Lane-Emden equation and stellar oscillations

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1 Lane-Emden equation

We integrate the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{1}$$

where θ and ξ are dimensionless quantities used to define the density and the radial distance: $\rho = \rho_c\theta^n$, $r = a\xi$ with $a = \left[\frac{(n+1)\kappa\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]^{\frac{1}{2}}$, $\rho_c = \rho(r=0)$.

Once we get the stellar model by integrating the system of ODEs obtained by rephrasing the Lane-Emden equation

$$\begin{aligned}\frac{d\theta}{d\xi} &= x \\ \frac{dx}{d\xi} &= -\theta^n - \frac{2}{\xi}x\end{aligned}$$

we can compute the pressure and the density with the aforementioned relations as well as their first derivatives as follows:

$$\frac{d\rho}{dr} = \frac{1}{a}\frac{d}{d\xi}(\rho_c\theta^n) = \frac{\rho_c}{a}n\theta^{n-1}\frac{d\theta}{d\xi} \tag{2}$$

$$\frac{dP}{dr} = \frac{1}{a}\frac{d}{d\xi}[\kappa(\rho_c\theta^n)^\Gamma] = \frac{\kappa}{a}\rho_c^\Gamma\Gamma n\theta^n\frac{d\theta}{d\xi} \tag{3}$$

where $\Gamma = 1 + \frac{1}{n}$.

Boundary conditions: $\theta(0) = 1$ and $\theta'(0) = 0$. We initially set $n = 1$ and $\kappa = 100$, $r_{start} = 10^{-4}\,\mathrm{km}$, $r_{end} = 100\,\mathrm{km}$, n_points=$5\times10^5$.

We decide to use the RK4 method to solve the system of ODEs and to write things in $c = G = M_\odot = 1$ units.

2 Master equation

Once we have the stellar model, we can solve the master equation for radial oscillations:

$$\frac{d^2\xi_r}{dr^2} + \left[\frac{2}{r} + \frac{1}{\Gamma_1 P}\frac{dP}{dr}\right]\frac{d\xi_r}{dr} + \left[\frac{\omega^2\rho}{\Gamma_1 P} - \frac{4}{r^2}\right]\xi_r = 0.$$

with $\Gamma_1 = \frac{\partial \ln P}{\partial \ln \rho}$.

Thus, we have to solve the following system of ODEs:

$$\begin{aligned}\frac{d\xi_r}{dr} &= y \\ \frac{dy}{dr} &= -\left(\frac{2}{r} + \frac{1}{\Gamma_1 P}\frac{dP}{dr}\right)y - \left(\frac{\omega^2\rho}{\Gamma_1 P} - \frac{4}{r^2}\right)\xi_r\end{aligned}$$

Boundary conditions: we set $\xi_r(r=0) = 0$ and $\frac{d\xi_r}{dr}|_{r=0} = 10^{-4}$. On the surface the following condition must be met:

$$\Delta P = 0 \implies \frac{d\xi_r}{dr}|_R + \frac{\rho'(R)}{\rho(R)}\xi_r(R) = 0.$$

In order to find the eigenvalue ω, we choose to implement the shooting method: we keep varying ω until the condition on the surface is satisfied.

