Double pendulum - Problem IX

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1 Equations of motion

We solve the following system of ordinary differential equations

$$\ddot{\theta}_1 = \frac{1}{l_1 F}\left(g m_2 \sin\theta_2 \cos\Delta\theta - g(m_1+m_2)\sin\theta_1 - (\omega_2^2 l_2 m_2 + \omega_1^2 l_1 m_2 \cos\Delta\theta)\sin\Delta\theta\right)$$
$$\ddot{\theta}_2 = \frac{1}{l_2 F}\left(g(m_1+m_2)\sin\theta_1 \cos\Delta\theta - g(m_1+m_2)\sin\theta_2 + (\omega_1^2 l_1(m_1+m_2) + \omega_2^2 l_2 m_2 \cos\Delta\theta)\sin\Delta\theta\right)$$

where $g = 9.81\,\mathrm{m/s^2}$ is the acceleration of gravity, $l1$ and $l2$ are the lengths of two strings, $m1$ and $m2$ are the two masses. The equations provided in the text do not yield the correct results. Then, I asked chat GPT for the correct equations. We initially set $l1 = l2 = m1 = m2 = 1$. Initial conditions: $\theta_1(0) = \theta_2(0) = \frac{\pi}{2}$ and $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$.

We decide to use the RK4 method to solve the system of ODEs.

1.1 Code structure

- Function
  - for the system of ODEs ("double_pendulum_derivatives")
  - performing one "RK4 step" in vectorial form ("rk4_step")
  - performing all the RK4 steps ("solve_double_pendulum")
- We set t_start=0, t_end=70, and n_points=25000 ($\delta t = 0.0028$).
- We solve the system and make an animated plot as well as the plot for the time evolution of θ_1 and θ_2.

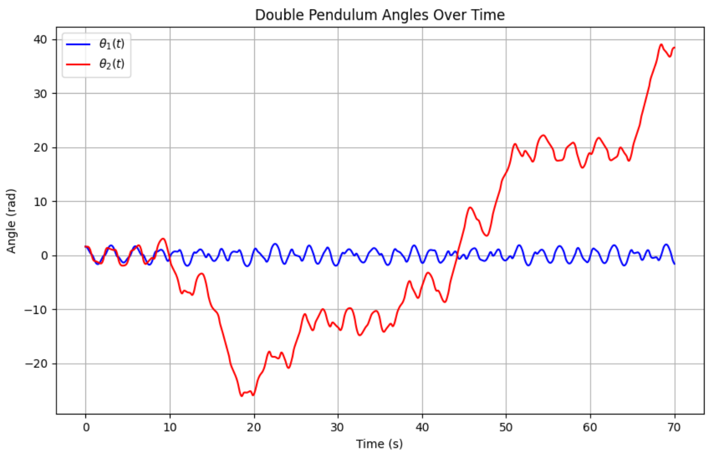


Figure 1: Time evolution of θ_1 and θ_2

2 Chaotic behavior

- We solve the equations considering a slightly different initial condition: $(\tilde{\theta}_1, \tilde{\theta}_2)(0) = (\theta_1, \theta_2 + 10^{-5})(0)$.
- We then plot the time evolution of the Cartesian coordinates.

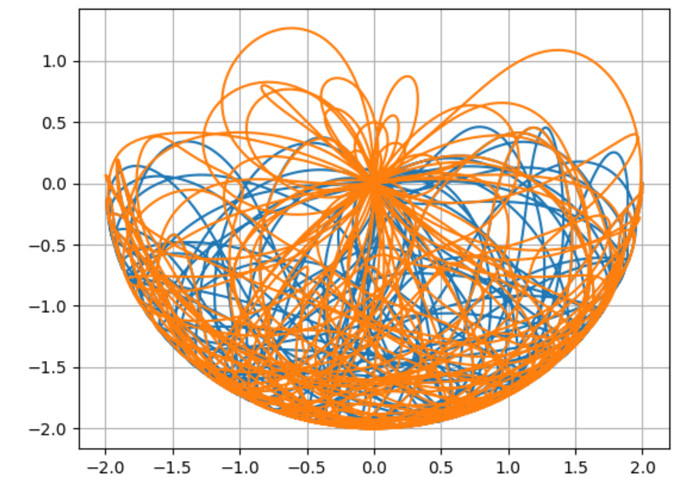


Figure 2: Comparison between the two time evolutions in the Cartesian plane.

- We compute the Lyapunov exponent λ at each step of the time evolution with the formula:

$$\lambda = \frac{1}{t[i] - t\_start} \log \frac{\sqrt{(x2[i] - x2\_2[i])^2 + (y2[i] - y2\_2[i])^2 + (z2[i] - z2\_2[i])^2}}{\sqrt{(x2[0] - x2\_2[0])^2 + (y2[0] - y2\_2[0])^2 + (z2[0] - z2\_2[0])^2}} \tag{1}$$

where $x2$, $y2$, and $z2$ are the Cartesian coordinates of the second mass of the first time evolution, while $x2_2$, $y2_2$, and $z2_2$ are the Cartesian coordinates of the second mass of the second time evolution.

