.

Example:

1. Use first principle of differentiation to find the derivative of the following functions

1. $f(x) = 3 - 2x - x^2$

By 1st principle of dif:

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \quad \text{—— (1)}$$

$$f(x) = 3 - 2x - x^2 \quad \text{—— (2)}$$

$$f(x+h) = 3 - 2(x+h) - (x+h)^2$$

$$f(x+h) = 3 - 2x - 2h - x^2 - 2xh - h^2 \quad \text{—— (3)}$$

Substitute eqn (2) and (3) in (1)

$$\lim_{h\to 0}\left[\frac{\cancel{3} - \cancel{2x} - 2h - \cancel{x^2} - 2xh - h^2 - (\cancel{3} - \cancel{2x} - \cancel{x^2})}{h}\right]$$

$$\lim_{h\to 0}\left[\frac{-2h - 2xh - h^2}{h}\right]$$

$$\lim_{h\to 0}\left[-2 - 2x - h\right]$$

put 0 in place of h

$$-2 - 2x - h = -2 - 2x - 0$$

$$\therefore \frac{df}{dx} = -2 - 2x$$

2. ~~State the rule to be used by~~ for

$$y(x) = \frac{1 - 3x}{-x + 5}$$

By 1st principle

$$y'(x) = \frac{dy}{dx} = \lim_{h\to 0}\left[\frac{y(x+h) - y(x)}{h}\right] \quad \text{—— (1)}$$

$$y(x) = \frac{1 - 3x}{-x + 5} \quad \text{—— (2)}$$

$$y(x+h) = \frac{1-3(x+h)}{-(x+h)+5} = \frac{1-3x-3h}{-x-h+5} \quad ③$$

Substitute ② & ③ in ①

$$\lim_{h\to 0}\left[\frac{1-3x-3h}{-x-h+5} - \frac{1-3x}{5-x}\right] \times \frac{1}{h}$$

Take the numerator first:

$$\frac{[5-x(1-3x-3h)]-[(1-3x)(5-x-h)]}{(5-x)(5-x-h)}$$

$$\frac{(5-x+3x-15x-15h+3x^2+3xh)-(5-x-h-15x+3x^2+3xh)}{(5-x)(5-x-h)}$$

Simplified $N = \frac{-14h}{(5-x)(5-x-h)}$

$D = h.$

$$\frac{-14\cancel{h}}{(5-x)(5-x-h)} \times \frac{1}{\cancel{h}} = \frac{-14}{(5-x)(5-x-h)}$$

$$y'(x) = \frac{dy}{dx} = \lim_{h\to 0} = \frac{-14}{(-x+5)(5-x)-h)}$$

put 0 in place of h.

$25-10x+x^2$

$$= \frac{-14}{(5-x)^2}$$

$\frac{-14x}{c+25x-5x^2+\frac{1}{3}x^3}$ $\frac{-14}{25-5x+\frac{1}{3}x^2}$

3. $$y(x) = \frac{1}{\sqrt{x}+2}$$

$$y'(x) = \frac{dy}{dx} = \lim_{h\to 0}\left[\frac{y(x+h)-y(x)}{h}\right] \quad ①$$

$$y(x) = \frac{1}{\sqrt{x}+2} \quad ②$$

$$y(x+h) = \frac{1}{\sqrt{x+h}+2} \quad ③$$

Substitute ② and ③ in 1

$$\lim_{h\to 0}\left[\frac{1}{\sqrt{x+h}+2} - \frac{1}{\sqrt{x}+2}\right] \times \frac{1}{h}$$

$$\left[\frac{(\sqrt{x}+2)-(\sqrt{x+h}+2)}{(\sqrt{x}+2)(\sqrt{x+h}+2)}\right] \times \frac{1}{h}$$

$$\left(\frac{\sqrt{x}\cancel{+2}-\sqrt{x+h}\cancel{-2}}{(\sqrt{x}+2)(\sqrt{x+h}+2)}\right) \times \frac{1}{h}$$

Put h=0

$$\frac{\sqrt{x}-\sqrt{x+h}}{(\sqrt{x}+2)(\sqrt{x+h})+2)} \times \frac{1}{h}.$$

$$= \lim_{h\to 0}\left(\frac{1}{h} \cdot \frac{\sqrt{x}-\sqrt{x+h}}{(\sqrt{x}+2)(\sqrt{x+h}+2)}\right)$$

By D/s = 0/0

Rationalize by $\sqrt{x}+\sqrt{x+h}$

$\cancel{\left(\frac{1}{h} \cdot (\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})\right)}$

Let $\sqrt{x}$ be a $\sqrt{x+h}$ be b.

$\cancel{\lim_{h\to 0}\left(\frac{1}{h} \cdot \frac{a-b}{\sqrt{x}-\sqrt{x+h}} \cdot \frac{a+b}{\sqrt{x}+\sqrt{x+h}}\right)}$ $(\sqrt{x+h}+2)(\sqrt{x}+2)$

$$\lim_{h\to 0}\left(\frac{1}{h}\frac{\sqrt{x}-\sqrt{x+h}}{(\sqrt{x+h}+2)(\sqrt{x}+2)} \times \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{x+h}+2)(\sqrt{x}+2)(\sqrt{x}+\sqrt{x+h})}$$

$$= \frac{-1}{(\sqrt{x+h}+2)(\sqrt{x}+2)(\sqrt{x}+\sqrt{x+h})}$$

Put 0 in place of h.

$$y'(x) = \frac{-1}{(\sqrt{x}+2)^2+2\sqrt{x}}$$

$y(x) = 6-x^3$

$y(x) = -2x^2-4x+5$

$y(x) = \frac{x}{1+3x}$ 2

$$y(x) = \frac{11-3x}{-2x-5}$$

$$y(x) = \frac{6+5x}{1-3x}$$

$$y(x) = \sqrt{x} - 5$$

$$y(x) = \frac{1}{\sqrt{x+5} - 3}$$

$$y(x) = \frac{1}{\sqrt{2x+1}}$$

Consider $y(x) = u(x)\,v(x)$
A product of 2 differentiable functions of x.
By 1^{st} Principle of differentiation

$$y'(x) = \frac{dy}{dx} = \lim_{h\to 0}\left[\frac{y(x+h) - y(x)}{h}\right] \quad \text{—(1)}$$

$$y(x) = u(x)\,v(x) \quad \text{—(2)}$$

$$y(x+h) = u(x+h)\,v(x+h) \quad \text{—(3)}$$

Substitute 2 & 3 to 1

$$y'(x) = \lim_{h\to 0}\left[\frac{u(x+h)\,v(x+h) - u(x)v(x)}{h}\right]$$

To the numerator + and − $u(x+h)\,v(x)$

Rearrange to get $u'(x)$ & $v'(x)$ by diff of

$$\frac{dy}{dx} = \left[\frac{u(x+h)v(x+h) + u(x+h)v(x) - u(x+h)v(x) - u(x)v(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[u(x+h)\left[\frac{v(x+h) - v(x)}{h}\right] + v(x)\left(\frac{u(x+h) - u(x)}{h}\right)\right]$$

Distribute the limits, by principle 1, 2 and 5 of limits.

$$\left(\lim_{h\to 0} u(x+h)\right)\cdot\left(\lim_{h\to 0}\frac{v(x+h) - v(x)}{h}\right) + \left(\lim_{h\to 0}(v(x))\right.$$

$\downarrow u(x)$ $\quad \downarrow v'(x)$ $\quad \downarrow v(x)$

$$\lim_{h\to 0}\left(\frac{u(x+h) - u(x)}{h}\right)$$

$\downarrow u'(x)$

By 1^{st} principle of diff.

$$y = u(x)v'(x) + u'(x)v(x)$$

In General:
If $y(x) = u\,v\,w\,k$

$$\frac{dy}{dx} = y'(x) = u'vwk + v'wku + w'uvk + k'uvw$$

↓ by Product rule

EXAMPLES
Find dy/dx of $y(x)$ given.

1. $y(x) = (1-3x^2)(1+4x)$

By product rule
$$\frac{dy}{dx} = y'(x) = u'(x)v(x) + u(x)v'(x)$$

Don't expand.

$$(-6x)(1+4x) + (4)(1-3x^2)$$

2. $y(x) = \left(\frac{1}{x^2} - \frac{1}{x^4}\right)(1 - x^3)(\sqrt[3]{x^2} - 10)$

$u = x^{-2} - x^{-4}$

$u' = -2x^{-3} + 4x^{-5}$

$V = 1 - x^3$
$V' = -3x^2$

$W = \sqrt[3]{x^2} - 10$

$W' = \frac{2}{3}x^{-1/3}$

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$y'(x) = uvw' + uv'w + u'vw$.

QUOTIENT RULE

- Consider $y(x) = \frac{u(x)}{v(x)}$ where u and v are different factors.

By 1st principle of differentiation

$$y'(x) = \frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$y(x) = \frac{u(x)}{v(x)}$$

$$y(x+h) = \frac{u(x+h)}{v(x+h)}$$

$$y'(x) = \left(\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)} \right) \times \frac{1}{h}$$

$$N \to \left(\frac{u(x+h)(v(x)) - u(x)(v(x+h))}{v(x)(v(x+h))} \right) \text{LCM}$$

$D \to h$

$$N/D = N \times \frac{1}{D} = \frac{v(x) \cdot u(x+h) - u(x) \cdot v(x+h)}{h(v(x) \cdot v(x+h))}$$

To the numerator + and − $u(x)\, v(x)$
- Rearrange to get $u(x)$ and $v(x)$ by the diff.

$$\lim_{h \to 0} \left[\frac{v(x) \cdot u(x+h) + u(x)v(x) - u(x) \cdot v(x) - u(x) \cdot v(x+h)}{h(v(x) \cdot v(x+h))} \right]$$

Separate (Disistribute limits)

$$\lim_{h \to 0} \left[\frac{1}{v(x)v(x+h)} \right] \left[\underbrace{\lim_{h \to 0} v(x)}_{v} \underbrace{\lim_{h \to 0} \left(\frac{u(x+h) - u(x)}{h} \right)}_{u'} \right.$$

$$\left. + \underbrace{\left(\lim_{h \to 0} u(x) \right)}_{u} \underbrace{\lim_{h \to 0} \left(\frac{v(x) - v(x+h)}{h} \right)}_{-v'} \right]$$

$$\lim_{h} \frac{1}{v(x)v(x)} \left(v(x)\, u'(x) + u(x)\left(-v'(x)\right) \right)$$

$$\therefore QR = \frac{u'(x)\,v(x) - v'(x)\,u(x)}{(v(x))^2}$$

$$\therefore \frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

$$\frac{dy}{dx}\left(\frac{f}{g}\right) = \frac{f'g - g'f}{g^2}$$

Example.

1. Find $y'(x)$ given. $y(x) = \frac{1+3x}{1+x^2}$ $\frac{u}{v}$

By QR $y'(x) = \frac{u'v - v'(u)}{v^2}$

$u' = 3$
$v' = 2x$

$$\frac{3(1+x^2) - 2x(1+3x)}{(1+x^2)^2}$$

2. $y(x) = \frac{1 - x^2 - 4x^3}{-x^2 + 4x + 1}$

By QR $y'(x) = \frac{u'v - v'u}{v^2}$

$u' = -2x - 12x^2$ $\quad$ $v' = -2x + 4$

$$y'(x) = \frac{-2x - 12x^2(-x^2+4x+1) - [-2x+4(1-x^2-4x^3)]}{(x^2+4x+1)^2}$$

CHAIN RULE BY SUBSTITUTION

- In this case, introduce a new variable to represent the interior function (A function which other operators act on ...
eg $(Q)^6$ $\cos(x)$

and to reduce the problem to a form which we can apply product rule, power rule or quotient rule.

Consider $y(x) = [f(x)]^n$

Let $p = f(x)$

$\therefore y = p^n$

$p = f(x)$

Diff p wr ~~x~~

$\frac{dp}{dx} = \frac{df}{dx}$

$y = p^n$

Diff y w r p

$\frac{dy}{dp} = n p^{n-1}$

By chain rule

$\frac{dy}{dx} = \frac{dp}{dx} \cdot \frac{dy}{dp}$

$= \cancel{dp} \frac{dy}{dx} (n)(p)^{n-1}$

$= n(f(x))^{n-1}$

Ex 1 $y(x) = \frac{1}{\sqrt{4x^2+5}}$ $\frac{1}{(4x^2+5)^{1/2}}$

$y(x) = (4x^2+5)^{-1/2}$

Let m be $4x^2+5$ $m = 4x^2+5$ $\frac{dm}{dx} = 8x$

$\frac{dm}{dx} = 8x$ $\frac{dy}{dm} = -\frac{1}{2} m^{-3/2}$

By chain rule

$-\frac{1}{2}(4x^2+5)^{-3/2}(8x)$

$\frac{dy}{dx} = \frac{dm}{dx} \cdot \frac{dy}{dm}$

$= 8x \left(-\frac{1}{2}\right)(m)^{-3/2}$

$= -4x(4x^2+5)^{-3/2}$

DIRECT CHAIN RULE

Consider

$y(x) = [f(x)]^n$

By D/C/R

$y'(x) = n(f(x))^{n-1}(f'(x))$

Termwise differentiate all terms

Example

$y(x) = (1-3x^2)^5$

$y'(x) = 5(1-3x^2)^4(0-6x)$

$= -30x(1-3x^2)^4$

$y(x) = (4x^2+5)^{-1/2}$

By D/C/R

$-\frac{1}{2}(4x^2+5)^{-3/2}(8x+0)$

$= -4x(4x^2+5)^{-3/2}$

$y(x) = (1-3x^2-x^4)^6$

By D/C/R.

$y'(x) = 6(1-3x^2-x^4)^5(0-6x-4x^3)$

$= -36x - 24x^3(1-3x^2-x^4)^5$

Given $y(x) = \underbrace{(1+x^4-2x^3)^4}_{u}\underbrace{(1-4x^2)^3}_{v}$

Find $y'(x)$

$y'(x) = u'v + v'u$

$u = (1+x^4-2x^3)^4$

$u' = 4(1+x^4-2x^3)^3(4x^3-6x^2)$

$v = (1-4x^2)^3$

$v' = 3(1-4x^2)^2(-8x)$

$= -24x(1-4x^2)^2$

$y'(x) = (16x^3-24x^2)(1+x^4-2x^3)^3 + -24x(1-4x^2)^2 \cdot (1+x^4-2x^3)^3$

DERIVATIVE OF TRIG. FUNCTIONS

SINE

1 Consider $y(x) = \operatorname{Sin} x$

By 1^{st} principle of differentiation

$$y'(x) = \frac{dy}{dx} = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$y(x) = \operatorname{Sin} x \quad \ldots ②$$

$$y(x+h) = \operatorname{Sin}(x+h) = \operatorname{Sin}x \operatorname{Cos}h + \operatorname{Sin}h \operatorname{Cos}x \ldots ③$$

Substitute 2 and 3 to 1

$$y'(x) = \frac{\operatorname{Sin}x \operatorname{Cos}h + \operatorname{Sin}h \operatorname{Cos}x - \operatorname{Sin}x}{h}$$

$$\underset{\text{constant}}{\operatorname{Sin}x}\frac{(\operatorname{Cos}h - 1)}{h} + \frac{\operatorname{Sin}h \underset{\text{const}}{\operatorname{Cos}x}}{h}$$

$$\cancel{y'(x) =}$$

$$= \operatorname{Sin}x \lim_{h \to 0}\left(\frac{\operatorname{Cos}h - 1}{h}\right) + \operatorname{Cos}x \lim_{h \to 0}\left(\frac{\operatorname{Sin}h}{h}\right)$$

$$= \operatorname{Sin}x \lim_{h \to 0}\left(\frac{\operatorname{Cos}h - 1}{h}\right) + \operatorname{Cos}x \lim_{h \to 0}\left(\frac{\operatorname{Sin}h}{h}\right)$$

$$\operatorname{Sin}x \cdot 0 \qquad\qquad \operatorname{Cos}x \cdot 0$$

$$\lim_{h \to 0}\left(\frac{\operatorname{Cosh} - 1}{h}\right) \quad DS = 0/0.$$

$$\lim_{h \to 0}\left(\frac{\operatorname{Sin}h}{h}\right) \quad DS = \cancel{0/0} \; 1$$

$$\lim_{h \to 0}\left(\frac{-\operatorname{Sin}h}{h}\right) \quad D/s = \frac{-\operatorname{Sin}0}{1} = 0$$

$$\frac{dy}{dx} = \frac{d \operatorname{Sin}x}{dx} = \operatorname{Cos}x$$

Similarly:

$$y = \operatorname{Sin} p$$

$$\frac{dy}{dp} = \operatorname{Cos} p$$

$$y = \operatorname{Sin} k$$

$$\frac{dy}{dk} = \operatorname{Cos} k.$$

$$d \cdot \operatorname{Sin} x^2 \neq \operatorname{Cos} x^2$$

$$d \cdot \operatorname{Sin} 3x \neq \operatorname{Cos} 3x$$

Example

1. $y = \operatorname{Sin} x^2$

Let $p = x^2 \qquad y = \operatorname{Sin} p$

$$\frac{dp}{dx} = 2x \qquad \frac{dy}{dp} = \operatorname{Cos} p.$$

By C.R.

$$\frac{dy}{dx} = \frac{dp}{dx} \cdot \frac{dy}{dp}$$

$$2x \cdot \operatorname{Cos} p$$

$$= 2x \operatorname{Cos} x^2$$

2. $y = \operatorname{Sin} 3x$

Let u be $3x$

$$\frac{du}{dx} = 3 \qquad \frac{dy}{du} = \operatorname{Cos} u.$$

By C.R.

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$$

$$= 3 \times \operatorname{Cos} u$$

$$= 3 \operatorname{Cos} 3x$$

3. $y(x) = \operatorname{Sin}^2 x = (\operatorname{Sin} x)^2$

$k = \operatorname{Sin} x \qquad y = k^2$

$$\frac{dk}{dx} = \operatorname{Cos} x \qquad \frac{dy}{dk} = 2k$$

By C.R

$$\frac{dy}{dx} = \frac{dk}{dx} \cdot \frac{dy}{dk}$$

$$\operatorname{Cos} x \cdot 2k$$

$$= 2 \operatorname{Cos} x \operatorname{Sin} x$$

OR

By D/C/R

$$2(\operatorname{Sin} x)^{2-1}(\operatorname{Cos} x) = 2 \operatorname{Sin}x \operatorname{Cos}x$$

In general:
If $y(x) = \sin f(x)$ then:

$$y'(x) = f'(x) \cos f(x)$$

Eg.

✓ $y(x) = \sin x^2$

$f = x^2$
$f' = 2x$

$y'(x) = f' \cos f$
$= 2x \cos x^2$

✓ $y(x) = \sin(6 + x^2 + x^3)$

$f = 6 + x^2 + x^3$

$f' = 2x + 3x^2$

$y'(x) = \underbrace{(2x + 3x^2)}_{f'} (\cos \underbrace{(6 + x^2 + x^3)}_{f})$

2. COSINE

2. $y(x) = \cos x$

By 1^{st} principle of diff.

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$y(x) = \cos x$

$y(x+h) = \cos(x+h) = \cos x \cos h - \sin x \sin h$

Substitute.

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{\overbrace{\cos x \cos h - \sin x \sin h}^{f(x+h)} - \overbrace{\cos x}^{f(x)}}{h} \right)$$

$$= \cos x \lim_{h \to 0} \left(\frac{\cos h - 1}{h} \right) - \sin x \lim_{h \to 0} \left(\frac{\sin h}{h} \right)$$

$$= \cos x \, (0) - \sin x \, (1)$$

$$\frac{dy}{dx} = \frac{d \cos x}{dx} = -\sin x$$

Therefore:

1. $y = \cos p'$ $\quad dy/dx = -\sin p$

2. $y = \cos w'$ $\quad \frac{dy}{dx} = -\sin w$

3. $y = \cos x^3$

Let p be x^3 $\qquad y = \cos p$

$\frac{dp}{dx} = 3x^2 \qquad \frac{dy}{dp} = -\sin p$

By C.R

$$\frac{dy}{dx} = \frac{dp}{dx} \times \frac{dy}{dp}$$

$$= 3x^2 \times -\sin p$$

$$= -3x^2 \sin p$$

$$= -3x^2 \sin x^3$$

4. $y = \cos \sqrt[3]{x^3 + 6}$.

Let w be $(x^3 + 6)^{1/3} = y$

$y = \cos w \qquad dy/dx = -\sin w$

$$\frac{dw}{dx} = \frac{1}{3}(x^3 + 6)^{-2/3}(3x^2)$$

$$= (x^3 + 6)^{-2/3}(x^2)$$

By C.R.

$$\frac{dy}{dx} = \frac{dw}{dx} \cdot \frac{dy}{dw}$$

$$= x^2 (x^3 + 6)^{-2/3} \cdot -\sin \sqrt[3]{x^3 + 6}$$

$$= -x^2 (x^3 + 6)^{-2/3} \sin \sqrt[3]{x^3 + 6}$$

In general:
If $y(x) = \cos f(x)$ then

$$y'(x) = -f'(x) \sin f(x).$$

1. $y = \cos(t^2+1)$

$f = t^2+1$

$f' = 2t$

$y' = -2t \sin(t^2+1)$

2. $y(x) = \cos(6-3x)$

$f' = -3$

$y'(x) = -3\sin(6-3x).$

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3. TANGENT.

$y(x) = \tan x = \dfrac{\sin x}{\cos x}$ in the form $\dfrac{u}{v}$.

By QR

$y'(x) = \dfrac{d \tan x}{dx} = \dfrac{u'v - v'u}{v^2}$

$u' = \cos x$

$v' = -\sin x$

$\dfrac{(\cos x)(\cos x) - (-\sin x)(\sin x)}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x} = \sec^2 x$

$y'(p) = \dfrac{d \tan p}{dp} = \sec^2 p'$

$\dfrac{d \tan k}{dk} = \sec^2 k.$

Examples

1 $y(x) = \tan(5x^2+1)$

Let p be $5x^2+1$

$y = \tan p$

$\dfrac{dp}{dx} = 10x \qquad \dfrac{dy}{dp} = \sec^2 p$

By C.R

$\dfrac{dp}{dx} \cdot \dfrac{dy}{dp} = \dfrac{dy}{dx}$

$10x \quad \sec^2 p$

$= 10x \sec^2(5x^2+1).$

2. $y(x) = \tan\sqrt{5x}.$

$p = (5x)^{1/2} \qquad y = \tan p.$

$\dfrac{dp}{dx} = \frac{1}{2}(5x)^{-1/2}(5) \qquad \dfrac{dy}{dp} = \sec^2 p$

By C.R

$\dfrac{dy}{dx} = \dfrac{dp}{dx} \cdot \dfrac{dy}{dp}.$

$= \dfrac{5}{2}(5x)^{-1/2} \cdot \sec^2 p.$

$= 5/2 (5x)^{-1/2} \sec^2(\sqrt{5x}).$

- In general:

If $y(x) = \tan f(x).$

$y'(x) = f'(x) \sec^2 f(x)$

4. COTANGENT

$y(x) = \cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x} \qquad \dfrac{u}{v}$

By QR $y'(x) = \dfrac{u'v - v'u}{v^2}$

$y'(x) = \dfrac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$

$y'(x) = \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$y'(x) = \dfrac{-1}{\sin^2 x} = -\operatorname{Cosec}^2 x$

$$\frac{d \cot p}{dp} = -\operatorname{Cosec}^2 p$$

$$\frac{d \cot u}{du} = -\operatorname{Cosec}^2 u.$$

EXEMPLES

1. $y(\theta) = \cot(2\theta+1)$

Let $P = 2\theta+1$ $\quad y = \cot p$

$\frac{dp}{d\theta} = 2$ $\quad \frac{dy}{dp} = -\operatorname{Cosec}^2 u$

$$\frac{dy}{d\theta} = \frac{dp}{d\theta} \cdot \frac{dy}{dp}$$

$$= 2 \cdot -\operatorname{Cosec}^2 p$$

$$= -2\operatorname{Cosec}^2 p$$

$$= -2 \operatorname{Cosec}^2 (2\theta+1)$$

2. $y(x) = \sqrt{x} \, \cot \sqrt{x}$

$u = x^{1/2}$

$u' = 1/2 x^{-1/2}$

$V = \cot x^{1/2}$

$V' \Rightarrow$ let $x^{1/2}$ be p

$V = \cot p$

$\frac{dv}{dp} = -\operatorname{Cosec}^2 P$ $\quad \frac{dp}{dx} = 1/2 x^{-1/2}$

$$\frac{dv}{dx} = \frac{dp}{dx} \cdot \frac{dv}{dp}$$

$$v' = 1/2 x^{-1/2} \cdot -\operatorname{Cosec}^2 p$$

$$y'(x) = u'v + uv'$$

In general:

If $y(x) = \cot f(x)$ Then:

$$f'(x) = f'(x) \operatorname{Cosec}^2(f(x)).$$

5. SECANT

$y(x) = \operatorname{Sec} x = \frac{1}{\cos x}$ $\quad \frac{u}{v}$

By Q.R

$$y'(x) = \frac{u'v - v'u}{v^2}$$

$u' = 0$

$v' = -\sin x$

$$y'(x) = \frac{0(\cos x) - (-\sin x)(\cancel{\tfrac{1}{\cos x}})}{(\cancel{\cos} x)^2}$$

$$y'(x) = \frac{\sin x \cancel{\cos x}}{\cos^2 x \cancel{\sin^2 x}}$$

$$y'(x) = \frac{\sin x}{\cos^2 x}$$

$$y'(x) = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$y'(x) = \tan x \operatorname{Sec} x$$

Therefore $\operatorname{Sec} w \Rightarrow \operatorname{Sec} w \tan w$

$$\frac{d \operatorname{Sec} p}{dp} = \operatorname{Sec} p \tan p$$

$y(x) = \operatorname{Sec} x^2$

$p = x^2$ $\quad \frac{dy}{dp} = \operatorname{Sec} p \tan p$

$\frac{dp}{dx} = 2x$

$$\frac{dy}{dx} = \frac{dp}{dx} \cdot \frac{dy}{dp}$$

$$= 2x \operatorname{Sec} x^2 \tan x^2$$

In general:
If $y(x) = \sec f(x)$

$y'(x) = f'(x) \sec f(x) \tan f(x)$

6 COSECANT

$y(x) = \operatorname{Cosec} x = \frac{1}{\sin x}$ $\quad \frac{u}{v}$.

