Fourier Series

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January 30, 2022

Defining the Fourier Series on $[-\pi, \pi]$

Given some function $f(x)$ defined from -π to π, we can represent it as the infinite sum of sines and cosines:

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty}(A_k \cos(kx) + B_k \sin(kx))$$

where

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

Recalling the inner product of functions will give some much needed context to these equations.

$$\langle f(x)\, ,\, g(x)\rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

But for real-valued functions, the conjugate operator does nothing.

$$\langle f(x)\, ,\, g(x)\rangle = \int_a^b f(x)g(x)\, dx$$

We see that our coefficients A_k and B_k can be reinterpreted.

$$A_k = \frac{1}{||\cos(kx)||^2}\langle f(x)\, ,\, \cos(kx)\rangle$$

$$B_k = \frac{1}{||\sin(kx)||^2}\langle f(x)\, ,\, \sin(kx)\rangle$$

The component of $f(x)$ in the direction of $\cos(kx)$ and $\sin(kx)$ respectively. Now let's check one more inner product, remembering that k is an integer.

$$\langle \sin(kx)\, ,\, \cos(kx)\rangle = \int_{-\pi}^{\pi} \sin(kx)\cos(kx)\, dx = 0$$

So they are orthogonal to eachother. Which gives us some intuition as to how $f(x)$ is formed by an infinite sum of sines and cosines with increasing frequency. We can also check that other sines and cosines of different frequencies will be orthogonal to eachother, so that all of these sines and cosines form a basis of the function space.
We can also approximate $f(x)$ with

$$f(x) \approx \frac{A_0}{2} + \sum_{k=1}^{N}(A_k \cos(kx) + B_k \sin(kx)), \qquad N \in \mathbb{N}$$

Defining the Fourier Series on $[0, L]$

These infinite sums actually give periodic functions, so for each move in the x direction equal to the length of the interval we gave, the function will repeat. We now extend our definitions to a period of $[0, L]$.

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(A_k \cos\left(\frac{2\pi kx}{L}\right) + B_k \sin\left(\frac{2\pi kx}{L}\right) \right)$$

$$A_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi kx}{L}\right) dx$$

$$B_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi kx}{L}\right) dx$$

Complex Fourier Series

Since now, we have taken $f(x)$ as a real-valued function. But what if we wanted to take $f(x)$ as a complex-valued function? We apply a slightly different, albeit simpler Fourier series. We restrict it for functions $[-\pi, \pi]$ to simplify notation.

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} (\alpha_k + i\beta_k)(\cos(kx) + i\sin(kx))$$

Noting that $c_k = \overline{c_{-k}}$ if $f(x)$ is real. Now we want to show that these functions of the form e^{ikx} are orthogonal to each other, and will produce a basis for our function space. First define,

$$e^{ikx} = \psi_k$$

Now computing:

$$\langle \psi_j \, , \, \psi_k \rangle = \int_{-\pi}^{\pi} e^{ijx} e^{-ikx} dx = \int_{-\pi}^{\pi} e^{i(j-k)x} dx = \frac{1}{i(j-k)} \left[e^{i(j-k)x} \right]_{-\pi}^{\pi}$$

$$= \begin{cases} 0, & \text{for } j \neq k \\ 2\pi, & \text{for } j = k \end{cases}$$

So we see that taking k from $-\infty$ to ∞ will provide us with an infinite basis for the function space in terms of $e^{ikx} = \psi_k$. Now we define the Fourier Series in a more full way, involving both complex functions and a domain of $-L$ to L.

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/L} \qquad c_k = \frac{1}{2L} \langle f(x) \, , \, e^{ik\pi x/L} \rangle = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-ik\pi x/L} dx$$

The Fourier Transform

While the Fourier series gives a representation of $f(x)$ that's $2L$ periodic, meaning repeating everytime we move $2L$ in any direction, the Fourier transform, keeps the tails of our function the same. So recalling our definition from above, we want to move L to infinity. We first define;

$$\omega_k = \frac{k\pi}{L} = k\Delta\omega, \qquad \Delta\omega = \frac{\pi}{L}$$

So as $L \to \infty, \Delta\omega \to 0$ which we can use to determine $f(x)$ as $L \to \infty$.

$$f(x) = \lim_{\Delta\omega \to 0} \sum_{-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-ik\Delta\omega\xi} d\xi e^{ik\Delta\omega x}$$

$$= \lim_{\Delta\omega \to 0} \sum_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-ik\Delta\omega\xi} d\xi e^{ik\Delta\omega x} \Delta\omega$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi e^{i\omega x} d\omega$$

From here, our actual Fourier transform pair is...

$$\hat{f}(\omega) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$f(x) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

In Fourier space, it is easier to compute derivatives so they are useful in solving differential equations, and the operator $\mathcal{F}$ is unitary, as well as linear. Also note that this Fourier transform pair only makes sense if our function decays to 0 as $L \to \infty$.

