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Fixed boundary Grad-Shafranov solver using finite difference method in nonhomogeneous meshgrid

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Abstract. In this work we present a numerical scheme to solve the Grad-Shafranov equation which correspond to magnetohydrodynamic equilibrium equation for a two-dimensional plasma. A typical case are the toroidal plasma in magnetic confinement devices used in thermonuclear fusion well known as Tokamaks. The proposed numerical scheme is based on the finite-difference method in nonhomogeneous meshgrid, which is adjusted to the fixed plasma boundary with "D-shape". The solution of the Grad-Shafranov equation is obtained using the successive over-relaxation method, usually applied to solve Poisson equation's problems. The values of the total plasma current and pressure in the magnetic axis are conserved in each iteration of the convergence process. The scheme is validated by direct comparison with the analytical result obtained by Soloviev.

1. Introduction

In thermonuclear fusion, magnetohydrodynamic equilibrium equation (MHD) equilibrium is the starting point to understand MHD instabilities [1]. Many numerical and analytical studies of confined plasmas in devices based on closed magnetic field lines as Tokamak have focused on this aspect [2, 3]. Usually, solvers have two ways to find the numerical equilibrium. In the first one way there are the "fixed boundary solvers", where the equilibrium state is obtained when the plasma boundary is analytically know [3, 4]. Then, the magnetic field, pressure and current density inside the plasma region is full characterized and the external region is not relevant [3,5]. In the second one way there are the "free boundary solvers", which consider that plasma doesn't have a well defined boundary, therefore it is necessary calculate profiles both inside and outside the plasma region [6]. In this kind of solvers, the border varies until its change is small compared to an established tolerance.

In the present paper, we show a computational Grad-Shafranov (GS) solver based on fixed plasma boundary with D-shape. This solver uses the finite differences method (FDM) in nonhomogeneous meshgrid and the successive over-relaxation scheme. Details of the meshgrid generation, the Grad-Shafranov equation in a finite difference scheme, and the conservation of the total plasma current and pressure on the magnetic axis are shown. We start from the main ideas presented in the Stephen's Jardin book [6]. The proposed solver is validated by direct comparison with the analytical result obtained by Soloviev [7].

2. Theoretical formalism

2.1. Grad-Shafranov equation

Ideal magnetohydrodynamics is the simplest MHD formulation where the plasma is considered as a charged fluid with very small resistivity (perfect conductor fluid) [6, 8]. In the static and stationary equilibrium, such formulation leads to:

$$\nabla p = \vec{J} \times \vec{B} \tag{1}$$

$$\nabla \times \vec{B} = \mu_o \vec{J} \tag{2}$$

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

Where p, $\vec{J}$ and $\vec{B}$ are the pressure, current density and magnetic field, respectively. Introducing the magnetic vector potential in cylindrical coordinates for the azimuthally symmetric system and defining the functions:

$$\psi = r A_\phi \tag{4}$$

$$g = r B_\phi = r\left[\partial_z(A_r) - \partial_r(A_z)\right] \tag{5}$$

The magnetic field can be written as:

$$\vec{B} = -\nabla\phi \times \nabla\psi + g\nabla\phi. \tag{6}$$

Combining Equations (6) and (2) and then (1); and taken into account that $\vec{B} \cdot \nabla p = \vec{B} \cdot (\vec{J} \times \vec{B}) = 0$ and $\vec{J} \cdot \nabla p = \vec{J} \cdot (\vec{J} \times \vec{B}) = 0$ it lead to the Grad-Shafranov equation, the most famous equilibrium expression in plasma fusion physics [1, 9–11]:

$$\Delta^* \psi = -\mu_o r^2 \frac{\mathrm{d}p}{\mathrm{d}\psi} - g\frac{\mathrm{d}g}{\mathrm{d}\psi} \tag{7}$$

Where $\Delta^*\psi$ is the elliptical operator, defined as:

$$\Delta^* \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \tag{8}$$

Because ψ has a close relationship with the poloidal magnetic flux, ψ is usually called the "poloidal flux function". Pressure $p(\psi)$ and the "poloidal current function" $g(\psi)$ are two free functions whose boundary conditions together with the corresponding boundary conditions for the poloidal flux determine the plasma equilibrium profile.