$y'(x) = \frac{u'v - v'u}{v^2}$

$y'(x) = \frac{0(\sin x) - (\cos x)\,1}{\sin^2 x}$

$= \frac{-\cos x}{\sin^2 x}$

$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$

$= -\cot x \operatorname{cosec} x$

$\frac{d \csc x}{dx} = -\cot x \csc x$

In general:

$y(v) = \operatorname{Cosec} v = -\cot y(v)$.

$y'(v) = -f(v) \operatorname{Cosec} f(v) \cot f(x)$

If $y(x) = \operatorname{Cosec} f(x)$

$y'(x) = -f'(x) \operatorname{Cosec} f(x) \cot f(x)$

Exercise

37 $y(x) = \sqrt{x}\,(x - \cos x)^3$.

By. P.R $\quad y'(x) = u'v + v'u$

$\cancel{y'(x)}\ u' = \frac{1}{2} x^{-1/2}$.

$v' = 3(x - \cos x)^2 (1 + \sin x)$

$y'(x) = \frac{1}{2} x^{-1/2} (x - \cos x)^3 + (v')(u)$.

38 $y = \sqrt{x} \sin(\sqrt{x + \sqrt{x}})$

$u = \sqrt{x} \qquad u' = \frac{1}{2} x^{-1/2}$

$v = \sin(x + x^{1/2})^{1/2}$

$v' = \frac{1}{2}(x + x^{1/2})^{-1/2}(1 + \frac{1}{2}x^{-1/2}) \cos(x + x^{1/2})$

Substitute in $y'(x) = u'v + v'u$.

20 $g(t) = \frac{1}{\sqrt{\sin^2 t + \sin^2 3t}}$ $\qquad g(t) = \frac{1}{\sqrt{p}}$.

$\frac{1}{\sqrt{\sin^2 t + \sin^2 3t}}$ $\quad (\sin 3t)^2$

Let P be $\sin^2 t + \sin^2 3t$

$g(t) = \frac{1}{p^{1/2}} = p^{-1/2}$ $\quad g'(t) = -\frac{1}{2} P^{-3/2}$ $\quad \frac{dg}{dp}$ $\qquad \frac{dg}{dp} \cdot \frac{dp}{dt}$

$\frac{dp}{dt} = 2 \sin t \cos t + \cancel{3 \cos}\ 2 \sin 3t\, 3 \cos 3t$

$= 2 \sin t \cos t + 6 \sin 3t \cos 3t$

$\frac{dg}{dt} = \frac{dg}{dp} \times \frac{dp}{dt}$

$= -\frac{1}{2} p^{-3/2} \times 2 \sin t \cos t + 6 \sin 3t \cos 3t$

$-\frac{1}{2}(\sin^2 t \sin^2 3t)^{-3/2} \cdot (2 \sin t \cos t + 6 \sin 3t \cos 3t)$

DERIVATIVE OF EXPONENTIAL FNS.

Consider $y(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

! Factorial

$0! = 1$

$\cancel{2}1! = 1$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

$4! = 4 \times 3 \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$n! = n \times (n-1) \times n-2 \dots 0!$

Differentiate each term with respect to x

$y(x) = e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \dots$

$\frac{dy}{dx} = \frac{de^x}{x} = 0 + \frac{1}{1!} + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!}$

$\frac{2}{2!} = \frac{2}{2 \times 1} = \frac{1}{1!}$

$\frac{3}{3!} = \frac{3}{3 \times 2 \times 1} = \frac{1}{2 \times 1} = \frac{1}{2!}$

$\frac{4}{4!} = \frac{4}{4 \times 3 \times 2 \times 1} = \frac{1}{3 \times 2 \times 1} = \frac{1}{3!}$

$\frac{5}{5!} = \frac{5}{5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{4!}$

$\Rightarrow 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$= e^x$

Therefore:

$\frac{de^x}{dx} = e^x$

$\frac{de^k}{dk} = e^k$

$\frac{de^p}{dp} = e^p$

$y = e^{x^2}$

let p be x^2

$y = e^p$

$\frac{dy}{dp} = e^p$

$p = x^2$

$\frac{dp}{dx} = 2x$

C.R

$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$

$= 2x e^p$

$= 2x e^{x^2}$

$y = e^{2x}$

Let $R = 2x$

$\frac{dR}{dx} = 2$

$y = e^R$

$\frac{dy}{dR} = e^R$

C.R

$\frac{dy}{dx} = \frac{dR}{dx} \cdot \frac{dy}{dR}$

$= 2e^R$

$= 2e^{2x}$

$y = e^{\sin 3x}$

Let $W = \sin 3x$

$\frac{dw}{dx} = 3 \cos 3x$

$y = e^w$

$\frac{dy}{dw} = e^w$

C.R

$\frac{dy}{dx} = 3 \cos 3x\, e^w$

$= 3 \cos 3x\, e^{\sin 3x}$

In general

If $y(x) = e^{f(x)}$ then

$y'(x) = f'(x)\, e^{f(x)}$

a) $y(x) = e^{x^3}$

$f(x) = x^3$

$f'(x) = 3x$

$y'(x) = 3x e^{x^3}$

8) $y = e^{\sqrt{x^2+1}}$

$f(x) = \sqrt{x^2+1} \quad = (x^2+1)^{1/2}$

$f'(x) =$ By D/C/R

$\frac{1}{2}(x^2+1)^{-1/2}(2x).$

$= x(x^2+1)^{-1/2}$

$g'(x) = \left(x(x^2+1)^{-1/2}\right)e^{\sqrt{x^2+1}}$.

9) $y = e^{\tan 7x}$

$f(x) = \tan 7x.$

$f'(x) = 7\sec^2 7x$

$y'(x) = 7\sec^2 7x \, e^{\tan 7x}$

DERIVATIVES OF LOGARITHMS FNS.

- Consider $y(x) = \ln X = \log_e X \rightarrow$ Natural log fxn.

Where $e = 2.718281828$

Natural log obeys the laws of logarithmic functions.

$\ln A - \ln B = \ln (A/B)$

$\ln A + \ln B = \ln (AB)$

$\ln A^n = n \ln A$

$\ln_e e = \log_e e = 1$

$\ln 1 = \ln_e 1 = 0$

To change from log notation to index

$\log_A B = X \qquad A^x = B$

$\log_{10} 100 = 2 \qquad 10^2 = 100.$

$\log_x 25 = 10 \qquad X^{10} = 25.$

If $y(x) = \ln x$ Then $\frac{dy}{dx} = y'(x) = \frac{d \ln x}{dx} = \frac{1}{x}$

Similarly $\frac{d \ln k}{dk} = \frac{1}{k}$

$\frac{d \ln p}{dp} = \frac{1}{p}.$

Examples

1. $y(x) = \ln x^2$

Let $p = x^2$ $\qquad y = \ln p.$

$\frac{dp}{dx} = 2x \qquad \frac{dy}{dp} = \frac{1}{p}.$

C.R

$\frac{dy}{dx} = \frac{2x}{p}. = \frac{2x}{x^2}$

$= \frac{2}{x}.$

2. $y(x) = \ln(6x+1).$

$u = 6x+1 \qquad y = \ln u$

$\frac{du}{dx} = 6 \qquad \frac{dy}{du} = \frac{1}{u}$

C.R

$\frac{dy}{dx} = \frac{6}{u} = \frac{6}{6x+1}$

3. $y(x) = \ln(\cos 6x)$

$k = \cos 6x \qquad y(x) = \ln k$

$\frac{dk}{dx} = -6\sin 6x \qquad y'(k) = \frac{1}{k}$

C.R

$\frac{dy}{dx} = \frac{-6\sin 6x}{\cos 6x}$

$= -6\tan 6x.$

In general:

$y(x) = \ln(fx).$ Then $y'(x) = \frac{f'(x)}{f(x)}$

a) $y(x) = Ln(x^2+2x+1)$

$f(x) = x^2+2x+1$
$f'(x) = 2x+2$

$y'(x) = \cancel{\dfrac{x^2+2x+1}{2x+2}} \quad \dfrac{2x+2}{x^2+2x+1}$

b) $y = Ln(\sqrt[3]{x^3-3x+1})$

$f(x) = \sqrt[3]{x^2-3x+1}$

$f'(x) \Rightarrow DCR$
$\frac{1}{3}(x^3-3x-1)^{-2/3}(3x^2-1)$

$= (x^2-1)(x^3-3x-1)^{-2/3}$

$y'(x) = \dfrac{(x^2-1)(x^3-3x-1)^{-2/3}}{(x^3-3x+1)^{1/3}}$

$= \dfrac{x^2-1}{(x^3-3x+1)}$ $\qquad a^{-n} = \dfrac{1}{a^n}$

c) $y(x) = Ln(\sin 5x)$

$\dfrac{f'(x)}{f(x)}$

$f = \sin x$
$f' = 5 \cancel{\sin} \cos 5x$

$y'(x) = \dfrac{5\cos 5x}{\sin 5x}$

$= 5 \cot 5x$

IMPLICIT FUNCTIONS.

- So far we have known how to differentiate explicit functions (functions where the dependent variable is the subject of the formula).
- To differentiate implicit functions (where making dependent variable the subject of the formula is difficult) Apply the following steps

1. Differentiate x normally

2. While differentiating y multiply the result by y' since y is a function of x also supported by D.C.R

y^2	$2(y)^{2-1}\, y' = 2y \cdot y'$
y^3	$3(y)^2\, y' = 3y^2 \cdot y'$
y'	$1\, y^0\, y' = y'$
y^{-5}	$-5(y^{-6})\, y' = -5y^{-6}\, y'$
$y^{1/2}$	$\frac{1}{2}(y)^{-1/2}\, y' = \frac{1}{2}y^{-1/2}\, y'$
$\sin y$	$f' \cos f$; $y' \quad y$; $= y' \cos y$
$\cos y$	$-f' \sin f$; $y' \quad y$; $= -y' \cos y$

3. ~~Mat~~ Collect like terms and make y' the subject.

Ex 1
Find dy/dx For:

$x^2 - 3\underset{u}{x}\underset{v}{y^2} = \sqrt[3]{y}$

$\downarrow$

$2x - 3[\overset{u'v + v'u}{1 \cdot y^2 + 2y \cdot y' x}] = \frac{1}{3}(y)^{-2/3}\, y'$

$2x - 3y^2 - 6y \cdot y' x = \frac{1}{3}y^{-2/3}\, y'$

$2x - 3y^2 = 6y \cdot y' x + \frac{1}{3}y^{-2/3}\, y'$

$2x - 3y^2 = y'(6xy + \frac{1}{3}y^{-2/3})$

$y' = \dfrac{2x - 3y^2}{6xy + \frac{1}{3}y^{2/3}}$

Ex 2
$\dfrac{1}{x} + \dfrac{1}{y} = 1$

$x^{-1} + y^{-1} = 1$

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Diff with respect to x.

$$-1x^{-2} + -1y^{-2}y' = 0$$

$$-x^{-2} = y'y^{-2}$$

$$y' = \frac{-x^{-2}}{y^{-2}}$$

EX 3

$$x + y^2 = e^{xy^3}$$

Diff w.r.t. x

$$x + y^2 = e^{xy^3}$$

$y = e^{f(x)}$

$y' = f'(x)\, e^{f(x)}$

$$1 + 2y \cdot y' = (y^3 + 3y^2 x y')\, e^{xy^3}$$

$$1 + 2y \cdot y' = e^{xy^3} \cdot y^3 + 3y^2 x y' e^{xy^3}$$

$$1 - y^3 e^{xy^3} = 3y^2 x y' e^{xy^3} - 2y \cdot y'$$

$$y'(3y^2 x e^{xy^3} - 2y) = 1 - y^3 e^{xy^3}$$

$$y' = \frac{1 - y^3 e^{xy^3}}{3y^2 x e^{xy^3} - 2y}$$

N/B in e^{xy^3}

$f = xy^3$

Ex 4

$$\underset{u}{\cos y}\ \underset{v}{\sin x} = \cos(x + y^2).$$

$u = \cos y$ $\qquad$ $v = \sin x$

$u' = -y' \sin y$ $\qquad$ $v' = 1 \cos x$

$-y' \sin y(\sin x) + 1\cos x(\cos y)$ Side 1.

for $\cos(x+y^2)$ $\qquad$ $f = x + y^2$

$f' = 1 + 2y \cdot y'$

$-f' \sin f$

$\Rightarrow -(1 + 2y \cdot y') \sin(x + y^2)$ Side 2

$$-y' \sin y \sin x + \cos x \cos y = -\sin(x+y^2) - 2y \cdot y' \sin(x+y^2)$$

$$\cos x \cos y + \sin(x + y^2) = y'(\sin y \sin x - 2y \sin(x+y^2))$$

$$y' = \frac{\cos x \cos y + \sin(x + y^2)}{\sin y \sin x - 2y \sin(x+y)^2}$$

EX 5

$$x + y^3 = \ln(x^2 + y).$$

$\frac{f'}{f}$

$$1 + 3y^2 y' = \frac{2x + y'}{x^2 + y}$$

$$1 + 3y^2 y'(x^2 + y) = 2x + y'$$

$$x^2 + y + 3x^2y^2y' + 3y^3y' = 2x + y'$$

$$y'(3x^2y^2 + 3y^3 - 1) = 2x - y - x^2$$

$$y' = \frac{2x - y - x^2}{3x^2y^2 + 3y^3 - 1}$$

DIFFERENTIATION OF OTHER FORMS OF EXPONENTIAL FUNCTIONS.

1. A constant raised to function.

$y = 2^{x^2}$ $\qquad$ $y = 5^{\sqrt{x}}$ $\qquad$ $y = 6^{\cos 3x}$

$y = a^{-9x}$ $\qquad$ $y = 2^{3x}$

$y = 9^{3\ln 2x}$ $\qquad$ $y = 5^{\ln x^2}$

2. A Function raised to a function

$y = x^{x^2}$ $\qquad$ $y = (\cos 2x)$

$y = (e^x)^{5x}$

$y = (\sin x)^x$

$y = (\ln x)^{\sin 3x}$

$y = (\ln x)^{x^2}$

- To differentiate the above introduce $\ln$ on both sides and recall the flwng

1. $\ln A^n = n \ln A$

2. $\ln f = \frac{f'}{f}$

3. $\ln y = \frac{y'}{y}$

4. $\ln 2 = 0$ Since it is a constant

$\ln 5 = 0$

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Examples

1- $y = 2^{x^2}$

$\ln y = \ln 2^{x^2}$

$x^2 \ln 2$

$\frac{y'}{y} = 2x \ln 2$ Diff w.r.t. x.

$y = ax^2$

$y' = 2ax$

$\frac{y'}{y} = 2x \ln 2$

$y' = 2x\, y \ln 2$

$y' = 2x \cdot 2^{x^2} \ln 2$

2. $y = 6^{\cos 3x}$

$\ln y = \cos 3x \ln 6.$

$\frac{y'}{y} = -3\sin 3x \ \ln 6.$

$y' = y(-3 \sin 3x) \ln 6.$

$y' = \cancel{\cos}\, 6^{\cos 3x} (-3 \sin 3x) \ln 6.$

3. $y = x^{x^2}$

$\ln y = \ln x^{x^2}$

$u = x^2 \quad u' = 2x$

$v = \ln x \quad v' = \frac{1}{x}$

$\frac{f'}{f}$

$\ln y = \underbrace{x^2}_{u} \underbrace{\ln x}_{v}$

$u'v + v'u$

$\frac{y'}{y} = \left[\overbrace{2x \ln x}^{u'v} + \overbrace{\frac{1}{x} \cdot x^2}^{v'u}\right]$

$y' = y(2x \ln x + x]$

$y' = x^{x^2}(2x \ln x + x)$

4. $y = (\sin 3x)^{\cos 2x}$

$\ln y = \ln (\sin 3x)^{\cos 2x}$

$\frac{y'}{y}\ \ln y = \underbrace{\cos 2x}_{u}\ \underbrace{\ln \sin 3x}_{v}$

$\frac{y'}{y} = \left[\overbrace{-2\sin 2x}^{u'} \cdot \overbrace{\ln \sin 3x}^{v} + \overbrace{\cos 2x}^{u}\ \overbrace{3\cot 3x}^{v'}\right.$

$y' = y\left(-2\sin 2x \ln \sin 3x + 3 \cot 3x \cos 2x\right)$

$y' = (\sin 3x)^{\cos 2x}\left[-2 \sin 2x \ln 3x + 3 \cot 3x\ (\cos 3x)\right]$

EXERCISE

Use logarithm to differentiation to find f' given.

$y(x) = \frac{(x+1)^2 \sqrt{x+1}}{(x-1)^3}$

Examples

1. $y = 3^x\, 2^{x^2}$

$\log y = \log 3^x 2^{x^2}$

$\frac{y'}{y} = \ln 3^x + \ln 2^{x^2}$

$\frac{y'}{y} = \cancel{x}\ln 3^x + \ln 2^{x^2}$

K $\qquad$ P

$K = x \ln 3. \qquad P = x^2 \ln 2$

$\frac{dK}{dx} = 1 \ln 3 \qquad \frac{dP}{dx} = 2x \ln 2$

$\frac{y'}{y} = \ln 3 + 2x \ln 2$

$y' = y \ln 3 + 2x \ln 2$

$y' = 3^x 2^{x^2}(\ln 3 + 2x \ln 2)$

2. $\log_3 (x\sqrt{x-1}) = y.$

N/B: $\log_B A = x$

$B^x = A$

Introduce Ln.

$Ln(x\sqrt{x-1}) = Ln\, 3^y$

$Ln(x\sqrt{x-1}) = y\, Ln\, 3$

Log A + log B = log AB

$Ln\, x + Ln\sqrt{x-1} = y\, Ln\, 3$

Diff w.r.t. x.

For any Ln $\frac{dy}{dx} = \frac{f'}{f}$.

$Ln\, x + Ln(x-1)^{1/2} = y\, ln\, 3$

$\left[\frac{1}{x} + \frac{1/2}{x-1}\right] = y'\, Ln\, 3.$

NOTE der. of $Ln(x-1)^{1/2} \to \frac{\frac{1}{2}(x-1)^{-1/2}(1)}{(x-1)^{1/2}}$

$a^m \cdot a^n = a^{m+n}$

$= \frac{1/2}{(x-1)^{1/2}(x-1)^{1/2}}$

$= \frac{0.5}{(x-1)}$

$y' = \frac{1}{Ln\, 3}\left(\frac{1}{x} + \frac{1/2}{x-1}\right)$

Exercise Note $Ln\left(\frac{AB}{C}\right) = Ln\, A + Ln\, B - Ln\, C$

$y(x) = \frac{(x+1)^2\sqrt{x+1}}{(x-1)^3}$ $a^m \times a^n = a^{m+n}$

$Ln(y) = Ln(x+1)^2 + Ln(x+1)^{1/2} - Ln(x-1)^3$

use D/C/R

$\frac{y'}{y} = \frac{2(x+1)(1)}{(x+1)^2} + \frac{\frac{1}{2}(x+1)^{-1/2}(1)}{(x+1)^{1/2}\cancel{(x+1)^{1/2}}} - \frac{3(x-1)^2(1)}{(x-1)^3}$

$y' = y\left(\frac{2}{x+1} + \frac{1/2}{(x+1)^{1/2}} - \frac{3}{x-1}\right)$

$y' = \left(\frac{(x+1)^2(\sqrt{x+1})}{(x-1)^3}\right)\left(\frac{2}{x+1} + \frac{1/2}{(x+1)^{1/2}} - \frac{3}{x-1}\right)$

30-10-19

DERIVATIVE OF INVERSE TRIG. FUNCTIONS

Consider $y = Sin^{-1}(x) \neq \frac{1}{\sin x}$

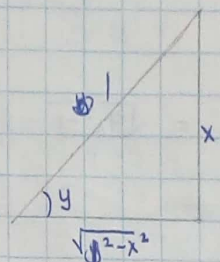
Write as: $Sin\, y = x$ —② $\left(Sin^{-1} 0.5 = 30;\ Sin\, 30 = 0.5\right)$.

Differentiate w.r.t. x both sides:

$y'\, Cos\, y$

$y'\, Cos\, y = 1$

$y' = \frac{1}{Cos\, y}$



$Cos\, y = \frac{Adj}{Hyp} = \frac{\sqrt{1^2 - x^2}}{1} = \frac{\sqrt{1-x^2}}{1}$

2. $y = Cos^{-1}(2x)$

Rewrite as

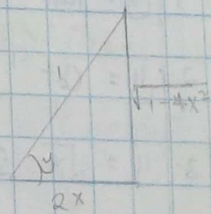
$Cos\, y = \frac{2x}{1}$ $Sin = \frac{Opp}{Hyp.}$

Diff. wr. to x

$-f'\, Sin\, f$

$-y'\, Sin\, y = 2$

$y' = \frac{-2}{Sin\, y}.$



$Sin\, y = \frac{\sqrt{1-4x^2}}{1}$

$y' = \frac{-2}{\sqrt{1-4x^2}}$

3 $y = Cosec^{-1}(3x)$

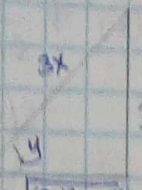
Rewritte as:

$Cosec\, y = 3x$

$-f'\, Cosec\, f\, Cot\, f$

$-y'\, \overset{3x}{cosec\, y}\, cot\, y = \cancel{y'} = 3$

By Pythagoras theorem,



$\sqrt{(3x)^2 - 1^2} = \sqrt{9x^2 - 1}$

$\text{Cosec } y = \dfrac{\text{Hypotenuse}}{\text{Opposite}}$

$= \dfrac{3x}{1}$

$y' = \dfrac{-3}{\text{Cosec } y \text{ Cot } y}$

$\text{Cot } y = \dfrac{\text{Adjacent}}{\text{Opposite}} = \dfrac{\sqrt{9x^2-1}}{1}$

$\text{Cosec } y = 3x$

$y' = \dfrac{-3}{(3x)(\sqrt{9x^2-1})}$

Note

$\text{Cot} = \dfrac{\text{Adj}}{\text{Opp}}$ $\quad \text{Sec} = \dfrac{\text{Hyp}}{\text{Adjacent}}$ $\quad \text{Cosec} = \dfrac{\text{Hyp}}{\text{Opposite}}$

Quiz.

