Phys10180C Analogue Circuits

This is the blended (demonstrated/online) version of the Circuits course. It combines experiments conducted by the demonstrator, with results you will be given, and simulations which you can do online. The default simulator package is `http://www.falstad.com/circuit` which you will be shown by your demonstrator. This is a web-based package which means that there are no installation problems. If you prefer to use SPICE you can do so, and there are instructions on Blackboard for using this.

The aims of Analogue Circuits are to teach both the experimental and theoretical aspects of electrical circuits and to form an introduction to some techniques used in electronics.

Objectives. By the end of the course, you should be able to: analyse simple electrical circuits and understand their response to sinusoidal waveforms, understand and be able to use the complex representation of voltage, current and impedance, understand the concept of phase difference of sinusoidal voltages and currents, understand what is meant by a low-pass and a high-pass filter, understand the decibel notation for expressing the gain of an electrical filter, understand the response of simple circuits to a step input.

Practical remarks. This script contains the experiments which you will see performed, and details of the simulations you should undertake. It also contains the theory of the experiments, which is described further in the associated video. You should keep a lab notebook (either on paper or online) which you will need to present as a single pdf file before the lab interview. You will be marked on criteria including error analysis (which you should perform on the experimental results) as well as physics understanding and how you have analysed the results.

1 General introduction

The package is assessed like a normal laboratory experiment. Some material overlaps with PHYS10302, Vibrations and Waves, where it is also examinable. There are also important links with PHYS10342 Electricity and Magnetism, with PHYS10071 Mathematics for Physicists, with PHYS10180 Digital Electronics. This lab runs from 11 am to 4.45 pm, with an hour for lunch, and your demonstrator will contact you with a Zoom/blackboard link which you should use for the session. You could lose credit for absence. Much of these notes is introduction, and theory; experiments and simulation activities are described in `typewriter font`, including basic activities (things you should certainly complete) and extra experiments (which you should complete if you can).

2 Equipment

Although the experiment is carried out remotely, your demonstrator will introduce you to the following important items of equipment:

- Waveform Generator, including how it can be used to deliver different types of signal (sine, square, triangular waves)
- Oscilloscope, including: correct use of earth connections, measurement of voltage and time intervals using the menus, triggering controls, DC/AC modes, averaging options, use of cursors.
- Resistor Colour Code: you should know how to read resistor value and tolerances from the colour codes (you will need the latter, as well as capacitor tolerances, for error analysis).

- `falstad.com/circuit` simulator: you should know how to set up basic circuits, adjust parameters of components, replace and add components, and carry out measurements of phases and amplitudes using the cursors.

3 Complex impedance - introduction

Ohm's law states that $V = IR$, where V (volts) is a steady voltage drop across a resistor, I (Amps) is a steady current and R (Ohms) is the resistance. This is called Direct Current. Our aim is to extend this to AC or Alternating Currents. Here the voltage and current oscillate sinusoidally at a fixed frequency, f. So there is also a definite angular frequency, $\omega = 2\pi f$ and a definite period, $T = 1/f$. Thus, for instance, $V(t) = V_0 \cos(\omega t + \phi_V)$ and $I(t) = I_0 \cos(\omega t + \phi_I)$. The manipulation of trig functions can be unwieldy and after a little practice it turns out that it is much simpler to use complex notation. In general with AC the current and the voltage are not in phase, so as well as specifying the magnitude of the voltage for a given current we must also specify the phase. A neat way to do this is to replace the resistance R, a real quantity, by a complex quantity Z. Z is called the Complex Impedance or just the Impedance. Being complex it has two parts, the real part and the imaginary part. More usefully the parts are the magnitude and the argument and indeed these give the amplitude and the phase of the voltage for a given current. Thus Ohm's law becomes[1]: $\tilde{V} = Z\tilde{I}$. We will measure both amplitude and phase and as well as using resistors we'll also study capacitors and inductors.

4 Complex Impedance of R,C,L - Measurements

4.1 General remarks

Consider the circuit as shown in Fig. 1 (which is for measuring a capacitor). The same current passes through the capacitor and the resistor. The resistor ($R_A = 2\Omega$) is used to measure the current through the capacitor. The current is given by $I = V_2/R_A$. We are using the resistor, R_A, and the scope as an ammeter. The voltage across the capacitor is given to a good approximation by V_1. We assume that the voltage across the resistor is much smaller than that across thec apacitor and this will be so, provided R_A is sufficiently small. Thus on the scope, channel 1 (V_1) is the voltage across the capacitor and channel 2 ($V_2 = IR_A$) is proportional to the current through the capacitor. From the two traces, we can observe both the relative amplitude and any phase difference. If you are simulating a circuit, then you can measure voltages across, and currents through, any component separately, so in principle this device is not needed BUT you are generally not allowed to have a capacitor or inductor loop with no resistance.

4.2 Amplitudes and phases

EXPERIMENT 1A. `You will be given some measurements of` V_1 `and` V_2 `for the circuit of Fig. 1, including the amplitude and phase difference, for two values of generator frequency. Make sure that you can understand these (including any necessary error analysis), calculate the value of` C `and compare with the actual value used.  Simulate the circuit, in which`

[1] We use a superscript tilde to signify that we are dealing with a complex representation of a wave (a wave represented by a complex number whose modulus is the amplitude and whose argument is the phase.)

