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On the Grad–Shafranov equation as an eigenvalue problem, with implications for q solvers

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It is shown that the Grad–Shafranov equation for toroidally symmetric ideal-magnetohydrodynamic (MHD) equilibria is a conventional albeit nonlinear eigenvalue problem. That this has been generally overlooked with limited consequences has been made possible by the existence of a scale-invariant transformation of the equation. If the safety factor q is chosen in place of the toroidal field as one of the free flux functions specifying the source (numerical Grad–Shafranov solvers with this capability are called "q solvers"), the eigenvalue is 1 and the scale-transformation factor drops out of the problem. It is shown how this is responsible for the numerical problems that have plagued a class of q solvers, and a simple remedy is suggested. This has been implemented in Livermore's toroidal equilibrium code (TEQ), and as an example, a quasistatically evolved vertical event is presented.

I. INTRODUCTION

As tokamaks have evolved over the past decade they have become increasingly complex. Interest in plasma surface shape, radial profiles, accuracy at finite aspect ratio, engineering constraints associated with superconducting coils, and the disposition of power along divertor surfaces, for example, all point to a narrower range of operating space and the importance of realistic, that is numerical as opposed to analytic, equilibria. The study of an actual device's start-up, vertical-stability-control system, current-profile evolution and control, volt-seconds consumption, etc., requires, furthermore, the ability to follow these equilibria on resistive timescales. The design of a new machine clearly benefits from the ability to do this rapidly: Parameter space is large, and simulation speed sets the constraint on how well it can be scoped out or submitted to optimization processes when inclusion of the full physics is important.

While it has long been appreciated that the disparity between the Alfvén, i.e., equilibrium, and resistive time scales must be exploited if one wants convenience or efficiency in the study of the slower processes of interest,[1] and a number of well-developed equilibrium codes are in use throughout the magnetic fusion energy program, the main computational tool for study of the resistive evolution of toroidal plasmas in the United States (U.S.) remains the TSC code,[2] which solves the time-dependent magnetohydrodynamic (MHD) equations using an artificial mass to lengthen the Alfvén time scale. Early attempts to carry out a formal separation-of-scales solution, quasistatically evolving equilibria *à la* Grad–Hogan, hinged on the availability of a robust equilibrium code capable of accepting the safety factor q as input (thereby determining the more usual input profile, $F\equiv RB_{\text{tor}}$). Since the plasma is to evolve self-consistently with the poloidal-field (PF) coils, the equilibrium code must also be "free boundary," that is it must solve for the plasma/vacuum boundary. Such a code was written as part of the PEST suite,[3] but the TSC procedure proved to be more practical. Since then, the DINA code developed at Kurchatov[4] has become available in the U.S. This code represents the first successful implementation of the Grad–Hogan approach to resistive-time-scale axisymmetric simulation; its Grad–Shafranov solver finds R and Z as functions of ψ and θ, (that is, it's an "inverse" solver) inside a fixed outer ψ surface, and self-consistency with the PF coils and an external solution is achieved by solving multiple equilibria and iterating on the plasma boundary.

The present authors, working with Livermore's "direct-solve" (i.e., for ψ as a function of R and Z) equilibrium code[5] and experiencing difficulty in convergence when trying to specify q, were led to a reexamination of the Grad–Shafranov equation and the realization that it is in fact a straightforward eigenvalue problem. Although this has been observed before,[6] it is not widely appreciated, and it has an important consequence for q solvers. In this paper, we present the argument and develop the associated variational formulation in Sec. II; this may be of general use, although we do not develop it further or investigate the properties of the eigenmode/value spectrum here. Concerning numerical solutions, we observe that the discreteness of the eigenvalue need not be taken into account when working in unnormalized coordinates: The overall constant multiplying the dimensionless p' and FF' profiles is equal to the eigenvalue times a continuous variable related to the normalization. We then look at the case when q rather than FF' is specified, and discuss the changes in the variational principle. Here, we find unavoidable implications of discreteness for numerical formulations: Even in unnormalized coordinates, the overall constant is 1, the eigenvalue. Since the numerical equations will have a solution only with a somewhat different value, failure to take the discreteness and distinction between the analytical and numerical eigenvalues into account will prevent convergence. We outline a particular simple solution to this problem. We conclude in Sec. III with an illustration of a quasistatically evolved vertical event as an example, comparing against an independent, linearized ideal-MHD calculation.