Grad-Shafranov equation is a nonlinear elliptic partial differential equation in (r, z) plane. Only for the particular case (Soloviev equilibrium):

$$\frac{\mathrm{d}p}{\mathrm{d}\psi} = -\frac{c_1}{\mu_o}, \qquad g\frac{\mathrm{d}g}{\mathrm{d}\psi} = -c_2 R_o^2 \tag{9}$$

Analytical solution can be found [7]

$$\psi(r, z) = \frac{1}{2}\left(c_2 R_o^2 + c_o r^2\right) z^2 + \frac{1}{8}(c_1 - c_o)\left(r^2 - R_o^2\right)^2 \tag{10}$$

Where c_o, c_1, c_2 are constants and R_o is the major radius. For Tokamak devices, usually the next set of values are used: $c_o = B_o/(R_o^2 \kappa_o q_o)$, $c_1 = B_o(\kappa_o^2 + 1)/(R_o^2 \kappa_o q_o)$ and $c_2 = 0$; where B_o, κ_o and q_o are the magnetic field on the axis, the ellipticity and safety factor value on the axis respectively.

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2.2. D-shape plasma boundary
There are two ways to define a good *D-shape* boundary in a toroidal plasma. The first one way use a parametric equations [12]. The second one way is based on Soloviev equilibrium, where the contour levels define a *D-shape*, given by the expression:

$$z_b = \pm \frac{1}{r}\sqrt{\frac{2R_o^2\kappa_o q_o}{B_o}\psi_b - \frac{\kappa_o^2}{4}\left(r^2 - R_o^2\right)^2} \tag{11}$$

3. Numerical scheme
3.1. Finite difference scheme
The Grad-Shafranov equation in a dimensionless form can be written as:

$$\tilde{\Delta}^*\tilde{\psi} = -\tilde{r}\tilde{J}_\phi, \quad \tilde{J}_\phi = \tilde{r}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{\psi}} + \frac{\tilde{g}}{\tilde{r}}\frac{\mathrm{d}\tilde{g}}{\mathrm{d}\tilde{\psi}} \tag{12}$$

Where: $\tilde{\psi} = \psi/\psi_c$, $\tilde{g} = g/g_c$, $\tilde{p} = p/p_c$, $\vec{\tilde{J}} = \vec{J}/J_c$, $\tilde{r} = r/R_o$, $\tilde{z} = z/R_o$ $\tilde{I} = I/I_p$. The subscript c refers to characteristic value for each variable, I_p is the total plasma current and finally: $J_c = I_p/R_o^2$, $\psi_c = \mu_o J_c R_o^3$, $p_c = \mu_o J_c^2 R_o^2$ and $g_c = \sqrt{\mu_o p_c}R_o$.

The GS equation can be expressed in a finite difference scheme by using a nonhomogeneous meshgrid, being $h_i = x_{i+1} - x_i$ the variable size step. For this case, using this meshgrid, expressions for the first and second derivative in the second-order approximation are given by:

$$f'(i) = \frac{h_{i-1}^2\left(f_{i+1} - f_i\right) - h_i^2\left(f_{i-1} - f_i\right)}{h_{i-1}^2 h_i + h_i^2 h_{i-1}}, \qquad f''(i) = \frac{h_{i-1}\left(f_{i+1} - f_i\right) + h_i\left(f_{i-1} - f_i\right)}{\frac{1}{2}\left(h_{i-1}h_i^2 + h_i h_{i-1}^2\right)} \tag{13}$$