1. $y = \tan^{-1}(3x)$ $\quad y' = \dfrac{3}{9x^2+1}$
2. $y = \text{Cot}^{-1}(4x)$
3. $y = \underset{\text{diff}}{\overset{u}{\sqrt{x}}}\ \underset{\text{diff}}{\overset{v}{\text{Sec}^{-1}(2x)}}$ Product rule
4. $y = \dfrac{1 + \text{Cos}^{-1}(3x)}{x^2 + \text{Sin}^{-1}(2x)}$ $\dfrac{u}{v}$ Product rule Quotient Introduce Ln, $y = y \ln x$
5. $y = 2^{\sin^{-1}(2x)}$ $\quad \ln y = \ln 2^{\sin^{-1}(2x)}$ $\quad \ln y = \sin^{-1}(2x) \ln 2$
5. $y = \cancel{a \text{ Cos}}\ a^{\cos^{-1}(x)} + b^{\sin^{-1}(4x)}$ $\quad y = k + p$ $\quad y' = k' + p'$
6. $y = [1 + \text{Sin}^{-1}(6x)]^2$

PARAMETRIC DIFFERENTIATION

If $y = y(t)$ or $y = y(\theta)$

$x = x(t)$ $\quad x = x(\theta)$

Where t and θ are the parameters then find $\dfrac{dy}{dx}$, Compute:

(i) $\dfrac{dx}{dt}$ or $\dfrac{dx}{d\theta}$

(ii) $\dfrac{dy}{dt}$ or $\dfrac{dy}{d\theta}$ $\quad$ * Chain rule concept.

(iii) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

OR

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$

Example.

If $x = t^3 + 2t - 4$.

$y = t^3 + 2t - 1$

Find $\dfrac{dy}{dx}$

$\dfrac{dx}{dt} = 3t^2 - 2$ $\qquad \dfrac{dt}{dx} = \dfrac{1}{3t^2 - 2}$

$\dfrac{dy}{dt} = 3t^2 + 2$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$\dfrac{dy}{dx} = 3t^2 + 2 \times \dfrac{1}{3t^2 - 2}$

$\dfrac{dy}{dx} = \dfrac{3t^2 + 2}{3t^2 - 2}$

2. If $x = \sin 3\theta$ $y = \cos 3\theta$

Find $\frac{dy}{dx}$

Soln

$\frac{dx}{dt} = f' \cos f = 3\cos 3\theta$

$\frac{dx}{dt} = \frac{1}{3\cos 3\theta}$

$\frac{dy}{d\theta} = -f' \sin f = -3\sin 3\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{dt}{dt}$

$\frac{dy}{dx} = \frac{3\sin 3\theta}{-3\cos 3\theta} = -\tan 3\theta$

3. If $x = \sin^{-1}(2t)$
$y = \cos^{-1}(2t)$.

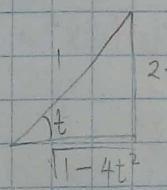
To find $\frac{dx}{dt}$ Rewrite as $\sin x = 2t$

$\sin x = 2t$

Diff wrt t

$x' \cos x = 2$

$x' = \frac{2}{\cos x}$



$x' = \frac{dx}{dt} = \frac{2}{\sqrt{1-4t^2}}$

To find $\frac{dy}{dt}$ Rewrite as $\cos y = 2t$

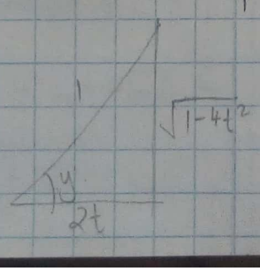
Diff wrt t

$\frac{dy}{dt} \Rightarrow -y' \sin y = 2$

~~$y \Rightarrow -2\sin 2t = 2$~~

$\sin y = \frac{\sqrt{1-4t^2}}{1}$

$y' = \frac{2}{\sin y}$



$\frac{dy}{dt} = y' = \frac{-2}{\sqrt{1-4t^2}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{-2}{\sqrt{1-4t^2}} \times \frac{\sqrt{1-4t^2}}{2}$

$\frac{dy}{dt} = -1$

QUIZ

1. $x = \frac{t^2}{2-3t}$ u v   2 $y = \frac{t^2 - 2t - 10}{2 - 3t}$ u v

2. $y = e^{-t}$ $\frac{dy}{dt} = e^{-t} = t'e^{t}$ $x = e^t$. $\frac{dx}{dt} = e^t$

3. $x = \ln(t^2 + 2t - 1)$ $y = \ln(t^3)$ t^3/t^3

4. $x = \sqrt{t}\,\sin 2t$ $y = t^2 \cos 2t$

5. $x = 2^t$ Introduce Ln $y = 2^{-t}$. Intro Ln

6. $x = \theta - \sin 2\theta$ $y = \theta + \cos 2\theta$.

1

7. $x = \cos^{-1}(4x)$ $y = \tan^{-1}(4x)$

Find $\frac{dy}{dx}$ for the following (above) pairs

HIGHER ORDER DERIVATIVES

H.O.D

If $y(x)$ is differentiated with respect to x i.e $\frac{dy}{dx} = y'(x)$ by 1^{st} Principle of diff. derivative of $y(x)$

- Redifferentiating $y'(x)$ w.r.t x

$\frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{(dx)^2} = y''$

We obtain the second derivative of $y(x)$ as $\frac{d^2y}{(dx)^2}$

- Redifferentiating $y''(x)$ w.r.t. x

$$\frac{d(y'')}{dx} = \frac{d\left(\frac{d^2y}{(dx)^2}\right)}{dx} = \frac{d^3y}{(dx)^3} = y'''$$

- Redifferentiating y'''

$$\frac{d(y''')}{dx} = \frac{d}{dx}\left(\frac{d^3y}{(dx)^3}\right) = \frac{d^4y}{(dx)^4} = \underbrace{y^{iv}}_{4^{th}\text{ derivative}}$$

- From y'', y''', y^{iv} .... are refered to as HOD

Given $y(x) = 2x^4 - 6x^3 + 12x^2 - 16x + 0$

Find

$y' = \cancel{18x^2 + 24}\ 8x^3 - 18x^2 + 24x - 16$

$y'' = 24x^2 - 36x + 24$

$y''' = 48x - 36$

$y^{iv} = 48$

$y^{v} = 0$

Example

1. If $y = e^{x^3}$ Find y' and y''

$y' = f' e^f$

$y' = 3x^2 . e^{x^3}$

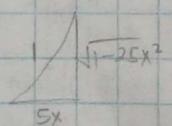
$y'' = u'v + v'u$

$y'' = 6x e^{x^3} + 3x^2 e^{x^3} 3x^2$

$y'' = 6xe^{x^3} + 9x^4 e^{x^3}$

2. If $y = \cos^{-1}(5x)$ Find y' and y''

$\cos y = 5x = \frac{Adj}{Hyp}$



Diff w.r.t x

$-y' \sin y = 5$

$y' = \frac{-5}{\sin y}$. But $\sin y = \frac{\sqrt{1-25x^2}}{1}$

$y' = \frac{-5}{\sqrt{1-25x^2}}$

$y' = -5(1-25x^2)^{-1/2}$

$y'' \Rightarrow$ D.C.R.

$-5\left(\frac{1}{2}\right)(1-25x^2)^{-3/2}(-50x)$

$= 125x(1-25x^2)^{-3/2}$

QUIZ

Find y' and y'' Given

1. $y(x) = 6$
2. $y^2 + x^2 = 25$
3. $y = \text{Cosec}^{-1}(2x)$
4. $y = 2^{x^2}$
5. $y = x^{x^2}$ $\quad \ln x = \frac{1}{x}$
6. $y = \ln(3x^3 + 4x - 1)$ $\quad \frac{y'}{y} = \frac{9x^2+4}{3x^3+4x-1}$ $\quad y' =$
7. $y = x e^{x^2}$
8. $y = \sin^2 x$
9. $y = \sin x^2$
10. $y = \frac{1+x}{1-2x}$
11. $y = (x^2+1)^4$ $\quad 4(x^2+1)^3(2x)$ $\quad y = 8x\cdot(x^2+1)^3$
12. $y = (3x - \sin 2x)^2$

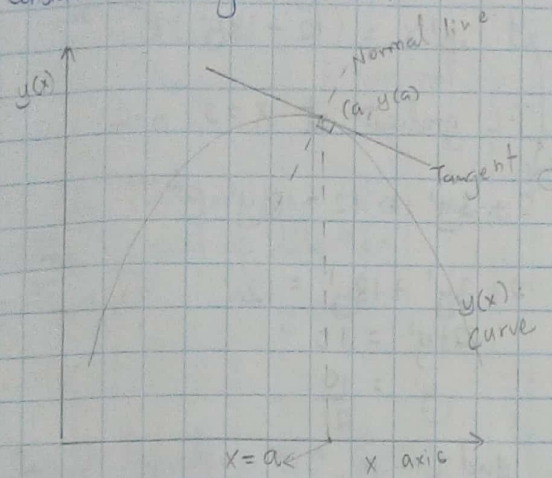
$y' = 2\underbrace{(3x - \sin 2x)}_{u}\underbrace{(3 - 2\cos 2x)}_{v}$

$y'' = (6x - 2\sin x)(3 - \cos 2x)$

Applications of Differentiation to :

(i) Equations of tangent line and normal line

- Consider the diagram below:



1. The rate of change in y with respect to x $(dy/dx) = y'(x)$ provides the gradient function to the curve $y(x)$.
2. If $y'(x)$ / dy/dx is evaluated at a point $x = a$ then the result is the gradient of the tangent line at the point $x = a$.
3. Since the normal line is ⊥ to the tangent line, the product of their gradients must be equal to -1.

   $m_1 m_2 = -1$
4. To find the equation of any straight line we require a known point (a,b) arbitrary point (x,y) all on the line and the gradient m ie $\frac{\Delta y}{\Delta x} = m$.

   $\frac{y-b}{y-a} = m$ equation of a straight line.

Ex

Find

1. Tangent line equation
2. Normal line equation for the following functions at the indicated points:

a) $2e^x + e^y = 3e^{x-y}$ $(0,0)$

b) $x^2 + y^2 = 25$ $(3, -4)$

$\underset{u\ \ v}{xy} = 6e^{2x-2y}$ product rule $(3,2)$

Soln

1. $x^2 + y^2 = 25$ $(3,-4)$

$3^2 + (4)^2 = 25 \rightarrow (3,-4)$ lies in the curve.

Diff w.r.t. x

$2x + 2yy' = 0$

$y' = -\frac{2x}{2y} = \frac{-x}{y}$

Tangent line gradient at $x = 3$ $y = -4$

$m_1 = y' = -\frac{3}{-4} = 3/4$

$m_1 = 3/4$. $\cancel{m_2 = -4/3}$

Taking $(3,-4)$ (x,y)

$\frac{y--4}{x-3} = 3/4$

$\frac{y+4}{x-3} = 3/4$ Linear equation.

N.L.E → Normal line equation.

$M_1 M_2 = -1$

$3/4\ M_2 = -1$

$M_2 = -4/3$

Taking $(3,-4)$ and (x,y)

$\frac{y--4}{x-3} = -\frac{4}{3}$

∴ Normal line ⇒ $\frac{y+4}{x-3} = \frac{-4}{3}$

∴ Tangent line = $\frac{y+4}{x-3} = 3/4$

Note $y = mx + c$

$\frac{x}{a} + \frac{y}{b} = 1$

Note:

e^f	$f'e^f$
e^{-x} $f=-x$, $f'=-1$	$-1e^{-x}$
e^{y} $f=y$, $f'=y'$	$y'e^{y}$
e^{x-y} $f=x-y$, $f'=1-y'$	$(1-y')e^{(x-y)}$
e^{2x-3y}	$(2-3y')e^{2x-3y}$

Quiz 2

$2e^{-x} + e^{y} = 3e^{(x-3)}$ (0,0)

- (0,0) lies on the line.
- Diff w.r.t x

$2e^{-x} + e^{y} = 3e^{(x-y)}$

$-2e^{-x} + y'e^{y} = 3(1-y')e^{(x-y)}$

T.L gradient at $x=0$ $y=0$

$-2e^{0} + y'e^{0} = 3(1-y')e^{0-0}$

$-2 + y' = 3 - 3y'$

$4y' = 5$

$y' = 5/4 \rightarrow M_1$.

$\frac{y-0}{x-0} = \frac{5}{4}$ (Tangent line)

N.L.E

$M_1 M_2 = -1$

$M_2 = -\frac{4}{5}$

$\frac{y-0}{x-0} = -\frac{4}{5}$ Normal line

Quiz 3

$xy = 6e^{2x-3y}$ (3,2)

u v

D.C.R

$u'v + v'u = 6(2x-3y')e^{2x-3y}$

$y + y'x = (12-18y')e^{2x-3y}$

T.L gradient at $x=3$ $y=2$

$2 + 3y' = 12 - 18y'e^{(6-6)}$

$2 + 3y' + 18y' = 12$

$21y' = 10$

$y' = \frac{10}{21}$

and $y' = m_1$

$m_2 = -\frac{21}{10}$

Tangent line $\Rightarrow$ $\frac{y-2}{x-3} = \frac{10}{21}$

Normal line $\Rightarrow$ $\frac{y-2}{x-3} = -\frac{21}{10}$

06/11/2019

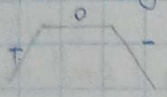
(ii) Stationary points, turning points, critical points

- Are the points where the rate of change in a dependant variable with respect to independent variable $(dy/dx)=0$ equals to 0.
- The points can be classified as:
  1. Maximum turning point
  2. Minimum turning point
  3. Point of inflection
- We use: to classify these points

(A) Sign test or first derivative test

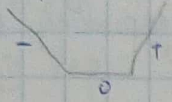
✓ Relies on the sign of dy/dx just to the left handside and just to the right handside of the turning point.

1 If sign dy/dx changes from + 0 −



Maximum pt

2. If sign of dy/dx changes from − 0 +



Minimum pt.

3. If sign changes from + 0 + or − 0 − it is a point of inflection.

(B) Second derivative test (y'')

✓ First compute $y'(x) = dy/dx$ and $y'' = \left(\frac{d^2y}{dx^2}\right)$

- If $y'' < 0$ at the point of the turning pt it is a maximum point
- If $y'' > 0$ .... minimum point
- If $y'' = 0$ at T.P then the test fails but it indicates a point of inflection hence we use sign to classify + 0 + or − 0 −

Ex 1

Identify and classify the stationary points of the following functions

a) $y = 3x^3 - 36x^2 + 135x - 13$

$$\frac{dy}{dx} = 9x^2 - 72x + 135$$

$$0 = \frac{9x^2}{9} - \frac{72x}{9} + \frac{135}{9} \quad \text{When } \frac{dy}{dx} = 0$$

$$x^2 - 8x + 15 = 0$$

$x = 3$ or $x = 5$

$$y = 3(3)^3 - 36(3^2) + 135(3) - 13$$
$(3, 149)$

$$y = 3(5)^3 - 36(5)^2 + 135(5) - 13$$
$(5, 137)$

Using second derivative

$$y' = 9x^2 - 72x + 135$$

$$y'' = 18x - 72$$

When $x = 3$ $y = -18$

when $x = 5$ $y = 18$

$(3, 149) \rightarrow$ Maximum pt

$(5, 137) \rightarrow$ Minimum pt.

Ex 2

$$y(x) = x^3 - \tfrac{3}{2}x^2$$
$$y' = 3x^2 - 3x$$
$$y'' = 6x - 3.$$

At $y = 0$

$$3x^2 - 3x = 0.$$

$$x = \frac{3 \pm \sqrt{9 - 0}}{6}$$

$$x = \frac{3 \pm 3}{6} \qquad x = 1 \text{ or } 0$$

At $x = 0$ $y = 0$ $(0, 0)$.

At $x = 1$ $y = 1 - \tfrac{3}{2}$ $(1, -\tfrac{1}{2})$

By 2nd derivative.

At $x = 0$ $y = 0 - 3 < 0$ Maximum pt $(0,0)$

At $x = 1$ $y = 3 > 0$ Minimum pt $(1, -\tfrac{1}{2})$

By sign test

$$y(x) = x^3 - \tfrac{3}{2}x^2$$

$$\frac{dy}{dx} = 3x(x-1) = 0$$

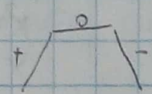
$3x = 0$ $x - 1 = 0$

$x = 0$ $x = 1$

$x = 0$ or $x = 1$.

$x = 0$

L	0	R
+	0	−

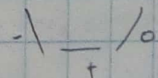


$(0, 0)$ Maximum pt.

Taking $x = 1$.

L	1	R
−	0	+

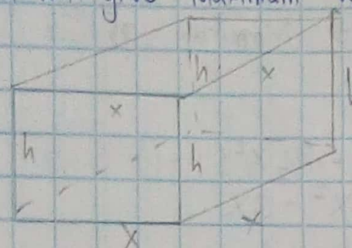
3x	(x − 1)	
−ve	+ve	L
+ve	+ve	R



Minimum turning pt.
$(1, -\tfrac{1}{2})$

Ex 3

A manufacture wants to design an open box having Square base and surface area of $108m^2$ Find the dimensions of the box that will give Maximum volume



$$\text{Volume} = x^2h \quad ----(i)$$

$$S.A = 108m^2$$

$$108 = x^2 + 4(xh) \quad ----(ii)$$

$$\frac{108-x^2}{4x} = h \quad \text{From (ii)}$$

$$V = x^2\left(\frac{108-x^2}{4x}\right).$$

$$V = \frac{108x - x^3}{4} \qquad V = 27x - \frac{x^3}{4}$$

$\frac{dv}{dx}$ should be maximum.

$$\frac{dv}{dx} = 27 - \frac{3}{4}x^2$$

$$27 - \frac{3}{4}x^2 = 0$$

$$\frac{3}{4}x^2 = 27$$

$$x^2 = 27 \times \frac{4}{3}$$

$$x^2 = 36$$

$$x = \pm\sqrt{36}$$

$$x = \pm 6$$

Ignore $x = -6$

Take $x = 6$ for max. Length.

$$V''(x) = -\frac{6}{4}x = -\frac{3}{2}x$$

When $x = 6$

$$V'' = -\frac{3}{2}(6) = -9 < 0 \text{ hence}$$

$x = 6$ leads to max volum.

$$h = \frac{108-6^2}{4\times 6}.$$

$$h = 3m.$$

(iii) Linear approximation / Small change

- For a small increment in x, say δx there exists a corresponding change in y, say δy.

$$x + \delta x \quad \exists \quad y + \delta y.$$

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \qquad \begin{array}{l}\delta - \text{Small delta}\\ d - \text{capital delta}\end{array}$$

$$\delta y \approx \frac{dy}{dx}(\delta x)$$

This technique is used to approximate values of some functions close to some known results. For example,

Ex 1

Use linear approximation / small changes to evaluate the following:

(a) $\sqrt{15}$

- $\sqrt{15} = \sqrt{16-1}$ $\qquad x + \delta x = 16 + (-1)$

- Let y represent a general root $\sqrt{x} = x^{1/2}$

- Find $dy/dx \Rightarrow \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

- Use approximation,

$$\delta y = \frac{dy}{dx}(\delta x)$$

$$\delta y = \frac{1}{2\sqrt{x}}(-1). \qquad x = 16, \ \delta x = -1$$

$$\frac{1}{2\sqrt{16}}(-1) = \frac{-1}{8} = -0.125.$$

- $y + \delta y$

$$\sqrt{x} + \delta y.$$

$$\sqrt{16} + -0.125$$

$$4 - 0.125$$

$$= 3.875.$$

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b) $\sqrt[4]{82}$

• Use $81 = 3^4$

$\sqrt[4]{81+1}$

$x + \delta x$

• Let y rep $\sqrt[4]{x} = x^{1/4}$

• $\frac{dy}{dx} = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$

• $\delta y = \frac{dy}{dx}(\delta x)$

$\delta y = \frac{1}{4x^{3/4}}(\delta x)$ where $x = 81$, $\delta x = 1$

$\delta y = \frac{1}{4\times 27} \times 1$

$\delta y = \frac{1}{108}$

• $y + \delta y \Rightarrow \sqrt[4]{x} + \delta y$

$= \sqrt[4]{81} + \frac{1}{108}$

$= 3 + 0.0093$

$= 3.0093$

c) Sin 29.

1. • Close to Sin 30.

• Sin (30 − 1) = Sin 29.

• $x + \delta x \rightarrow 30° + -1°$

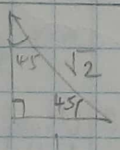
2. • $y = \sin 30$.

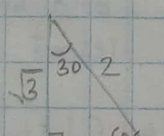
3. $\frac{dy}{dx} = \cos 30$

4. $\delta y = \frac{dy}{dx}(\delta x)$

$dy \cos 30 (-1°)$

$dy = (-1)° \cos 30°$





$\cos 60 = \frac{1}{2}$

$\sin 30 = 1/2$

$\cos 30 = \frac{\sqrt{3}}{2}$

$\sin 60 = \sqrt{3}/2$

$\cos 45 = \frac{1}{\sqrt{2}}$

Sin 45 =

5. ~~$y + dy$~~

~~Sin 30 − Cos 30~~

~~$\frac{1}{2}$~~

$\pi^c = 180°$

$= 1°$

$\frac{1\times\pi^c}{180} = 0.0175$

$\delta y = \left(\frac{\sqrt{3}}{2}\right)(-0.0175)$

In rad — rads

$\delta y = -0.0152$

5. $y + dy$

Sin 30 − 0.0152

0.5 − 0.0152

= 0.4848

Assignment

Cos 31°

Sin 61°

Cos 59°

Tan 44°

Cos 28°

Sin 46°

$(26)^{2/3}$

$(27)^{2/3} =$

Note

Ln 1 = 0

$e^0 = 1$

$\log_{10} 10 = 1$

(iv) Cost function, revenue function

- Cost function, C(x) is the cost incurred in producing x items

- Marginal cost (mc) is the rate of change of the cost function w.r.t. the number of units produced, x

$MC = \frac{dc}{dx}$.

It represents the extra cost incurred in producing:

(1) Additional items when the level of production is already at x.

eg x = 10 | 11 } produced
x = 100 | 101 }

Average cost = $\frac{\text{Total cost } (x)}{x}$

Revenue ~~cost~~ fxn R(x) is the revenue received when x units of a given commodity are produced and sold at a unit price, $P(x)$ or Demand function.

Marginal ~~funct~~ revenue $= \frac{dR}{dx}$

Revenue function: $R(x) = x \cdot P(x) - C(x)$

Profit function $P(x) = T_R - T_C = \Pi(x)$

Marginal profit $MP = \frac{d\Pi}{dx}$.

At max / Minimum

T_R Total revenue
T_C - Total cost

$MC = \frac{dC}{dx} = 0$

$MR = \frac{dR}{dx} = 0$

$MP = \frac{d\Pi}{dx} = 0$.

Critical points.

Example

In marketing a certain item, a business has discovered that the demand/for the item is represented by: unit price

$p(x) = \frac{50}{\sqrt{x}}$

The cost of producing x items is given by $C(x) = 0.5x + 500$. Find the price per unit that will yield maximim profit.

$P(x) = \Pi(x) = \text{Total Rev} - \text{Total cost}$

$\Pi(x) = x\left(\frac{50}{\sqrt{x}}\right) - 0.5x - 500$ Revenue.

$= 50x^{1/2} - 0.5x - 500$

Marginal profit $MP = \frac{d\Pi}{dx}$

Π Profit
$= T_R - T_C$

$MP = 0$

$0 = 25x^{-1/2} - 0.5$

$0.5 = 25x^{-1/2}$

$x^{-1/2} = \frac{0.5}{25}$

$x^{1/2} = \frac{25}{0.5}$

$\sqrt{x} = 50$

$x = 2500$ at max.

