Lorenz attractor - Problem V and X

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We solve the following system of ordinary differential equations

$$
\begin{aligned}
x' &= \sigma(y - x) \\
y' &= x(r - z) - y \\
z' &= xy - bz
\end{aligned}
$$

where $x = x(t)$, $y = y(t)$, $z = z(t)$ and σ, r, and b are positive constants. In our case, we set $\sigma = 10$, $r = 28$, $b = 8/3$ and $(x, y, z)(0) = (1, 1, 1)$.

We decide to use the RK4 method to solve the system od ODEs.

1 Code structure

- Function
  - for the system of ODEs (”lorenz_eq”)
  - performing one ”RK4 step” in vectorial form (”rk4_step”)
  - performing all the RK4 steps (”solve_Lorenz_eq”)
- We set initially t_start=0, t_end=60, and n_points=25,000 ($\delta t = 0.0024$).
- We solve the system and plot the solution.

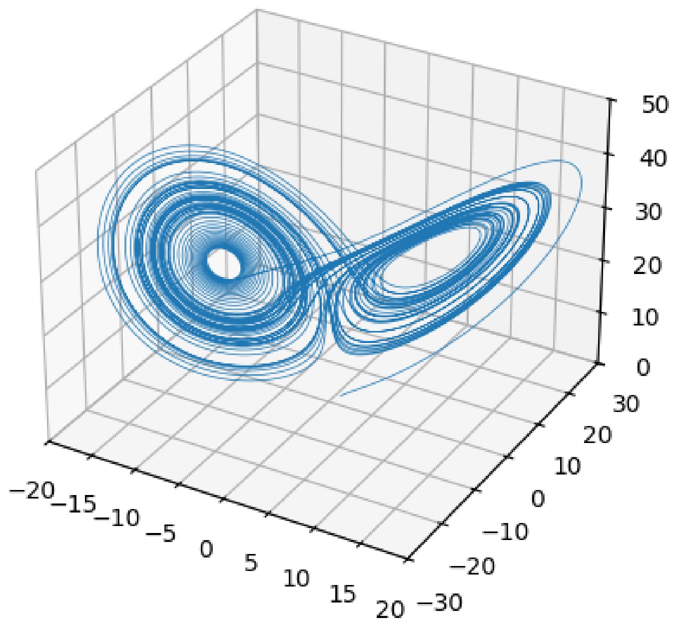
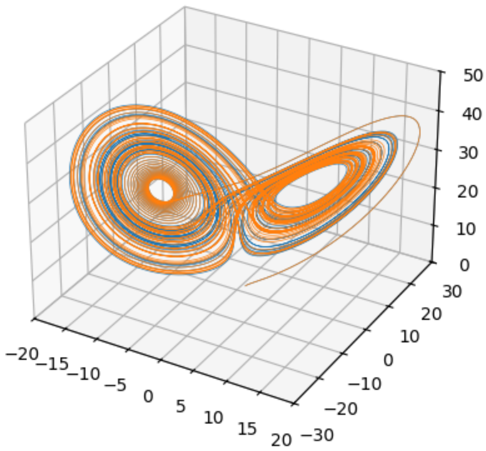


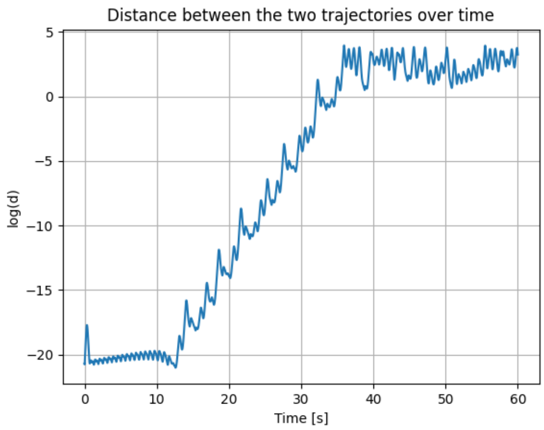
Figure 1: Solution of the system of ODEs with $h = 0.0024$.