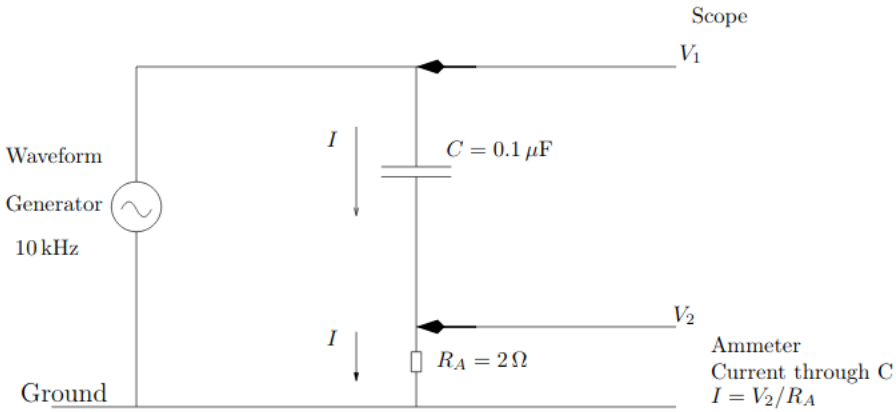

Figure 1. Measurement of voltage across and current through a capacitor. The 2Ω resistor with channel 2 of the scope constitutes an ammeter for measuring the current. V_1 is the voltage across the capacitor, provided R_A is small so that $V_2 \ll V_1$.

case you can measure the voltages across C and current through C directly. Again make sure that you can numerically verify the results for at least two different combination of f and C (you may need to use a lower f for accurate results).

Experiment 1B. You will be given some measurements of V_1 and V_2 for the circuit of Fig. 1 but with an inductor instead of C, including the amplitude and phase difference, for two values of generator frequency. Calculate the value of L (with error). Simulate the circuit for at least two values of f and L and measure the amplitudes of the voltages across the resistor and inductor; compare with theory.

Experiment 1C. You will be given one measurement of V_1 and V_2, including the amplitude and phase difference, for Fig. 1 with a resistor of unknown resistance instead of C. Calculate the resistance with error.

Draw up a table summarising your results for resistors, capacitors and inductors. In each of these three cases give the amplitude ratio (i.e. the magnitude of the voltage divided by the magnitude of the current in terms of the component values) and give the phases of the voltage relative to the phase of the current.

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5 Complex Notation - more theory

Suppose the voltage across and the current through a component are given by: $V(t) = V_0 \cos(\omega t + \phi_V)$ and $I(t) = I_0 \cos(\omega t + \phi_I)$. The ratio V_0/I_0 and the phase difference $\Delta\phi = \phi_V - \phi_I$ are determined by the particular component. The component might be a resistor R, a capacitor C or an inductor L, as you have already studied. More generally it might be any combination of these.

We use complex notation here again, as the algebra is much easier. Recall that $e^{j\theta} = \cos\theta + j\sin\theta$.

Thus the above is equivalent to

$$V(t) = \mathrm{Re}(V_0 e^{j(\omega t + \phi_V)}) \quad \text{and} \quad I(t) = \mathrm{Re}(I_0 e^{j(\omega t + \phi_I)}).$$

In general, $V(t) = \mathrm{Re}(\tilde{V}(t))$ and $I(t) = \mathrm{Re}(\tilde{I}(t))$ where

$$\tilde{V}(t) = V_0 e^{j(\omega t + \phi_V)} \quad \text{and} \quad \tilde{I}(t) = I_0 e^{j(\omega t + \phi_I)}.$$

$\tilde{V}(t)$ and $\tilde{I}(t)$ are usually referred to as phasors. They can each be represented by a vector in the complex plane, the Argand diagram, as described in the supplementary notes. As time increases these vectors rotate (anticlockwise) at the same rate, given by ω . They have constant length and the angle between them is constant. Therefore, one is just a constant (a constant complex number) multiplied by the other. This complex number is called the Complex Impedance and is usually denoted by Z. Thus: $\tilde{V} = Z\tilde{I}$ and this is the AC form of Ohm's law. The modulus of Z gives the ratio of the amplitudes of the voltage and thecurrent: $|V| = |Z||I|$. The argument of Z (i.e. its phase) gives the relative phase of the voltage and the current: arg$V$-arg$I$= arg$Z$ or equivalently $\phi_V - \phi_I = \phi_Z$. As the tangent of the phase of Z is the Imaginary part over the Real part this is:

$$\frac{\mathrm{Im}Z}{\mathrm{Re}Z} = \tan\phi_Z = \tan(\phi_V - \phi_I) = \tan\Delta\phi.$$

This is an essential feature of the complex impedance, Z. It gives both the relative amplitude and the relative phase of the voltage and current.

6 Complex Impedance of R C and L - Theory

The complex impedance of a resistor is $Z_R = R$. This merely states that for sinusoidal waveforms, the current and voltage are in phase and that $V = RI$.