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II. EIGENVALUE PROBLEM

A. p', FF' profiles

The Grad–Shafranov equation in cgs units can be written in the form

$$\Delta^*\psi = -\frac{4\pi}{c} R J_{\mathrm{tor}} = -\frac{4\pi\alpha}{\Delta\psi}\,[\beta R^2 \nu_p' + (1-\beta) R_0^2 \nu_F'], \tag{1}$$

where $\mathbf{B}_{\mathrm{pol}} = \nabla\psi \times \nabla\phi$ with ϕ the toroidal angle introduces our poloidal flux variable, and $\Delta\psi \equiv \psi_{\mathrm{edge}} - \psi_{\mathrm{axis}}$. Also, $\nu_p(x)$, where $x \equiv (\psi - \psi_{\mathrm{axis}})/(\psi_{\mathrm{edge}} - \psi_{\mathrm{axis}})$, is a dimensionless free function of x in terms of which the plasma pressure profile is given: $p'(\psi) = \alpha\beta\nu_p'/\Delta\psi$, where the prime indicates derivative with respect to argument; similarly ν_F determines F: $FF' = 4\pi\alpha(1-\beta)R_0^2\nu_F'/\Delta\psi$. By convention, $\nu_p'(0) = \nu_F'(0) = -1$. Among the constants α, β, and R_0, the latter is chosen for convenience, β is chosen with an eye to finding an equilibrium with a desired $\beta_{\mathrm{pol}} \equiv 8\pi p/B_{\mathrm{pol}}^2$, and α plays the role of a normalization, as will be explained below. Fixed-boundary equilibrium problems require, in addition, a $Z_{\mathrm{edge}}(R)$ to specify the outer ψ=constant surface, and they look for ψ inside this surface; whereas free-boundary equilibria require currents from a PF-coil set (these may be either fixed, in which case the plasma/vacuum boundary falls where it may, or solved for, for example, by demanding the boundary pass through a number of specified R-Z points), and require $\mathbf{B}_{\mathrm{pol}} \to 0$ at R, $Z \to \infty$. We suppress writing the PF-coil contribution $-(4\pi/c)\Sigma I_c R\delta(R-R_c)\delta(Z-Z_c)$ in Eq. (1); note that fixed-boundary problems can also include such currents, if desired, as long as the R_c, Z_c points fall inside the plasma boundary.

As is well known, the Grad–Shafranov equation possesses a scale-invariance transformation as follows:

$$\psi \to \lambda\psi, \quad \alpha \to \lambda^2\alpha, \quad R \to R, \quad Z \to Z, \tag{2}$$

and, in the case of free-boundary problems, $I_{\mathrm{coils}} \to \lambda I_{\mathrm{coils}}$. As a result, the toroidal plasma current I_{tor} will also scale as λ. If an iterative method is employed to find the equilibrium, this transformation can be used to converge on a solution with a desired I_{tor} or alternatively $\Delta\psi$: within the iteration loop, compute $\lambda = I_{\mathrm{tor}}^{\mathrm{desired}}/I_{\mathrm{tor}}^{\mathrm{present\ iterate}}$ and apply the transformation. (Unless one's aim is actually to scale up one converged solution to find another, the PF currents are left unscaled.) Thus the free constant α becomes part of the solution, traded for I_{tor} or $\Delta\psi$, one of which is now input.

With this in mind, we transform the dependent variable from ψ to $\tilde{x} \equiv 1 - x$ (note that $\tilde{x}=0$ is at $\psi=\psi_{\mathrm{edge}}$), obtaining

$$\Delta^*\tilde{x} = -\tilde{\alpha}[\beta R^2 \tilde{\nu}_p' + (1-\beta) R_0^2 \tilde{\nu}_F']\Theta, \tag{3}$$

where $\tilde{\alpha} \equiv 4\pi\alpha/(\Delta\psi)^2$ and $\tilde{\nu}_p(\tilde{x}) \equiv \nu_p(x)$. We have explicitly included the step function $\Theta(\rho_{\mathrm{edge}} - \rho)$ as a reminder that in free-boundary problems the source is zero outside the first $\psi=\psi_{\mathrm{edge}}$ surface encountered from the magnetic axis, even though $\tilde{x}$ might lie between 0 and 1 as it does inside the plasma (e.g., consider the quadrant straight across the X point in a fully diverted plasma); ρ is any appropriate minor-radius-like variable.

