School of Electronics and Computer Science
University of Southampton

COMP6216: Report on Coursework Assignment II

Worth: 70%

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Contents

1 Introduction

This report is about solving a modeling problem by using both agent based model and differential equation based approach. The problem is called "Group Dynamics of Hard Workers and Lazy Workers", it is described in the section below.

1.1 Problem Description

In one university, all the students is working on group projects for every courses, they randomly form into groups and regroup after finish projects then prepare for next course:

1. Student can follow two strategies (S) in the beginning : work hard ($S = H$) or lazy ($S = L$).
2. The effort (e) of the group is the sum of the strategies for all group members (n).
3. When course finished, each group member gets the same mark (m) as $m = e/n$.
4. Then every student rethink his strategy by randomly select another student and comparing a measure (π) based on $\pi = m - a * S$. If he had a lower π than the another student, he will imitate the another student's strategy S with a probability proportional to the difference in measures π, otherwise stay the same strategy.

If all the students take courses forever and follow the same procedure above, then we have a dynamic model of hard and lazy students working in groups.

The research question is to assess how these parameters below affect the results, such as how long it takes to reach equilibrium state.

- Initial composition of the population
- Group size (n)
- Cost of effort a
- Contribution of hard workers to group effort ($H\ and\ L$)

2 Analytical Solution

This section mainly include how to develop a model based on differential equations addressing this problem, and also explored the equilibrium solutions and its stability. Then, try to mathematically solve the differential equation by integration.

2.1 Differential Equations

As it described in the section above, there is only one variable needed for this particular problem which is the population with strategy H or L, we use the density ζ_H to represents the portion of H's population, therefore the density of L's population will be $1 - \zeta_L$.

For the evolution of ζ_H, certain rule is needed. lets say we have a population of u agents $\{\zeta_i\}_{i=1}^{u}$ in which each agents makes payoff π_i and agents evolve according to the update rule below:

$$
\begin{aligned}
\dot{\zeta}_i &= \zeta_i \Big\{ -\sum_{j,\pi_j>\pi_i} \zeta_j(\pi_j - \pi_i) + \sum_{j,\pi_i>\pi_j} \zeta_j(\pi_i - \pi_j) \Big\} \\
&= \zeta_i \sum_j \zeta_j(\pi_i - \pi_j) \\
&= \zeta_i(\pi_i \sum_j \zeta_j - \sum_j \zeta_j \pi_j) \quad where \sum_j \zeta_j = 1 \; and \; \sum_j \zeta_j \pi_j = \bar{\pi} \\
&= \zeta_i(\pi_i - \bar{\pi})
\end{aligned} \tag{1}
$$

In our problem, the $\bar{\pi}$ can be obtained as the equation (2) below:

$$
\begin{aligned}
\bar{\pi} &= \zeta_H \pi_H + \zeta_L \pi_L \\
&= \zeta_H \pi_H + (1 - \zeta_H)\pi_L \\
&= \zeta_H(\pi_H - \pi_L) + \pi_L
\end{aligned} \tag{2}
$$

Next, we substitute the equation (2) into (1) given:

$$
\begin{aligned}
\dot{\zeta}_H &= \zeta_H\{\pi_H - (\zeta_H(\pi_H - \pi_L) + \pi_L)\} \\
&= \zeta_H\{\pi_H - \pi_L - \zeta_H(\pi_H - \pi_L)\} \\
&= \zeta_H(\pi_H - \pi_L)(1 - \zeta_H)
\end{aligned} \tag{3}
$$

The equation above can still be simplified by the function below as it described in the problem:

$$
\begin{aligned}
\pi_H - \pi_L &= (m - a * H) - (m - a * L) \\
&= a(L - H)
\end{aligned} \tag{4}
$$

Finally, the differential equation for the problem is obtained as this function below:

$$
\dot{\zeta}_H = a(L - H)(\zeta_H - \zeta_H^2) \tag{5}
$$

2.2 Classification

2.3 Analyses System Behavior

In order to analyses the stability of the above differential equation, we need to differentiate it based on ζ_H as below:

$$
\begin{aligned}
f(\zeta_H) &= a(L - H)(\zeta_H - \zeta_H^2) \\
\frac{df}{d\zeta_H} &= a(L - H)(1 - 2\zeta_H)
\end{aligned} \tag{6}
$$