The complex impedance of a capacitor is $Z_C = 1/j\omega C$. The basic property of a capacitor is $Q = CV$. Differentiating and using $I = dQ/dt$ gives $I(t) = CdV(t)/dt$. If there is a current then the voltage is changing. For sinusoidal waveforms, this is just the real part of $\tilde{I}(t) = Cd\tilde{V}(t)/dt$. Using the expression $\tilde{V}(t) = V_0 e^{j(\omega t + \phi_V)}$ (see section 5), this gives $\tilde{I}(t) = Cj\omega\tilde{V}(t)$ and so $Z_C = 1/j\omega C$.

The complex impedance of an inductor is $Z_L = j\omega L$. The basic property of inductors is $V(t) = LdI(t)/dt$. There is a voltage if the current is changing (i.e. the opposite of a capacitor). For sinusoidal waveforms, this is just the real part of $\tilde{V}(t) = Ld\tilde{I}(t)/dt$. Using the expression $\tilde{I}(t) = I_0 e^{j(\omega t + \phi_I)}$ (see section 5), this gives $\tilde{V}(t) = Lj\omega\tilde{I}(t)$ and so $Z_L = j\omega L$. Your experimental results (sections 4.2 - 4.5) should be consistent with these expressions - both the amplitudes and phases. What is the meaning of the j's in these expressions? Recall that $j = e^{j\pi/2}$ and $1/j = -j = e^{-j\pi/2}$. So the j in $j\omega L$ implies a phase of $\pi/2$ and the $-j$ in $1/j\omega C$ implies a phase of $-\pi/2$. For a capacitor the current leads the voltage by 90° and for an inductor the voltage leads the current by 90°.

7 Components in Series and Parallel

If two components, (any combination of resistors, capacitors and inductors) are connected in series then the same current passes through both of them and the voltage across the combination is the sum of the separate voltages, see Fig. 2. It follows that $Z_{\text{Series}} = Z_1 + Z_2$. Likewise, if they are connected in parallel (Fig. 2), the total current is the sum of the individual currents and the voltage across the combination is the same as the voltage on either. It follows that $1/Z_{\text{Parallel}} = 1/Z_1 + 1/Z_2$. Use these relations to obtain expressions for how R's, C's and L's combine in series and parallel.

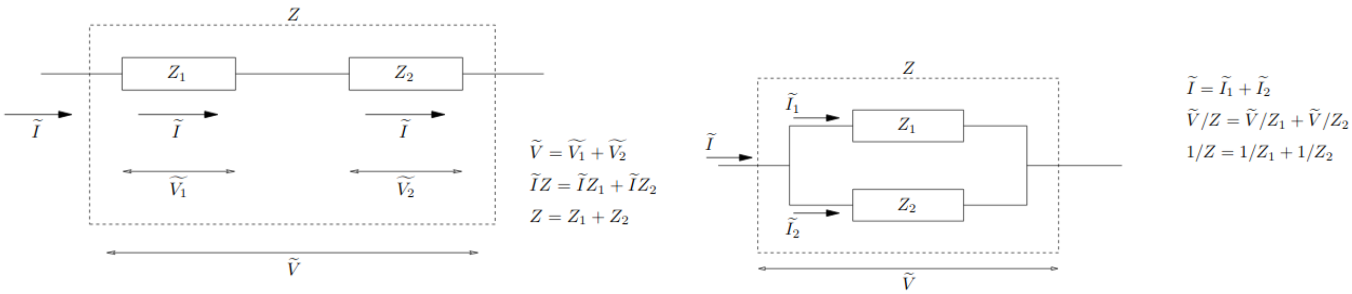

Figure 2. Impedances in series and parallel

8 Impedance of a Combination

The introduction of Complex Impedances makes it very easy to calculate the impedance of any combination of components. Without the complex formalism the algebra is long and tedious. To illustrate how it works, here are some simple examples.

8.1 Complex impedance of R and C in series

The complex impedance of a resistor and capacitor in series is just $Z = Z_R + Z_C = R + 1/j\omega C = R + j(-1/\omega C)$. Thus the modulus of Z is: $|Z| = (R^2 + 1/\omega^2 C^2)^{1/2}$. As $|V| = |Z||I|$ (see section 5) this is just the amplitude of the voltage divided by the amplitude of the current. The complex impedance also tells us the phase difference, $\Delta\phi$, (i.e the phase of the current relative to the phase of the voltage). $\tan(\Delta\phi)$ is given by (see section 5) the imaginary part over the real part, i.e. $\tan(\Delta\phi) = (-1/\omega C)/R = -1/\omega RC$. Make a note of the units (dimensions) of the product RC as this combination often occurs and it's well worth remembering. Likewise, what are the units of the combination ωRC? You can obtain these units by inspecting the above formulae.