Notice that under an application of the scale transformation, nothing happens in Eq. (3): In particular, $\tilde{\alpha}$ is a unique constant, independent of scale (e.g., I_{tor}). With linear profiles and fixed-boundary conditions (recall the boundary conditions are homogeneous), Eq. (3) is manifestly an eigenvalue equation. For nonlinear profiles, as long as the functional dependence of the $\tilde{\nu}(\tilde{x})$''s is such that as $\tilde{x}$ goes to zero, $\tilde{\nu}'$ also goes to zero, the problem remains homogeneous, so that it is only due to a normalization or equivalent constraint that the trivial solution is ruled out; we expect here, as in the linear case, that Eq. (3) will have solutions only for special values of $\tilde{\alpha}$.

For further insight, proceeding with the general case, we develop a variational formulation of the problem. Multiply Eq. (3) by $dR\,dZ\,\delta\tilde{x}/R$, where $\delta\tilde{x}$ is an arbitrary infinitesimal function of R and Z which satisfies the appropriate free- or fixed-boundary conditions, and integrate. On the left-hand side, integrate by parts (the homogeneous boundary conditions play their key role here); and on the right-hand side, observe that $\delta\tilde{x}\tilde{\nu}' = \delta\tilde{\nu}$. Define $K \equiv \int dR\,dZ[\beta R\tilde{\nu}_p + (1-\beta)\tilde{\nu}_F R_0^2/R]\Theta$, and the result is

$$\delta\left(\int dR\,dZ\tfrac{1}{2}|\nabla\tilde{x}|^2/R - \frac{4\pi}{c\Delta\psi}\sum I_c\tilde{x}(R_c,Z_c) - \tilde{\alpha}K\right) = 0.$$

It is now apparent that the Grad–Shafranov Eq. (3) is the Euler–Lagrange equation for a homogeneous system which minimizes the (scale-invariant) poloidal field plus coil energy $\widetilde{W} \equiv \widetilde{W}_B + \widetilde{W}_c$, subject to a constraint, $K = K_0$. We have defined $\widetilde{W}_B \equiv \frac{1}{2}\int dR\,dZ\,R B_{\mathrm{pol}}^2/(\Delta\psi)^2 = \frac{1}{2}\int dR\,dZ|\nabla\tilde{x}|^2/R$ and $\widetilde{W}_c \equiv -(4\pi/c\Delta\psi)\Sigma I_c\tilde{x}(R_c,Z_c)$. At this point we recognize $\tilde{\alpha}$ as the Lagrange multiplier associated with the constraint in this variational formulation, and as the eigenvalue of the differential Eq. (3).

Repeating the steps above with the solution $\tilde{x}$ rather than $\delta\tilde{x}$ shows that

$$\tilde{\alpha} = (2\widetilde{W}_B + \widetilde{W}_c)^{\min} \Big/ \int dR\,dZ\,\tilde{x}[\beta R\tilde{\nu}_p' + (1-\beta) \times \tilde{\nu}_F' R_0^2/R]\Theta.$$

If the current profile is linear in flux, this becomes $\tilde{\alpha} = (\widetilde{W}_B + 0.5\widetilde{W}_c)^{\min}/K_0$.

The constraint constant K_0 bears some discussion. In linear eigenvalue problems, K_0 is purely a normalization, chosen for convenience, and affects neither the eigenvalue nor function. Here, it is clear that for nonlinear $\tilde{\nu}$''s, $\tilde{\alpha}$, as well as the form of $\tilde{x}$, depends continuously on K_0. But in fact, only one of these variational solutions is admissible, and the value of the constraint is determined as part of the final solution: Namely, we select that K_0 whose variational solution has $\tilde{x}=1$ at the magnetic axis. (At the edge, on the other hand, $\tilde{x}=0$ is satisfied by construction in the variational formulation: In fixed-boundary problems, it is the boundary condition required of the trial functions. In free-boundary problems, the edge is identified as that contour which passes through a fixed R-Z "limiter" point, or through the X point, etc. Alternatively, we can assign a value of $\tilde{x}$ to the symmetry axis, look for that K_0 which has $\tilde{x}=1$ at the magnetic axis, and identify the edge with whatever $\tilde{x}=0$ emerges.)