Test if $x = 2500$ will lead to max pt

$\Pi''(x) = -\frac{1}{2} \times 25(x)^{-3/2} = -\frac{25}{2}x^{-3/2}$

$x = 2500$

$\Pi''(2500) = -\frac{25}{2}(2500)^{-3/2}$

$-0.0001 < 0.$

Hence the business will reduce max. profit if 2500 units of the item are produced.

Price per unit $= \frac{50}{\sqrt{2500}} = \frac{50}{50}$

$= 1$ (unit)

18/11/2019. Monday.

Relative rates

1. Give symbols for those that are known and those unknown.
2. Write an equation involving all the variables whose rates of change are given or are to be determined.
3. Using chain rule, implicitly differentiate both sides of the equation with respect to time, T.
4. Substitute into the resulting equation all the known variables and their rates of change. Then solve for the required rate of chage.

1. Sand is falling in a conical pipe at a rate of $100 m^3/min$. Find the rate of change of the height when the height is 10m. (Assume that the coasance of the sand is such that the sand is equal to the radius.

Step 1
Let V be the vol of Conical pipe, h the height, r The radius

Given $dv/dt = 100 m^3/min$.

Required to find $\frac{dh}{dt}$ when $h=10$ and $r=h$.

Step 2:

$V = \frac{1}{3}\pi r^2 h \quad h = r$

$V = \frac{1}{3}\pi (h)^2 h$

$V = \frac{1}{3}\pi h^3$

Step 3; diff implicitly w.r.t. T.

$1(v)^{1-1} \frac{dv}{dt} = \frac{1}{3}\pi\, 3(h)^{3-1}\frac{dh}{dt}$

$\frac{dV}{dt} = \pi h^2 \frac{dv}{dt}$

$\frac{dv}{dt} = 100 \quad h = 10$

$100 = \pi(10)^2 \frac{dh}{dt}$

$\frac{100}{100\pi} = \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{\pi} = 0.318 m/min$

The rate of change of height is increasing

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$5 = 4\pi(2)^2\frac{dr}{dt}$

$\frac{5}{4(4)\pi} = \frac{dr}{dt} = \frac{5}{16\pi} = 0.3995$

2. Air is pumped into a spherical baloon at a rate of $5cm^3/min$ Find the rate of change in The radius, when The radius is 2 cm

① Let V of spherical baloon be V
let r be radius of spherical baloon
Given $\frac{dV}{dt} = 5cm^3/min$

Required to find $\frac{dr}{dt}$ when $r=2$

②
$V = \frac{4}{3}\pi r^3$

③ Differentiate implicitly w.r.t T.

$1(v)^{1-1} \cdot \frac{dv}{dt} = \frac{4}{\cancel{3}}\pi\, \cancel{3}(r)^{3-1}\frac{dr}{dt}$

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$5 cm^3/min = 4\pi(2)^2\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{4\pi \times 4}$

$=$

Kinematics

If the displacement of a particle after time, t is $S(t)$ then its velocity V will be $\frac{dS}{dt}$ and acceleration $\frac{dV}{dt}$

$a = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{(dt)^2}$

N/B: When an object is momentarily at rest $V=0$ also sometimes The initial velocity is 0.

Ex 1
If $S(t) = 3t^3 - 36t^2 + 135t - 13$ meters
(i) Find The distance when the object is momentarily at rest
(ii) The acceleration when $t = 7$ seconds

(i) $V = \frac{ds}{dt} = 9t^2 - 72t + 135$

At $V = 0$. Momentarily at rest.

$9t^2 - 72t + 135 = 0$

$t^2 - 8t + 15 = 0$.

$t = \frac{8 \pm \sqrt{64 - 60}}{2}$

$t = \frac{8 \pm 2}{2}$ $\quad t = 3$ sec or $t = 5$ sec

S at 3 sec $= 3(3)^3 - 36(9) + 135(3) - 13$

$S = 1$

S at 5 sec $= 3(5)^3 - 36(25) + 135(5) - 13$

$S =$

(ii) $V = 9t^2 - 72t + 135$

$a = 18t - 72$

at 7 sec

$18(7) - 72$

$= 54$ m.

INTEGRATION

- Is the reverse process of differentiation
- Given a differentiated function

$\frac{dy}{dx} = f(x)$

$dy = dx\,(f(x))$

Intergrate

$\int dy = \int dx\, f(x)$

$y(x) = \int f(x)\, dx + K$

Symbol — Integrad — Constant.

Techniques of Integration.

1. Power rule of integration.

If $\int x\, dx = \frac{x^{n+1}}{n+1} + C$

Examples

1. $\int x^3\, dx = \frac{x^4}{4} + K$

2. $\int x^{-7}\, dx = \frac{x^{-6}}{-6} + K$

3. $\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \frac{x^{3/2}}{3/2} + K$

$= \frac{2}{3} x^{3/2} + K$

4. $\int x^6 - 2x + 6\, dx = \int x^6 - 2\int x + \int 6$

$= \frac{x^7}{7} - \frac{2x^2}{2} + \frac{6x}{1} + K.$

$= \frac{x^7}{7} - x^2 + 6x + K$

5. $\int \frac{x^6 - 2x - 4}{x^8}\, dx$

$= \int x^{6-8} - 2x^{1-8} - 4(x)^{8-8}\, dx$

$= \int x^{-2} - 2x^{-7} - 4x^{-8}\, dx$

$= \frac{x^{-1}}{-1} - \frac{2x^{-6}}{-6} - \frac{4x^{-7}}{-7} + K$

N/B.

$\frac{a^m}{a^n} = a^{m-n}$

Exercise

1. $\int 2x^3 - 6\frac{1}{x^6} + 2x\, dx$

a. $\int \sqrt[3]{x^2} \int 2x^3 - x^{-6} + 2x \, dx$

$$= \frac{2x^4}{4} - \frac{x^{-5}}{-5} + \frac{2x^2}{2} + k$$

$$= \frac{1}{2}x^4 + \frac{x^{-5}}{5} + x^2 + k$$

$$= \frac{x^4}{2} + \frac{1}{5x} + x^2 + k$$

2. Substitution.

This technique requires that a new variable is introduced to represent the interior function and to reduce the problem to a form in which power rule of integration can be applied.

Example:

a) $\int x(3-5x^2)^4 dx$ .... ①

Let p be $(3-5x^2)$ ②

Diff p w.r.t. x make dx the subject

$$p = 3 - 5x^2$$

$$\frac{dp}{dx} = -10x$$

$$\frac{dp}{-10x} = dx \quad .... ②$$

Sub 2 and 3 in 1

$$\int x(p)^4 \frac{dp}{-10x} = \int -\frac{1}{10} p^4 \, dp$$ Apply power rule on p.

$$= -\frac{1}{10} \frac{p^5}{5} + K$$

$$= -\frac{1}{10} \frac{(3-5x^2)^5}{5} + K$$

$$= \frac{-(3-5x^2)^5}{50} + K$$

$$= -\frac{1}{50}(3-5x^2)^5 + K$$

b) $\int \frac{x+1}{\sqrt[5]{x^2+2x+7}} dx$ .... (i)

Let u be $x^2 + 2x + 7$ .... (ii)

$$\frac{du}{dx} = 2x + 2$$

$$dx = \frac{du}{2x+2} \quad .... (iii)$$

Substitute (ii) and (iii) in (i)

$$\int \frac{x+1}{\sqrt[5]{u}} \cdot \frac{du}{2x+2}$$

$$= \int \frac{\cancel{x+1}}{\sqrt[5]{u}} \frac{du}{2(\cancel{x+1})}$$

$$= \int \frac{1}{2} \frac{du}{\sqrt[5]{u}} = \frac{1}{2} \int \frac{du}{u^{1/5}} = \frac{1}{2}\left(\int u^{-1/5} du\right)$$

Apply power rule to u.

$$\frac{1}{2} \cdot \frac{5}{4} u^{4/5} + k$$

$$= \frac{5}{8}(x^2+2x+7)^{4/5} + K$$

c) $\int x(2+3x)^4 \, dx$ .... (1)

Let $M = 2 + 3x$ .... ②

$$\frac{dM}{dx} = 3$$

$$\frac{dM}{3} = dx \quad ...... ③$$

Substitute 2 and 3 in 1

$$\int x(M)^4 \frac{dM}{3} = \frac{1}{3}\int x(M)^4 dM$$

??

Use equation ② to eliminate x

$$M = 2 + 3x$$

$$\frac{M-2}{3} = \frac{3x}{3} \qquad x = \frac{M-2}{3}$$

$$= \frac{1}{3}\int \frac{M-2}{3}(M^4) \, dM$$

Apply power rule

$$\frac{1}{3}\int \frac{M^5}{3} - \frac{2}{3}\cdot M^4 \, dM$$

Integrate

$$\frac{1}{3}\left(\frac{M^6}{3\cdot 6} - \frac{2}{3}\left(\frac{M^5}{5}\right)\right) + K$$

$$\frac{1}{3}\left(\frac{M^6}{18} - \frac{2M^5}{15}\right) + K$$

Replace M.

$$\frac{1}{3}\left(\frac{(2+3x)^6}{18} - 2\frac{(2+3x)^5}{15}\right) + K$$

Exercise

1. $\int \frac{x^2+3}{(x^3+4x+17)^4}\, dx$

2. $\int (x+2)\sqrt[3]{x^2+4x-5}\, dx$

3. $\int \frac{x^2}{(5x^3+6)^7}\, dx$

4. $\int x\sqrt{5-3x}\, dx$

5. $\int \frac{-x}{4}(5+3x)^6\, dx$ $\quad \frac{-x}{4}(p^6)\,dx$

$p = 5+3x$

$\frac{dp}{dx} = 3$

$dx = \frac{dp}{3}$

$\frac{-x}{4}\left(\frac{dp}{3}\right)$

6. $\int 5x(1-2x)^3\, dx$

7. $\int \frac{x}{-7}(6+8x)^5\, dx$

8. $\int \frac{x^2}{(x^3+4)^7}\, dx$

Semester 2. 14/01/2020

CALCULUS II.

1. Parametric and implicit differentiation involving H.O.D. ✓

2. Applications of Differentiation to:
   - Tangent line equation and Normal line equation. ✓

3. Curve sketching and assymptotes ✓

4. Hyperbolic functions, their definition, their differentiation and Integration.

5. Introduction to Integration and techniques of Integration:
   - ✓ Power rule ✓
   - ✓ Substitution ✓
   - ✓ Integration by parts ✓
   - ✓ Integration via partial fractions ✓
   - ✓ Integration via t-substitution. ✓

6. Numerical Integration ✓
   - Trapezidal rule ✓
   - Simpson's rule

7. Applications of Integration to:
   - Area
   - Volume
   - Arc length
   - Cost
   - Revenue
   - Profit
   - Kinematics

8. Solution of Ordinary differential eqns by separation of variables methods ✓

References

i) Calculus I notes
ii) Calculus & Analytical Geometry

Review of parametric differentiation involving H.O.D.

- Consider $y = t^3 + 6t + 1$

$x = 2t^2 + 4$.

Find dy/dx, d^2y/dx^2, d^3y/dx^3
y' y'' y'''

Done in calc I

$\frac{dx}{dt} = 4t$

$\frac{dy}{dt} = 3t^2 + 6$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dt} \times \frac{dt}{dx}$

$dy/dx = (3t^2 + 6) \times \frac{1}{4t}$

$dy/dx = \frac{3t^2 + 6}{4t}$

$dy/dx = \frac{3}{4}t + \frac{3}{2}t^{-1}$

To find $y'' = \frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ → A function of the Parameter t

Cannot be diff. w.r.t. x.

Using chain rule.

$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$

$= \frac{d}{dt}\left(\frac{3}{4}t + \frac{3}{2}t^{-1}\right) \cdot \left(\frac{1}{4t}\right)$

$= \left(3/4 - 3/2\, t^{-2}\right) \cdot \left(\frac{1}{4t}\right)$

$= 3/16t - \frac{3}{8}\frac{t^{-2}}{t}$

$= 3/16\, t^{-1} - 3/8\, t^{-3}$

To find $y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

↓ a fxn of t
$3/16\,t^{-1} - 3/8\,t^{-3}$

Use chain rule.

$= \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) \cdot \frac{dt}{dx}$

$= \frac{d}{dt}\left(\frac{3}{16}t^{-1} - \frac{3}{8}t^{-3}\right) \cdot \left(\frac{1}{4t}\right)$

Power rule - Bring power down reduce power by 1

$= \left[-\frac{3}{16}t^{-2} + \frac{9}{8}t^{-4}\right] \cdot \left(\frac{1}{4t}\right)$

$y''' = -\frac{3}{64}t^{-3} + \frac{9}{32}t^{-5}$

Example 2.

Given $y = \cos 2\theta$ $x = \sin 2\theta$.

Find y', y'' and y'''.

a) $y' = \frac{dy}{dx}$ $-f' \sin f$

$\frac{dy}{d\theta} = -2 \sin 2\theta$

$\frac{dx}{d\theta} = 2 \cos 2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$= -\cancel{2} \sin 2\theta \times \frac{1}{\cancel{2} \cos 2\theta}$

$dy/dx = -\tan 2\theta$

b) $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ A function of θ

By chain rule:

$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$

$\frac{d}{d\theta}\left(-\tan 2\theta\right) \cdot \frac{d\theta}{dx}$ $\quad$ Tan f : f' $\sec^2$ f

$= -2\sec^2 2\theta \cdot \frac{1}{2\cos 2\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\quad$ $\sec^2\theta = \frac{1}{\cos^2\theta}$

$= -\frac{1}{\cos^2 2\theta} \cdot \frac{1}{\cos 2\theta}$

$y'' = -\frac{1}{\cos^3 2\theta} = -\sec^3 2\theta$

c) $y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ — A fxn of θ

Chain rule

$y''' = \frac{d}{d\theta}\left(\frac{d^2y}{dx^2}\right) \cdot \frac{d\theta}{dx}$

$= \frac{d}{d\theta}\left(-\sec^3 2\theta\right) \cdot \left(\frac{1}{2\cos 2\theta}\right)$

f' sec f tan f

~~Sec θ → y'~~

Apply direct chain rule.

$-(\sec 2\theta)^3$, By D/C/R

$-3(\sec 2\theta)^2(2\sec 2\theta \tan 2\theta)$

$y''' = -3(\sec 2\theta)^2(2\sec 2\theta \tan 2\theta) \cdot \frac{1}{2\cos 2\theta}$

$y''' = -3(\sec^2 2\theta)(2\sec 2\theta \tan 2\theta)\left(\frac{1}{2\cos 2\theta}\right)$

Simplified

$= \frac{-3\sec^3 2\theta \tan 2\theta}{\cos 2\theta}$

$= \frac{-3\tan 2\theta}{\cos^4 2\theta}$ $\quad$ $\tan = \frac{\sin}{\cos}$

$= -\frac{3\sin 2\theta}{\cos^5 2\theta} = -\frac{3\sin 2\theta}{\cos 2\theta} \cdot \frac{1}{\cos^4 2\theta}$

$= -3\tan 2\theta \cdot \sec^4 2\theta$

REVIEW OF IMPLICIT DIFFERENTIATION INVOLVING H.O.D.

Find y', y'', y''' Given $x^2 + y^2 = 1$

a) $y' = \frac{dy}{dx}$

Diff w.r.t. x

$x^2 + y^2 = 1$

$2x + 2y \cdot y' = 0$

$\frac{-2x}{2y} = y'$

$y' = \frac{-x}{y}$

b) $y'' = \frac{d^2y}{dx^2}$

Redifferentiate y' w.r.t. x.

$y'' = \frac{d}{dx}(y')$

$= \frac{d}{dx}\left(-\frac{x}{y}\right) \rightarrow f(x)$

i) Quotient rule
ii) Product rule

Quotient rule	Product rule
$u = -x$ $\quad$ $u' = -1$	$u = -x$ $\quad$ $u' = -1$
$v = y$ $\quad$ $v' = y'$	$v = \frac{1}{y} = y^{-1}$ $\quad$ $v' = -y^{-2} \cdot y'$
$\frac{u'v - v'u}{v^2}$	$u'v + v'u$
$\frac{-1(y) - y'(-x)}{y^2}$	$-1(y^{-1}) + (-x)(-y^{-2} \cdot y')$
$= \frac{-y + xy'}{y^2}$	$-y^{-1} + (-x)\left(-y^{-2} \cdot \frac{-x}{y}\right)$
$= \frac{-y + x\left(\frac{x}{y}\right)}{y^2}$	$-y^{-1} + \left(\frac{x}{y^2} \cdot \frac{-x}{y}\right)$
$= \frac{\frac{-y^2 - x^2}{y}}{y^2}$	$= -\frac{1}{y} - \frac{x^2}{y^3}$
$= \frac{-y^2 - x^2}{y^3}$	$= \frac{-y^2 - x^2}{y^3}$

c) $y''' = \frac{d^3y}{dx^3}$ Diff. w.r.t x $\frac{d^2y}{dx^2}$

Scanned by CamScanner

Quotient rule

$$\frac{d}{dx}\left(\frac{-y^2-x^2}{y^3}\right)$$

$u = -y^2 - x^2$, $\quad u' = -2y \cdot y' - 2x$

$v = y^3$ $\quad v' = 3y^2 \cdot y'$

Q.R $\frac{u'v - v'u}{v^2}$ $\quad$ But $y' = \frac{-x}{y}$

$$= \frac{\left(-2y\left(\frac{-x}{y}\right) - 2x\right) \cdot y^3 - \left(3y^2\left(\frac{-x}{y}\right)\right) \cdot -y^2 - x^2}{y^6}$$

$$= \frac{4xy^3 + 3xy(-y^2 - x^2)}{y^6}$$

$$= \frac{4x}{y^3} + \frac{3x(-y^2 - x^2)}{y^5}$$

$$\frac{-2y^4 y' - 2xy^3 + 3y^4 y' + 3y^2 y' x^2}{y^6}$$

Find y' and y'' $\quad$ Exercise

1. $2e^{-x} + e^y = 3e^{x+y}$
2. $x^{2/3} + y^{2/3} = 5$
3. $xy^2 + x^2y = 2$
4. $\sqrt{xy} = x - 2y$
5. $y^2 = \frac{x^2 - 9}{x^2 + 9}$
6. $x^{1/2} + y^{1/2} = 9$

No. 3

$x^2y + xy^2 = 2$

$1 \cdot y^2 + x \cdot 2yy' + 2xy + x^2 \cdot y' = 0$

$y'(2xy + x^2) = -2xy - y^2$

$$y' = \frac{-2xy - y^2}{2xy + x^2}$$

$$y'' = \frac{u'v - v'u}{v^2}$$

$u = -2xy - y^2$

$u' = -2(y) + (-2x)(y') - 2y \cdot y'$

$v = 2xy + x^2$

$v' = 2y + 2x(y') + 2x$

$$\frac{(-2y + -2x(y') - 2y(y') \cdot 2xy + x^2) - 2y + 2x(y') + 2x \cdot (-2xy - y^2)}{(2xy + x^2)^2}$$

CURVE SKETCHING

To sketch any curve $y = f(x)$

i) Identify the x-intercepts (Points where $y = 0$)

(ii) Identify the y-intercepts (Pts where $x = 0$)

(iii) Identify the turning points (Pts where $y'\left(\frac{dy}{dx}\right) = 0$.

(iv) Classify the turning points as maximum, minimum or points of inflection by Sign test (or first derivative test) or by second derivative test.

(v) Mark the above points on an unscaled cartesian plane. Use free-hand to join the points.

Example

Sketch the following curve

1. $y(x) = x^2 + 5x + 6$

(i) x-intercept at $y = 0$

$x^2 + 5x + 6 = 0$

$x = -3 \qquad x = -2$

$(-3, 0) \qquad (-2, 0)$

ii) y-intercept at $x = 0$

$y = 6. \qquad (0, 6)$

iii) $\frac{dy}{dx} = 2x + 5$

At $y' = 0 \quad 2x + 5 = 0$

$2x = -5$

$x = -5/2$ $\qquad (-5/2, -1/4)$

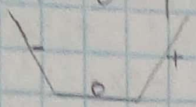
$y = \left(\frac{-5}{2}\right)^2 + 5\left(\frac{5}{2}\right) + 6$

$y = -\frac{1}{4}$

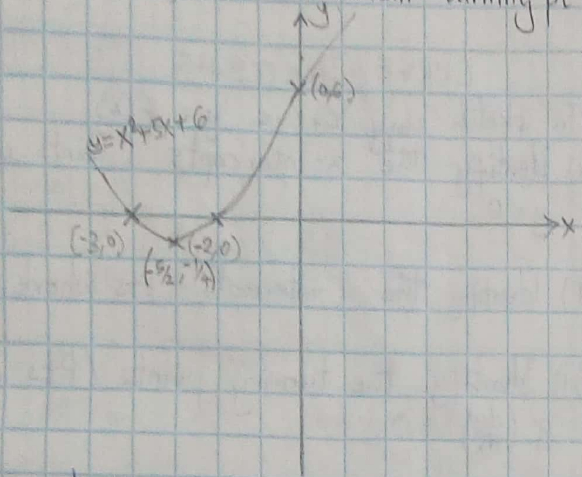
iv) Classify pts (Turning)

• By sign test

x	L	-2.5	R.
dy/dx	-	0	+



$(-5/2, -1/4)$ is a minimum turning pt.



Example 2

$y(x) = \frac{2}{3}x^3 - 2x^2$

Step 1 x intercept $(y=0)$

$0 = 2x^2\left(\frac{x}{3} - 1\right)$

$2x^2 = 0$ $\qquad x/3 - 1 = 0$

$x = 0$ $\qquad x = 3$

$(0,0)$ $\qquad (3,0)$

Step 2 y-intercept

$y = 2/3(0)^3 - 2(0)^2 = 0$

$(0,0)$

Step 3 T.pts $dy/dx = 0$

$dy/dx = 2/3(3x^2 - 4x)$

$0 = 2x^2 - 4x$

$0 = 2x(x-2)$

$x = 0$ $\qquad x = 2$

$y = 2/3(0^3) - 2(0)^2$
$= 0$
$(0,0)$

$y = 2/3(2)^3 - 2(2)^2$
$\frac{16}{3} - 8$
$= -8/3$ $\qquad (2, -8/3)$

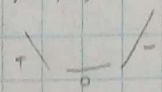
Step 4 classify T. pts $(0,0)$ & $(2,-8/3)$

$y'(x) = 2x^2 - 4x$

$y''(x) = 4x - 4$

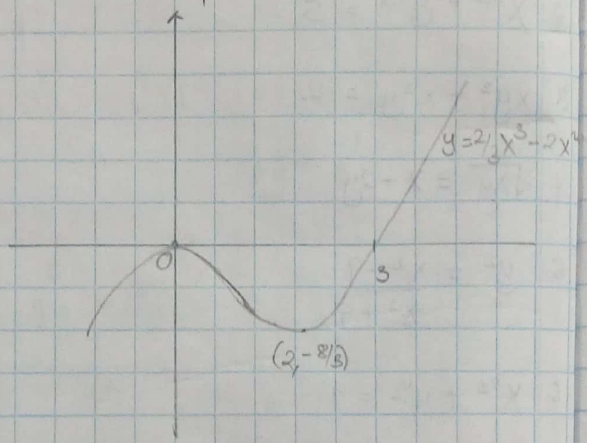
When $x=0$ $y''(0) = 4(0) - 4 = -4 < 0$.
hence $(0,0)$ is max. T.pt

When $x = 2$ $y''(2) = 4(2) - 4 = 4 > 0$
hence $(2, -8/3)$ is a minimum t.pt



Step 5

Mark $(0,0)$, $(2,0)$ $(0,0)$, $(0,0)$ $(2,-8/3)$ on unscaled plane.