9 Complex impedance of R and C in series

EXPERIMENT 2A. `Replace the capacitor of experiment 1A with a capacitor and resistor in series. Simulate the circuit for at least two different values of` C, R `and` f `and measure the voltages across the capacitor and resistor directly. Check they are what you expect. You will be given a number of amplitude and phase measurements with a setup as for section 4, but with a resistor and capacitor in series instead of just a single component. You will get measurements for several values of capacitor and/or resistor. You should understand these by calculating the expected values of amplitude and phase, and comparing with the measured values (taking account of errors). You will also get a set of measurements for several different frequencies for a given` $R$ `and` $C$. `In this case, plot` $\tan(\Delta\phi)$ `versus the inverse frequency,` $1/\omega$. `Explain why this should be a straight line and interpret the slope, including errors.`

EXPERIMENT 2B. (EXTRA) `Simulate a number of amplitude and phase measurements with a setup as for section 4, but with a resistor and inductor in series instead of just a single component. You should understand these by calculating the expected values of amplitude and phase, and comparing with the measured values. In this case, plot` `a suitable graph with a function of phase on the` $y$ `axis and a function of` $R$ `on the x axis in order to get a straight line and interpret the slope.`

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10 Power in AC Circuits - theory

10.1 Power dissipated by a circuit

The instantaneous power, $P(t)$, dissipated in a component (or a group of components) is $P(t) = V(t)I(t)$, where $V(t)$ is the voltage across the component and $I(t)$ is the current flowing through the component. For DC and a resistance this is just $P = VI$ where V and I are the steady DC voltage and current. For AC (i.e. sinusoidal waveforms) care is needed, as in general the current is not in phase with the voltage. This is the case if there are reactive components (i.e. capacitors or inductors).

10.1.1 Power for a resistor - voltage and current in phase

First consider a pure resistance. The voltage and current are in phase and so $V(t) = V_0 \cos \omega t$ and $I(t) = I_0 \cos \omega t$. The power averaged over a complete cycle is $\langle P \rangle = \langle V_0 I_0 \cos^2 \omega t \rangle = \frac{1}{2} V_0 I_0$. (It is worth remembering that the average value over a complete cycle of both $\sin^2 \omega t$ and of $\cos^2 \omega t$ is 1/2. You can work this out by integrating over a whole cycle). But it's easier to get the result from the fact that $\sin^2 \omega t + \cos^2 \omega t = 1$. Since sin and cos differ only by a phase of $\pi/2$, the averages of $\sin^2$ and $\cos^2$ must be the same, and so each average is 1/2.)

AC voltages and currents (e.g. the mains) are usually quoted as root meansquare (rms) values: $V_{\rm rms} = V_0/\sqrt{2}$ and $I_{\rm rms} = I_0/\sqrt{2}$. Notice that a mains rms voltage of 240 V actually has an amplitude of $\sqrt{2}$ times this, i.e., about 340 V! When using rms values the average power in a resistance is given by

$$\langle P \rangle = \frac{1}{2} V_0 I_0 = V_{\rm rms} I_{\rm rms}.$$

This equation applies, to a good approximation, to an electric kettle, but not, for example, to an electric motor, which contains coils which have significant inductance.

10.1.2 Power when there is a phase difference between voltage and current

When there is a phase difference between the voltage and the current, the average power depends on the phase difference, $\Delta\phi$, as well as on the rms voltage and current. Let $V(t) = V_0 \cos \omega t$ and $I(t) = I_0 \cos(\omega t + \Delta\phi)$, then the instantaneous power is $P(t) = V(t)I(t)$ and the average power is

$$\begin{aligned} \langle P \rangle &= \langle V_0 I_0 \cos(\omega t + \Delta\phi) \cos \omega t \rangle \\ &= \frac{1}{2} V_0 I_0 \cos \Delta\phi \\ &= V_{\rm rms} I_{\rm rms} \cos \Delta\phi. \end{aligned}$$

(It is left as an exercise for the student to show that this equation is correct.) The term $\cos \Delta\phi$ in the expression for average power is called the power factor. Note that in the case of a resistor,

$\Delta\phi = 0$, $\cos\Delta\phi = 1$ and so $\langle P \rangle = V_{\rm rms} I_{\rm rms}$ as above. However, for a pure capacitor or inductor, $\Delta\phi = \pm\pi/2$, $\cos\Delta\phi = 0$ and so the average power is zero. This is a very important property. Of course, capacitors and inductors do store energy ($\frac{1}{2}CV^2$ and $\frac{1}{2}LI^2$) but with AC in one part of the cycle energy flows from the power supply to the component and in another part of the cycle energy flows from the component back into the power supply. In a resistor electrical energy is converted to heat (this is sometimes called Joule heating), but this does not happen with capacitors and inductors. In capacitors and inductors there is no mechanism to convert electrical energy to heat so the average power is zero. (Note, when considering power, the voltages and currents must be represented by real quantities and complex numbers should not be used. It is incorrect to use complex voltages and currents because the power depends on the product of voltage and current. The imaginary parts have no physical meaning. This does not matter when we are doing circuit analysis because we only add or subtract currents and voltages. The real and imaginary parts do not interfere with one another, and the complex representation is a very neat way of keeping track of phases. But when working out power the voltage and current are multiplied. The product of two imaginary parts is real, and since the imaginary parts do not correspond to physical voltages or currents, this term is spurious.)

11 Power in a resonant LCR circuit

EXPERIMENT 3. `Simulate a set of measurements for an LCR circuit with` L`=15mH,` C`=1`μ`F, R=20`Ω`. Measure the amplitudes of the current through the circuit and the generator voltage, and the phase difference between the generator voltage and the current in the circuit. Plot suitable graphs (see theory below) for the amplitude and phase, verify the resonance curve has the expected width, and find the Q factor.`

The impedance in the circuit is $Z = R + i\omega L + 1/i\omega C$, and so the power dissipated in the resistor is I^2R, where the current $I = V_{\rm gen}/|Z|$. The phase difference between the generator voltage and the current in the circuit is the same as the phase angle of Z.