We emphasize that the profiles—the functional dependences of $\tilde{v}_p$ and $\tilde{v}_F$ upon their arguments—are *not* varied in the variational process. This is in distinction to the derivation of general ideal-MHD equilibria from an energy principle, where the plasma plus magnetic energy as a function of fluid coordinate $\mathbf{x}(\mathbf{x}_0,t)$ is extremized, subject to differential constraints conserving mass, flux, and entropy. Our variational principle is also distinct from equilibrium formulations which specify the constraints, that is the profiles (there is no concept of K), as functions of flux (rather than as initial conditions in space), but still minimize the total energy.[7]

For free-boundary problems, it is the Θ function, which plays the role of a localizing potential, which provides the bound-state solutions with their characteristic discrete eigenvalues.

We give a simple example: If the current profile is linear in flux, the differential equation is

$$\Delta^*\tilde{x}=-\tilde{\alpha}\tilde{x}[\beta R^2+(1-\beta)R_0^2]\Theta; \tag{4}$$

and the variational problem is $\tilde{W}=\min$, while $\frac{1}{2}\int dR\,dZ\,\tilde{x}^2[\beta R+(1-\beta)R_0^2/R]\Theta=\text{constant}$. As discussed above, because the source is linear, any value of the constant will do; K_0 and $\tilde{x}$ can be scaled appropriately when done to achieve $\tilde{x}=1$ on axis. We remark here on the possibility of higher modes. Certainly doublets and more ornate plasma shapes are known to exist; but it is also clear that Eq. (4) supports higher modes with internal separatrices and X points for given, simple plasma boundaries; e.g., at large aspect ratio, with circular boundaries, $\psi\sim$Bessel functions in $\rho\times$sin's and cos's in θ.

Next we discuss finite current density at the plasma edge. Note that current-density jumps, in general, can be accommodated by Eq. (3). We consider adding to the profiles $\tilde{v}_j'$ terms, which approach the limiting form $j_0\Theta(\tilde{x}-\tilde{x}_0)$. As long as $\tilde{v}_j(\tilde{x})'$ still goes smoothly to zero as $\tilde{x}$ goes to zero, homogeneity of the equation in $\tilde{x}$ is maintained. In the variational formulation, K will now have contributions to the $\tilde{v}$'s of the form $\tilde{v}=0$ for $\tilde{x}<\tilde{x}_0$; $\tilde{v}=\tilde{x}-\tilde{x}_0$ for $\tilde{x}>\tilde{x}_0$. Finite current density at the edge, then, is simply the case $\tilde{x}_0\to 0$; free-boundary problems and fixed-boundary problems that locate their boundary on the zero side of the step function retain their status as eigenvalue problems.

We conclude the general discussion with some remarks on numerical implications. Given that even one-dimensional eigenvalue problems require special methods [one expects that any numerical method that attempts to find ϕ, where $\phi''+\kappa^2\phi=\phi(0)=\phi(L)=0$, but does not take into account that there are only special values of κ for which solutions exist, will fail], and that two-dimensional eigenvalue problems with general boundaries are even more difficult, the question arises: How have direct-solve codes succeeded in working? In our case (which is relevant to any code that uses the standard functional-iteration/scale-transformation technique), we conclude that it is only by virtue of solving Eq. (1) rather than working in normalized variables, as in Eq. (3), that we have escaped difficulty: α is a continuous variable [because λ is; assuming Eq. (3) has a solution, for *every* value of α a solution of Eq. (1) exists]; in adjusting α to satisfy the constraint I_{tor}, we are converging on $\tilde{\alpha}$ times a continuous variable, masking the presence of the discrete eigenvalue. Finally, we point out that the standard functional-iteration/scale-transformation technique often employed by Grad–Shafranov codes should easily carry over, at least for finding the lowest mode, to linear eigenvalue problems with arbitrary boundaries. [To wit, a fixed-boundary solution to $\mathscr{L}\phi=-\kappa\phi$ is obtained iteratively as follows, assuming one has a method for solving the inhomogeneous problem $\mathscr{L}\phi=S$ inside the desired boundary, with homogeneous boundary conditions: invert $\mathscr{L}\phi^{n+1}=-(\kappa\phi_0)^n\phi^n/\phi_0^n$; compute ϕ_0^{n+1}, where ϕ_0 is any convenient normalization for ϕ, e.g., $\phi_0\equiv\phi(\mathbf{x}_0)$; compute the scale factor $\lambda=N^{\text{desired}}/N^n$, where N is the normalization constraint, e.g., $N=\int\phi^2$ or $N=\phi_0$; set $(\kappa\phi_0)^{n+1}=\lambda^p(\kappa\phi_0)^n$, where p is chosen appropriately considering the choice for N.]