Then, we apply the stable state of the ζ_H as $\zeta_H^* = 0 \; and \; 1$ to get the system behavior.

$$
\begin{gathered}
\frac{df}{d\zeta_H}|_{\zeta_H^*=0} = a(L - H) \\
due\ to\ a > 0,\ stable\ if\ L < H,\ unstable\ if\ L > H
\end{gathered} \tag{7}
$$

$$\begin{aligned} &\frac{df}{d\zeta_H}|_{\zeta_H^*=1} = -a(L-H) \\ &due\ to\ a>0,\ stable\ if\ L>H,\ unstable\ if\ L<H \end{aligned} \tag{8}$$

Therefore, the behavior of the system depends on the relationship between L and H.

$$\begin{aligned} &if\ L>H: \qquad population\ dominated\ by\ all\ H's \\ &if\ H>L: \qquad population\ dominated\ by\ all\ L's \end{aligned} \tag{9}$$

2.4 Integration

In this section, we aim to get the solution by integrate the differential equation. Firstly, we rewrite the equation (5) as below:

$$\begin{aligned} \frac{d\zeta}{dt} &= a(L-H)(\zeta-\zeta^2) \\ \frac{d\zeta}{\zeta-\zeta^2} &= a(L-H)dt \end{aligned} \tag{10}$$

Then we apply integration method on both sides with different conditions to get:

$$\begin{aligned} \int_{\zeta_0}^{\zeta} \frac{d\zeta}{\zeta(1-\zeta)} &= \int_{t_0}^{t} a(L-H)dt \\ \int_{\zeta_0}^{\zeta} \frac{(1-\zeta)+\zeta}{\zeta(1-\zeta)}d\zeta &= a(L-H)t \\ \int_{\zeta_0}^{\zeta} (\frac{1}{\zeta}+\frac{1}{1-\zeta})d\zeta &= a(L-H)t \\ ln\frac{\zeta}{\zeta_0} + ln\frac{1-\zeta_0}{1-\zeta} &= a(L-H)t \\ ln\frac{\zeta/(1-\zeta)}{\zeta_0/(1-\zeta_0)} &= a(L-H)t \\ \frac{\zeta/(1-\zeta)}{\zeta_0/(1-\zeta_0)} &= e^{a(L-H)t} \\ \frac{\zeta}{1-\zeta} &= \frac{\zeta_0}{1-\zeta_0}e^{a(L-H)t} \end{aligned} \tag{11}$$

We replace the RHS in equation (6) as A, then the function obtained can be derived as:

$$\begin{aligned} \frac{\zeta}{1-\zeta} &= A \\ \zeta &= A(1-\zeta) \\ \zeta(1+A) &= A \\ \zeta &= \frac{A}{1+A} \\ \zeta &= \frac{1}{1+\frac{1}{A}} \end{aligned} \tag{12}$$

Then, we substitute the A into equation (7):

$$
\begin{aligned}
\zeta &= \frac{1}{1+\frac{1}{\frac{\zeta_0}{1-\zeta_0}e^{a(L-H)t}}} \\
\zeta &= \frac{1}{1+\frac{1-\zeta_0}{\zeta_0}\frac{1}{e^{a(L-H)t}}} \\
\zeta &= \frac{1}{1+\frac{1-\zeta_0}{\zeta_0}e^{a(H-L)t}}
\end{aligned}
\tag{13}
$$

Finally, the analytical solution for the differential equation is obtained as above, it applies to both H and L with their initial variables.

3 Program Implementation

This section described how to use computer code for modeling the specific student problem, two methods were used to design this system.