- We also create the animated plot (with the help of chat GPT).
- We then focus on the chaotic behavior of the system.
  - Initial conditions: $(x, y, z)(0) = (1, 1, 1)$ and $(x, y, z)(0) = (1 + 10^{-9}, 1, 1)$.
  - We plot the two solutions, compute the distance d between the two trajectories at each point and plot $\log(d)$ against time.
  - Then, we compute and plot the Lyapunov exponent λ at each step of the time evolution with the formula:

$$\lambda = \frac{1}{t[i] - t\_start} \log \frac{\sqrt{(x[i] - x2[i])^2 + (y[i] - y2[i])^2 + (z[i] - z2[i])^2}}{\sqrt{(x[0] - x2[0])^2 + (y[0] - y2[0])^2 + (z[0] - z2[0])^2}}. \tag{1}$$



(a) Solutions of the system of ODEs for two slightly different initial conditions.



(b) Distance between the two trajectories over time,

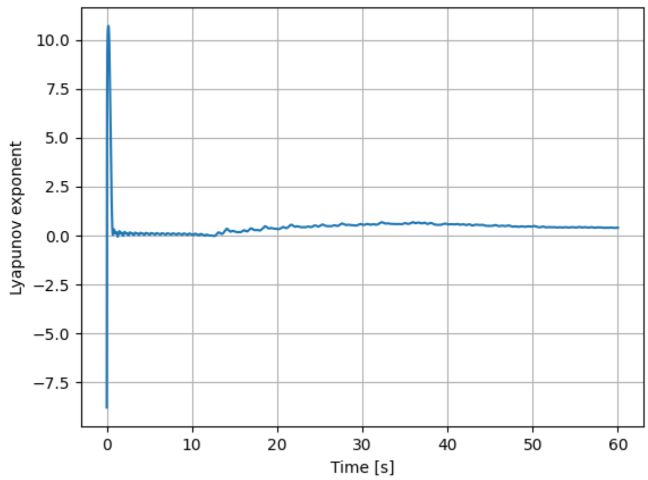


Figure 3: Lyapunov exponent

We find a positive Lyapunov exponent and this confirms that we are dealing with a chaotic system.

- We change the step size: t_start=0, t_end=60, n_points=$6 \times 10^6 \implies h = 10^{-6}$

- We change also the different initial condition: $(x, y, z)(0) = (1 + 5 \times 10^{-15}, 1, 1)$.
- We compute $\log(d)$.
- We repeat these steps for $h = 10^{-3}$.
- We plot the two distances against time.

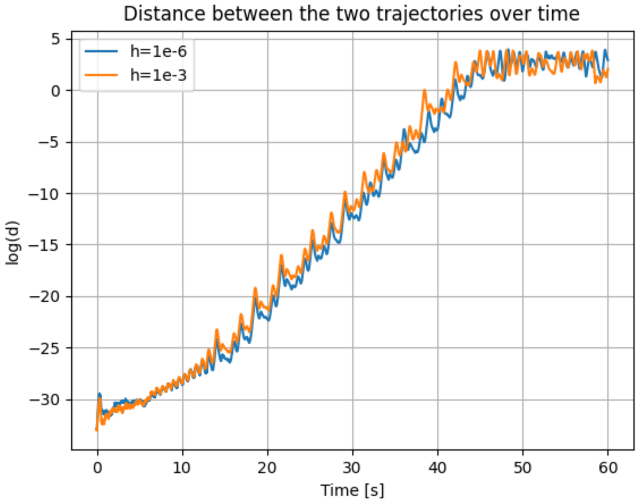


Figure 4: Distances between the two trajectories for two different step size.

This proves that the chaotic behavior is not an artificial effect due to the step size considered.