MA202
Numerical Methods

Numerical Analysis of 2 Body Problem

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G9

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1 Introduction to the Problem

The N-body problem is the problem of predicting the individual motions of a group of celestial objects interacting with each other gravitational forces. Solving this problem is necessary for planning space expeditions since the satellites' path needs to be set before the mission. Also, this problem helps us understand the orbital motions of the planets and these orbital motions are discussed extensively in great eccentric problems like Barker's equation, parabolic eccentricity, etc. By using mathematical technique of Lagrange's problem we can solve the equations that predict the orbital motions.

First, we express the weak gravitational interaction in the form of a Lagrangian for the N-body system. Using this Lagrangian, we derived the equations we need using the Euler-Lagrange equation. Finally, we get an equation from which we can determine the position of a body with respect to the amount of time passed from a chosen position. In this analysis, we don't consider the effects of relativity which can induce several discrepancies in our calculations.

A special case of this N-body problem is a 2 body problem which is solved for Kepler's equation. For the numerical analysis of this equation, we use the Newton method and Laguerre's Iterative method. We compare these methods in terms of error and accuracy.

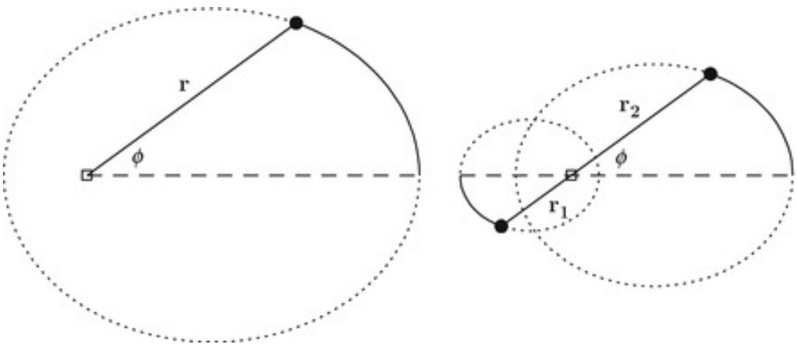


Figure 1: Two-Body Gravitational Problem

2 Mathematical Formulation

2.1 Notations

Following notations are followed throughout the text to be consistent
F : Force
$\mathcal{L}$: Lagrangian
L : Angular Momentum
ϕ : Trajectory (here, dependent on r)
$\mathcal{M}$: Mean Anomaly
ξ : Eccentric Anomaly
f : True anomaly
τ : Time of pericentric passage
μ : Gravitational Mass
m : Reduced Mass
r : Distance from one focus
e : Eccentricity of the ellipse
T : Time period of motion
l : Length of semi-latus rectum
a : Length of semi-major axis
b : Length of semi-minor axis
n : Mean motion

2.2 Assumptions

- In this case we have assumed that since the mass of the sun is much greater than the mass of earth, sun is at rest.
- Both the bodies involved in the problem are spherically symmetric and should be treated as point masses.
- We assume that the bodies have no other force acting between them other that the one due to gravity.
- We also do not consider the effects relativity as they can can induce several discrepancies in our calculations.

2.3 Kepler's Equation for 2-body Problem

The potential energy of a single particle moving through an external field depends only on the distance r from a fixed point (origin),

$$\mathbf{F} = -\frac{\partial V}{\partial \mathbf{r}}$$

In terms of Cartesian coordinates,

$$F_x = -\frac{\partial V}{\partial x} = -\frac{dV}{dr}\frac{\partial r}{\partial x}$$

Now,

$$\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2 + y^2 + z^2}}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

Therefore the force along x becomes,

$$F_x = -\frac{dV}{dr}\frac{x}{r}$$

Taking the center of the field as origin, the angular momentum can be expressed as,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Also,

$$\dot{\mathbf{r}} \times \mathbf{p} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$$

and

$$\mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F} = 0$$

Therefore the rate of change of angular momentum becomes,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = 0$$

Thus angular momentum L is conserved and is orthogonal to r and thus r stays in the same plane. Motion is planar.

