Problem 1

a. $f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 2x \qquad f''(x) = 2$$

for $\forall x \in R$, $f''(x) \geq 0$, $f'(x)$ is monotonic increasing fuction in $x \in R$.

$\therefore$ $f(x)$ is convex function

b. $f(x) = \ln x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

for $\forall x \in R$, $f''(x) < 0$, $f(x)$ is concave function. in $x \in R$

c. $f(x) = \dfrac{1}{1+e^{-x}}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$f''(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \frac{e^{-x}(e^{-x} - 1)}{(1+e^{-x})^3}$$

$\because$ for $x < 0$, $f''(x) > 0$

$x > 0$, $f''(x) < 0$

$\therefore$ $f(x)$ is convex when $x < 0$, $f(x)$ is concane when $x > 0$.

Problem 2

Since X is drawn from uniform distribution between 0 and θ

$$f(x) = \frac{1}{\theta} \qquad 0 \leq x \leq \theta$$

a. $E_x[X]$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_0^{\theta} x \cdot \frac{1}{\theta}\, dx$$

$$= \frac{x^2}{2\theta} \Big|_0^{\theta}$$

$$= \frac{\theta}{2}$$

b. $Var(X)$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \int_0^{\theta} x^2 \frac{1}{\theta}\, dx - \left(\frac{\theta}{2}\right)^2$$

$$= \frac{\theta^2}{12}$$

c. $H(X)$

$$H(X) = -\int_{-\infty}^{+\infty} f(x) \cdot \log f(x)\, dx$$

$$= -\int_0^{\theta} \frac{1}{\theta} \log\left(\frac{1}{\theta}\right) \cdot dx$$

$$= -\frac{\log(\theta)}{\theta} x \Big|_0^{\theta} = \frac{\log(\theta)}{\theta} \cdot \theta = \log(\theta)$$

Problem 3

We know $X \sim U(0,\theta)$, M is drawn from X, $x_1, x_2 \cdots x_M$ is from M.

Let $x_{(1)} = \min\{x_1, x_2, x_3 \cdots x_m\}$; $x_{(m)} = \max\{x_1, x_2 \cdots x_m\}$

$$L(\theta|X) = \prod_{i=1}^{M} \frac{1}{\theta} = \frac{1}{\theta^M}$$

Find a $\hat{\theta}$ makes $L(\theta|x)$ maximum.

$$\therefore \hat{\theta} = \underset{\theta}{\arg\max} \frac{1}{\theta^M}$$

$$= \underset{\theta}{\arg\min}\ \theta^M$$

$$L(\theta) = \begin{cases} \frac{1}{\theta^M}, & 0 \le x_1, x_2, \cdots x_m \le \theta \\ 0, & \text{else} \end{cases}$$

$\because 0 \le x_1, x_2 \cdots x_m \le \theta$ equals to $0 \le x_{(1)}, x_{(m)} \le \theta$

$$\therefore L(\theta) = \begin{cases} \frac{1}{\theta^M}, & 0 \le x_{(1)}, x_{(m)} \le \theta \\ 0, & \text{else} \end{cases}$$

So $L(\theta) = \frac{1}{\theta^M} \le \frac{1}{(x_{(m)} - x_{(1)})^M}$

Only when $\theta = x_{(m)}$, $L(\theta)$ has max value $\frac{1}{x_{(m)}^n}$

$\therefore \hat{\theta} = \text{Max}\{x_1, x_2, x_3 \cdots x_m\}$

To make sure $0 \le x_1, x_2 \cdots x_m \le \theta$

$\therefore \hat{\theta} \ne 0$

Problem 4

Let R be the event "1M cash behind the door I initially choose", and W be the event "I win the 1M cash by switching doors."

$$P(W|R) = 0$$

$$P(W|R^c) = 1$$

based on Bayes' Rule

$$P(W) = P(W|R)\cdot P(R) + P(W|R^c)\cdot P(R^c)$$

$$= 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}$$

$\because$ The probability of winning by switching doors is $\frac{2}{3}$ and the probability of winning without switching doors is $\frac{1}{3}$

$\therefore$ I should switch doors to maximize the winning probability.

Problem 5

$$\Sigma = E[(X - E[X])(X - E[X]^T)]$$

To illustrate Σ is positive semi-definite, we need to prove

$$u \Sigma u^T \geq 0$$

Use defination of Σ

$$u E[(X - E[X])(X - E[X]^T)] u^T$$

$$= E[(u(X - E(X))][(u(X - E(X))^T]$$

$$= E[u(X - E(X))^2]$$

An expectation of a positive number is positive.

$\therefore \Sigma$ is positive semidefinite