The Fourier Transform and Derivatives

$$\mathcal{F}\left(\frac{d}{dx} f(x)\right) = \int_{-\infty}^{\infty} \frac{df}{dx} e^{-i\omega x} dx$$

Now integrating by parts we get

$$\left[f(x) e^{-i\omega x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega e^{-i\omega x}) dx$$

Now, recalling that the Fourier transform pair only makes sense if our function decays to 0 as $L \to \infty$, as well as that the norm of $e^{-i\omega x}$ is 1, so the first term is evaluated at 0, giving us:

$$i\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} = i\omega \mathcal{F}(f(x))$$

So, we have found how easy it is to compute derivatives in the Fourier space, rather than in the physical space!

From PDEs to ODEs with the Fourier Transform

The Fourier Transform and Convolution Integrals

Remember the convolution of two functions is given by

$$(f * g) = \int_{\infty}^{\infty} f(x-\xi)g(\xi)d\xi$$

Now, the Fourier transform can actually simplify this very convoluted expression:

$$\mathcal{F}(f * g) = \mathcal{F}(f) \times \mathcal{F}(g)$$

We will endeavour to show this, by performing the inverse Fourier transform on $\mathcal{F}(f)\mathcal{F}(g)$.

$$\mathcal{F}^{-1}\left(\mathcal{F}(f) \times \mathcal{F}(g)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{i\omega x}d\omega$$

Which is achieved simply by using the formula. Now we will replace $\hat{g}$ with the expanded Fourier transform of g.

$$\mathcal{F}^{-1}\left(\mathcal{F}(f) \times \mathcal{F}(g)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\left(\int_{-\infty}^{\infty} g(y)e^{-i\omega y}dy\right)e^{i\omega x}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\omega)g(y)e^{i\omega(x-y)}d\omega\, dy$$

We arrive at the final step by moving $\hat{f}(\omega)$ and $e^{-i\omega x}$ inside the $\hat{g}(\omega)$ in which we represent with the Fourier transform with respect to a dummy variable y since those functions of ω and x may be treated like constants. We then switch the $d\omega$ and dy due to commutativity. Now moving $g(y)$ into the outer integral...

$$= \int_{-\infty}^{\infty} g(y)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega(x-y)}d\omega\right)\, dy$$

Now recalling the fact that:

$$f(x) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

We see that,

$$f(x-y) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega(x-y)}d\omega$$

So,

$$\mathcal{F}^{-1}\left(\mathcal{F}(f) \times \mathcal{F}(g)\right) = \int_{-\infty}^{\infty} g(y)f(x-y)dy = (f * g)$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 d\omega = 2\pi\int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Which gives us a relation, more or less, between the “norm" or “energy" of these functions. Which then tells us that if we approximate $f(x)$ and there are some small, negligible Fourier coefficients, we may set them equal to zero, and our function approximation will still be fine due to the proportionality.

The Discrete Fourier Transform (DFT)

Motivating the DFT, we do not always have a rule for a continuous function, but rather data points, which, may have underlying function. So instead of a function, we have a vector of data points: $[f_0\ f_1\ ...,\ f_{n-1}]^T$. With this vector, we would apply a matrix transformation, the DFT, to attain a new vector of Fourier coefficients:

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix} \qquad \text{where } \hat{f}_k = \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}$$

Similarly,

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_j e^{i2\pi jk/n}$$

Now noticing the exponential term in the formula for the Fourier coefficients,

$$\text{Let } \omega_n = e^{-2\pi i/n}$$

Now, we attempt a bottom up approach to find the matrix which takes us from f_k to $\hat{f}_k$. What if $k = 0$? $\hat{f}_k$ would simply be a sum of all f_j, as our exponential goes to 1. So, our first row would be a row of 1's.

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

What if $k = 1$?

$$\hat{f}_1 = \sum_{j=0}^{n-1} f_j e^{-i2\pi j/n} = f_0 \times 1 + f_1 \times e^{-i2\pi/n} + f_2 \times e^{(i4\pi/n)} + \ldots + f_{n-1} \times e^{i2\pi(n-1)/n}$$

$$= f_0 \times 1 + f_1 \times \omega_n + f_2 \times \omega_n^2 + \ldots + f_{n-1} \times \omega_n^{n-1}$$

Giving us our second row,

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ & & & & \\ & & & & \end{bmatrix}$$

Continuing these steps give us the following matrix: The DFT..

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix}$$