The full width at half maximum (FWHM) of the power curve is the distance between the frequencies where it falls to half its maximum value above and below resonance and is $\Delta f = R/2\pi L$. The sharpness of the resonance is measured by the Q-factor:

$$Q = \frac{f_0}{\Delta f} = \frac{2\pi f_0 L}{R} = \frac{\omega_0 L}{R},$$

where ω_0 is the angular frequency at resonance. Measure the FWHM of your graph and compare the Q-factor from this measurement with the theoretical value $\omega_0 L/R$. (In a real experiment, how would you take account of the internal resistance $R_{\rm int}$ of the inductance?)

12 Integrating and Differentiating Circuits

Next you will investigate how some simple circuits modify the shape of input waveforms and signals. This is of great practical importance, since any circuit necessarily has capacitance and inductance. Circuits always contain loops, and a loop has self-inductance. Similarly there is a capacitance between any pair of wires. At mains frequency, these unavoidable stray impedances are completely negligible,but at very high frequency their effects cannot be ignored.

Digital signals operate at very high frequency - the clock rate of a PC is a few GHz. Fourier analysis of digital pulses shows that the frequencies present in a pulse are many times the repetition rate. It is important to know how digital signals are affected by intended or unwanted circuit impedances, since the signals may bedegraded by their presence.

We are going to deal with circuits that integrate or differentiate a waveform, the response of circuits to a step function input and LCR circuits that may ring or oscillate and may be lightly, critically or heavily damped.

Make sure that you are fully clear that a current **passes through** a wire or component and a voltage is **across** a component.

Note: These integrating and differentiating circuits apply to any waveform. (The Complex Impedance that we studied last week (Day 1) applies only to sinusoidal waveforms.)

It turns out, perhaps surprisingly, that a circuit as simple as a resistor and capacitor can be used to integrate or differentiate waveforms. The first experiments demonstrate this.

12.1 Integrating circuits

In the RC circuit shown in Fig. 3 the same current $I(t)$ passes through both components. This is a very good approximation because the oscilloscope has a high input impedance and so draws negligible current. The voltage drop across R is IR, and across C it is Q/C, where $Q(t)$ is the charge stored on the capacitor at time t. Hence $V_{\text{in}}(t) - V_{\text{out}}(t) = IR$ and $C(V_{\text{out}}(t) - V_{\text{gnd}}) = Q$.

But current is rate of change of charge, so $I = dQ/dt$. Eliminating I and Q gives

$$\frac{V_{\text{in}}(t) - V_{\text{out}}(t)}{R} = C\frac{dV_{\text{out}}(t)}{dt} \qquad \text{or} \qquad \frac{dV_{\text{out}}(t)}{dt} - \frac{V_{\text{in}}(t)}{RC} + \frac{V_{\text{out}}(t)}{RC} = 0. \tag{1}$$

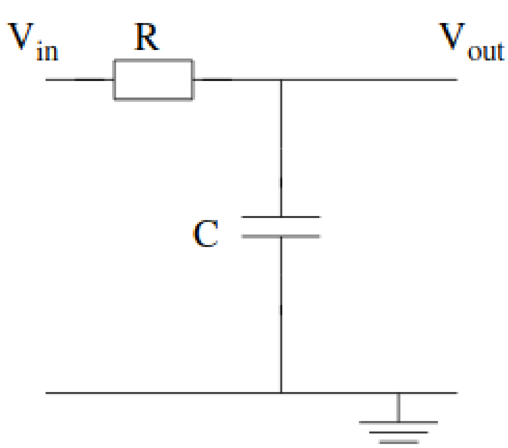

Figure 3. RC integrating circuit

Notice that the quantity RC, (Ohms Farads) has the same dimensions as time and is measured in seconds. RC is often called a **time constant**.

If $V_{in}(t)$ is known one can solve for $V_{\rm out}(t)$. Unless the form of $V_{\rm in}(t)$ is simple, numerical methods may be needed to solve the equation. (One soluble case is for sinusoidal signals; we studied that case on Day 1.) Here, we consider the case when the output voltage is much smaller than the input. The third term in the above equation is then negligible and with this approximation the equation is:

$$\frac{dV_{\rm out}(t)}{dt} \simeq \frac{V_{\rm in}(t)}{RC}.$$

This is easily solved (integrated) giving:

$$V_{\rm out}(t) - V_{\rm out}(0) = \frac{1}{RC}\int_{t'=0}^{t'=t} V_{\rm in}(t')dt'.$$

(Notice that in order for the output voltage to remain small the time average ofthe input voltage must be zero.) We can choose $t = 0$ to be at any convenient time. Thus, provided that the output voltage is small compared to the input, the output voltage is just the integral of the input. This condition is effectively that the time constant RC be long compared with the period of the input voltage.The circuit is generally known as an integrating circuit.