Tuesday 21/01/2020

ASYMPTOTES

An asymptote is a straight line which the graph of $y=f(x)$ approaches as the values of x increases or decreases in the interval $-\infty$ to $+\infty$.

i) The vertical asymptotes

The line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either

- $\lim_{x \to a^+} f(x) = +\infty$ or $-\infty$

or

- $\lim_{x \to a^-} f(x) = +\infty$ or $-\infty$

NOTE: For rational functions such as $f(x) = \frac{p(x)}{g(x)}$ the vertical asymptotes are given by the values of x for which $g(x) = 0 \Rightarrow$ The denominator. $D(x)=0$

Example:

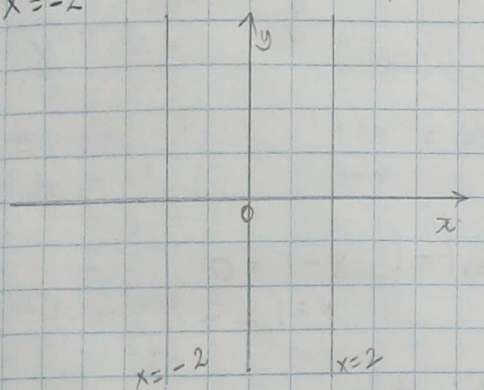
$f(x) = \frac{x^2}{4-x^2}$

In this case the vertical asymptotes are

$4-x^2=0$

$x = \pm 2$

$x=2$, $x=-2$ > are vertical assymptotes.



(ii) Horizontal assymptotes

The line $y=b$ is a horizontal asymptote of the graph of the function $y=f(x)$ if either

- $\lim_{x \to +\infty} f(x) = b$

or

- $\lim_{x \to -\infty} f(x) = b$

Example

$$\lim_{x \to \infty} \left(\frac{x^2}{4-x^2}\right)$$

$$\frac{\frac{x^2}{x^2}}{\frac{4}{x^2} - \frac{x^2}{x^2}} = \frac{1}{\frac{4}{x^2} - 1} = \frac{1}{0-1} = \frac{1}{-1}$$

$y = -1$

$y=-1$ is a horizontal asymptote

2. $f(x) = \frac{1+x}{x-1}$

$$\lim_{x \to \infty} \left(\frac{1+x}{x-1}\right)$$

Divide by x.

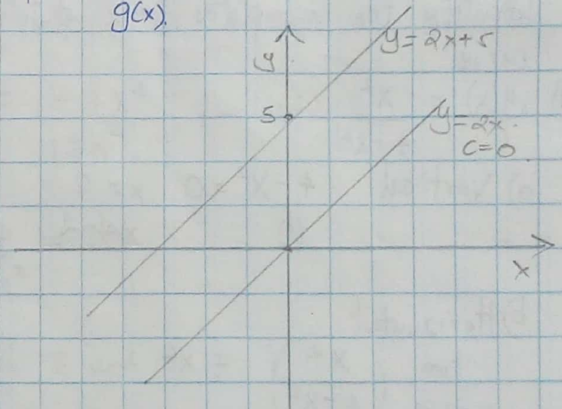
$$\lim_{x \to \infty} \left(\frac{\frac{1}{x}+1}{1-\frac{1}{x}}\right)$$ Substitute x with ∞

$$\frac{0+1}{1-0} = \frac{1}{1}$$

$y=1$ is a H/A.

(iii) Slant asymptotes (oblique)

If the function $f(x) = \frac{P(x)}{g(x)}$ can be written as $(mx+c) + \frac{h(x)}{x}$ via long division & in case the degree of in p is greater than that in g, then the part that represent the straight line equation $(mx+c)$ is called the slant assymptote of $y=f(x) = \frac{P(x)}{g(x)}$.



Example.

Identify assymtotes in the function

$y(x) = \frac{x^2+x-1}{x-1}$

Vertical asymptote

$x-1=0$

$x=1$

H/A $\lim_{x \to \infty} f(x)$

$$\lim_{x \to \infty} \left(\frac{x^2+x-1}{x-1}\right) = \lim_{x \to \infty} \left(\frac{x+1-\frac{1}{x}}{1-\frac{1}{x}}\right)$$

$$\lim_{x \to \infty} \frac{(x+1)}{1} = \infty \text{ (No horizontal asymptote)}$$

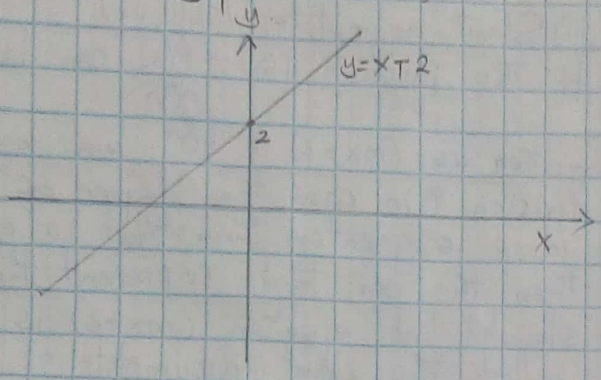
Slant asymptote.

$$\begin{array}{r|l} & x + 2 \\ \hline x-1 & x^2 + x - 1 \\ & -x^2 \mp x \\ \hline & \quad 2x - 1 \\ & \quad -2x - 2 \\ \hline & \qquad 1 \end{array}$$

$$mx + c + h(x)$$

$$(x+2) + \frac{1}{x-1}$$

Slant asymptote $= (x+2)$



EXERCISE

Identify the asymptotes and sketch the curve

① $y(x) = \dfrac{x^2}{4-x^2}$

a) Vertical $\quad 4 - x^2 = 0 \quad x = 2$
$x = -2$.

b) Horizontal

$$\lim_{x \to \infty} \left(\frac{x^2}{4-x^2}\right) \div x^2$$

$$\lim_{x \to \infty} \left(\frac{1}{4/x^2 - 1}\right) = -1$$

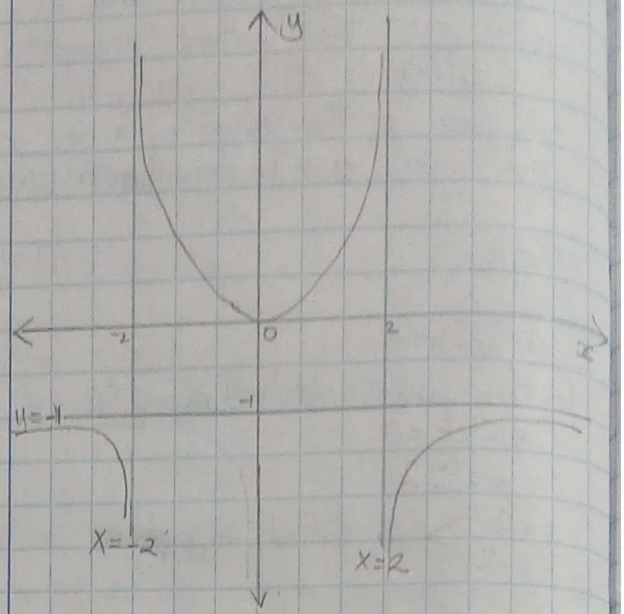
$y = -1$

c) No slant asymptotes since the degree of x in N and D are the same.

NOTE:

(i) The graph of $y = f(x)$ will not intercept any of the above asymptotes since the asymptotes are the break points for the function $y = f(x)$.

(ii) The asymptotes are not part of the graph of the function $y = f(x)$ but they are tools necessary for sketching the graph.



② $y(x) = \dfrac{x+1}{x-1}$

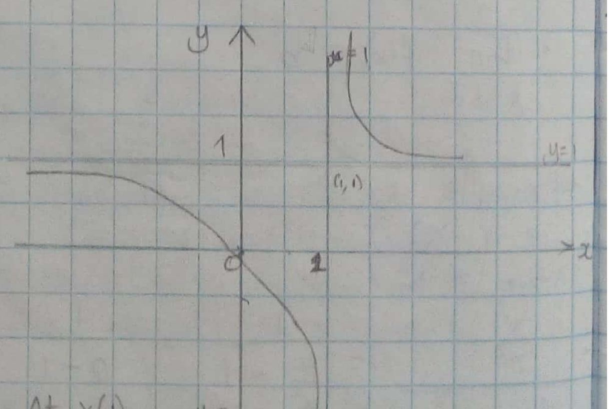
a) Vertical $\quad x - 1 = 0$
$x = 1$

b) Horizontal

$$\lim_{x \to \infty} \left(\frac{x+1}{x-1}\right) = \lim_{x \to \infty} \left(\frac{1 + \frac{1}{x}}{\cancel{1} - \frac{1}{x}}\right) = 1$$

$y = 1$

c) No slant asymptotes since degree of x is same in both D and N.



At x(1) $\quad y = \dfrac{2}{0}$

undefined.

Introduction to Integration

Given a derived fxn or a differentiated function

$$\frac{dy}{dx} = f(x)$$

$$dy = f(x)\ dx$$

Integrate both sides

$$\int dy = \int f(x)\ dx$$

$$y\ f(x) = \int f(x)\ dx + C$$

Symbol for Integrant; The integrand; constant of integrand

Techniques of Integration

1. Power rule.

Given $I = \int x^n\ dx$. By power rule

$$\frac{x^{n+1}}{n+1} + k$$

Examples

1. $\int x^2\ dx = \frac{x^3}{3} + C$

2. $\int x^{-5}\ dx = \frac{x^{-4}}{-4} + k$

3. $\int \sqrt{x}\ dx = \int (x^{1/2})\ dx$

$$= \frac{x^{1+1/2}}{1+1/2} + M$$

$$= 2/3\ x^{3/2} + M.$$

4. $\int \frac{x^6 - 2x + 4}{x^4} = \int x^2 - 2x^{-3} + 4x^{-4}\ dx$

$$\frac{x^3}{3} - \frac{2x^{-2}}{-2} + \frac{4x^{-3}}{-3} + C$$

$$= \frac{x^3}{3} + x^{-2} - \frac{4}{3}x^{-3} + C$$

5. $\int \sqrt{x} - 3\sqrt[4]{x^5}\ dx$

$$= \int x^{1/2} - 3x^{5/4}\ dx$$

$$\frac{x^{1 1/2}}{1 1/2} - \frac{3x^{9/4}}{9/4} + k$$

6. $\int (2x^3 - 4x - 14x - 1)\ dx$

7. $\int (x^3 + 6)^2\ dx$

8. $\int \frac{\sqrt[3]{x} - 6\sqrt{x^3} + 1}{x^3}\ dx.$

9. $\int \frac{x^{1/3} - 6\sqrt{x^3} + 1}{x^3}\ dx$

2. Substitution

This technique require that a new variable is introduced in the integrand to reduce the problem to a form in which power rule of integration can be applied.

Example.

$$\int x^2(1 + 4x^3)^3\ dx \quad \text{.....(i)}$$

$$p = 1 + 4x^3 \quad \text{.....(2)}$$

$$\frac{dp}{dx} = 12x^2$$

$$\frac{dp}{12x^2} = dx \quad \text{.....(3)}$$

Subst. 2 and 3 in i

$$\int x^2 p^3 \frac{dp}{12x^2}$$

$$\int \frac{1}{12} p^3\ dp$$

Power rule

$$\frac{1}{12}\left(\frac{p^4}{4}\right) + k$$

$$= \frac{1}{48} p^4 + k$$

$$= \frac{1}{48}(1 + 4x^3)^4 + k$$

$$\int \frac{x+1}{\sqrt{2x^2+4x-1}}\,dx \quad \text{......(1)}$$

$u = 2x^2 + 4x - 1$ ....(2)

$\frac{du}{dx} = 4x + 4 = 4(x+1)$

$\frac{du}{4(x+1)} = dx$ .....(3)

Subst 2 and 3 in 1

$$\int \frac{\cancel{x+1}}{\sqrt{u}} \cdot \frac{du}{4\cancel{(x+1)}} = \frac{1}{4}\int \frac{1}{u^{1/2}}\,du$$

$$= \frac{1}{4}\int u^{-1/2}\,du$$

Power rule:

$$\frac{1}{4}\left(\frac{u^{1/2}}{1/2}\right) + c$$

$$= \frac{2}{4}u^{1/2} + c$$

$$= \frac{1}{2}(u^{1/2}) + c$$

$$= \frac{1}{2}\sqrt{2x^2+4x-1} + c$$

3. $\int x(1+3x)^5\,dx$ ....(1)

$u = 1 + 3x$ .....(2)

$\frac{du}{dx} = 3$

$\frac{du}{3} = dx$ .....(3)

Subst 2, 3 in 1

$$\int x(u)^5 \frac{du}{3}$$

$$\int \frac{1}{3} x u^5\,du$$

Use eqn 2 to eliminate x

$u = 1 + 3x$

$\frac{u-1}{3} = \frac{3x}{\cancel{3}}$

$x = \frac{u-1}{3}$

$$\int \frac{1}{3}\left(\frac{u-1}{3}\right)u^5\,du$$

$$\int \frac{1}{3}\left(\frac{u^6}{3} - \frac{u^5}{3}\right)du$$

$$\frac{1}{3}\left(\frac{u^7}{7}\left(\frac{1}{3}\right) + \frac{u^6}{6}\left(\frac{1}{3}\right)\right) + k$$

$$\frac{1}{3}\left(\frac{1}{21}(1+3x)^7 - \frac{1}{18}(1+3x)^6\right) + k$$

Substitution based on the reverse definition of integration / based on the def of integration as the reverse process of differentiation.

(1) Consider

$I = \int e^x\,dx$

Quiz ⇒ What function if diff will give e^x? e^x (ans).

$= e^x + k$

- Similarly $\int e^p\,dp = e^p + k$

$\int e^u\,du = e^u + k$

$\int e^v\,dv = e^v + k$

Example

1. Evaluate the integral $\int e^{3x}\,dx$ ___(1)

Let $u = 3x$ ___(2)

$\frac{du}{dx} = 3 \qquad \frac{du}{3} = dx$ ___(3)

Subs 2, 3 in 1

$$\int e^u \cdot \frac{du}{3} = \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + k$$

$$= \frac{1}{3}e^{3x} + k$$

2. $\int x^2 e^{2x^3-4}\,dx$ ___(1)

let p be $2x^3 - 4$ ___(2)

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$$\frac{dp}{dx} = 6x^2$$

$$\frac{dp}{6x^2} = dx \quad \_\_\_ \; 3$$

Subst 2 & 3 in 1

$$\int x^2 e^p \frac{dp}{6x^2} = \int \frac{1}{6} e^p dp$$

$$\frac{1}{6} \int e^p \, dp$$

$$= \frac{1}{6} e^p + k$$

$$= \frac{1}{6} e^{2x^3+4} + k$$

3. $\int \cos 4x \, e^{\sin 4x} \, dx \quad \_\_\_ \; 1$

let $u = \sin 4x \quad \_\_\_ \; 2$

$$\frac{du}{dx} = 4 \cos 4x$$

$$\frac{du}{4 \cos 4x} = dx \quad \_\_\_ \; 3$$

Subst 2 & 3 in 1

$$\int \cancel{\cos 4x} \, e^u \frac{du}{4 \cancel{\cos 4x}}$$

$$\int \tfrac{1}{4} e^u \, du$$

$$= \tfrac{1}{4} e^u + k$$

$$= \tfrac{1}{4} e^{\sin 4x} + k.$$

In general if $f(x)$ is a function, then $I = \int e^{f(x)} dx$

$$= \frac{1}{f'(x)} e^{f(x)} + c$$

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + c \qquad f = 2x, \; f' = 2$$

$$\int e^{-5t} dt = \frac{1}{-5} e^{-5t} + c \qquad f = -5t, \; f' = -5$$

Tuesday 28/01/2020

2. $\int \frac{1}{x} dx$

Q. Which function if differentiated will give $\frac{1}{x}$? $\ln|x|$

$$\Rightarrow \int \frac{1}{x} dx = \ln|x| + k$$

$$\int \frac{1}{p} dp = \ln|p| + k$$

$$\int \frac{1}{v} dv = \ln|v| + k$$

Power rule fails.

To show that power rule fails.

$$\int \frac{1}{x} = \int x^{-1} dx$$

Therefore use $\ln|x|$.

$$\frac{x^{-1+1}}{-1+1} = \frac{1}{0} \text{ undefined.}$$

Example

1. $\int \frac{x^3}{5+x^4} dx \quad 1$

$u = 5 + x^4 \quad 2$

$$\frac{du}{dx} = 4x^3$$

$$\frac{du}{4x^3} = dx \quad 3$$

Sub 2 and 3 in 1

$$\int \frac{\cancel{x^3}}{u} \frac{du}{4\cancel{x^3}}$$

$$\int \frac{1}{4u} du$$

$$\frac{1}{4} \int \frac{1}{u} du$$

$$\frac{1}{4} \ln|u| + k$$

$$= \frac{1}{4} \ln|5 + x^4| + k.$$

2. $\int \frac{e^{3x}}{2+e^{3x}} dx \qquad u = 2 + e^{3x} \qquad \tfrac{1}{3} \ln|2+e^{3x}| + c$

$\cancel{\int \cos 3}$

3. $\int \frac{\cos 3x}{1+\sin 3x} dx \qquad u = 1 + \sin 3x$

$\tfrac{1}{3} \ln|1 + 3x| + c$

4. $\int \frac{\sec^2 2x}{1+\tan 2x} dx$ $\quad u = 1+\tan 2x$

$\frac{1}{2}\ln|1+\tan 2x|$

5. $\frac{x+1}{x^2+2x-5} dx$ $\quad u = x^2+2x-5$

6. $\int \frac{x}{1-3x^2} dx$ $\quad u = 1-3x^2$

③ $\int \sin dx$

Q - Which function if diff. gives $\sin x$?

$-\cos x$

$\int \sin x \, dx = -\cos x + c$

$\int \sin p \, dp = -\cos p + c$

$\int \sin 6x \, dx$

$p = 6x$

$\frac{dp}{dx} = 6$

$\frac{dp}{6} = dx$

$\int \sin p \frac{dp}{6} = \frac{1}{6}\int \sin p \, dp$

$\int \sin p \frac{dp}{6} = -\frac{1}{6}\cos p + k$

$\frac{1}{6}\int \sin p \, dp = -\frac{1}{6}\cos 6x + k$

Quiz

1. $\int x \sin(6-3x^2) dx$

$p = 6-3x^2$

$\frac{dp}{dx} = -6x$

$dx = \frac{dp}{-6x}$

$\int x \sin p \frac{dp}{-6x} = \frac{1}{6}\int -\sin p \, dp$

$= \frac{1}{6}\cos p$

$= \frac{1}{6}\cos(6-3x^2)$

2. $\int (\pi x+1)\sin(\pi x^2+2x-5) dx$ $\quad p = \pi x^2+2x-5$

3. $\int e^{2x}\sin(5-e^{2x}) dx$ $\quad p = 5-e^{2x}$

4. $\int (x^2+1)\sin(x^3+3x-5) dx$ $\quad p = x^3+3x-5$

④ $\int \cos x \, dx$

Derivative of ? $\underline{\sin x} = \cos x$

$\int \cos x \, dx = \sin x$

$\int \cos p \, dp = \sin p$

$\int \cos(4x+5) dx$ — ①

$u = 4x+5$ — ②

$\frac{du}{dx} = 4$

$dx = \frac{du}{4}$

$\int \cos u \frac{du}{4} = \frac{1}{4}\int \cos u \, du = \frac{1}{4}\sin u + k$

$= \frac{1}{4}\sin(4x+5) + k$

Eg. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

~~$\int x \cos$~~ $\left(\frac{1}{2\sqrt{x}}\right)$

$p = \sqrt{x} = x^{1/2}$

$\frac{dp}{dx} = \frac{1}{2}x^{-1/2}$ $\quad dx = dp \, 2\sqrt{x}$

$\int \frac{\cancel{\cos\sqrt{x}}}{\cancel{\sqrt{x}}} \cancel{\frac{dp}{2\sqrt{x}}}$ $\quad \int \frac{\cos p}{\sqrt{x}} \cdot dp \, 2\sqrt{x}$

$= 2\sin(\sqrt{x})$

① $\int x \cos(3x^2-4) dx$

② $\int e^{-3x}\cos(4+e^{-3x}) dx$

3. $\int (x^2+1)\cos(3x^3+9x-5) dx$

⑤ $\int \sec^2 x \, dx$

$\int \sec^2 x \, dx = \tan x + k$

$\int \sec^2 p \, dp = \tan p + c$

$\int \sec^2(7x-1)\,dx$ — ①

$u = 7x-1$

$\frac{du}{dx} = 7 \quad dx = \frac{du}{7}$ — ②

$\int \sec^2 u \,\frac{du}{7} = \frac{1}{7}\int \sec^2 u\,du$

$= \frac{1}{7}\tan u + k$

$= \frac{1}{7}\tan(7x-1) + k$

$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$ . . . ①

$u = \sqrt{x}$ . . . ②

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$2\sqrt{x}\,du = dx$ . . . ③

$\int \frac{\sec^2 u}{\cancel{\sqrt{x}}} \cdot 2\cancel{\sqrt{x}}\,du$ $\quad \frac{1}{f'}\tan f$

$= 2\int \sec^2 u\,du$

$= 2\tan u + k$

$= 2\tan\sqrt{x} + k.$

⑥ $\int \csc^2 x\,dx$

$\int \csc^2 x\,dx = -\cot x + c$

$\int \csc^2 p\,dp = -\cot p + c$

$\int \csc^2 4x\,dx$ ①

$u = 4x$ . . . ②

$\frac{du}{dx} = 4 \quad dx = \frac{du}{4}$ . . . ③

$\int \csc^2 u\,\frac{du}{4}$ ② and ③ in ①

$= \frac{1}{4}\int \csc^2 u\,du$ $\quad \frac{1}{f'}\cot f$

$= -\frac{1}{4}\cot 4x + k$

Exercise

1. $\int \frac{\csc^2\sqrt{x}}{\sqrt{x}}\,dx$

2. $\int e^{3x}\csc^2(e^{3x}+1)\,dx$

3. $\int x^2\csc^2(1-3x^3)\,dx$

4. $\int \sin 2x\,\csc^2(1+\cos 2x)\,dx$

5. $\int (x+1)\csc^2(2x^2+4x-5)\,dx$

⑦ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ — ①

$u = \cos x$ — ②

$\frac{du}{dx} = -\sin x \quad dx = \frac{du}{-\sin x}$ — ③

Subs. 2 and 3 in 1

$\int \frac{\cancel{\sin x}}{u}\cdot\frac{du}{-\cancel{\sin x}}$

$= -\int \frac{1}{u}\,du$

$= -\ln|u| + k$ $\quad$ NOTE $n\ln x = \ln x^n$

$= \ln|u^{-1}| + k$

$= \ln\left|\frac{1}{u}\right| + k$

$= \ln\left|\frac{1}{\cos x}\right| + k$

$= \ln|\sec x| + k$

Therefore

$\int \tan p\,dp = \ln|\sec p| + k$

$\int \tan v\,dv = \ln|\sec v| + k.$

Example

1. $\int \tan(5x-2)\,dx \qquad u = 5x-2$

$\frac{du}{dx} = 5$ ——— ③ — $dx = \frac{du}{5}$

Sub 2 & 3 in 1

$\int \tan u \cdot \frac{du}{5} = \frac{1}{5}\int \tan u \, du$

$= \frac{1}{5} \ln |\sec u| + k$

$= \frac{1}{5} \ln |\sec (\cancel{5x-8})| + k$

Questions

1- $\int \frac{\tan \sqrt{x}}{\sqrt{x}} dx$ ② $\int x^2 \tan (5 - x^3) dx$

3- $\int e^{7x} \tan (e^{7x} - 5) dx$ ④ $\int (x^2 - 1) \tan (3x^3 - 9x - 7) dx$

⑧ $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$ .... ①

$u = \sin x$ — ②

$\frac{du}{dx} = \cos x$ $dx = \frac{du}{\cos x}$ ——— ③

Subst 2 & 3 in 1

$\int \frac{\cancel{\cos x}}{u} \cdot \frac{du}{\cancel{\cos x}} = \int \frac{1}{u} du$

$= \ln |u| + k$

$= \ln |\sin x| + k$

$\Rightarrow \int \cot p \, dp = \ln |\sin p| + c$

$\int \cot w \, dw = \ln |\sin w| + c$

Example:

1- $\int \cot 5x \, dx$

2. $\int \cot (2 - x) dx$

3. $\int (x - 1) \cot (x^2 - 2x - 1) dx$

4. $\int \frac{\cot \sqrt{x}}{\sqrt{x}} dx$

5. $\int e^{4x} \cot (1 - e^{4x}) dx$

Assignment

⑨ $\int \sec x \, dx$ With examples

⑩ $\int \operatorname{cosec} x \, dx$

⑨ $\int \sec x \, dx$

~~$\sec = \frac{1}{\cos x}$ — ①~~

~~$\int \frac{1}{\cos x}$ $u = \cos x$ — ②~~

~~$-\frac{1}{f_1} \sin f$~~

~~$\frac{du}{dx} = -\sin x$~~

~~$\frac{du}{-\sin x} = dx$ — ③~~

~~$\int \frac{1}{u} \cdot \frac{du}{-\sin x}$~~

Multiply through the numerator and denominator by $(\sec x + \tan x)$

$\int \sec x \, dx \Rightarrow \int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx$ ...①

Let u be $\sec x + \tan x$ .... ②

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$\frac{du}{\sec x \tan x + \sec^2 x} = dx$ ..... ③

Sub 2 and 3 in 1

$\int \frac{\cancel{\sec x} (\cancel{\sec x + \tan x})}{u} \cdot \frac{du}{\cancel{\sec x \tan x} + \sec^2 x}$

$\int \frac{1}{u} du$

$= \ln |u| + k$

$= \ln |\sec x + \tan x| + k$

Examples

a) $\int 2x \sec(4x^2+5)\, dx$

Let $w = 4x^2+5$

$\frac{dw}{dx} = 8x$

$\frac{dw}{8x} = dx$

$\int \cancel{2x} \sec w \cdot \frac{dw}{\cancel{8x}_4}$

$\frac{1}{4}\int \sec w \cdot dw$

$= \frac{1}{4} \ln |\sec w + \tan w + k$

$= \frac{1}{4} \ln |\sec(4x^2+5) + \tan(4x^2+5)| + k$

b) $\int 2x^2 \sec(4\pi x^3+6)\, dx$ ....①

Let $v = 4\pi x^3 + 6$ ....②

$\frac{dv}{dx} = 12\pi x^2$

$\frac{dv}{12\pi x^2} = dx$ ....③

Sub 2 and 3 in 1

$\int \cancel{2x^2} \sec v \cdot \frac{dv}{\cancel{12}_6 \pi \cancel{x^2}}$

$\int \frac{1}{6\pi} \sec v \cdot dv$

$= \frac{1}{6\pi}\int \sec v \cdot dv$

$= \frac{1}{6\pi} \ln |\sec v + \tan v| + k$

$= \frac{1}{6\pi} \ln |\sec(4\pi x^3+6) + \tan(4\pi x^3+6)| + k$

⑩ $\int \operatorname{cosec} x\, dx$

Multiply through the numerator and denominator by $(\csc x + \cot x)$.