B. p', q profiles

We turn now to the case when q is used to specify one of the free flux functions. (Nothing special happens when the entropy is used in place of the pressure profile; for this discussion, we continue to use p.) Noting that

$$F=2\pi q/\oint dl/(R^2B), \tag{5}$$

the Grad–Shafranov equation is

$$\Delta^*\psi=-\alpha\beta R^2 v_p'-\frac{d}{d\psi}\frac{1}{2}\left(\frac{2\pi q}{\oint dl/(R^2B)}\right)^2. \tag{6}$$

Note that $\alpha\beta$ now appears as a single product. It can be input, as β alone was before, or can be adjusted to control the ratio of the p' to FF' terms on axis; but we have lost one parameter, the overall scale-setting multiplier [$q(0)$ cannot be factored out and regarded as that parameter: Equilibria with different $q(0)$ are different, unrelated by the scale transformation]. The scale-invariant equation is

$$\Delta^*\tilde{x}=-\tilde{\alpha}\beta R^2\tilde{v}_p'-\frac{d}{d\tilde{x}}\frac{1}{2}\left(\frac{2\pi\tilde{q}}{\oint dl_{\text{pol}}/|R\nabla\tilde{x}|}\right)^2. \tag{7}$$

Note that the new form for FF' causes no trouble with the scale transformation: q does not scale and the powers of ψ in FF' work out correctly to yield Eq. (7). [However, since q does not scale and we require Eq. (5), we can no longer imagine $F_{\text{vac}}^2+F_{\text{plas}}^2\to F_{\text{vac}}^2+\lambda^2F_{\text{plas}}^2$; now $F\to\lambda F$ and the vacuum toroidal field must change with the scale.]

The extremized integral is

$$\delta(\tilde{W}-1K_q)=0, \tag{8}$$

where

$$K_q\equiv\int dR\,dZ\left(\tilde{\alpha}\beta R\tilde{v}_p+\frac{1}{2R}\frac{(2\pi\tilde{q})^2}{(\oint dl_{\text{pol}}/|R\nabla\tilde{x}|)^2}\right)\Theta.$$

The loss of an overall multiplying parameter on the right-hand side of Eq. (6) possibly indicates the presence of an eigenvalue equal to 1, as suggested in Eq. (8). But how is $\delta K_q=0$ as a constraint with Lagrange multiplier 1 to be distinguished from $\delta K_q=0$ with K_q now contributing to the energy to be minimized? We argue in favor of the former: The equilibrium equation (7) remains homogeneous [in the sense that for a given $\tilde{x}(R,Z)$ one can expand the source in positive powers of $\tilde{x}$], with homogeneous boundary conditions. If δK_q were to be extremized, the solution to Eq. (8) is $\tilde{x}=0$. It is only the normalization requirement ($K_q=K_{q0}$, or equivalently $\tilde{x}_{\rm axis}=1$) that prevents eigenfunctions from vanishing everywhere. Also we adduce numerical evidence: Simply differentiating Eq. (5) to obtain FF' turns out to be numerically unstable (even when simply trying to reproduce converged equilibria), indicating that the solution is in fact not at a minimum.

Before proceeding with the implications for q solvers of this recast variational problem, we briefly discuss the instability just alluded to. We are uncertain how general a problem it is (although we suspect the free boundary itself is key) and simply offer here our method for treating it at present. Each time FF' is needed to update the current, we solve the following ordinary differential equation, obtained by surface-averaging $1/R\times$ the Grad–Shafranov equation:

$$FF'=-F^2\frac{[\oint dl\, B_{\rm pol}^2/(BF)]'+4\pi(p'/F)\oint dl/B}{2\pi q+\oint dl\, B_{\rm pol}^2/(BF)}. \tag{9}$$

[To be complete, we note it was further necessary for numerical stability to replace F^2 on the right-hand side of Eq. (9) with a weighted combination of F computed from Eq. (5) and from integrating (9).]

This brings us to the final numerical problem: Because the scale-transformation variable no longer multiplies the eigenvalue, but rather the powers of λ are confined to the numerical ψ itself ($d/d\psi$ and $\nabla\psi$), and because the numerical eigenvalue never equals the analytic, a numerical solver which uses the scale-transformation technique (2) to satisfy the normalization constraint will at best get near a solution with the desired $I_{\rm tor}$ but be unable to converge, and at worst be driven unstable. This was indeed the long-standing source of the difficulties in our attempts at a q solver. A simple solution, appropriate for such codes, is to provide something in the way of an eigenvalue to solve for, e.g., λ itself.