Respectively; which can be used to express the GS Equation (12) in a finite difference way as:

$$\tilde{\psi}(i,j) = C_1\tilde{\psi}(i+1,j) + C_2\tilde{\psi}(i-1,j) + C_3\tilde{\psi}(i,j+1) + C_4\tilde{\psi}(i,j-1) + C_5\tilde{J}_\phi(i+1,j) \tag{14}$$

Where: $C_1 = A_r/C$, $C_2 = B_r/C$, $C_3 = A_z/C$, $C_4 = B_z/C$, $C_5 = \tilde{r}_i/C$, $A_r = \frac{hr_{i-1}}{D_{1r}} - \frac{hr_{i-1}^2}{\tilde{r}_i D_{2r}}$, $B_r = \frac{hr_i}{D_{1r}} + \frac{hr_i^2}{\tilde{r}_i D_{2r}}$, $A_z = \frac{hz_{i-1}}{D_{1z}}$, $B_z = \frac{hz_i^2}{D_{2z}}$, $C = \frac{hr_{i-1}}{D_{1r}} + \frac{hr_i}{D_{1r}} - \frac{hr_{i-1}^2}{\tilde{r}_i D_{2r}} + \frac{hr_i}{\tilde{r}_i D_{2r}} + \frac{hz_{j-1}}{D_{1z}} + \frac{hz_j}{D_{1z}}$
and
$D_{1r} = \frac{1}{2}\left(hr_{i-1}hr_i^2 + hr_i hr_{i-1}^2\right)$, $D_{2r} = hr_{i-1}^2 hr_i + hr_i^2 hr_{i-1}$, $D_{1z} = \frac{1}{2}\left(hz_{j-1}hz_j^2 + hz_j hz_{j-1}^2\right)$.

Note that $\tilde{\psi}(i,j)$ depends on the current density while current density depend on the $p(\tilde{\psi})$ and $g(\tilde{\psi})$ functions. From these expressions and choosing the adequate functions to p and g we can build a Poisson solver to find $\tilde{\psi}$ inside the plasma.

With the aim of keep some parameters constant, we introduce a new function:

$$\psi_n = \frac{\tilde{\psi}_l - \tilde{\psi}}{\tilde{\psi}_l - \tilde{\psi}_o} \tag{15}$$

here, $\tilde{\psi}_l$ and $\tilde{\psi}_o$ are the value of $\tilde{\psi}$ at the boundary and magnetic axis respectively, so, ψ_n is between 0 and 1 for all $\tilde{\psi}$ values inside the plasma. Now is possible write to p function as $p(\tilde{\psi}) = p_o\hat{p}(\psi_n)$, being p_o the pressure on the magnetic axis. If we choose $\hat{p}(\psi_n)$ in such way that its value in the plasma edge is 0, and equal to 1 on the magnetic axis, then the pressure on the magnetic axis is thereby held fixed. For the g function is appropriate to use

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$\frac{1}{2}g^2(\tilde{\psi}) = \frac{1}{2}g_o^2\left[1 + \alpha_g \hat{g}(\psi_n)\right]$, where α_g is a constant which is adjusted in each iteration to keep constant the plasma current value; which is given by:

$$I_p = \sum_{i,j} \tilde{J}_\phi(i,j) hr_i hz_j = \sum_{i,j} \left[-\frac{\tilde{r}p_o}{\Delta\tilde{\psi}}\hat{p}'(\psi_n) - \frac{\alpha g_o^2}{2\tilde{r}\Delta\tilde{\psi}}\hat{g}'(\psi_n) \right] hr_i hz_j \tag{16}$$

$$\alpha_g = -2\frac{\left[I_p * \Delta\tilde{\psi} + p_o \sum_{i,j} \tilde{r}\hat{p}'\right]}{g_o^2 \sum_{i,j} \hat{g}'/\tilde{r}} \tag{17}$$

Where: $\Delta\tilde{\psi} = \tilde{\psi}_l - \tilde{\psi}_o$, $\hat{g}(0) = 0$ and $\hat{g}(1) = 1$ are the values of $\hat{g}$ at the border and on the axis, respectively.