- We plot the Lyapunov exponent:

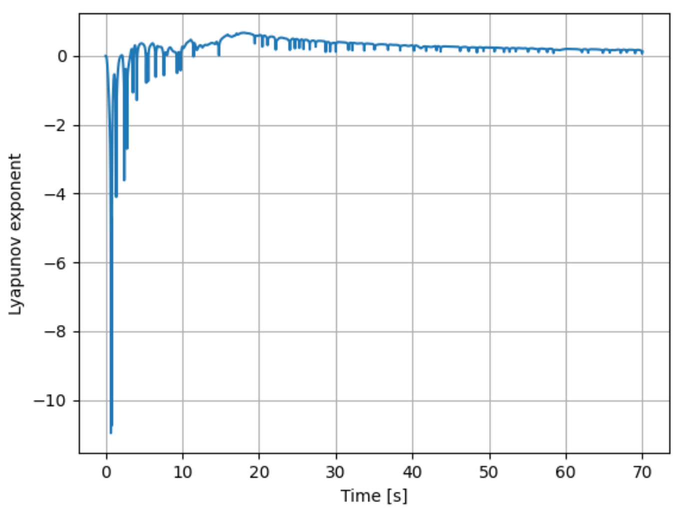


Figure 3: The Lyapunov exponent calculated at each step of the time evolution.

We find a positive Lyapunov exponent and this confirms that we are dealing with a chaotic system.

- We now change the mass ratio $m1/m2$ and compute the Lyapunov exponent at the end of the time evolution for each mass ratio considered.
- Plot of the Lyapunov exponent for different mass ratios:

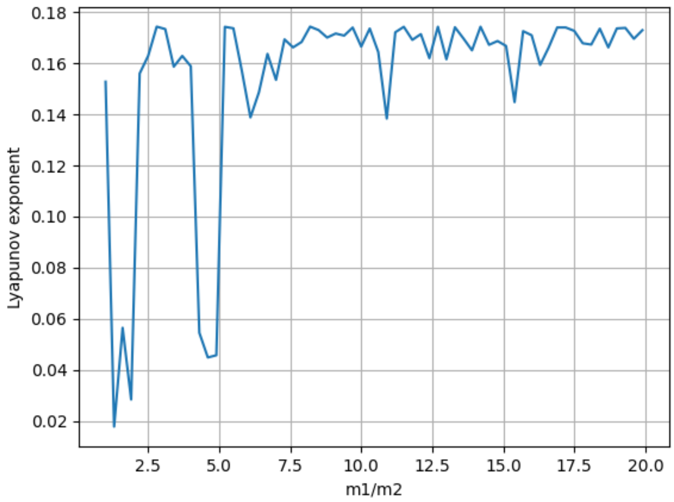
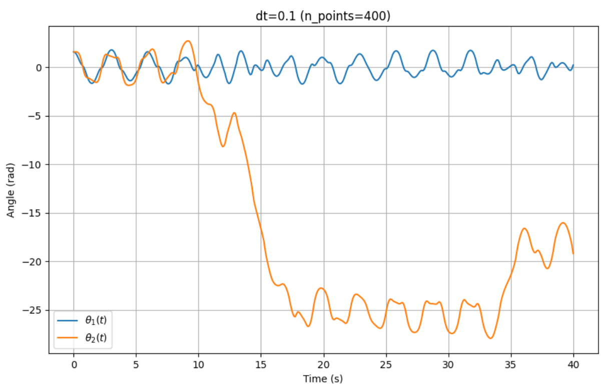
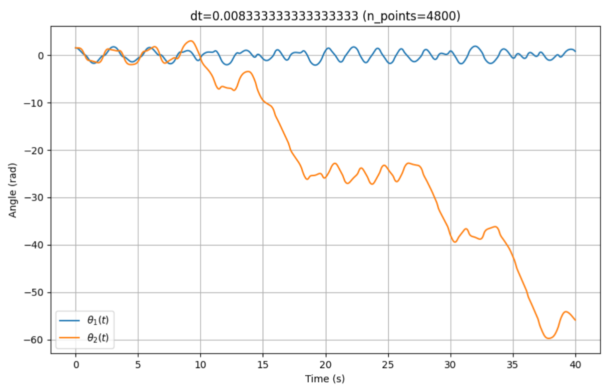


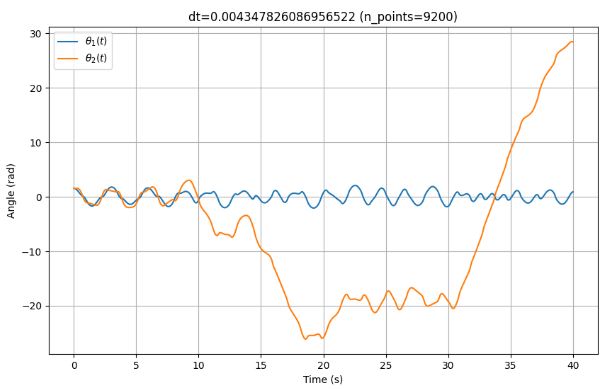
Figure 4: The Lyapunov exponent for different mass ratios.