12.2 Experiment: The integrating circuit

EXPERIMENT 4. `You will get a measurement of amplitude and phase for the output of the integrating circuit with` $R = 5\mathrm{k}\Omega$, $C = 0.1\mu\mathrm{F}$ `at 20` kHz `(these values ensure that` $V_{\rm out} \ll V_{\rm in}$`) for a sine wave. Determine the functional form of the output waveform. Check the shapes, time scales and sizes of the signal and` **compare measurements with calculations including errors where necessary**. `To do this, write an explicit algebraic function for the input waveform and integrate it.`

12.3 Differentiating circuits

Consider a CR circuit, i.e. Fig. 3 with the R and C interchanged.The expressions for the voltages across the two parts of the circuit are now $V_{\rm in}(t) - V_{\rm out}(t) = Q/C$ and $V_{\rm out}(t) - V_{\rm gnd} = IR$. Differentiating Q w.r.t. t and substituting for $I = dQ/dt$ gives

$$\frac{dV_{\rm in}(t)}{dt} - \frac{dV_{\rm out}(t)}{dt} = \frac{V_{\rm out}(t)}{RC}. \tag{2}$$

Now assuming $dV_{\rm out}(t)/dt$ is small compared to the other terms we have

$$V_{\rm out}(t) \simeq RC\frac{dV_{\rm in}(t)}{dt}.$$

Hence this circuit is called a **differentiating** circuit.

The approximation requires that the output voltage is small compared to the input or equivalently that the period of the input waveform (and its harmonics) is long compared to the time constant RC.

EXPERIMENT 5A. (EXTRA)`Simulate a differentiating circuit for your choice of frequency and component values, with an input sine wave. What should be the functional form of the output waveform. Check the amplitude and phase of the output signal and numerically compare your expectations with the simulation.`

EXPERIMENT 5B (EXTRA). `Consider how` RL `and` LR `circuits behave. These are similar to` RC `and` CR `circuits, but with the capacitor replaced by an inductor. Do they differenti or integrate?`

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13 Transients

13.1 Steps in voltage and current

A step function (Fig. 4) consists of a sudden change from one value to another. For example a voltage which is zero before $t = 0$ and has a constant value V_s after $t = 0$ is a model step function. Consider the three defining equations for C, R, and L. They tell us what occurs if we try to make instantaneous changes in V or I.

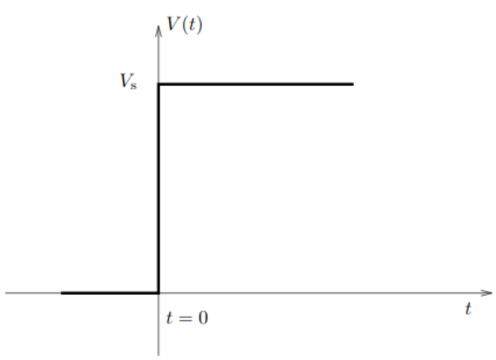

Figure 4. A step in voltage.

Capacitor $I = CdV/dt$ Since I must be finite then dV/dt is finite; so the voltage across a capacitance cannot change instantaneously. This is consistent with the property that the energy stored in a capacitor is $E = \frac{1}{2}CV^2$.

Resistor $V = IR$ This just says that V and I are linearly related - neither can be infinite so there is nothing specific here about instantaneous changes. When a voltage is switched on, the current flows immediately. No energy is stored in a resistor.

Inductor $V = LdI/dt$ Suppose we put a voltage (finite) across L, then this equation tells us that dI/dt is finite and so the current through an inductance cannot change instantaneously. This is consistent with the property that the energy stored in an inductor is $E = \frac{1}{2}LI^2$.

13.2 Transient response of differentiating circuit to a step

How does the differentiating circuit used in expt 2.4 respond to a step input? Equation (2) applies to this circuit at all times:

$$\frac{dV_{\text{in}}(t)}{dt} - \frac{dV_{\text{out}}(t)}{dt} = \frac{V_{\text{out}}(t)}{RC}.$$

We shall solve this equation with the conditions (a)$V_{\text{in}}(t) = 0$ for $t \leq 0$ and (b) $V_{\text{in}}(t) = +V_s$ for $t \geq 0$ .For $t < 0$,V_{out} has reached a steady value. No current flows through R and $V_{\text{out}} = 0$. At $t = 0$ the input voltage is suddenly changed from 0 to $+V_s$, i.e. it jumps by V_s. Since the voltage across a capacitor cannot change instantaneously, V_{out} also changes by V_s, from 0 to V_s. From $t = 0$ onwards V_{in} is constant, the first term in the differential equation is zero, and we are left with

$$\frac{-dV_{\text{out}}(t)}{dt} = \frac{V_{\text{out}}(t)}{RC}, \quad \text{or} \quad \frac{dV_{\text{out}}(t)}{V_{\text{out}}(t)} = \frac{-dt}{RC},$$

with the solution

$$\ln[V_{\text{out}}(t)] = -\frac{t}{RC} + \text{constant}.$$

At $t = 0$,$V_{\text{out}}(t) = V_s$, and finally

$$V_{\text{out}}(t) = V_s \exp\left(-\frac{t}{RC}\right).$$

The voltage decays exponentially with time. The time constant is RC.What we've calculated is in effect what happens when you connect a charged capacitor to a resistor: the charge decays away exponentially with time.

