The algorithms used in this section on integration and the next section on function recovery are all based on quadratic splines on $[a,b]$. The node set and the quadratic spline algorithm using $n+1$ function values are defined for $n \in \mathcal{I} := \{4,6,8,\dots\}$ as follows:

$$t_i = \frac{(i-1)(b-a)}{n}, \qquad i = 1,\dots,n+1, \tag{1a}$$

$$\begin{aligned} A_n(f)(t) := \frac{n^2}{2(b-a)^2}\,[f(t_i)(t-t_{i+1})(t-t_{i+2}) \\ -2f(t_{i+1})(t-t_i)(t-t_{i+2}) + f(t_{i+2})(t-t_i)(t-t_{i+1})] \\ \text{for } t_i \le x \le t_{i+2}. \end{aligned} \tag{1b}$$

The problem to be solved is univariate integration on the unit interval, $\mathrm{INT}(f) := \int_a^b f(t)\,\mathrm{d}t \in \mathbb{R}$. The fixed cost building blocks to construct the adaptive integration algorithm are the composite Simpson's rules based on n intervals:

$$\begin{aligned} S_n(f) &:= \int_a^b A_n(f)\,\mathrm{d}t \\ &= \frac{(b-a)}{3n}[f(t_1)+4f(t_2)+2f(t_3)+4f(t_4)+2f(t_5)\cdots+4f(t_{n-1})+f(t_n)]. \end{aligned} \tag{2}$$

Given any partition, define an approximation to $\mathrm{Var}(f''')$ as:

$$\widehat{V}(f''',\{x_i\}_{i=0}^n) = \sum_{i=2}^{n-1} |f'''(x_i) - f'''(x_{i-1})| \le \mathrm{Var}(f''').$$

If we consider:

$$\begin{aligned} \widetilde{V}_n(f) &= \sum_{i=1}^{n-3} |f'''(x_i) - f'''(x_{i-1})|\,, \\ &= \frac{n^3}{(b-a)^3}\sum_{i=1}^{n-3} |f(t_{3i-3}) - 3f(t_{3i-2}) + 3f(t_{3i-1}) - 2f(t_{3i}) + 3f(t_{3i+1}) - 3f(t_{3i+2}) + f(t_{3i+3})|\,. \end{aligned}$$

Since

$$\frac{n^3}{(b-a)^3}\,|f(t_{3i-3}) - 3f(t_{3i-2}) + 3f(t_{3i-1}) - f(t_{3i})| = f'''(x_{i-1}),$$

for some $x_{i-1} \in [t_{3i-3}, t_{3i}]$, then

$$\widetilde{V}_n(f) = \sum_{i=1}^{n-3} |f'''(x_i) - f'''(x_{i-1})| = \widehat{V}(f'''),$$

for some $x_i \in [t_{3i}, t_{3i+3}]$ and for some $x_{i-1} \in [t_{3i-3}, t_{3i}]$. Then we can use $\widetilde{V}_n(f)$ to approximate $\mathrm{Var}(f''')$ by just using function values.

Define the cone:

$$\mathcal{C}_\tau := \left\{ f \in \mathcal{V}^3, \mathrm{Var}(f''') \leq \mathrm{C}(\mathrm{size}\{x_i\}_{i=0}^n) \widehat{V}(f''', \{x_i\}_{i=0}^n) \right\}. \tag{3}$$

Similar Lemma: $\widetilde{V}_n(f) \leq \mathrm{Var}(f''') \leq \mathrm{C}(2(b-a)/n)\widetilde{V}_n(f)$, then the error bound:

$$\mathrm{err}(f, n) \leq \frac{(b-a)^4}{36n^4} \mathrm{Var}(f''') \leq \frac{(b-a)^4}{36n^4} \mathrm{C}(2(b-a)/n)\widetilde{V}_n(f).$$

Upper bound of computational cost:

Denote $N(f, \varepsilon)$ as the computational cost, which is the number of points used for Simpson's rule:

$$(b-a)\left(\frac{\mathrm{Var}(f''')}{36\varepsilon}\right)^{1/4} \leq N(f, \varepsilon) \leq (b-a)\left(\frac{\mathrm{C}(2(b-a)/n)\widetilde{V}_n(f)}{36\varepsilon}\right)^{1/4} + 1.$$