3.1 Numerical Integration

We use both Euler and RungeKutta's method to numerically solve the integration. By setting the time of every course to be 1, the Euler's equation can be written as:

$$
\begin{aligned}
\frac{\zeta_H^{n+1}-\zeta_H^n}{t^{n+1}-t^n} &= a(L-H)(\zeta_H^n-(\zeta_H^n)^2) \\
\zeta_H^{n+1} &= a(L-H)(\zeta_H^n-(\zeta_H^n)^2)+\zeta_H^n
\end{aligned}
\tag{14}
$$

Therefore, the density of H can be iteratively calculated by equation (14) above until it converged into 1 or 0. The essential code for both methods are listed as figure 1 below:

```
to euler-method
  let slope (a * (1 - high - high) * (temp_h_density1 - temp_h_density1 ^ 2))
  let temp_h_density2 (temp_h_density1 + 1 * slope)
  set h_population (population * temp_h_density2)
  set temp_h_density1 temp_h_density2
end

to Runge-Kutta-method
  let k1 (a * (1 - high - high) * (temp_h_density1 - temp_h_density1 ^ 2))
  let k2 (a * (1 - high - high) * (temp_h_density1 + 0.5 * k1 - (temp_h_density1 + 0.5 * k1) ^ 2))
  let k3 (a * (1 - high - high) * (temp_h_density1 + 0.5 * k2 - (temp_h_density1 + 0.5 * k2) ^ 2))
  let k4 (a * (1 - high - high) * (temp_h_density1 + k3 - (temp_h_density1 + k3) ^ 2))
  let temp_h_density2 temp_h_density1 + 1 / 6 * (k1 + 2 * k2 + 2 * k3 + k4)
  set h_population (population * temp_h_density2)
  set temp_h_density1 temp_h_density2
end
```

Figure 1: Essential code for numerical integration

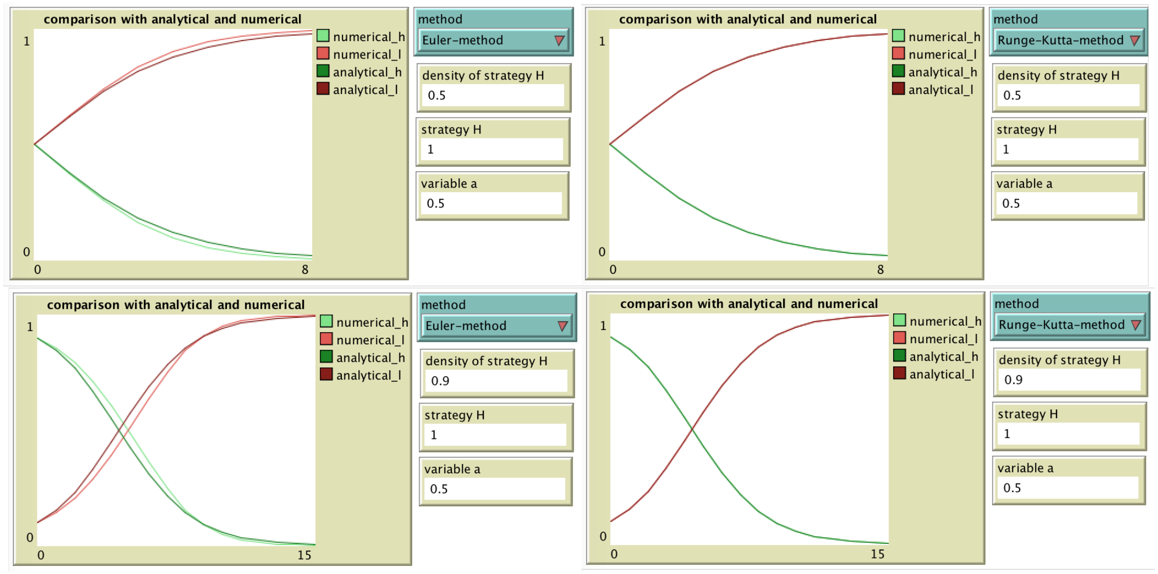

Figure 2: comparisons between integration methods

3.2 Comparison between analytical and numerical integration

3.3 Agent-based Model

An agent-based model has been implemented based on Netlogo open source software,

4 Extension