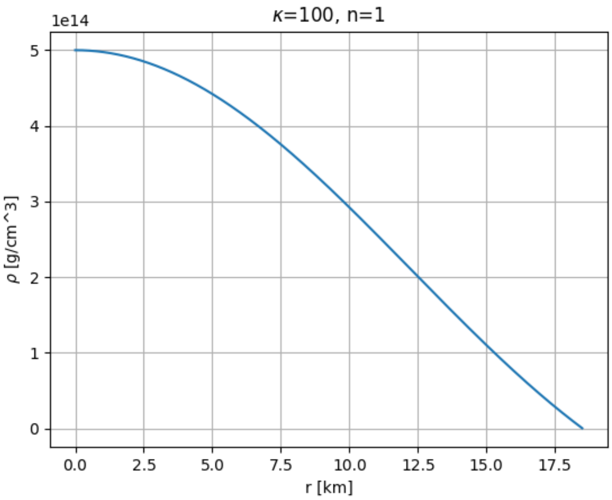
3 Code structure

3.1 Stllar model

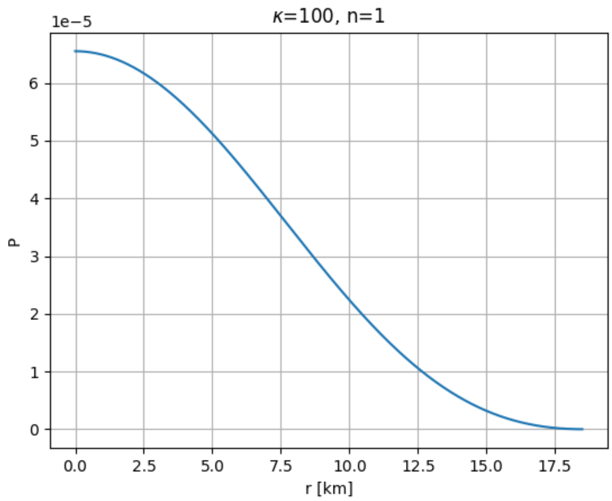
- Function
  - for the polytropic EoS ("EoS")
  - for the system of ODEs ("LE_eq")
  - for constructing the stellar model ("make_star")
    - It performs the "RK4 steps". The integration stops when $\theta(\xi) < 10^{-3}$.
  - for computing the total mass M of the star
    - It solves the following integral with the Simpson 1/3 rule:

$$M = 4\pi\rho_c a^3 \int_0^R d\xi \theta^n \xi^2. \tag{4}$$

- We get the stellar model and find $M = 2.03\,\mathrm{M}_\odot$ and $R = 18.49\,\mathrm{km}$.



(a) Density profile



(b) Pressure profile in $c = G = \mathrm{M}_\odot$=1

3.2 Radial oscillations

- Function
  - for the system of ODEs.
  - performing one RK4 step ("rk4_step").
  - performing all the RK4 steps and checking the condition on the stellar surface after each integration: the loop stops when $\frac{d\xi_r}{dr}|_R + \frac{\rho'(R)}{\rho(R)}\xi_r(R)$ changes sign.
  - for computing $\Gamma_1 = \frac{\partial \ln P}{\partial \ln \rho}$
- We then compute the first derivative of the pressure and the density with eq. (2) and (3):

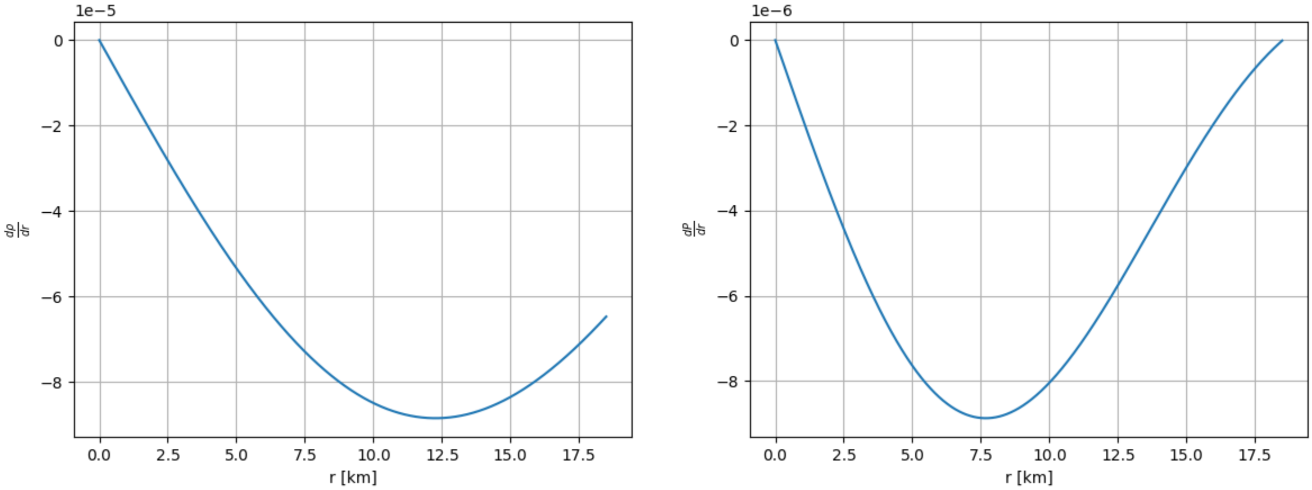
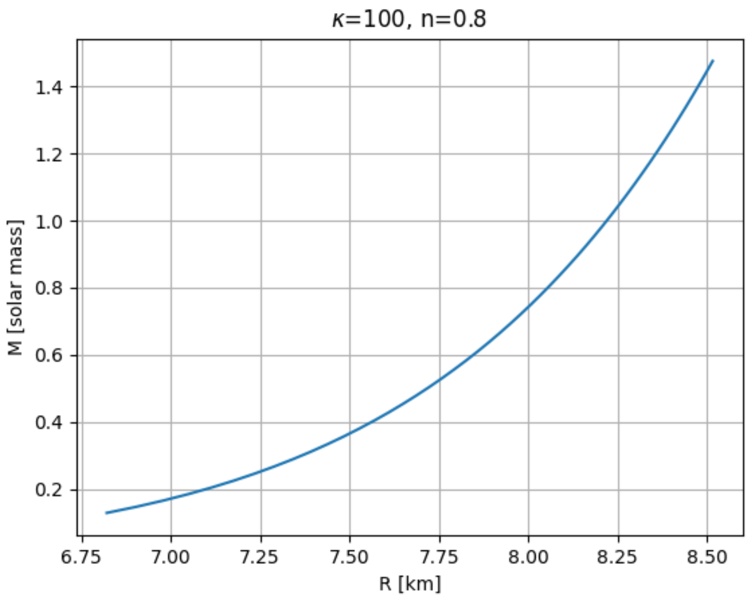


