Fourier Series

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Defining the Fourier Series on $[-\pi, \pi]$

(Definition 1.1) Given a continuous function $f(x)$ defined on $[-\pi, \pi]$

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty}(A_k \cos(kx) + B_k \sin(kx))$$

where

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \qquad B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

To give some motivation for these equations, we recall the standard inner product of functions.

$$\langle f(x)\,,\, g(x)\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

Though for now, we deal with real-valued functions, so we may ignore the conjugate operator.

$$\langle f(x)\,,\, g(x)\rangle = \int_a^b f(x)g(x)\,dx$$

We see that our coefficients A_k and B_k can be reinterpreted.

$$A_k = \frac{1}{||\cos(kx)||^2}\langle f(x)\,,\,\cos(kx)\rangle \qquad B_k = \frac{1}{||\sin(kx)||^2}\langle f(x)\,,\,\sin(kx)\rangle$$

The component of $f(x)$ in the direction of $\cos(kx)$ and $\sin(kx)$ respectively. We now attempt to solidify, and justify our formula by examining an inner product for m and n integers.

$$\langle \sin(mx)\,,\,\cos(nx)\rangle = \int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx = 0$$

So we can see that $\sin(mx)$ is orthogonal to $\cos(nx)$, no matter the value for n or m. So, as we change m and n, we begin to form an infinite dimensional basis for what we can call our function space. Giving us the intuition to describe all functions as an infinite sum of sines and cosines of increasing frequency.

(Definition 1.2) We approximate some $f(x)$ on $[-\pi, \pi]$ with

$$f(x) \approx \frac{A_0}{2} + \sum_{k=1}^{N}(A_k \cos(kx) + B_k \sin(kx)), \qquad N \in \mathbb{N}$$

Defining the Fourier Series on $[0, L]$

If we are to examine our function given by the Fourier series outside our given domain, we actually see the Fourier series gives a representation for periodic functions, a property we should expect from the inclusion of periodic functions $\sin$ and $\cos$. We now extend our definitions to a period of $[0, L]$.

(Definition 2.1) For any continuous function $f(x)$ on $[0, L]$

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty}\left(A_k \cos\left(\frac{2\pi kx}{L}\right) + B_k \sin\left(\frac{2\pi kx}{L}\right)\right)$$

$$A_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi kx}{L}\right)dx \qquad B_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi kx}{L}\right)dx$$

Complex Fourier Series

Since now, we have taken $f(x)$ as a real-valued function. But what if we wanted to take $f(x)$ as a complex-valued function? We apply a slightly different, albeit simpler Fourier series. We restrict it for functions $[-\pi, \pi]$ to simplify notation.

(Definition 3.1) For any complex function $f(x)$ on $[-\pi, \pi]$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} (\alpha_k + i\beta_k)(\cos(kx) + i\sin(kx))$$

Noting that $c_k = \overline{c_{-k}}$ if $f(x)$ is real. Now we want to show that these functions of the form e^{ikx} are orthogonal to each other, and will produce a basis for our function space. First define,

$$e^{ikx} = \psi_k$$

Now computing:

$$\langle \psi_j\,,\,\psi_k\rangle = \int_{-\pi}^{\pi} e^{ijx}e^{-ikx}dx = \int_{-\pi}^{\pi} e^{i(j-k)x}dx = \frac{1}{i(j-k)}\left[e^{i(j-k)x}\right]_{-\pi}^{\pi}$$

$$= \begin{cases} 0, & \text{for } j \neq k \\ 2\pi, & \text{for } j = k \end{cases}$$

So we see that taking k from $-\infty$ to ∞ will provide us with an infinite basis for the function space in terms of $e^{ikx} = \psi_k$. Now we define the Fourier Series in a more full way, involving both complex functions and a domain of $-L$ to L.

(Definition 3.1) For any complex function $f(x)$ on $[-L, L]$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/L} \qquad c_k = \frac{1}{2L}\langle f(x)\,,\,e^{ik\pi x/L}\rangle = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}dx$$

The Fourier Transform

While the Fourier series gives a representation of $f(x)$ that's $2L$ periodic, meaning repeating everytime we move $2L$ in any direction, the Fourier transform, keeps the tails of our function the same. So recalling our definition from above, we want to move L to infinity. We first define;

$$\omega_k = \frac{k\pi}{L} = k\Delta\omega, \qquad \Delta\omega = \frac{\pi}{L}$$

So as $L \to \infty, \Delta\omega \to 0$ which we can use to determine $f(x)$ as $L \to \infty$.

$$f(x) = \lim_{\Delta\omega \to 0} \sum_{-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-ik\Delta\omega\xi} d\xi e^{ik\Delta\omega x}$$

$$= \lim_{\Delta\omega \to 0} \sum_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-ik\Delta\omega\xi} d\xi e^{ik\Delta\omega x} \Delta\omega$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi e^{i\omega x} d\omega$$

(Definition 4.1) The Fourier Transform Pair

$$\hat{f}(\omega) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$f(x) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

In Fourier space, it is easier to compute derivatives so they are useful in solving differential equations, and the operator $\mathcal{F}$ is unitary, as well as linear. Also note that this Fourier transform pair only makes sense if our function decays to 0 as $L \to \infty$.

The Fourier Transform and Derivatives

$$\mathcal{F}\left(\frac{d}{dx} f(x)\right) = \int_{-\infty}^{\infty} \frac{df}{dx} e^{-i\omega x} dx$$

Now integrating by parts we get

$$\left[f(x) e^{-i\omega x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega e^{-i\omega x}) dx$$

Now, recalling that the Fourier transform pair only makes sense if our function decays to 0 as $L \to \infty$, as well as that the norm of $e^{-i\omega x}$ is 1, so the first term is evaluated at 0, giving us:

(Definition 5.1) Derivative of the Fourier Transform

$$\mathcal{F}\left(\frac{d}{dx} f(x)\right) = i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} = i\omega \mathcal{F}(f(x))$$

So, we have found how easy it is to compute derivatives in the Fourier space, rather than in the physical space!