Compute $\lambda=\Delta\psi^{\rm desired}/\Delta\psi^{\rm present\ iterate}$, setting the scale, as usual. Scale only the plasma-current array, that is, the source term for the solver (and, if not conserving entropy and trying to attach significance to β, scale α). Now λ no longer goes to one at convergence (it did before) but approaches some constant whose departure from one is a measure of the numerical method's error. Note that we have not scaled the flux itself: Once the source term is scaled, scaling ψ only affects the grid-boundary condition for the elliptic solver. If this is set by summing Green's functions$\times$currents (the usual method for satisfying the boundary condition at infinity for free-boundary codes), it will never converge to $\lambda\psi(R_{\rm BC},Z_{\rm BC})^{\rm prev.\ iterate}$. Our modified procedure amounts to restoring an overall multiplier on the right-hand side of Eq. (6), and λ at convergence is precisely the numerical eigenvalue.

We give a warning concerning the use of Eq. (8). The variational principle we have given will lead to solutions of the differential equation only if the $\tilde{x}$ dependence of the profiles is not varied (we again used $\delta\tilde{x}\, d/d\tilde{x}=\delta$). Thus it is required that the $\tilde{x}$ dependence of $\Im\equiv\oint dl_{\rm pol}/|R\nabla\tilde{x}|$ be that of the solution, which is not known. (If it were known and used in K_q, this would be the same as specifying ν_F.) We suggest without proof an iterative solution to Eq. (8), assuming $\tilde{x}$ is represented by given functions of R and Z with parameters to vary. (Due to the nonlinearity of the problem, some sort of iteration scheme is likely in any event.) Define $K_q^n\equiv\int dR\, dZ\,[\tilde{\alpha}\beta R\tilde{\nu}_p+1/2R(2\pi\tilde{q}/\Im^n)^2]$, where n is the iteration index. At each iteration, $\tilde{x}$ is computed as a function of R and Z, the contours of constant $\tilde{x}$ are found, and thus $\Im^n$ as a function of $\tilde{x}$ can be computed. This functional dependence [as well as $\tilde{\nu}_p(\tilde{x})$ and $\tilde{q}(\tilde{x})$] is to be preserved as K_q^n supplies the constraint on the parameters while carrying out the next iteration. If this procedure converges, it has satisfied the conditions of the variational principle, and has taken as the specified profiles only $\tilde{\nu}_p'$ and $\tilde{q}$. (A preferable formulation might emerge if an equation relating $\Im'$ and $\delta\Im=\Im[\tilde{x}(R,Z)+\delta\tilde{x}(R,Z)]-\Im(\tilde{x})$ were pursued. However, developing a variational principle for practical use is not our main concern here. We wish merely to be able to cast Eq. (7) as the Euler–Lagrange equation for any variational problem, exhibiting the eigenvalue as a Lagrange multiplier equal to 1. Although we lack the extensive apparatus and knowledge available for linear problems, we would suspect something amiss with our understanding of Eq. (7) if this could not be done—thus we are interested in a demonstration from the variational point of view.)

Finally, we remark that if the q profile never diverges, there can be no internal separatrices: It appears that the lowest mode is the only solution. If the q profile has the logarithmic divergence associated with an X point, it appears that higher modes, if they exist, have a degenerate eigenvalue spectrum.

III. NUMERICAL EXAMPLE: A VERTICAL EVENT

We have carried out these ideas [namely, the modified scaling procedure and the use of Eq. (9)] in our equilibrium code. In addition, we conserve the entropy S, an easy extension: $pV'^{\gamma}=S$, where $S(\psi)$ is given and $V'=\oint dl/B$, is solved simultaneously with Eq. (9). Convergence rates, even for fully diverted plasmas, are comparable to other profiles', although as with any profile a penalty is paid for not using the coils to pin any plasma-boundary points.