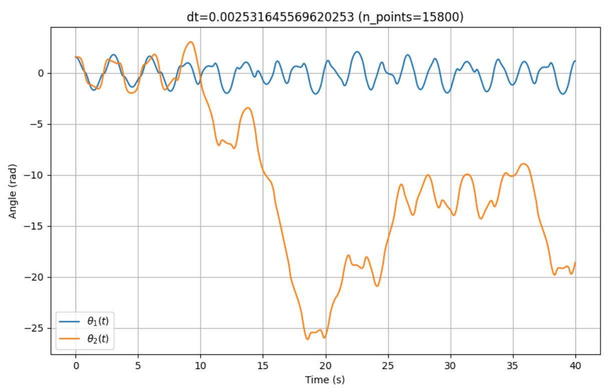
Thus, we find that the Lyapunov exponent increases as the mass ratio increases.

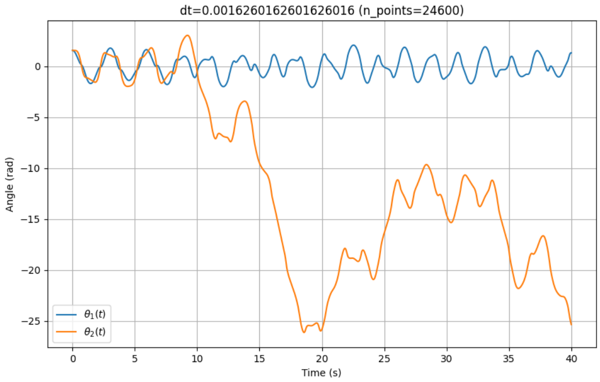
- We then change the step size δt by varying n_points to see how the time evolution is affected by this parameter.
- Some plots we got:

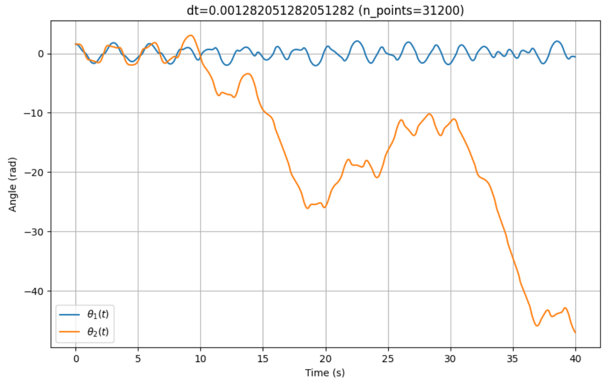


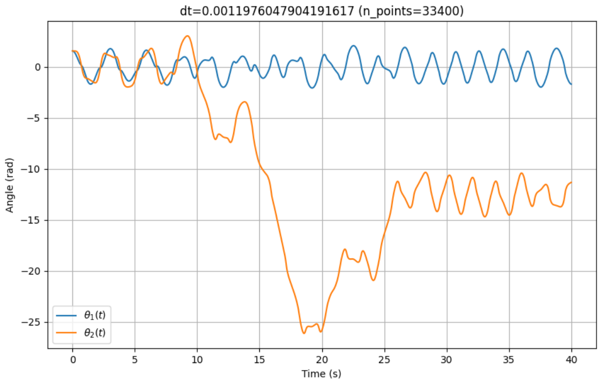


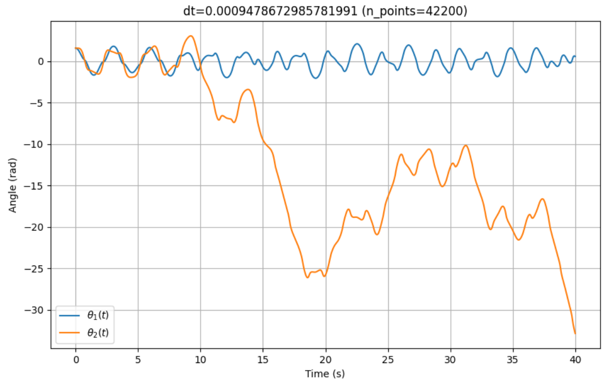





dt=0.0016260162601626016 (n_points=24600)
Angle (rad)
Time (s)
$\theta_1(t)$
$\theta_2(t)$


dt=0.001282051282051282 (n_points=31200)
Angle (rad)
Time (s)
$\theta_1(t)$
$\theta_2(t)$


dt=0.0011976047904191617 (n_points=33400)
Angle (rad)
Time (s)
$\theta_1(t)$
$\theta_2(t)$


dt=0.0009478672985781991 (n_points=42200)
Angle (rad)
Time (s)
$\theta_1(t)$
$\theta_2(t)$