$\int \csc x\, dx \Rightarrow \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} dx$ ....①

Let $z = \csc x + \cot x$ ....②

$\frac{dz}{dx} = -\cot x \csc x - \csc^2 x$

$dx = \frac{dz}{-\cot x \csc x - \csc^2 x}$ ....③

Sub 2 and 3 in 1

$\int \frac{\cancel{\csc x(\csc x + \cot x)}}{z} \cdot \frac{dz}{\cancel{-\cot x \csc x - \csc^2 x}_{-1}}$

$\int -\frac{1}{z} \cdot dz = -1 \int \frac{1}{z} \cdot dz$

$= -\ln |z| + k$

$= -\ln |\csc x + \cot x| + k$

Examples

a) $\int (x+2) \csc(x^2+4x-3)\, dx$ ....(i)

Let $p = x^2+4x-3$ ....②

$\frac{dp}{dx} = 2x+4$

$dx = \frac{dp}{2x+4}$ ....③

Sub 2 and 3 in 1

$\int \cancel{(x+2)} \csc p \cdot \frac{dp}{2\cancel{(x+2)}}$

$\frac{1}{2}\int \csc p \cdot dp$

$= \frac{1}{2} \ln |\csc p + \cot p| + k$

$= \frac{1}{2} \ln |\csc(x^2+4x-3) + \cot(x^2+4x-3)| + k$

b) $\int \csc(2x+3)\,dx$

Let $n = 2x+3$

$\frac{dn}{dx} = 2$ $\quad dx = \frac{dn}{2}$

$\int \csc n \left(\frac{dn}{2}\right)$

$\frac{1}{2}\int \csc n \cdot dn$

$= \frac{1}{2} \ln |\csc n + \cot n| + c$

$= \frac{1}{2} \ln |\csc(2x+3) + \cot(2x+3)| + c.$

- In general Therefore:

For $\int \sec x\, dx$:

If $\int \sec f(x)\, dx = I$ then

$I = \frac{1}{f'(x)} \ln |\sec f(x) + \tan f(x)| + k$

For $\int \operatorname{cosec} x\, dx$:

If $I = \int \csc f(x) \cdot dx$ then.

$I = \frac{1}{f'(x)} \ln |\csc f(x) + \cot f(x)| + k$

Where k is a constant.

Integrals containing products of Cos x Sin x

- In such cases you use the following identities to rewrite the integrand

(i) $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

The restriction is that A is associated with Sine and B is associated with Cosine

(ii) $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

(iii) $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

(iv) $\sin(-A) = -\sin A$

(v) $\cos(-A) = +\cos A$

Example:

1. Evaluate

$\int \cos 5x \sin 3x\, dx$

- Use identity (i) - There is restriction.

$A = 3x$ $\quad \underset{(5x+3x)}{(A+B)} \quad \underset{(3x-5x)}{(A-B)}$

$B = 5x$

$= \frac{1}{2}\int \sin 8x + \sin(-2x)$

Rule (iv) $\sin -A = -\sin A$

$= \frac{1}{2}\int \sin \underset{f}{8x}\, dx - \frac{1}{2}\int \sin \underset{f}{2x}\, dx$

for Sine. → $\left(-\frac{1}{f'}\cos f\right)$

$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x\right] - \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right] + k$

$= -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + k.$

2. $\int \sin \underset{A}{2x} \sin \underset{B}{6x}\, dx$

$= \frac{1}{2}\int \cos(-4x) - \cos 8x\, dx$

Rule (v)

$= \frac{1}{2}\int \cos 4x\, dx - \frac{1}{2}\int \cos 8x\, dx$

Integrating $\cos \to \frac{1}{f'} \sin f$

$= \frac{1}{2}\left[\frac{1}{4}\sin 4x\right] - \frac{1}{2}\left[\frac{1}{8}\sin 8x\right] + c$

$= \frac{1}{8}\sin 4x - \frac{1}{16}\sin 8x + c$

OR - To show no restriction

$\int \underset{B}{\sin 2x}\ \underset{A}{\sin 6x}\ dx$

$\frac{1}{2}\int \left[\cos(6x-2x) - \cos(6x+2x)\right] dx$

$= \frac{1}{2}\int (\cos 4x - \cos 8x)\, dx$

$= \frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{8}\sin 8x\right] + k$

$= \frac{1}{8}\sin 4x - \frac{1}{16}\sin 8x + k.$

Questions

1. $\int \sin 4x \cos 9x\, dx$
2. $\int \cos 6x \sin 7x\, dx$
3. $\int \cos 3x \cos 4x\, dx$
4. $\int \sin 7x \sin 8x\, dx$
5. $\int \sin 5u \cos 9u\, du$
6. $\int \sin 3m \cos 4m\, dm$
7. $\int \sin 11x \sin 2x\, dx$
8. $\int \cos 2x \cos 8x\, dx$

Integrals involving Even powers of $\sin x$ & $\cos x$.

- In such cases, we use the identity :

(i) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

(ii) $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$	$\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$
$\cos^2 5x = \frac{1}{2}(1 + \cos 10x)$	$\sin^2 6t = \frac{1}{2}(1 - \cos 12t)$
$\cos^2 9x = \frac{1}{2}(1 + \cos 18x)$	$\sin^2 21x = \frac{1}{2}(1 - \cos 42x)$
	$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$

Example

Evaluate:

1. $\int \sin^2 4x\, dx$

$= \frac{1}{2}\int 1 - \cos 8x\, dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 8x\, dx$

$x^0 \qquad \frac{1}{f'}\sin f$

By power rule.

$= \frac{1}{2}\left(\frac{x^{0+1}}{0+1}\right) - \frac{1}{2}\left(\frac{1}{8}\sin 8x\right) + k.$

$= \frac{x}{2} - \frac{1}{16}\sin 8x + k$

2. $\int \cos^4 2x\, dx$

$m^4 = mmmm = (m^2)^2$

$\cos^4 2x = (\cos^2 2x)^2$

$\left[\frac{1}{2}(1 + \cos 4x)\right]^2$

From $(a+b)^2 = a^2 + 2ab + c$

$= \frac{1}{4}(1 + 2\cos 4x + \cos^2 4x)$

$= \frac{1}{4}\left[1 + 2\cos 4x + \frac{1}{2}(1 + \cos 8x)\right]$

$\left(\frac{1}{4} \times \frac{1}{2}\right)$

$\int \cos^4 2x\, dx = \frac{1}{4}\int dx + \frac{2}{4}\int \cos 4x + \frac{1}{8}\int dx + \frac{1}{8}\cos 8x\, dx$

$= \frac{x}{4} + \frac{1}{2}\left(\frac{1}{4}\sin 4x\right) + \frac{1}{8}x + \frac{1}{8}\left(\frac{1}{8}\sin 8x\right) + c$

$= \frac{x}{4} + \frac{1}{8}\sin 4x + \frac{x}{8} + \frac{1}{64}\sin 8x + c.$

Exercise

a) $\int \sin^2 2x\, dx$ d) $\int \cos^2 6x\, dx$

b) $\int \sin^4 3x\, dx$ e) $\int \cos^4 5x\, dx$

c) $\int \sin^6 x\, dx$ f) $\int \cos^6 x\, dx$

replace $\frac{1}{2}(1+\cos 4x)$

g) $\int \sin 3x \cos^2 2x\, dx$

h) $\int \cos 5x \sin^2 3x\, dx$ $\frac{1}{2}(1-\cos 6x)$

i) $\int \cos^2 x \sin 7x\, dx$
$\frac{1}{2}(1+\cos 2x)$

j) $\int \sin^2 x \cos 3x\, dx$
$\frac{1}{2}(1-\cos 2x)$

Tuesday 4th Feb 2020

Integrals involving odd powers of $\sin x$, $\cos x$

In such cases use the identity $\cos^2 x + \sin^2 x = 1$ after breaking the power into even and odd.

eg $\int \sin^3 2x\, dx = \int \sin^2 2x \sin 2x\, dx$

$= \int (1 - \cos^2 2x) \sin 2x\, dx$

for $\sin$
$\frac{1}{f'}\cos f$

$= \int \sin 2x\, dx - \int \sin 2x \cos^2 2x\, dx$

NOTE

$= \left(-\frac{1}{2}\cos 2x\right) - \left(-\frac{1}{6}\cos^3 2x\right) + k$

$= -\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x + k$

NOTE

$u = \cos 2x$ (1)

$\frac{du}{dx} = -2\sin 2x$

$\frac{du}{-2\sin 2x} = dx$ (2)

$\int u^2 \sin 2x \left(\frac{du}{-2\sin 2x}\right)$

$-\frac{1}{2}\int u^2\, du = -\frac{1}{2}\left(\frac{u^3}{3}\right) = -\frac{1}{6}\cos^3 2x$

Example 2 $\int \cos^5 3x\, dx = \int \cos^4 3x \cos 3x\, dx$

$\int \cos^4 3x \cos 3x\, dx$

$\int (\cos^2 3x)^2 \cos 3x\, dx$

$\int (1 - \sin^2 3x)^2 \cos 3x\, dx$

$(a+b)^2 = a^2 - 2ab + b^2$

$\int (1 - 2\sin^2 3x + \sin^4 3x) \cos 3x\, dx$

Expand

$\int \cos 3x\, dx - 2\int \sin^2 3x \cos 3x\, dx +$

$\frac{1}{f'}\sin f$

$u = \sin 3x$

$\int \sin^4 3x \cos 3x\, dx$ $\frac{du}{dx} = 3\cos 3x$

$dx = \frac{du}{3\cos 3x}$

$\left(\frac{1}{3}\sin 3x\right) - 2\left(\frac{1}{9}\sin^3 3x\right) + \left(\frac{1}{15}\sin^5 3x\right) + k$

NOTE

$\int u^2 \cos 3x \frac{du}{3\cos 3x}$ | $\int u^4 \cos 3x \cdot \frac{du}{3\cos 3x}$

$\frac{1}{3}\int u^2 \cdot du$ | $\frac{1}{3}\int u^4 \cdot du$

$= \frac{1}{3}\frac{u^3}{3} = \frac{1}{9}u^3 + k$ | $= \frac{1}{3}\cdot\frac{u^5}{5} + k$

$= \frac{1}{15}u^5 + k$

$= \frac{1}{9}\sin^3 3x + k$ | $= \frac{1}{15}\sin^5 3x + k$

Exercise

$\int \sin^3 6x\, dx$ $\int \cos^3 8x\, dx$

$\int \sin^5 3x\, dx$ $\int \cos^5 2x\, dx$

$\int \sin^7 x\, dx$ $\int \cos^7 x\, dx$

NOTE :

i) For integrals involving powers of $\tan x$ and $\sec x$ use $1 + \tan^2 x = \sec^2 x$

ii) For integrals involving powers of $\cot x$ and $\operatorname{cosec} x$ use $1 + \cot^2 x = \operatorname{cosec}^2 x$

iii) For definite integrals eg $\int_a^b f(x)\, dx$

* Ignore constant of intergration c or k

* The limits of integration only act on the result eg

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = 7/3$$

$$\int_0^1 e^{2x}dx = \frac{1}{2}e^{2x}\Big|_0^1 = \tfrac{1}{2}e^{2(1)} - \tfrac{1}{2}e^{2(0)} =$$

$$\int_0^{\pi/3} \cos 2x\,dx = \tfrac{1}{2}\sin 2x\Big|_0^{\pi/3} = \cancel{\tfrac{1}{2}\sin}$$

$$= \tfrac{1}{2}\sin 2\cdot(\pi/3) - \tfrac{1}{2}\sin 2(0).$$

Integrals containing radicals of the form....

Form | Suggested soln | Result

Notes Next page.

EXAMPLES

Evaluate the following integrals.

1. $\int \frac{dx}{\sqrt{49-x^2}}$ — ① $\quad \int \frac{1}{\sqrt{49-x^2}}\,dx$

Use (i) $a^2 - x^2 \to s/s \to x = a\sin u$.

$a^2 = 49$

$a = 7 \longrightarrow x = 7\sin u$ ②

$x^2 = 49\sin^2 u$ ③ $\quad \frac{dx}{du} = 7\cos u$

$dx = 7\cos u\cdot du$ ④

Substitute 3 and 4 in ①.

$$\int \frac{7\cos u\cdot du}{\sqrt{49 - 49\sin^2 u}} \qquad \sqrt{49(1-\sin^2 u)} \quad \sqrt{49\cos^2 u} = 7\cos u$$

$$\int \frac{7\cos u}{7\cos u}\,du$$

$$= \int 1\,du \qquad 1 = u^0$$

$$= \int u^0\,du$$

By Power rule

$$\frac{u^{0+1}}{0+1} + k = u + k$$

From eqn ②

$$= \sin^{-1}\left(\frac{x}{7}\right) + k$$

2. $\int \frac{dx}{\sqrt{4-3(x+2)^2}}$ — ①

$a^2 - b^2x^2$

$a^2 = 4 \qquad b^2 = 3 \qquad x^2 = x+2$

$a = 2 \qquad b = \sqrt{3}$

$s/s \to x = a/b \sin p$

differentiate

$(x+2) = \frac{2}{\sqrt{3}}\sin p$ — ②

$(x+2)^2 = \frac{4}{3}\sin^2 p$ — ③

$\frac{dx}{dp} + \frac{d(2)}{dp} = \frac{2}{\sqrt{3}}\cos p$

$\frac{d(2)}{dp} \to 0$

$\frac{dx}{dp} = \frac{2}{\sqrt{3}}\cos p$

$dx = \frac{2}{\sqrt{3}}\cos p\cdot dp$ — ④

Substitute 3 & 4 in 1

$$\int \frac{\frac{2}{\sqrt{3}}\cos p\,dp}{\sqrt{4 - \cancel{3}\cdot \frac{4}{\cancel{3}}\sin^2 p}}$$

$$\int \frac{2/\sqrt{3}\,\cancel{\cos p}}{2\,\cancel{\cos p}}\,dp$$

$$\frac{1}{\sqrt{3}}\int p^0\,dp$$

$$= \tfrac{1}{3}p + k$$

use eqn 2 to get P.

$$= \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}}{2}(x+2)\right) + k.$$

NOTE

- $\sqrt{4 - 4\sin^2 p}$

$= \sqrt{4(1-\sin^2 p)} = \sqrt{4\cos^2 p}$

$= 2\cos p$

- $x + 2 = \frac{2}{\sqrt{3}}\sin p$

$\frac{\sqrt{3}(x+2)}{2} = \sin p$

$p = \sin^{-1}\left(\frac{\sqrt{3}}{2}(x+2)\right)$

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INTEGRALS CONTAINING RADICALS OF THE FORM...

Form of the radical	Suggested substitution.	Result
1. $a^2 - x^2$ or $\sqrt{a^2 - x^2}$	$x = a \sin u$ $x^2 = a^2 \sin^2 u$ $a^2 - a^2 \sin^2 u = a^2 - x^2$ $a^2(1 - \sin^2 u)$ $= a^2 \cos^2 u$	$a^2 - x^2 = a^2 \cos^2 u$ $\sqrt{a^2 - x^2} = a \cos u$
2. $a^2 - b^2x^2$ or $\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b} \sin u$ $x^2 = \frac{a^2}{b^2} \sin^2 u$ $a^2 - b^2x^2 = a^2 - \cancel{b^2} \cdot \frac{a^2}{\cancel{b^2}} \sin^2 u$ $= a^2(1 - \sin^2 u)$ $= a^2 \cos^2 u$	$a^2 - b^2x^2 = a^2 \cos^2 u$ $\sqrt{a^2 - b^2x^2} = a \cos u$
3. $a^2 + x^2$ or $\sqrt{a^2 + x^2}$ $x^2 + a^2$ or $\sqrt{x^2 + a^2}$	$x = a \tan t$ $x^2 = a^2 \tan^2 t$ $a^2 + x^2 = a^2 + a^2 \tan^2 t$ $= a^2(1 + \tan^2 t)$ $= a^2 \sec^2 t$	$a^2 + x^2 = a^2 \sec^2 t$ $\sqrt{a^2 + x^2} = a \sec t$
4. $a^2 + b^2x^2$ or $\sqrt{a^2 + b^2x^2}$ $b^2x^2 + a^2$ or $\sqrt{b^2x^2 + a^2}$	$x = \frac{a}{b} \tan t$ $x^2 = a^2/b^2 \tan^2 t$ $a^2 + b^2x^2 = a^2 + \cancel{b^2} \cdot \frac{a^2}{\cancel{b^2}} \tan^2 t$ $= a^2(1 + \tan^2 t)$ $= a^2 \sec^2 t$	$a^2 + b^2x^2 = a^2 \sec^2 t$ $\sqrt{a^2 + b^2x^2} = a \sec t$
5. $x^2 - a^2$ or $\sqrt{x^2 - a^2}$	$x = a \sec t$ $x^2 = a^2 \sec^2 t$ $x^2 - a^2 = a^2 \sec^2 t - a^2$ $a^2(\sec^2 t - 1)$ $= a^2 \tan^2 t$	$x^2 - a^2 = a^2 \tan^2 t$ $\sqrt{x^2 - a^2} = a \tan t$
6. $b^2x^2 - a^2$ or $\sqrt{b^2x^2 - a^2}$	$x = a/b \sec t$ $x^2 = a^2/b^2 \sec^2 t$ $b^2x - a^2 = b^2 (a^2/b^2) \sec^2 t - a^2$ $= a^2(\sec^2 t - 1)$	$b^2x^2 - a^2 = a^2 \tan^2 t$ $\sqrt{b^2x^2 - a^2} = a \tan t.$

Examples.

3. $\int \frac{\sqrt{x^2-36}}{x}\,dx$ — ①

Case 5 $x^2 - a^2$ s/s $x = a \operatorname{Sec} z$

$a^2 = 36$ $x = 6 \operatorname{Sec} z$ ②

$a = 6$ $x^2 = 36 \operatorname{Sec}^2 z$ ③

$$\frac{dx}{dz} = 6 \operatorname{Sec} z \tan z$$ ④

$$dx = 6 \operatorname{Sec} z \tan z \cdot dz$$ ④

Sub 2 & 3 & 4 in 1 $\sqrt{36(\operatorname{Sec}^2 z - 1)} = 6 \tan t$

$$\int \frac{\sqrt{36 \operatorname{Sec}^2 z - 36}}{6 \operatorname{Sec} z} \cdot 6 \operatorname{Sec} z \tan z \cdot dz$$

$$= \int 6 \tan t \tan t \, dt = 6 \int \tan^2 z \, dz.$$

$$= 6 \int (\operatorname{Sec}^2 z - 1)\, dx$$

$$6 \int \operatorname{Sec}^2 z \, dz - 6 \int z^0 dz$$

Reverse difn.

$$6 \tan z - 6z + k$$

use eqn 2

$$= 6 \tan\left(\operatorname{Sec}^{-1}\left(\frac{x}{6}\right)\right) - 6 \operatorname{Sec}^{-1}\left(\frac{x}{6}\right) + k$$

4. $\int \frac{dx}{4 + 3(x-1)^2}$ —— ①

Case 4 $a^2 + b^2x^2$ s/s $x = a/b \tan p$

$a^2 = 4$ $b^2 = 3$ $x' = x - 1$

$a = 2$ $b = \sqrt{3}$

$$x - 1 = \frac{2}{\sqrt{3}} \tan p$$ —— ②

$$(x-1)^2 = \frac{4}{3} \tan^2 p$$ —— ③

$$\frac{dx}{dp} - 0 = \frac{2}{\sqrt{3}} \operatorname{Sec}^2 p$$

$$dx = \frac{2}{\sqrt{3}} \operatorname{Sec}^2 p \, dp$$ —— ④

Sub 3 & 4 in 1

$$\int \frac{\frac{2}{\sqrt{3}} \operatorname{Sec}^2 p \cdot dp}{4 + 3 \cdot \frac{4}{3} \tan^2 p}$$

$4 + 4\tan^2 p$
$4(1 + \tan^2 p)$
$4(\operatorname{Sec}^2 p)$

$$\int \frac{\frac{2}{\sqrt{3}} \operatorname{Sec}^2 p \, dp}{4 \operatorname{Sec}^2 p}$$

$\frac{2}{\sqrt{3}} \times \frac{1}{4_2} = \frac{1}{2\sqrt{3}}$

$$\frac{1}{2\sqrt{3}} \int p^0 dp.$$

$$= \frac{1}{2\sqrt{3}} p + k$$

use eqn 2

$\sqrt{3}(x-1) = 2 \tan p$

$p = \tan^{-1} \frac{\sqrt{3}}{2}(x-1)$

$$= \frac{1}{2\sqrt{3}} \left\{ \tan^{-1}\left(\frac{\sqrt{3}}{2}(x-1)\right) \right\} + k$$

5. $\int \frac{dx}{x\sqrt{4x^2-1}}$ —— ①

Case 6 $b^2 = 4$ $a^2 = 1$ s/s $a/b \operatorname{Sec} u$.