As a benchmark, we choose a particularly simple problem, testing the ability of the code to evolve the plasma adiabatically: We compare with an independent simulation of a vertical event. An up–down asymmetric ideal-MHD plasma, unstable on the resisitive timescale of the surrounding passive structure, is given a 1 cm displacement; a proportional feedback voltage is applied to a single active

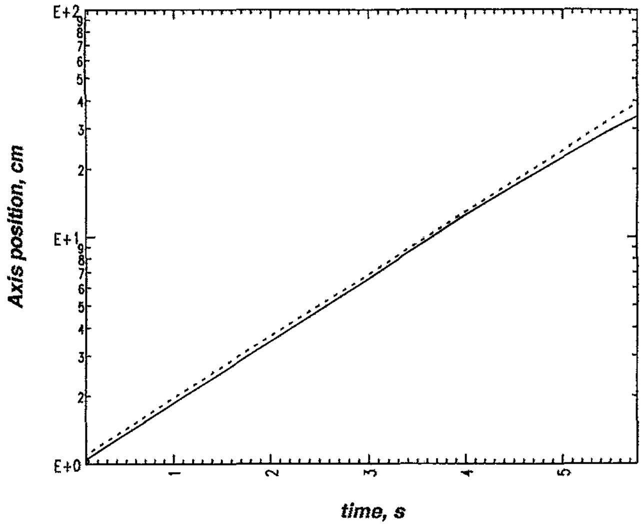

FIG. 1. Displacement of the magnetic axis versus time for a quasistatically evolved ideal-MHD ITER plasma in the presence of a resistive passive-stabilization structure (solid curve), compared to the linear growth rate (dashed curve). The main plasma parameters are $I_{tor}=25$ MA, $F_{vac}=46.5$ mT, $R_{axis}=8.35$ m, $Z_{axis}=1.58$ m, minor radius$=2.99$ m, $q_{axis}=0.866$, $q_{edge}=4.29$, elongation and triangularity at the edge $=1.65$ and 0.310, $\beta_{pol}\equiv 8\pi\langle p\rangle(\oint dl/BB_{pol}/\oint dl/BB_{pol}^2)^2_{edge}=0.703$, $l_i\equiv\langle B_{pol}^2\rangle(\oint dl/BB_{pol}/\oint dl/BB_{pol}^2)^2_{edge}=1.06$, $(\psi_{xpt}^{upper}-\psi_{xpt}^{lower})/\psi_{xpt}^{lower}=0.158$ (a measure of the up–down asymmetry), and $(\psi_{xpt}^{lower}-\psi_{edge})/\psi_{xpt}^{lower}=0.01$ (ψ_{xpt}^{lower} is the nearer to the plasma edge for this equilibrium). Here we have defined the volume average $\langle\cdots\rangle\equiv\int d^3x\cdots/\int d^3x$.

coil, and the displacement of the plasma axis is returned to zero. The passive structure is modeled as a set of one hundred or so filaments. The equilibrium parameters are for an International Thermonuclear Experimental Reactor (ITER)-like plasma.

The appropriate time-dependent equations for the quasistatically evolved Grad–Shafranov equilibria here consist merely of a set of the form $\partial\psi(R_c,Z_c)/\partial t=V_c-I_c\mathscr{R}_c$ for each coil c, where V_c, I_c, and $\mathscr{R}_c$ are the voltage (zero, except for the active coil), current and resistance of the coil (or, in the case of the passive structure, filament). The PF coils obey $\partial\psi(R_c,Z_c)/\partial t=0$ (the PF coils are actually made up of filaments modeling each coil's finite extent, but these filament currents are kept in fixed ratio and only one flux per PF coil is conserved). Plasma q, S, and $\Delta\psi$ are conserved. Time stepping can be centered to fully implicit (the shortest $L_c/\mathscr{R}_c$ time is four orders of magnitude faster than the instability), and we have devised a technique that requires only one equilibrium per time step.

In Fig. 1, the feedback voltage is zero, and we plot the displacement of the axis versus time (solid curve). The dashed curve represents the linear growth rate—the result of a linearized-plasma (δW) calculation,[8] normalized to the 1 cm initial displacement.

In Figs. 2(a) and 2(b) we turn on the active coil and plot the displacement at the axis and the voltage, current, and power of the active coil versus time. Figure 2(a) is the quasistatically evolved equilibrium and Fig. 2(b) the linearized-plasma calculation. The former was run with