3.2. Adaptable meshgrid

To calculate numerically $\tilde{\psi}(i,j)$ from Equation (14), a simplest method to create a non homogeneous meshgrid which is fitted to the fixed plasma boundary was used. This method is described below.

We define 4 regions on the plasma border $\Re_1 : [R_{min}, R_o - a\delta] \times [0, z_{top}]$, $\Re_2 : [R_o - a\delta, R_{max}] \times [0, z_{top}]$, $\Re_3 : [R_o - a\delta, R_{max}] \times [-z_{top}, 0]$ and $\Re_4 : [R_{min}, R_o - a\delta] \times [-z_{top}, 0]$. To locate meshgrid points on the plasma boundary, first we consider $\Re_1$ to solve the Equation (18) by using any root solver, e.g., the bisection method. For this case r_o and z_o starts in the values R_{min} and 0.0, respectively.

$$f(r_i) = (r_o - r_i)^2 + [z_o - z_i(r_i)]^2 - (ds)^2 = 0 \tag{18}$$

ds is the distance between the points (r_o, z_o) and (r_i, z_i), which is the order of 10^{-2}. Once the first r_i value is found then z_i is obtained from $z_i(r_i)$. Next, the values of r_o and z_o are updated from r_i and z_i, respectively. Then the process start again to obtain the following meshgrid point value and so on. It should be noted that $z_i(r_i)$ is the function defining the plasma boundary whit D-shape, given by Equation (11) in Soloviev equilibrium (see Figure 1).

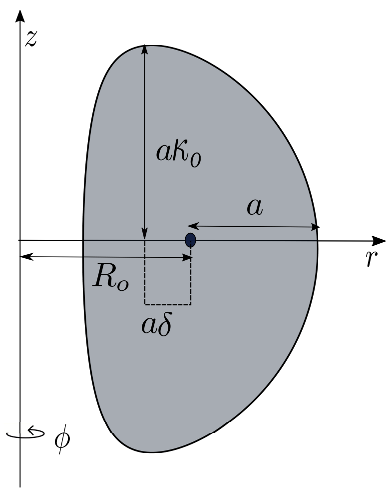

Figure 1. Geometrical parameters in a D-shape plasma boundary.

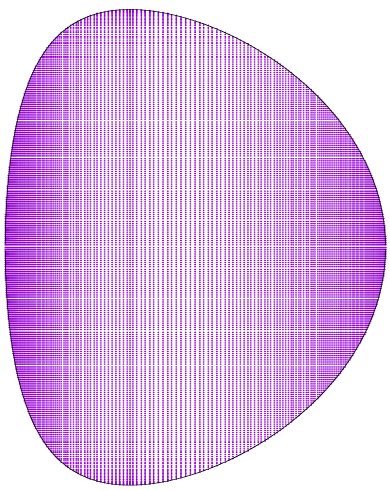

Figure 2. Meshgrid points generated inside plasma region with D-shape by analytical Soloviev equilibrium.

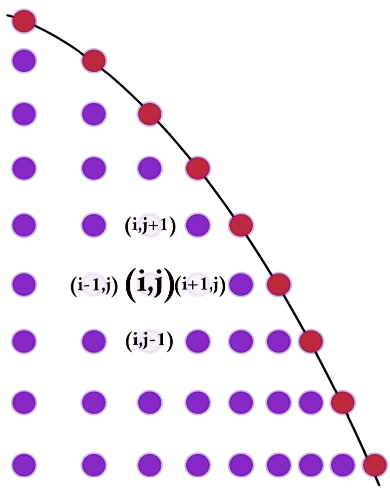

Figure 3. Meshgrid points (red) adjusted to the plasma border and computational molecule scheme.