$b = 2$ $a = 1$

$$x = \tfrac{1}{2} \operatorname{Sec} u$$ — 2

$$x^2 = \tfrac{1}{4} \operatorname{Sec}^2 u$$ — 3

$$\frac{dx}{du} = \tfrac{1}{2} \operatorname{Sec} u \tan u$$

$$dx = \tfrac{1}{2} \operatorname{Sec} u \tan u \cdot du$$ — 4

Sub 2, 3 & 4 in 1

$$\int \frac{\frac{1}{2} \operatorname{Sec} u \tan u \, du}{\frac{1}{2} \operatorname{Sec} u \sqrt{4(\frac{1}{4} \operatorname{Sec}^2 u - 1)}}$$ $\sqrt{\tan^2 u} = \tan u$

$$\int \frac{\frac{1}{2} \operatorname{Sec} u \tan u}{\frac{1}{2} \operatorname{Sec} u \tan u} du$$

$2x = \operatorname{Sec} u$

$u = \operatorname{Sec}$

$$\int u^0 du = u + k$$

$$= \operatorname{Sec}^{-1}(2x) + k$$

11/2/2020

Evaluation of integrals by completing square process.

- Completing square process is applied on an expression to rewrite the expression to one of the radical forms.

- Completing square method is applied on an equation to solve the unknown.

Example:

$x^2 - 5x + 6$ or $-3x^2 - 4x + 6$

Step ①

· Make the coeffecient of x^2 to be $+1$

$x^2 - 5x + 6$ | $-3\left(x^2 - \frac{4}{3}x \mp 2\right)$

Step ②.

Get ½ coefficient of X, square, add and subtract, sign inclusive.

$\frac{1}{2}x - 5 = -\frac{5}{2}$

$\left(-\frac{5}{2}\right)^2 = \frac{25}{4}$

$x^2 - \left(\frac{5}{2}\right)^2 + \left(-\frac{5}{2}\right)^2 + 6 - 5x$

$\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$ Case 5

$x^2 - a^2$

$x^* = \left(x - 5/2\right)$

$a = \frac{1}{2} = \sqrt{\frac{1}{4}}$

$x = a \sec z$

$\left(x - \frac{5}{2}\right) = \frac{1}{2} \sec z$

$\frac{1}{2}x - \frac{4}{3} = -\frac{4}{6}$

$\left(-\frac{4}{6}\right)^2$

$-3\left(x^2 - \frac{4}{3}x - (-\frac{2}{3})^2 + (-\frac{2}{3})^2 - 2\right)$

$-3\left(x^2 - \frac{4}{3}x - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 - 2\right)$

$\frac{4}{9} - 2 = \frac{-4-18}{9}$

$-3\left(\left(x + 2/3\right)^2 - \frac{22}{9}\right)$

$-3\left(x + 2/3\right)^2 + \frac{22}{3}$

$= \frac{22}{3} - 3\left(x + 2/3\right)^2$

$a^2 - b^2 x^2$

(Case 2)

$x = a/b \sin p$

$x^{\cdot} = \left(x + 2/3\right)$

$a = \sqrt{22/3}$ $b = \sqrt{3}$

$x^2 + a^2$ $x = a \tan z$

$x^{\cdot} = x + 3$

$a = 2$

$$= \int_{-3}^{-1} \frac{dx}{4 + (x+3)^2} \quad ①$$

$x + 3 = 2 \tan z$ ②

$(x+3)^2 = 4 \tan^2 z$ ③

$\frac{dx}{dz} + 0 = 2 \sec^2 z$

$dx = 2 \sec^2 z \cdot dz$ ④

Subst. 3 & 4 in 1

$$\int_{-3}^{-1} \frac{2 \sec^2 z \cdot dz}{4 + 4 \tan^2 z}$$

$$\int_{-3}^{-1} \frac{\cancel{2} \cancel{\sec^2 z}\, dz}{\cancel{4}2 \cancel{\sec^2 z}}$$

$$\int_{-3}^{-1} \frac{1}{2} \cdot dz = \frac{1}{2} \int_{-3}^{-1} dz$$

$= \frac{1}{2}(z)$

Put x

Use equation 2

$$= \frac{1}{2} \tan^{-1}\left(\frac{x+3}{2}\right) \Big|_{-3}^{-1}$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{-1+3}{2}\right) - \frac{1}{2} \tan^{-1}\left(\frac{-3+3}{2}\right)$$

$= \frac{1}{2} \tan^{-1}(1) - \frac{1}{2} \tan^{-1}(0)$

$= 22.5 -$

EVALUATE THE FOLLOWING INTEGRALS.

1. $$\int_{-3}^{-1} \frac{dx}{x^2 + 6x + 13}$$

By C.S.P

$x^2 + 6x + 13$

$\frac{1}{2} \times 6 = 3$ $x^2 + 6x + 3^2 - 3^2 + 13$

$(x + 3)^2 + 4 \rightarrow$ Case 3

2. $$\int \frac{dx}{\sqrt{-x^2 - 4x}} \quad ①$$

By C.S.P. $-x^2 - 4x$

$-\left[x^2 + 4x\right]$

$-\left[x^2 + 4x + 2^2 - 2^2\right]$

$-[(x+2)^2 - 4]$

$a^2 - x^2$ – Case 1.

$-(x+2)^2 + 4 = 4 - (x+2)^2$

s/s $\Rightarrow x = a \sin u$

$a^2 = 4 \qquad x' = (x+2)$

$a = 2 \qquad x'^2 = (x+2)^2$

$x+2 = 2\sin u$ ——— ②

$(x+2)^2 = 4\sin^2 u$ ——— ③

$\frac{dx}{du} = 2\cos u$

$dx = 2\cos u \cdot du$ ——— ④

Sub. 3 & 4 in 1

$$\int \frac{2\cos u\, du}{\sqrt{4 - 4\sin^2 u}}$$

$$\int \frac{\cancel{2}\cos u}{\cancel{2}\cos u}\, du = \int \frac{1}{\cancel{\cos u}}\, du$$

$\int u^0\, du = u + K$

↓ use equation 2.

$= \sin^{-1}\left(\frac{x+2}{2}\right) + K.$

3. $\int \frac{dx}{(x-1)\sqrt{x^2 - 2x}}$

By C.S.P $\quad x^2 - 2x$

$x^2 - 2x + (-1)^2 - (-1)^2$

↓

$(x-1)^2 - 1 \qquad x^2 - a^2$

case 5.

s/s $= a \sec p = X$

$X' = (x-1) \qquad x-1 = \sec p$ ——— ②

$X'^2 = (x-1)^2 \qquad (x-1)^2 = \sec^2 p$ ——— ③

$a = 1$

$a^2 = 1$

$$\int \frac{dx}{(x-1)\sqrt{(x-1)^2 - 1}} \text{ ——— ①}$$

$\frac{dx}{dp} = \sec p \tan p$

$dx = \sec p \tan p \cdot dp$ ——— ④

Sub 2, 3 & 4 in 1

$$\int \frac{\cancel{\sec p}\tan p\, dp}{\cancel{\sec p}\sqrt{(\sec^2 p - 1)}}$$

$\int p^0\, dp$

$= p + K$

↓

$= \sec^{-1}(x-1) + K$

INTEGRATION BY PARTS

- Integration by parts is traced/derived from product rule of differentiation.

Recall: uv

$$\frac{d}{dx}(uv) = v\cdot\frac{du}{dx} + u\frac{dv}{dx}$$

$$u\cdot\frac{dv}{dx} = \frac{d}{dx}(uv) - v\cdot\frac{du}{dx}$$

Integrate with respect to x

$$\int u\cdot\frac{dv}{dx}\cdot dx = \int \frac{d}{dx}(uv)\, dx - \int v\cdot\frac{du}{dx}\cdot dx$$

$\boxed{\int u\cdot dv = uv - \int v\cdot du}$ ⟶ Integration by parts formula.

To identify u:

(i) Use LIATEC rule

- L – logarithmic function
- I – Inverse functions
- A – Algebraic functions eg x^2, x^3
- T – Trigonometric functions
- E – Exponential functions
- C – Constant terms eg 1, 5, 6

(ii) The remaining terms and dx forms dV

Examples

1. $\int x^3 \ln x \, dx \quad u = \ln x \quad dv = x^3 dx$

2. $\int x^2 \sin x \, dx \quad u = x^2 \quad dv = \sin x \, dx$

3. $\int e^x \cos x \, dx \quad u = \cos x \quad dv = e^x dx$

4. $\int 1 \sin^{-1}(x) \, dx = \int \arcsin x \, dx$

$u = \sin^{-1} x \quad dv = 1 \, dx$

iii) Differentiate u w.r.t x and make du the subject

iv) Integrate dV w.r.t. x and disregard the constant, because a single constant will be added to the last step.

v) Substitute the above results in the parts formula.

- Terminate the process if the last integral can be integrated or repeat the process if the last integral still possess the product of functions OR

Collect like terms if the last integral is the same as the starting integral

vi) If the LIATE rule fails then "u" must be a group of terms that can be differentiated easily, dV must be a group of terms that can be integrated easily.

EXAMPLES.

1. Evaluate the following integrals

$\int x^3 \ln x \, dx$ LIATEC

$u = \ln x \qquad \int dV = x^3 dx$

Differentiate $\qquad$ Integrate

$\frac{du}{dx} = \frac{1}{x} \qquad V = \frac{x^4}{4}$ (ignore K

$du = \frac{1}{x} \cdot dx$

Parts formula:

$$\int u \cdot dV = uv - \int v \cdot du$$

$$\int x^3 \ln x \, dx = u \cdot v - \int v \cdot du$$

$$\int x^3 \ln x \, dx = (\ln x)\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4} \cdot \frac{dx}{x}$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx$$

By power rule

$$= \frac{x^4}{4} \ln x - \frac{1}{4}\left(\frac{x^4}{4}\right) + K$$

$$= \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + K$$

2. $\int x^2 \sin x \, dx$ LIATEC

$u = x^2 \qquad \int dV = \sin x \, dx$

$\frac{du}{dx} = 2x \qquad V = -\cos x$ ignore K

$du = 2x \cdot dx$

$$\int u \cdot dV = uv - \int v \cdot du$$

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int -\cos x \cdot 2x \, dx$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

Product

Reapply parts formula

$u = x \qquad \frac{du}{dx} = 1$

$dV = \cos x \, dx \qquad V = \int \cos x \, dx$

$du = dx \qquad V = \sin x$ (ignore K)

$$= -x^2 \cos x + 2\left[x \sin x - \int \sin x \, dx\right]$$

$$= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x - 2(-\cos x) + K$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + K$$

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3. $\int e^x \cos x \, dx$ LIATEC

$u = \cos x$ $\qquad \int dv = \int e^x \, dx$

$\frac{du}{dx} = -\sin x$ $\qquad v = e^x$ (ignore k)

$du = -\sin x \, dx$

$$\int u \cdot dv = uv - \int v \cdot du$$

$$\int e^x \cos x \, dx = \cos x \cdot e^x - \int e^x \cdot -\sin x \, dx$$

$$= \cos x \cdot e^x + \int e^x \sin x \, dx$$

Product ∴ Reapply

$\int e^x \sin x \, dx$ $\quad u = \sin x$ $\quad \int dv = \int e^x \, dx$

$\frac{du}{dx} = \cos x$ $\quad v = e^x$ (ignore k)

$du = \cos x \, dx$

$$\int u \, dv = uv - \int v \, du$$

$$\int e^x \sin x \, dx = \sin x \cdot e^x - \int e^x \cos x \, dx$$

Same as starting one

Collect like terms

$$\int e^x \cos x \, dx = \cos x \cdot e^x + \sin x \cdot e^x - \int e^x \cos x \, dx$$

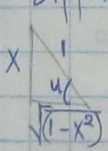
$$\frac{2\int e^x \cdot \cos x \, dx}{2} = \frac{e^x \cos x + e^x \sin x}{2}$$

$$\int e^x \cdot \cos x \cdot dx = \frac{1}{2}\left(e^x \cdot \cos x + e^x \sin x\right) + k$$

4. $\int \arcsin x \, dx = \int \sin^{-1} x \, dx$ LIATEC

$u = \sin^{-1}(x)$ $\qquad dV = 1 \, dx$

Differentiate $\qquad$ Integrate



$\sin u = x$

$u' \cos u = 1$

$u' = \frac{1}{\cos u}$ $\qquad V = x$ ignore k

$\cos u = \sqrt{1-x^2}$

$u' = \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$du = \frac{dx}{\sqrt{1-x^2}}$

$$\int \sin^{-1} x \, dx = (\sin^{-1}(x)) \cdot (x) - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

$$= x \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

↳ • Radicals
• Normal substitution

$\int \frac{x}{\sqrt{1-x^2}} \, dx$ — ①

$u = 1 - x^2$ — ②

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$ — ③

$$\int \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x} = -\frac{1}{2}\int \frac{1}{u^{1/2}} \, du$$

$$= -\frac{1}{2}\int u^{-1/2} \, du$$

$$= -\frac{1}{2} \frac{u^{1/2}}{1/2}$$

$$= -\frac{1}{2} \cdot \frac{2}{1} u^{1/2}$$

$$= -u^{1/2} = -(1-x^2)^{1/2}$$

$$= x \sin^{-1} x - -(1-x^2)^{1/2} + k$$

$$= x \sin^{-1}(x) + (1-x^2)^{1/2} + k$$

18/02/2020

Evaluation of Integrals via partial fractions

- This technique is used to decompose/separate a converged integrand into forms that can be integrated.

Consider $\int \frac{p(x)}{q(x)} \, dx$.

Case I

- If the power of x in the numerator is greater or equal to that of x in the denominator, use long division to rewrite the integrand.

Example

1. Separate $\frac{5x+2}{3x+1}$ into partial fractions hence evaluate $\int \frac{5x+2}{3x+1}dx$

Soln. Power of x in Num = power of x in Den.

By long division.

$$\begin{array}{r|l} & \frac{5}{3} \\ 3x+1 & 5x+2 \\ & -(5x+\frac{5}{3}) \\ \hline & \frac{1}{3} \end{array}$$

$$= \frac{5}{3} + \frac{\frac{1}{3}}{3x+1}$$

$$\int \frac{5x+2}{3x+1} = \int \frac{5}{3}dx + \frac{1}{3}\int \frac{dx}{3x+1}$$

$$= \frac{5}{3}x + \frac{1}{3}\cdot\frac{1}{3}\ln|3x+1| + k$$

ANSWER

N/B.

$$\int \frac{1}{3}\cdot\frac{dx}{3x+1} = \frac{1}{3}\int \frac{dx}{3x+1} \qquad u = 3x+1 \quad \frac{du}{3} = dx \quad \frac{du}{dx} = 3$$

$$\int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\int \frac{1}{u}du$$

$$= \frac{1}{3}\ln|u| = \frac{1}{3}\ln|3x+1|$$

2. Separate $\frac{2x^2-5}{x+3}$ into partial fractions hence solve $\int \frac{2x^2-5}{x+3}$

Power of x in N > power of x in D.

$$\begin{array}{r|l} & 2x-6 \\ x+3 & 2x^2-5 \\ & -(2x^2+6x) \\ \hline & -6x-5 \\ & -(-6x-18) \\ \hline & 13 \end{array}$$

$$\frac{2x^2-5}{x+3} = 2x-6 + \frac{13}{x+3}$$

$$\int \frac{2x^2-5}{x+3}dx = \int 2x\,dx - 6\int dx + 13\int \frac{dx}{x+3}$$

$u = x+3$

$\int \frac{1}{u} = \ln|u|$

$$= \frac{2x^2}{2} - 6x + 13\ln|x+3| + c$$

$$= x^2 - 6x + 13\ln|x+3| + c$$

CASE II Distinct linear factors

- If there exists distinct linear factors of the form $ax \pm b$, $cx \pm d$ in the denominator, then the corresponding partial fraction is of the form:

$$\frac{A}{ax \pm b} + \frac{B}{cx \pm d}$$

where A and B are constants to be determined.

Example

Evaluate:

1. $\int \frac{5}{(2x+1)(x-2)}dx$

By partial fractions

$$\frac{5}{(2x+1)(x-2)} = \frac{A}{(2x+1)} + \frac{B}{(x-2)}$$

$$= \frac{A(x-2) + B(2x+1)}{(2x+1)(x-2)}$$

Equate the numerators and compare the coefficents.

$$5 = A(x-2) + B(2x+1)$$

LHS RHS

Coef of x: $0 = A + 2B$

Constant term: $5 = -2A + B$

$$2A + 4B = 0 \qquad 2A + 4B = 0$$
$$B - 2A = 5 \qquad + \; -2A + B = 5$$
$$5B = 5$$
$$B = 1$$

$$5 = -2A + 1$$
$$4 = -2A$$
$$2 = -A$$
$$A = -2$$

$$\int \frac{5}{(2x+1)(x-2)}dx = \int \frac{-2}{2x+1}dx + \int \frac{1}{(x-2)}dx$$

$u = 2x+1$ $\quad$ $u = x-2$

$\frac{du}{dx} = 2$ $\quad$ $\int \frac{1}{u} =$ $\ln|u|$, $\ln|x-2|$

$$= -2\int \frac{1}{u}\cdot\frac{du}{2}$$

$$= -\ln|2x+1| + \ln|x-2| + k$$

$\Rightarrow \ln a - \ln b = \ln\left|\frac{a}{b}\right|$

$$\therefore = \ln\left|\frac{x-2}{2x+1}\right| + k$$

2. Evaluate.

$$\int \frac{x+5}{x^2+5x+6}dx = \int \frac{x+5}{(x+3)(x+2)}dx$$

By p.Fs

$$\frac{x+5}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2} = \frac{A(x+2)+B(x+3)}{(x+3)(x+2)}$$

Equate numerators

$x+5 = A(x+2) + B(x+3)$

$x: \quad 1 = A + B$

Constant: $5 = 2A + 3B$

$$\begin{aligned} 2A + 2B &= 2 \\ -\ 2A + 3B &= 5 \\ \hline 0 \quad -B &= -3 \end{aligned}$$

$B = 3$

$A = 1 - B$

$A = 1 - 3$

$A = -2$

$$\int \frac{-2}{x+3}dx + \int \frac{3}{x+2}dx$$

$u = x+3$ $\quad$ $u = x+2$

$\frac{du}{dx} = 1$ $\quad$ $\frac{du}{dx} = 1$

$$-2\int \frac{1}{u}\cdot du + 3\int \frac{1}{u}\cdot du$$

$$= -2\ln|x+3| + 3\ln|x+2| + k$$

CASE II Repeated linear factors

If there exists repeated linear factors of the form $(ax \pm b)^n$ in the Denom. then the corresponding partial fraction is of the form:

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots\cdots \frac{K}{(ax+b)^n}$$

For example:

① $(x+2)^3(x-1)^2$ ------- $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{D}{(x-1)} + \frac{E}{(x-1)^2}$

② $x^4(x+2)^2$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+2} + \frac{F}{(x+2)^2}$$

Example.

Evaluate:

1. $\int \frac{x^2+1}{x^3+2x^2+x}dx$

$$= \int \frac{x^2+1}{x(x^2+2x+1)} = \int \frac{x^2+1}{x(x+1)^2}$$

By P.Fs

$$\frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + B(x(x+1)) + Cx}{x(x+1)^2}$$

Equate numerators

$x^2+1 = A(x+1)^2 + B(x^2+x) + Cx$

Coef of x^2: $1 = A + B$

Coef of x: $0 = 2A + B + C$

Constant: $1 = A$

$B = 0$

$C = -2$

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{1}{x} + \frac{0}{x+1} + \frac{-2}{(x+1)^2}$$

$$= \int \frac{1}{x}\,dx + \int \frac{-2}{(x+1)^2}$$

$u = x+1$

$\frac{du}{dx} = 1 \quad dx = du$

$$= \left[\ln|x|\right] + \left[\int \frac{-2}{u^2}\,du = -2\int u^{-2}\,du = -\frac{2u^{-1}}{-1} = +2u^{-1}\right]$$

$$= \ln|x| + \frac{2}{(x+1)} + K$$

u^{-1}

2. Solve:

$$\int \frac{4x^4+x+1}{x^5+x^4}\,dx = \int \frac{4x^4+x+1}{x^4(x+1)}\,dx$$

By P.Fs

$$\frac{4x^4+x+1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$$

$$= \frac{Ax^3(x+1) + B(x^2)(x+1) + C(x+1)x + D(x+1) + Ex^4}{x^4(x+1)}$$

Equate numerators:

$$4x^4+x+1 = A(x^4+x^3) + B(x^3+x^2) + C(x^2+x) + D(x+1) + Ex^4$$

Coefficient

x^4	$4 = A+E$	$E = 4$
x^3	$0 = A+B$	$A = 0$
x^2	$0 = B+C$	$B = 0$
x	$1 = C+D$	
Constant	$1 = D$	$C = 1-D$, $C = 1-1 = 0$

$$= \int \frac{1}{x^4}\,dx + \int \frac{4}{x+1}\,dx$$

$$= \int x^{-4}\,dx + 4\int \frac{1}{u}\,dx$$

$$= \frac{x^{-3}}{-3} + 4\ln|x+1| + K$$

CASE 4

If there exists a quadratic factor of the form (ax^2+bx+c) in the denom, then the corresponding partial fractions is of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

- If (ax^2+bx+c) and (cx^2+dx+e) in the D ...... $\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{cx^2+dx+e}$

- What is considered as a quadratic factor cannot be factorised further.

Eg: x^2-4 is not a quadratic factor because $x^2-4 = (x+2)(x-2)$

$\frac{A}{x-2} + \frac{B}{x+2}$

x^2-4x+4 is not a q/f $\Rightarrow (x-2)^2$

$\frac{A}{x-2} + \frac{B}{(x-2)^2}$

x^2+3x+5 is a q/f $\frac{Ax+B}{x^2+3x+5}$

x^2+4 is a q/f $\frac{Ax+B}{x^2+4}$

$$x^4-16 = (x^2-4)(x^2+4)$$
$$= (x+2)(x-2)(x^2+4)$$

Linear linear quadratic

$$\frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+4}$$

Example.

1. $\int \frac{1}{(x^2+2)(x^2+4)}\,dx$

By Pfs

$$\frac{1}{(x^2+2)(x^2+4)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+4}$$

$$= \frac{Ax+B(x^2+4) + (Cx+D)(x^2+2)}{(x^2+2)(x^2+4)}$$

(A = 0, B = 1/2, C = 0, D = -1/2)

Equate numerators

$$1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 2Cx + Dx^2 + 2D.$$

Coeff of:

$x^3 \quad 0 = A + C \qquad A = -C$

$x^2 \quad 0 = B + D$

$x \quad 0 = 4A + 2C$

Constant $1 = 4B + 2D$

$0 = 4B + 4D$

$1 = -2D$

$D = -\frac{1}{2} \qquad 1 = 2B$

$B = \frac{1}{2}$

$$\frac{0 + \frac{1}{2}x}{x^2+2} + \frac{0 + -\frac{1}{2}x}{x^2+4}$$

$$\int \frac{1}{(x^2+2)(x^2+4)} = \int \frac{\frac{1}{2}}{x^2+2}\,dx + \int \frac{-\frac{1}{2}}{x^2+4}\,dx$$

Case 3 radicals cases

$x = a\tan u$

$x^2 = a^2 \tan^2 u$

$x = 2\tan u$ ②

$x^2 = 4\tan^2 u$ ③

$x = \sqrt{2}\tan u$ ②

$dx = \sqrt{2}\sec^2 u \cdot du$ ④

$dx = 2\sec^2 u \cdot du$

$$\frac{1}{2}\int \frac{\sqrt{2}\,\cancel{\sec^2 u}\,du}{2\,\cancel{\sec u}}$$

$$-\frac{1}{2}\int \frac{2\sec^2 u\,du}{4\sec^2 u}$$

$$-\frac{1}{4}\int du$$

$$= \frac{\sqrt{2}}{4}u$$

$$= -\frac{1}{4}u$$

$$= -\frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)$$

$$= \frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{x}{\sqrt{x}}\right) - \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + k.$$

CASE 5

If there exists a repeated quadratic factor of the form $(ax^2+bx+c)^2$ in D, then the corresponding partial fraction is of the form:

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}$$

If it is: $(ax^2+bx+c)^3$ then.

$$= \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{(ax^2+bx+c)^3}$$

Evaluate

$$\int \frac{x^2+4}{(x^2+1)^2(x^2+2)}\,dx$$

By P.F's.

$$\frac{x^2+4}{(x^2+1)^2(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{x^2+2}$$

$$= \frac{Ax+B(x^2+1)(x^2+2) + Cx+D(x^2+2) + Ex+F(x^2+1)}{(x^2+1)^2(x^2+2)}$$

Expand

$$= \frac{Ax+B(x^4+3x^2+2) + (Cx^3+2Cx+Dx^2+2D) + Ex+F(x^4+2x+1)}{(x^2+1)^2(x^2+2)}$$

$= Ax^5 + 3Ax^3 + 2Ax + Bx^4 + 3Bx^2 + 2B + Cx^3 + 2Cx + Dx^2 + 2D + Ex^5 + 2Ex^2 + Ex + Fx^4 + 2Fx^2 + F$

Numerator.