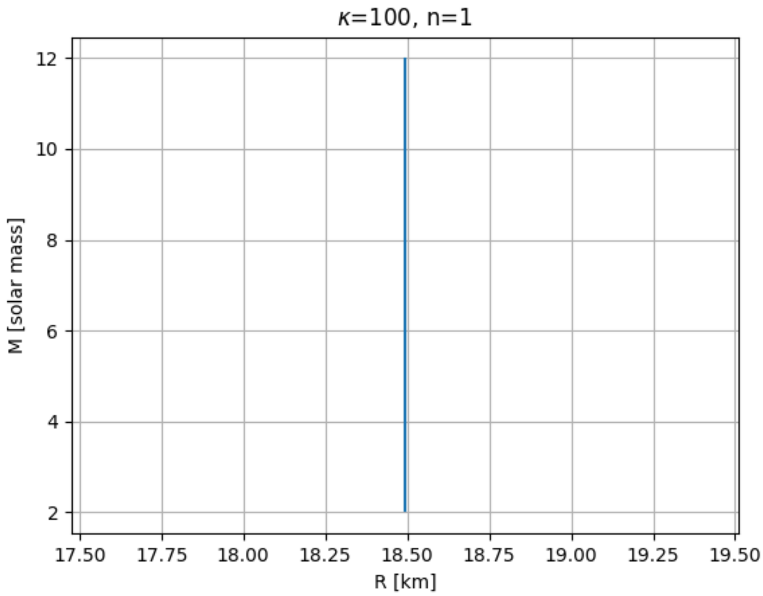
Figure 2: $c = G = \mathrm{M}_{\odot} = 1$ units.

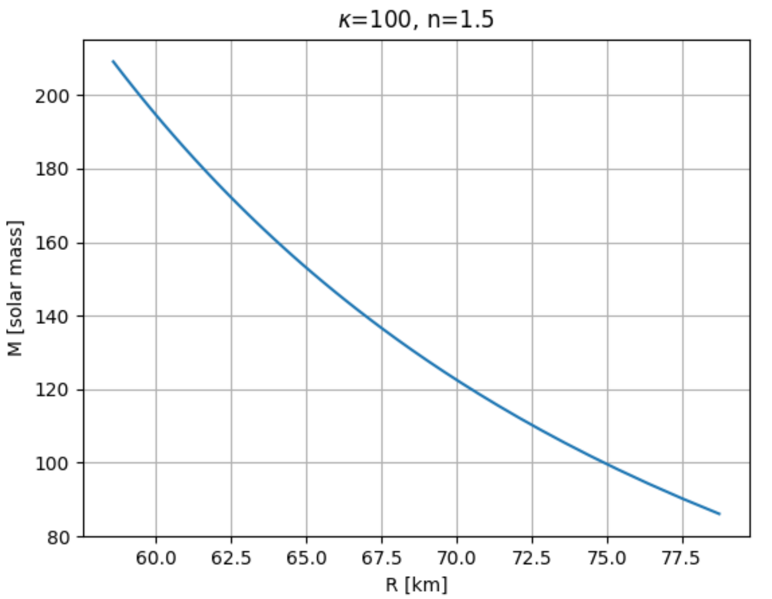
- We set $\omega \in [1, 3]$ (natural units).
- We get $\omega = 1.0014 \implies f \approx 32\,\mathrm{kHz}$.

3.3 M-R diagrams

- We build M-R diagrams for different polytropic indices: $n \in (0.8, 1, 1.5)$.
- $\kappa = 100$ is fixed.
- We find the correct behavior for each n considered:




κ=100, n=1
M [solar mass]
R [km]
17.50
17.75
18.00
18.25
18.50
18.75
19.00
19.25
19.50
2
4
6
8
10
12


κ=100, n=1.5
M [solar mass]
R [km]
60.0
62.5
65.0
67.5
70.0
72.5
75.0
77.5
80
100
120
140
160
180
200