13.3 Response of CR circuit to a step input

EXPERIMENT 6.You will get data with the differentiating circuit but with components $R = 5\text{k}\Omega$ and $C = 0.1\mu F$. You can apply what is effectively a step input by taking one step from a square wave. Explain the height of the voltage steps on the input and output waveforms. The output voltage will be measured at four or five equally spaced times from the start of the step to a time when the voltage has fallen to about 1/10 of its initial value. Rearrange the equation for $V_{\text{out}}(t)$ so that you can plot a straight line graph. Deduce the time constant from your graph and compare with the value calculated from R and C, including consideration of errors.

13.4 LR circuit: integration and stepresponse

EXPERIMENT 7 (EXTRA). How can you make an integrating circuit with a coil and a resistor? Write down the differential equation for this circuit, consider the case `when` V_{out} `is small compared to` V_{in} `and derive its time constant. Simulate this circuit and demonstrate that it integrates. Consider the response to a step in voltage. The initial conditions follow from the application of the rule that one cannot change the current through an inductance instantaneously. Measure the time constant and compare with the calculated value.`

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14 Ringing and Damping in a Series LCR Circuit

14.1 Theory of Ringing and Damping

In PHYS10071 Mathematics 1 you met the theory of damped simple harmonic motion via the solutions of a general second order differential equation. You have studied (or will soon do so) the physics of a mechanical resonant system in the laboratory experiment on Forced Oscillations. And you have met, or you will meet, the theory of damped simple harmonic motion in PHYS10302 Vibrations and Waves.Here we study the damping in an electrical circuit, a resonant system containing an inductance, a capacitor and a resistor. First we summarise the theory. It is essential to realise that the same physical phenomena occur in apparently different physical systems and that the mathematics is the same for each. Thus a study of one particular physical system has immediate application in others. The electrical circuit (see Fig. 5) provides a very good way of observing different damping effects, since it is easy to change the circuit parameters and to display the different forms of damping. The theory starts in the same way as for the RC circuits we have already considered. The actual solution depends on how a step is applied. We shall consider the case when a voltage V_s is applied at $t = 0$. For $t < 0$, we assume that no current is flowing anywhere in the circuit and that the voltage is everywhere zero. At all times, the same current, $I(t)$, flows through the three components. The sum of the voltages across the three components is (for $t > 0$) just the input voltage,V_s. Thus,in obvious notation:

$$V_L = LdI/dt, V_C = Q/C, V_R = IR, V_L + V_R + V_C = V_s.$$

If we are to observe the voltage across R, we need to work out the current. Thisis done by deriving a differential equation for the current:

$$dV_L/dt = Ld^2I/dt^2, I = CdV_C/dt, dV_Rdt = RdI/dt, dV_L/dt + dV_R/dt + dV_C/dt = 0.$$

Eliminating all V's we have

$$Ld^2I/dt^2 + RdI/dt + I/C = 0.$$

This differential equation determines how the current $I(t)$ varies with time. This differential equation appears in many places in physics: the parameters are different but the behaviour, i.e. variation with time, is the same. In particular, you've met this same equation in the Mathematics for Physicists course PHYS10071 in the form: $\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = 0$. You recognise this as the equation for damped harmonic motion. In our case, the solution has to satisfy the appropriate boundary conditionswhich are the initial conditions at $t = 0$. These are firstly that $I(t = 0) = 0$. This is because when the voltage step is applied at the input the current through the inductor cannot change instantaneously and as it was zero just before $t = 0$ it must also be zero just after.

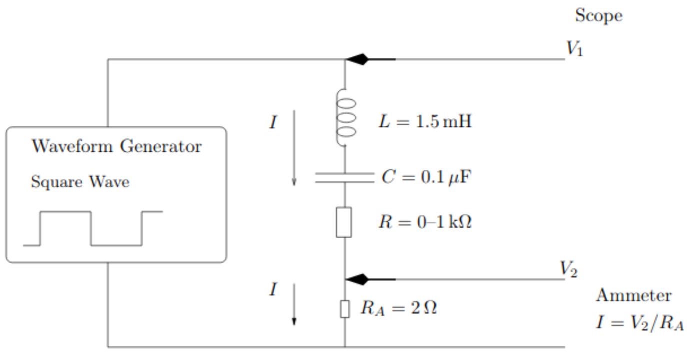

Figure 5. LCR circuit.

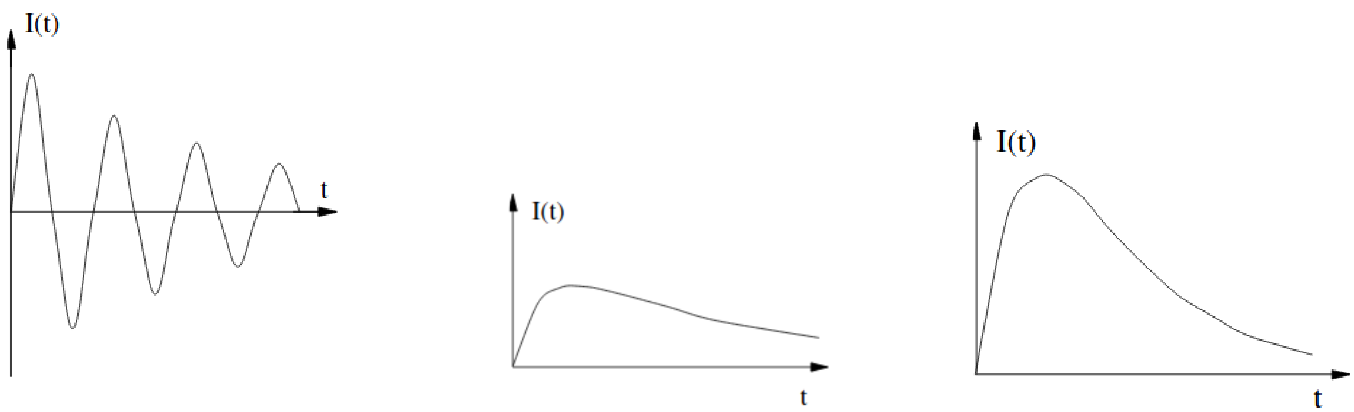

Figure 6. Light (left), heavy (middle) and critical damping (right - note the tail falls rapidly)