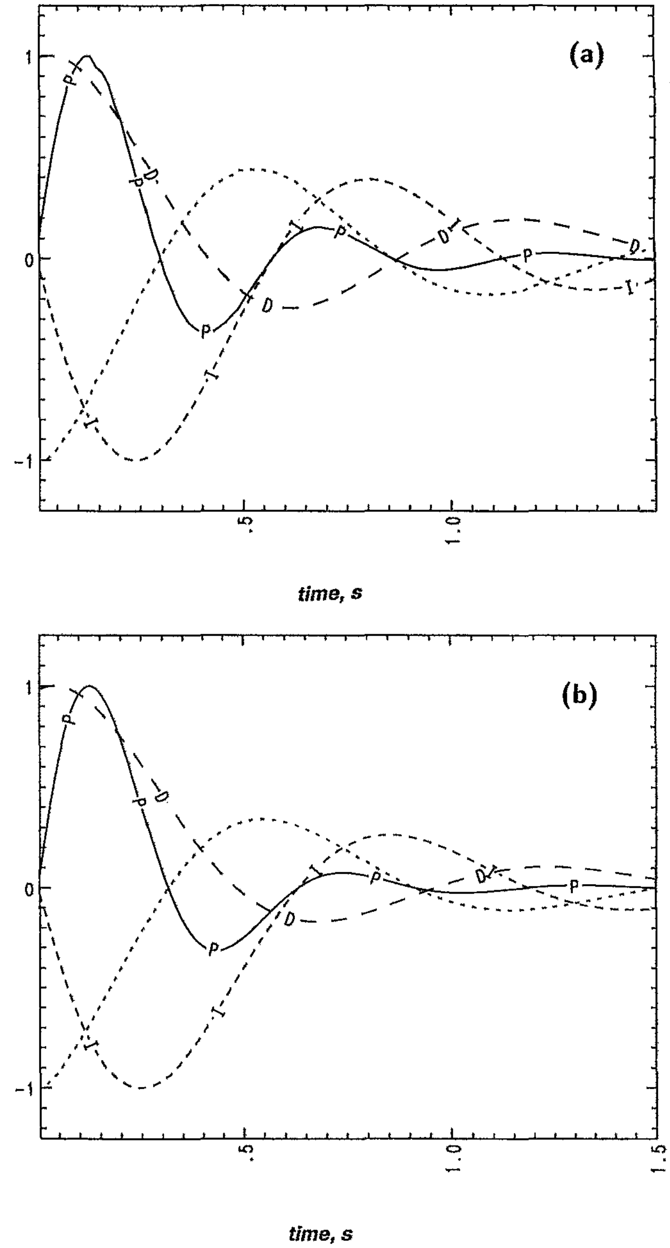

FIG. 2. Simulation of active feedback (on the magnetic axis) control of the vertical instability of Fig. 1. Plotted are the displacement (— — —) and the power (—), voltage (---), and current (– – –) of the active coil. (a) is the quasistatically evolved and (b) the linearized-plasma calculation.

100 equal time steps and used 3.5 c-90 cpu min at NERSC.

We experienced initial surprise at such good agreement: We have not yet included surface currents in TEQ; our algorithm is not yet robust enough to handle finite current density at the edge in a time-dependent run (we set FF' smoothly to zero at the edge in the case shown); and we have not conserved $\psi_{edge}-\psi(0,0)$; all of which effects are of course properly included in the linearized calculation. However, monitoring the errors associated with relaxing these constraints [respectively, $F_{vac}-F(\psi_{edge})$, $q(\psi,t)-q(\psi,0)$ near the edge, and $\psi_{edge}(t)-\psi_{edge}(0)$] shows them to be small, and this is to a certain extent to be expected in an ITER plasma: single-null (the case with up–down asymmetry) equilibria force $\xi \cdot \nabla\psi \to 0$ at the X point,[8] so that (since the equilibrium J_{tor} also vanishes at the edge) the edge cannot play a strong role. The slight difference in the displacements at $t=0$ is a measure of the numerical error associated with reproducing a p', FF' equilibrium (used in conjunction with the linearized code to calculate the currents and fluxes around the passive structure associated with the 1 cm displacement) using the (displaced) S, q calculation.

The first two sections of this work were first presented (under a different title) at the November 1992, American Physical Society meeting. While preparing for publication this spring, the authors were made aware of related work by Turkington *et al.*[9] The latter, in the tradition of Woltjer, and Kruskal and Kulsrud, regard the q and S profiles as infinite families of constraints of the zero-resistivity time-dependent MHD equations (and so find Lagrange-multiplier functions, as opposed to the single scalar associated with the constrained minimization problem of the present work); they develop an associated numerical variational problem, have written a full-scale time-dependent code, and present sample solutions. The authors thank J. Manickam and A. Lifschitz for bringing this to our attention.

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