In $\Re_2$ the same idea is used; however in this region z_i is not a function of r_i. Therefore we must use the z_i values obtained in $\Re_1$. The next equation is used to fit the meshgrid point which comes from Equation (11):

$$f(r_i) = (z_b r_i)^2 - \frac{2R_o^2\kappa_o q_o}{B_o}\psi + \frac{\kappa_o^2}{4}\left(r_i^2 - R_o^2\right)^2 = 0 \tag{19}$$

Similarly than $\Re_1$, the bisection method is used to solve it. Because the symmetry, the meshgrid points in $\Re_3$ and $\Re_4$ are obtained in straightforward from those previously obtained in both $\Re_1$ and $\Re_2$, respectively. Figures 2 and 3 show the the meshgrid points obtained from this method.

4. Results

To check the proposed solver, we compare the obtained results with those obtained from the Soloviev's solution. To do this, a mesh with Soloviev's contours is constructed (See Equation (11)). We choose the next parameters for the test: $B_o = 0.5T$, $R_o = 0.95m$, $a = 0.6m$, $\kappa_o = 2.2$, $q_o = 1.1$, $\psi_b = 0.09T/m^2$. In this calculation we use a meshgrid of 241×241 points.

Figures 4(a) and 4(b) show the poloidal flux in the (r, z) plane obtained from both the Soloviev's solution and the numerical solution, respectively. These graphics show an excellent agreement. The error along the r axis does not exceed 10^{-5} (See Figure 4(c)).

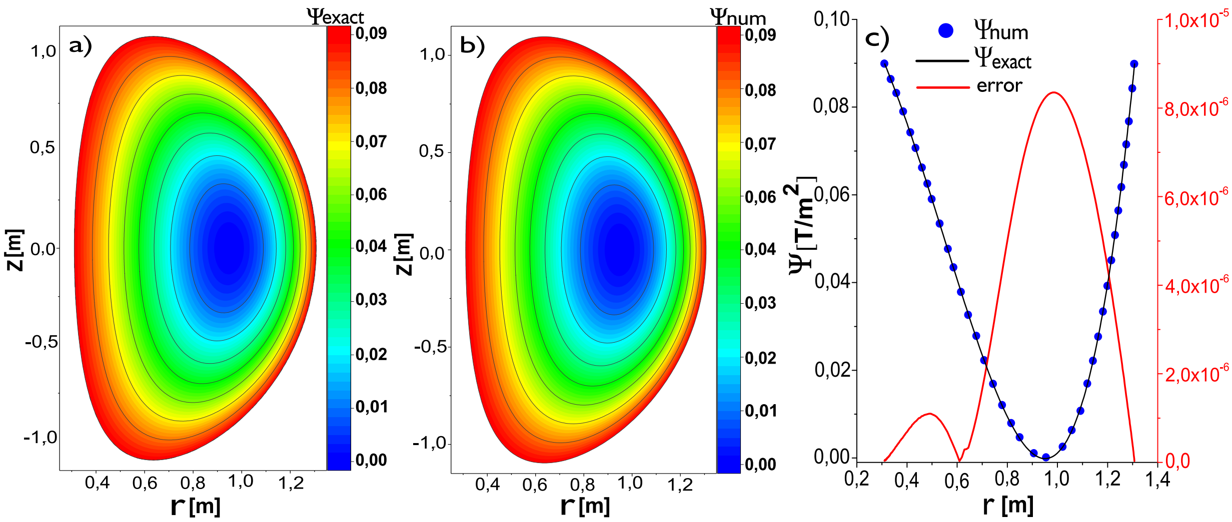

Figure 4. (a) Poloidal flux by analytical Soloviev equilibrium, (b) poloidal flux obtained from proposed scheme and (c) numerical and exact equilibrium and absolute error along r axis ($z = 0$).

5. Conclusions

A fixed Grad-Shafranov solver to obtain equilibrium profiles in toroidal devices with azimuthal symmetry was both presented and validated. The maximum error obtained in Soloviev's equilibrium is about 10^{-6}. The method presented in this paper doesn't require any approximation for p and g functions and hold fixed both the pressure on the magnetic axis and the total plasma current.

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