Matrix Factorisations and Decompositions

Harry Xi

February 2, 2022

LU Decomposition

$LU decomposition$ is a type of decomposition usually applied to square matrices A, though can be applied aswell to rectangular matrices. It comes in the form,

$$A = LU$$

Where L is a lower triangular matrix, and U is an upper triangular matrix.

Related Factorisations

LDU factorisation - where L is lower triangular, U is upper triangular, and D is diagonal. This factorisation exists whenever the LU factorisation exists.
LUP factorisation - where L is lower triangular, U is upper triangular, and P is the permutation matrix. This factorisation exists for any square matrix A.

Applications

- Solving $A\mathrm{x} = \mathrm{b}$
- Matrix Inversion
- Computing Determinants

Rank Factorisation

A rank factorisation is a type of non-unique factorisation for $m \times n$ matrices of rank r given in the form,

$$A = CF$$

Where C is an $m \times r$ full column rank matrix and F is $r \times n$ full row rank matrix.

QR Decomposition

QR decomposition is a decomposition applicable to all $m \times n$ rectangular matrices A, with $m \geq n$, so where $m = n$, i.e. square matrices, such a decomposition exists. The matrix must also have linearly independent columns. It comes in the form,

$$A = QR$$

Where Q is an $m \times m$ unitary ($A^\star A = AA^\star = I$) matrix, or also orthonormal when we are under the real field. Here R is an $m \times n$ upper triangular matrix.

Applications

- Solving Linear Least Squares
- QR Algorithm for Eigenvalues

Eigendecomposition (Spectral Decomposition)

Eigendecomposition is a very nice type of decomposition, turning a square matrix A with linearly independent eigenvectors into the form,

$$A = X\Lambda X^{-1}$$

In which X is a matrix of eigenvectoes and Λ the matrix of its corresponding eigenvalues. Only diagonalisable matrices can be factored this way, and every unitary matrix can be factored this way.

For the case of a real symmetric matrix, we can decompose it into,

$$A = Q\Lambda Q^T$$

Singular Value Decomposition

The Singular Value Decomposition is a generalised version of the eigendecomposition for any matrix A, whether it be rectangular or square. It comes in the form,

$$A = U\Sigma V^\star$$

Where U and V are matrices of the left and right singular vectors corresponding to their singular values in Σ.

Finding U, V and Singular Values

The connection between the SVD and eigendecomposition come into use for finding our values for the SVD. Given a matrix A,

$$A^\star A = V\left(\Sigma^\star\Sigma\right)V^\star \qquad AA^\star = U\left(\Sigma\Sigma^\star\right)U^\star$$

Where for each we have found the eigendecompositions of each left-hand matrix multiplication. Here we build our SVD:

- The columns of V (right-singular vectors) are eigenvectors of $M^\star M$
- The columns of U (left-singular vectors) are eigenvectors of $MM^\star$
- The non-zero elements of Σ (non-zero singular values) are the square roots of the non-zero eigenvalues of $M^\star M$ or $MM^\star$

Polar decomposition

Applicable to any square, real or complex matrix A, comes in the forms,

$$A = UP \qquad A = P'U$$

Where the first defines the right polar decomposition and the second defines the left polar decomposition. Here, P and P' are positive semi-definite Hermitian matrices, and U is a unitary matrix.

Relation to the SVD

Given the SVD $A = U\Sigma V^\star$ for the right polar decompositions,

$$P = V\Sigma V^\star \qquad U = WV^\star$$

For the left polar decompositions,

$$P' = U\Sigma U^\star$$