Secondly, as the current is zero at $t = 0$, there is no voltage drop across the resistor and no charge has built up on the capacitor so that there is no voltage across the capacitor, therefore the whole of the applied voltage appearsacross the inductor. This requires $LdI/dt(t = 0) = V_s$.The nature of the solution depends on the amount of damping. There are three situations to consider:light damping, heavy damping and critical damping. The solutions given below, for the three cases, should be checked by direct substitution into the differential equation.

14.1.1 Light damping

This occurs when $\gamma < \omega_0$ which in our case is $R/2L < \sqrt{1/LC}$. This means that the resistance is smaller than some critical value. The solution is $x(t) = e^{-\lambda t}(A \cos \omega t + B \sin \omega t)$ which becomes, for our case, $I(t) = (V_s/\omega L)e^{-\gamma t} \sin \omega t$. ω is given by:$\omega^2 = \omega_0^2 - \gamma^2 = (1/LC) - (R^2/4L^2)$. The system is said to be lightly damped.

The system oscillates (Fig. 6) at an angular frequency, ω, close to the resonant angular frequency, ω_0, with an amplitude which decays exponentially with the time constant $R/2L$. This an example of lightly damped harmonic motion analogous to the damped SHM of a pendulum or an oscillating spring, which both have diminishing amplitudes as their energy ebbs away.

14.1.2 Heavy damping

This occurs when $\gamma > \omega_0$ which in our case is $R/2L > \sqrt{1/LC}$. This means that the resistance is larger than some critical value. The solution is $x(t) = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t}$ which becomes, for our case, $I(t) = (V_0/R)(e^{-\lambda_1 t} - e^{-\lambda_2 t})$.The λs are given by:$\lambda = (-R/2L) \pm \frac{R}{2L}\sqrt{1 - 4L/R^2C}$. In the limit of large R we can approximate: $\lambda \simeq -R/2L \pm R/2L(1 - \frac{1}{2}\frac{4L}{R^2C})$ giving $\lambda_1 \simeq -\frac{R}{2L} + \frac{R}{2L} - \frac{R}{2L}(1/2)(4L/R^2C) = 1/RC$ and $\lambda_2 \simeq -\frac{R}{2L} - \frac{R}{2L} + \frac{R}{2L}(1/2)(4L/R^2C) = -R/L$. The solution becomes

$$I(t) \simeq V_s R[e^{-t/RC} - e^{-Rt/L}].$$

The second term causes an initial rise but quickly becomes small so the dominant term over longer times is parametrised by the RC time constant. The largest current is much smaller than in the lightly damped (oscillating) case.The system is said to be heavily damped (Fig. 6).

14.1.3 Critical damping

This occurs when $\gamma = \omega_0$ which in our case is $R/2L = \sqrt{1/LC}$. This means that the resistance is exactly equal to a critical value. The roots of the auxiliary equation are equal and in this case the solution takes a different form and is $x(t) = \exp(-\gamma t(At + B))$ which becomes, for our case, $I(t) = (V_s/L)te^{-\gamma t}$.This is the condition for the current to die away most rapidly (Fig. 6).The system is said to be critically damped.

14.2 Experiment: Damping (simulation only)

EXPERIMENT 8A. `Simulate a series` LCR `circuit with a capacitor of` $0.1\mu\mathrm{F}$, `an inductor of` $1.5\mathrm{mH}$ `and a resistance set to` 20Ω `(you do not need the 2-ohm resistor as you can measure the current directly). Apply a step function (e.g. with a switch as in the default falstad.com circuit). Observe the oscillations in the voltage across` R. `This phenomenon is known as ringing. Measure the frequency of the oscillations and compare with calculation. Measure the attenuation of the amplitude over a reasonable number of cycles,and compare with expectations. Find a way of plotting your results in a straight line.`

EXPERIMENT 8B (EXTRA). `Observe the variation of the shape and size of the transient voltage as` R `is increased. Increase` R `so that the circuit becomes heavily damped. Measure the time for the transient to reach its maximum and the time for the amplitude of the transient to drop from its maximum by a factor 1/e. Compare your results with theory. Assume that the maximum occurs while` $e^{-t/RC}$ `is almost 1 and show that` $t_{\max} \sim (L/R)\ln(CR^2/L)$

EXPERIMENT 8C (EXTRA). `Find the value of` R `for which oscillations just disappear (critical damping). Note that for critical damping the trace returns to its initial value more rapidly than for other values of` R. `Compare your value of the resistance for critical damping with that predicted by calculation.`
