Fourier Series

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Defining the Fourier Series on $[-\pi, \pi]$

Given some function $f(x)$ defined from -π to π, we can represent it as the infinite sum of sines and cosines:

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos(kx) + B_k \sin(kx))$$

where

$$A_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$B_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

Recalling the inner product of functions will give some much needed context to these equations.

$$\langle f(x)\,,\, g(x) \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

But for real-valued functions, the conjugate operator does nothing.

$$\langle f(x)\,,\, g(x) \rangle = \int_a^b f(x) g(x)\, dx$$

We see that our coefficients A_k and B_k can be reinterpreted.

$$A_k = \frac{1}{||\cos(kx)||^2} \langle f(x)\,,\, \cos(kx) \rangle$$

$$B_k = \frac{1}{||\sin(kx)||^2} \langle f(x)\,,\, \sin(kx) \rangle$$

The component of $f(x)$ in the direction of $\cos(kx)$ and $\sin(kx)$ respectively. Now let's check one more inner product, remembering that k is an integer.

$$\langle \sin(kx)\,,\, \cos(kx) \rangle = \int_{-\pi}^{\pi} \sin(kx) \cos(kx)\, dx = 0$$

So they are orthogonal to eachother. Which gives us some intuition as to how $f(x)$ is formed by an infinite sum of sines and cosines with increasing frequency. We can also check that other sines and cosines of different frequencies will be orthogonal to eachother, so that all of these sines and cosines form a basis of the function space.
We can also approximate $f(x)$ with

$$f(x) \approx \frac{A_0}{2} + \sum_{k=1}^{N} (A_k \cos(kx) + B_k \sin(kx)), \qquad N \in \mathbb{N}$$

Defining the Fourier Series on $[0, L]$

These infinite sums actually give periodic functions, so for each move in the x direction equal to the length of the interval we gave, the function will repeat. We now extend our definitions to a period of $[0, L]$.

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(A_k \cos\left(\frac{2\pi kx}{L}\right) + B_k \sin\left(\frac{2\pi kx}{L}\right) \right)$$

$$A_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi kx}{L}\right) dx$$

$$B_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi kx}{L}\right) dx$$

Complex Fourier Series