Equate numerators

Coeff of:

$x^5 \quad 0 = A + E \qquad A = -E$

$x^4 \quad 0 = B + F \qquad B + F = 0$

$x^3 \quad 0 = 3A + C + 2E \qquad -3E + C + 2E = 0 \rightarrow C - E = 0$

$x^2 \quad 1 = 3B + 2F + 2D$

$x \quad 0 = 2A + 2C + E \qquad C = E$

$-2E + 2C + E = 0 \rightarrow 2C - E = 0$

Constant $4 = 2B + 2D + F \qquad 2C - C = 0$

$4 = 2B + 2D + F \qquad C = 0$

$E = 0$

$B = -F$

$-3F + D + 2F = 1 \qquad -F + D = 1$

$-2F + 2D + F = 4 \qquad -F + 2D = 4$

$-D = -3$

$D = 3$

$F = 2$

$B = -2$

$\therefore A = 0,\ B = -2,\ C = 0,\ D = 3,\ E = 0$

$$\int \frac{x^2+4}{(x^2+1)^2(x^2+2)}\,dx = \int \frac{-2\,dx}{x^2+1} + \frac{3}{(x^2+1)^2}\,dx + \int \frac{2}{x^2+2} \rightarrow \text{(Radicals Case 3)}$$

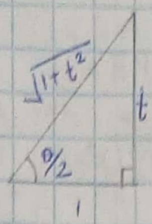
25/02/2020

T-Method of Substitution.

Let $t = \tan\frac{\theta}{2}$

or

$\rightarrow \tan\frac{\theta}{2} = \frac{t}{1}$ $\rightarrow$ Opp, $\rightarrow$ Adj.



$\cos\theta/2 = \frac{1}{\sqrt{1+t^2}}$ $\quad \sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$

i) $\cos\theta = \cos(\theta/2 + \theta/2)$

$\cos A\cos B - \sin A \sin B = \cos(A+B)$

$= \cos\theta/2 \cos\theta/2 - \sin\theta/2 \sin\theta/2$

$= \frac{1}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} - \frac{t}{\sqrt{1+t^2}} \cdot \frac{t}{\sqrt{1+t^2}} = \frac{1-t^2}{1+t^2}$

ii) $\sin\theta = \sin(\theta/2 + \theta/2)$ $\quad \sin(A+B)$

$\sin A\cos B + \cos A \sin B$

$= \sin\theta/2 \cos\theta/2 + \cos\theta/2 \sin\theta/2$

$= \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} + \frac{1}{\sqrt{1+t^2}} \cdot \frac{t}{\sqrt{1+t^2}}$

$= \frac{2t}{1+t^2}$

iii) If $t = \tan\theta/2$ $\quad f' \sec^2 f$

$\frac{dt}{\theta} = \frac{1}{2}\sec^2\theta/2$

$\frac{2dt}{\sec^2\theta/2} = d\theta$

But $\sec\theta/2 = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{\sqrt{1+t^2}}}$

$\sec\theta/2 = \sqrt{1+t^2}$

$\sec^2\theta/2 = 1+t^2$

$d\theta = \frac{2dt}{\sec^2\theta} = \frac{2dt}{1+t^2}$

In Summary:

$\cos x = \cos\theta = \frac{1-t^2}{1+t^2}$

$\sin x = \sin\theta = \frac{2t}{1+t^2}$

$dx = d\theta = \frac{2dt}{1+t^2}$

$t = \tan x/2$

$t = \tan\theta/2$

Examples

$\int \frac{d\theta}{1+\sin\theta}$

Soln. t-substitution.

$d\theta = \frac{2dt}{1+t^2}$

$D \Rightarrow 1 + \sin\theta = 1 + \frac{2t}{1+t^2} = \frac{1+t^2+2t}{1+t^2}$

$= \frac{t^2+2t+1}{1+t^2}$

$N \Rightarrow \frac{2dt}{1+t^2}$

$N/D = \frac{2dt}{\cancel{1+t^2}} \times \frac{\cancel{1+t^2}}{t^2+2t+1} = \frac{2dt}{t^2+2t+1}$

$\int \frac{2dt}{t^2+2t+1} = \int \frac{2dt}{(1+t)^2}$

$u = t+1$

$\frac{du}{dt} = 1 \quad du = dt$

$\int \frac{2}{u^2} du = 2\int u^{-2} du$

$= 2\frac{u^{-2+1}}{-2+1} + k$

$= -\frac{2}{u} + k$

$$\frac{-2}{t+1} + k \qquad t = \tan \theta/2$$

$$= \frac{-2}{\tan \theta/2 + 1} + k$$

2. $\int \frac{dx}{1+\sin x + \cos x} \qquad t = \tan x/2$

$N \Rightarrow dx = \frac{2dt}{1+t^2}$

$D \Rightarrow 1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$

$$= \frac{1+t^2+2t+1-t^2}{1+t^2} = \frac{t^2+2t+2-t^2}{1+t^2}$$

$$= \frac{2t+2}{1+t^2}$$

$$N/D = \frac{2dt}{\cancel{1+t^2}} \times \frac{\cancel{1+t^2}}{2t+2} = \frac{2dt}{2t+2}$$

$$\int \frac{\cancel{2}}{\cancel{2}(t+1)} dt = \int \frac{dt}{t+1} \qquad u = t+1 \quad du/dt = 1$$

$du = dt$

$$\int u^{-1} du = \int \frac{1}{u} \cdot du$$

$$= \ln |u| + k$$

$$= \ln |t+1| + k$$

$$= \ln |\tan x/2 + 1| + k$$

3 $\int \frac{d\theta}{5+4\cos\theta}$

$N \Rightarrow d\theta = \frac{2dt}{1+t^2}$

$D \Rightarrow 5 + 4\left(\frac{1-t^2}{1+t^2}\right)$

$$= \frac{5+5t^2+4+-4t^2}{1+t^2} = \frac{9+t^2}{1+t^2}$$

$$N/D \Rightarrow \frac{2dt}{\cancel{1+t^2}} \cdot \frac{\cancel{1+t^2}}{9+t^2}$$

$$\Rightarrow \int \frac{2dt}{9+t^2} \quad \text{— ①} \qquad a^2 + x^2$$

Radical II $\quad t = 3\tan u$ — 2

$t^2 = 9\tan^2 u$ — 3

$dt = 3\sec^2 u \, du$ — 4

$$2\int \frac{3\sec^2 u \, du}{9\tan^2 u + 9} = \int \frac{\cancel{3}\sec^2 u}{\frac{\cancel{9}}{3}\sec^2 u} du$$

$9\sec^2 u$

$1 + \tan^2 x = \sec^2 x$

$$= \frac{1}{3}\int du = \frac{2}{3}u + k$$

use ② equ.

$$= 2/3 \tan^{-1}\left(\frac{t}{3}\right) + k$$

$$= 2/3 \tan^{-1}\left(\frac{1}{3}\tan \theta/2\right) + k$$

Questions

$\int \frac{d\theta}{1+\cos\theta}$

$\int \frac{d\theta}{\sin\theta + \cos\theta}$

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Solution of ordinary differential equations by separation of variables method

Consider: $\frac{dy}{dx} = \frac{A}{B}$ ----- (1)

If equation (1) can be rearranged /separated such that:

A is a function of x alone
B is a function of y alone

Then we have

$B\,dy = A\,dx$.

Integrate both sides.

$\int B\,dy = \int A\,dx + C$ ______ (2)

Equation (2) is a solution of equation (1)

Solve

$\frac{dy}{dx} = \frac{1+x^2}{1-y^3}$ $y(0) = 2 \Rightarrow$ Initial condition
- used to find value of k
- Means that when $x=0$ $y=2$

$(1-y^3)dy = (1+x^2)dx$

Integrate.

$\int (1-y^3)\,dy = \int (1+x^2)\,dx$.

$\frac{y^{0+1}}{0+1} - \frac{y^4}{4} = x + \frac{x^3}{3} + k$

$y - \frac{y^4}{4} = x + \frac{x^3}{3} + k$.

$2 - \frac{2^4}{4} = 0 + \frac{0}{3} + k$

$2 - 4 = k$

$k = -2$

$y - \frac{y^4}{4} = x + \frac{x^3}{3} - 2$.

Solve: $y(5) = 10$.

$\frac{dy}{dx} = \frac{y^2-4}{x^2-1}$

$\frac{dy}{y^2-4} = \frac{dx}{x^2-1}$

Integrate:

$\int \frac{dy}{y^2-4} = \int \frac{dx}{x^2-1}$

✓ Partial fractions case II
✓ Radicals case 5

LHS

$\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2} = \frac{A(y+2)+B(y-2)}{(y-2)(y+2)}$

$1 = A(y+2) + B(y-2)$.

Coeff y: $0 = A + B$

Constant: $1 = 2A - 2B$

$2A + 2B = 0$

$-2A - 2B = 1$

$0 \quad 4B = -1$

$B = -1/4$ $A = 1/4$.

$\int \frac{1}{y^2-4}dy = \int \frac{\frac{1}{4}dy}{y-2} + \int \frac{-\frac{1}{4}}{y+2}dy$.

$= \frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2|$

$= \ln a - \ln b = \ln(a/b)$.

$= \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right|$.

RHS

$\frac{1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$

$1 = Ax - A + Bx + B$

$0 = A + B$

$1 = B - A$

$\frac{1}{2} = \frac{2A}{2}$ $A = 1/2$ $B = -1/2$

$\int \frac{1/2}{x-1}dx + \int \frac{1/2}{x+1}dx$

$\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$

$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + k$

$x=5$
$y=10$

$\frac{1}{4}\ln\left|\frac{8}{12}\right| = \frac{1}{2}\ln\left|\frac{4}{6}\right| + k$

$\frac{1}{4}\ln\left(\frac{2}{3}\right) - \frac{1}{2}\ln\left(\frac{2}{3}\right) = k$

Exercise

Verify that the given equation represents the solution to the stated differential equations.

a) $y = ce^{4x}$ $\quad y' = 4y$ or $y' - 4y = 0$

$y' = 4ce^{4x}$

$y' - 4y = 4ce^{4x} - 4ce^{4x} = 0$

hence it is a solution.

b) $y = c_1 \cos x + c_2 \sin x \rightarrow y' = c' \sin x + c_2 \cos x$

$y'' = -c_1 \cos x - c_2 \sin x$

$y'' + y = 0$

$c' \cos x + c_2 \sin x$

$-c_1 \cos x - c_2 \sin x$

$-c_1 \cos x - c_2 \sin x + c_1 \cos x + c_2 \sin x = 0$

hence it is a solution.

c) $t^2 + y^2 = cy$; $y' = \frac{2xy}{x^2 - y^2}$

d) $y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$, $y'' + 2y' + 2y = 0$

e) $y = 3\sin 3x$, $y^4 - 16y = 0$

f) $y = e^{-2x}$; $y^4 - 16y = 0$

g) $y = c_1 e^{2x} + c_2 e^{-2} + c_3 \sin 2x + c_4 \cos 2x$; $y^4 - 16y = 0$

h) $y = 5e^{-2x} + 3\cos 2x$

i) $y = \sin x$

$\Big\}\ y^4 - 16y = 0.$

f) $y = e^{-2x}$

$y' = -2e^{-2x}$

$y'' = 4e^{-2x}$

$y''' = -8e^{-2x}$

$y^{IV} = 16e^{-2x}$

$y^4 - 16y = 16e^{-2x} - 16e^{-2x} = 0$ hence it is a solution

REMARK:

– Given any rate of change:

$s'(t) = p \rightarrow \frac{ds}{dt} = p \quad \int ds = \int p \cdot dt$

$S(t) = \int p \cdot dt + k$

$N'(t) = q$

$\frac{dN}{dt} = q$

$\int dN = \int q \cdot dt$

Integrate

$N = \int q \cdot dt + c$ → constant depends on initial condition provided.

$g'(r) = f$

$\frac{dg}{dr} = f \quad \int dg = \int f \cdot dr$

$g = \int f \cdot dr + k$

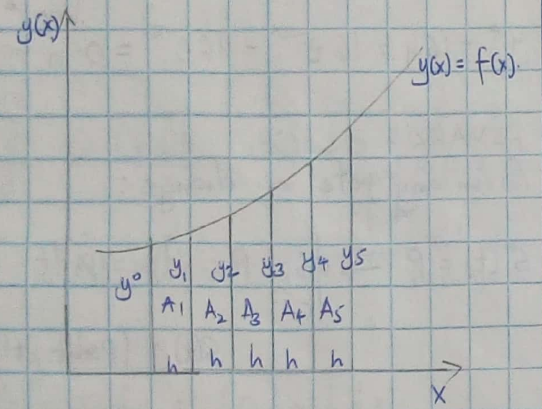
The constant, k or c is obtained from the initial condition given.

Numerical Integration

- This technique of integration is used to approximate the value of the integral in cases where the exact methods discussed above fails.
- The techniques include:
  - Trapezoidal rule
  - Simpson's rule.

1 TRAPEZOIDAL RULE

Consider approximating the area of the region bounded by the curve $y = f(x)$ between $x = a$ and $x = b$ by subdividing the region into equall strips of width h as shown below.



- Since the shape of each strip resembles that of a trapezium, the formula for finding the area of a trapezium can be used to estimate the area of each strip

$A_1 = h/2\,(y_0 + y_1)$ $\qquad A_4 = h/2\,(y_3 + y_4)$

$A_2 = h/2\,(y_1 + y_2)$ $\qquad A_5 = h/2\,(y_5 + y_4)$

$A_3 = h/2\,(y_2 + y_3)$

Total area $= A_1 + A_2 + A_3 + A_4 + A_5$

$= h/2\left(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + y_3 + y_4 + y_4 + y_5\right)$

$= h/2\left(y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5\right)$

In general:

$$T_{Area} = \frac{h}{2}\left(y_0 + 2(y_1 + y_2 + \cdots y_{(n-1)}) + y_n\right)$$

where $h = \dfrac{b-a}{n}$ is called the step size

b = upper limit

a = lower limit

n = number of strips.

- $y_0\ y_1\ y_2 \cdots y_n$ are called the ordinates in the above case we have:

✓ 6 Ordinates $(y_0\ y_1\ y_2\ y_3\ y_4\ y_5)$ and

✓ 5 Strips $(A_1\ A_2\ A_3\ A_4\ A_5)$

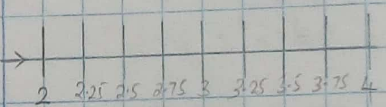
Example:

Approximate the value of the following intergrals, use trapezoidal rule with 9 ordinates correct to 4 d.p.

1. $$\int_2^4 \frac{5\ln 2x}{2 + \ln 2x}\,dx.$$

Soln → 9 Ordinates means $n = 8$ (8 strips)

$h = \dfrac{b-a}{n} = \dfrac{4-2}{8} = \dfrac{2}{8} = \dfrac{1}{4} = 0.25$ Step size



$y(x) = \dfrac{5\ln 2x}{2 + \ln 2x}.$ $\qquad y_2 = 2.229$

When $x = 2$ $\quad y(0) = \dfrac{5\ln(2\times 2)}{2 + \ln 2\times 2} = 2.0469$

$y(1) = 2.1461$

$y(2) = 2.2295$

$y(3) = 2.3008$

$y(4) = 2.3627$

$y(5) = 2.472$

$y(6) = 2.4657$

$y(7) = 2.5082$

$y(8) = 2.5487$

$$A = \frac{h}{2}\left(y_0 + 2(y_1 + \dots + y_7) + y_8\right)$$

$$A = \frac{0.25}{2}\left(2.0469 + 32.8604 + 2.5487\right)$$

$$= 4.6820$$

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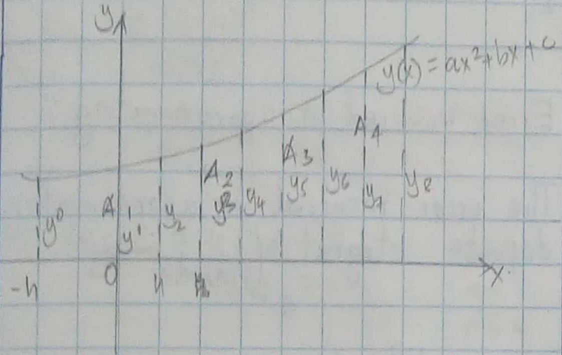
2. SIMPSON'S RULE

- Consider approximating the area of the region bounded by the curve $y(x) = ax^2 + bx + c$ between $x = -h$ to $x = +h$ shown below.



By direct integration:

$$A_1 = \int_{-h}^{h} (ax^2 + bx + c)\,dx$$

$$A_1 = \frac{ax^3}{3} + \frac{bx^2}{2} + cx \Big|_{-h}^{h}$$

$$A_1 = \frac{a(h^3)}{3} + \cancel{\frac{b(h)^2}{2}} + c(h) - \left[\frac{a(-h^3)}{3} + \cancel{\frac{b(-h)^2}{2}} - ch\right]$$

$$A_1 = \frac{2ah^3}{3} + 2ch \quad \text{——} (1)$$

When $x = -h$

$$y_0 = a(-h^2) + b(-h) + c = ah^2 - bh + c \quad (2)$$

When $x = 0$

$$y_1 = a(0)^2 + b(0) + c \qquad y_1 = c \quad \text{——} (3)$$

When $x = h$

$$y_2 = a(h)^2 + b(h) + c = ah^2 + bh + c \quad \text{——} (4)$$

Substituting eq 3 $(y_1 = c)$ into eqn 2 and 4 then add

$$y_0 = ah^2 - bh + y_1$$

$$y_2 = ah^2 + bh + y_1$$

$$y_0 + y_2 = 2ah^2 + 2y_1$$

$$2ah^2 = y_0 + y_2 - 2y_1 \quad \text{——} (5)$$

Subst eq 3 $(y_1 = c)$ and eqn 5 in eq 1

$$A_1 = \frac{h}{3}(2ah^2) + 2ch$$

$$= \frac{h}{3}(y_0 + y_2 - 2y_1) + \frac{2y_1 h}{1}$$

$$= \frac{h(y_0 + y_2 - 2y_1) + 6y_1 h}{3}$$

$$A_1 = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly: $A_2 = \frac{h}{3}(y_2 + 4y_3 + y_4).$

$$A_3 = \frac{h}{3}(y_4 + 4y_5 + y_6).$$

$$A_4 = \frac{h}{3}(y_6 + 4y_7 + y_8)$$

Total Area, $A = A_1 + A_2 + A_3 + A_4.$

$$A = \frac{h}{3}\left(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + y_6 + 4y_7 + y_8\right.$$

$$A = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) + y_8\right].$$

In general:

$$SR = \frac{h}{3}\left[y_0 + 4\begin{pmatrix}\text{All odd} \\ \text{ordinates btwn} \\ \text{extremes}\end{pmatrix} + 2\begin{pmatrix}\text{All even} \\ \text{ordinates}\end{pmatrix} + y_{2n}\right]$$

Example

1. Use Simpson's rule with 9 ordinates correct to 4 d.p to estimate

$$\int_2^4 \frac{5\ln 2x}{5+\ln 2x}\,dx$$

9 Ordinates means $n = 8$, $h = \frac{b-a}{n}$

$$h = \frac{4-2}{8} = \frac{2}{8} = \frac{1}{4}$$

$$h = 0.25$$

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6$

$2 \quad 2.25 \quad 2.5 \quad 2.75 \quad 3 \quad 3.25 \quad 3.5 \quad 3.75 \quad 4$

When $x = 2$ $\quad y_0 = \frac{5\ln 4}{5+\ln 4} = 1.0853$

When $x = 2.25$ $\quad y_1 = \frac{5\ln 4.5}{5+\ln 4.5} = 1.1563$

When $x = 2.5$ $\quad y_2 = \frac{5\ln 5}{5+\ln 5} = 1.2175$

When $x = 2.75$ $\quad y_3 = \frac{5\ln 5.5}{5+\ln 5.5} = 1.2713$

$$y_4 = \frac{5\ln 6}{5+\ln 6} \doteq 1.3191$$

$$y_5 = \quad = 1.3619$$

$$y_6 = \quad = 1.4007$$

$$y_7 = \quad = 1.4362$$

$$y_8 = \quad = 1.4686$$

$$A = \frac{h}{3}\left[y_0 + 4(\text{All odd}) + 2(\text{Evens}) + y_{2n}\right]$$

$$A = \frac{0.25}{3}\left[1.0853 + 4\left(y_1, y_3, y_5, y_7\right) + 2\left(y_2, y_4, y_6\right) + 1.4686\right]$$

$$A = \frac{0.25}{3}\left[1.0853 + 4(5.2257) + 7.8748 + 1.4686\right]$$

$$= 2.6110$$

2. Use i) TR ii) SR with 9 ordinates correct to 4 d.p to estimate:

2. $\displaystyle\int_0^1 \frac{5e^{\sqrt{x}}}{1+2x}\,dx$

3. $\displaystyle\int_2^4 \frac{10e^x}{\sqrt{10+\ln 2x}}\,dx$

4. $\displaystyle\int_0^{\pi/4} \sqrt{5-\sin^2\theta}\,d\theta$ — set calc to radians

5. $\displaystyle\int_{10°}^{20°} \frac{5+\cos 4x}{2-\sin 4x}\,dx$

6. $\displaystyle\int_2^3 \frac{5x^3+6x+1}{x+e^x}\,dx$

Error incurred in approximating

- The error incurred in approximating a definite integral $\int_a^b f(x)\cdot dx$ by

i) Trapezidal rule with n strips is:

$$E_T = \frac{m(b-a)^3}{12n^2} \quad \text{where } m = \left|f''(x)\right|_{\max\,(a,b)}$$

$f''(a)$, $f''(b)$ } pick the largest absolute value to be m

ii) Simpson's rule with n strips is:

$$E_S = \frac{k(b-a)^5}{180n^4}$$

$$\text{Where } k = \left|f^4(x)\right| \quad \max \in (a,b)$$

$f^4(a)$, $f^4(b)$ } pick the largest absolute value to be K

Example:
Find the error incured if the following integrals are estimated by: i) TR ii) SR with $n=4$

a) $\int_1^2 \frac{1}{x}\,dx$

By TR: $E_T = \frac{m(b-a)^3}{12n^2}$

$b=2$
$a=1$
$n=4$

$m = |f''(x)|_{\max\,(a,b)}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

$f''(a) = \frac{2}{1} = 2 \rightarrow M$

$f''(b) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$

$E_T = \frac{2(2-1)^3}{12(4)^2}$

$= \frac{2}{12 \times 16} = \frac{1}{96} = \underline{0.0104}$

By SR: $E_S = \frac{k(b-a)^5}{180n^4}$

$b=2$
$a=1$
$n=4$

$k = |f^4(x)|_{\max \in (a,b)}$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = -6x^{-4}$

$f^{iv}(x) = 24x^{-5}$
$= \frac{24}{x^5}$

For $a = \frac{24}{1^5} = 24$ (K)
For $b = 24/2^5 = 0.75$

$E_S = \frac{24(2-1)^5}{180(4^4)}$

$= \frac{24}{46080}$

$= \underline{5.2083 \times 10^{-4}}$

b) $\int_1^2 \frac{dx}{1+2x}$

c) $\int_0^1 e^{\sqrt{2x}}\,dx$ ~~$e^{\sqrt{x}}\,dx$~~

d) $\int_2^4 \frac{dx}{1+\ln 2x}$

e) $\int_1^2 \frac{dx}{\sqrt{1+2x}}$

f) $\int_0^1 \frac{dx}{1+x^2}$