The Fourier Transform and Convolution Integrals

(Definition 6.1) Convolution of two functions

$$(f * g) = \int_{\infty}^{\infty} f(x - \xi) g(\xi) d\xi$$

Now, the Fourier transform can actually simplify this very convoluted expression:

(Definition 6.2) Convolution of two functions under the Fourier Transform

$$\mathcal{F}(f * g) = \mathcal{F}(f) \times \mathcal{F}(g)$$

(Proof 6.3)
We will endeavour to show this, by performing the inverse Fourier transform on $\mathcal{F}(f)\mathcal{F}(g)$.

$$\mathcal{F}^{-1}\left(\mathcal{F}(f) \times \mathcal{F}(g)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) e^{i\omega x} d\omega$$

Which is achieved simply by using the formula. Now we will replace $\hat{g}$ with the expanded Fourier transform of g.

$$\mathcal{F}^{-1}\left(\mathcal{F}(f) \times \mathcal{F}(g)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \left(\int_{-\infty}^{\infty} g(y) e^{-i\omega y} dy \right) e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega) g(y) e^{i\omega(x-y)} d\omega \, dy$$

We arrive at the final step by moving $\hat{f}(\omega)$ and $e^{-i\omega x}$ inside the $\hat{g}(\omega)$ in which we represent with the Fourier transform with respect to a dummy variable y since those functions of ω and x may be treated like constants. We then switch the $d\omega$ and dy due to commutativity. Now moving $g(y)$ into the outer integral...

$$= \int_{-\infty}^{\infty} g(y) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega(x-y)} d\omega \right) \, dy$$

Now recalling the fact that:

$$f(x) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

We see that,

$$f(x - y) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega(x-y)} d\omega$$

So,

$$\mathcal{F}^{-1}\left(\mathcal{F}(f) \times \mathcal{F}(g)\right) = \int_{-\infty}^{\infty} g(y) f(x - y) dy = (f * g)$$

Parseval's Theorem

(Definition 7.1) Parseval's Theorem

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

Which gives us a relation, more or less, between the "norm" or "energy" of these functions. Which then tells us that if we approximate $f(x)$ and there are some small, negligible Fourier coefficients, we may set them equal to zero, and our function approximation will still be fine due to the proportionality.

The Discrete Fourier Transform (DFT)

Motivating the DFT, we do not always have a rule for a continuous function, but rather data points, which, may have underlying function. So instead of a function, we have a vector of data points: $[f_0\, f_1\, ..., f_{n-1}]^T$. With this vector, we would apply a matrix transformation, the DFT, to attain a new vector of Fourier coefficients:

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix} \qquad \text{where } \hat{f}_k = \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}$$

Similarly,

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_j e^{i2\pi jk/n}$$

Now noticing the exponential term in the formula for the Fourier coefficients,

$$\text{Let } \omega_n = e^{-2\pi i/n}$$

Now, we attempt a bottom up approach to find the matrix which takes us from f_k to $\hat{f}_k$. What if $k = 0$? $\hat{f}_k$ would simply be a sum of all f_j, as our exponential goes to 1. So, our first row would be a row of 1's.

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

What if $k = 1$?

$$\hat{f}_1 = \sum_{j=0}^{n-1} f_j e^{-i2\pi j/n} = f_0 \times 1 + f_1 \times e^{-i2\pi/n} + f_2 \times e^{(i4\pi/n)} + \ldots + f_{n-1} \times e^{i2\pi(n-1)/n}$$

$$= f_0 \times 1 + f_1 \times \omega_n + f_2 \times \omega_n^2 + \ldots + f_{n-1} \times \omega_n^{n-1}$$

Giving us our second row,

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ & & & & \\ & & & & \end{bmatrix}$$

Continuing these steps give us the following matrix: The DFT..

(Definition 8.1) The Discrete Fourier Transform Matrix

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix}$$

The Fast Fourier Transform (FFT)

In practice, we do not want to compute the DFT matrix and then do an expensive matrix by vector multiplication when our data gets bigger and bigger. Doing so would be an $O(n^2)$ operation, while the FFT is an $O(nlogn)$ operation.
To analyse how the algorithm works, we take the case $n = 1024$.

$$\hat{f} = F_{1024}f = \begin{bmatrix} I_{512} & -D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & 0 \\ 0 & F_{512} \end{bmatrix} \begin{bmatrix} f^{(0)}_{even} \\ f^{(0)}_{odd} \end{bmatrix}$$

$$\text{where,} \qquad f^{(0)}_{even} = \begin{bmatrix} f_0 \\ f_2 \\ \vdots \\ f_{1022} \end{bmatrix} \quad f^{(0)}_{odd} = \begin{bmatrix} f_1 \\ f_3 \\ \vdots \\ f_{1023} \end{bmatrix} \quad D_{512} = \begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{511} \end{bmatrix}$$

Even and odd here refer to the index of the vector, not the value of subscript. With this, we can actually continue to break down F_k, in this case breaking down the F_{512} into F_{256} and so on, also taking the even/odd indices of f_{even} as well as f_{odd}.

$$F_n \to F_{n/2} \ldots \to F_2$$

This process is so incredibly fast, as we are practically halving the Fourier matrix each iteration, and taking advantage of diagonal matrices, which are extremely fast to multiply.