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r).$$

E-L equation for coordinate $\phi \Longrightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}=\frac{\mathrm{d}\left(mr^2\dot{\phi}\right)}{\mathrm{d}t}=0$$

where

$$L_z=L=mr^2\dot{\phi}=\text{ const}$$

The total energy E can be written as,

$$E=2T-\mathcal{L}=T+V=\text{ const}$$

$$E=\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)+V(r)=\tfrac{1}{2}m\dot{r}^2+\tfrac{L^2}{2mr^2}+V(r),$$

$$\tfrac{\mathrm{d}r}{\mathrm{d}t}=\sqrt{\tfrac{2}{m}\left[E-V_{\text{eff}}(r)\right]}$$

$$\mathrm{d}t=\tfrac{\mathrm{d}r}{\sqrt{\frac{2}{m}\left[E-V_{\text{eff}}(r)\right]}}$$

where

$$V_{\text{eff}}(r)=V(r)+\frac{L^2}{2mr^2}$$

Trajectory $\phi=\phi(r)$ (angle ϕ as a function of r),
We know that the angular momentum can be written as $mr^2\ \mathrm{d}\phi/L=\mathrm{d}t$,

$$\mathrm{d}\phi=\frac{L\ \mathrm{d}r}{r^2\sqrt{2m\left[E-V_{\text{eff}}\left(r\right)\right]}}$$

Now for a two-body problem the Lagrangian can be expressed as,

$$\mathcal{L}=\frac{1}{2}m_1\left|\dot{\mathbf{r}}_1\right|^2+\frac{1}{2}m_2\left|\dot{\mathbf{r}}_2\right|^2-V\left(\left|\mathbf{r}_1-\mathbf{r}_2\right|\right)$$

Taking origin to be the center of mass, so $\mathbf{r}_1m_1+\mathbf{r}_2m_2=0$, and define $\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$, so

$$\mathbf{r}_1=\frac{m_2}{m_1+m_2}\mathbf{r}$$
$$\mathbf{r}_2=-\frac{m_1}{m_1+m_2}\mathbf{r}$$

Substituting the values of r1 and r2 from the above equation we get,

$$\mathcal{L}=\frac{1}{2}m|\dot{\mathbf{r}}|^2-V(r)$$

where

$$m \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

Here m is the reduced mass and $M \equiv m_1 + m_2$ is the total mass.

$$V\left(\left|\mathbf{r}_1 - \mathbf{r}_2\right|\right) = -\frac{G m_1 m_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|}$$

$$V(r) = -\frac{G m_1 m_2}{r} = -\frac{G m M}{r}$$

In celestial mechanics μ, the gravitational mass is taken as GM or $G\left(m_1 + m_2\right)$, Note that m and M have units of mass, while μ has units of length^3/time^2.

For $V = -\alpha/r$ with constant $\alpha > 0$. We are describing the motion of a particle m moving in a central potential $V(r) = -\alpha/r$.

For the gravitational two-body problem with reduced mass $m, \alpha = G\left(m_1 + m_2\right) m = \mu m$. So we will take $\alpha = m\mu$.

We saw that radial motion in a central field is similar to 1-D motion with effective potential energy.

$$V_{\text{eff}}\left(r\right) = -\frac{m\mu}{r} + \frac{L^2}{2mr^2},$$

Minimum of V_{eff} at $r = L^2/m^2\mu . V_{\text{eff ,min}} = -\mu m^2/2L^2$. When $E = V_{\text{eff ,min}}$ the orbit is circular.

Motion is possible only when $E > V_{\text{eff.}}$ If $E < 0$ motion is finite. If $E > 0$ motion is infinite.

Path: $\phi = \phi(r)$. Take

$$\mathrm{d}\phi = \frac{L \ \mathrm{d}r}{r^2\sqrt{2m\left[E - V_{\text{eff}}(r)\right]}} = \frac{L \ \mathrm{d}r}{r^2\sqrt{2m[E - V(r)] - \frac{L^2}{r^2}}}$$

and substitute $V = -m\mu/r$. We get

$$\mathrm{d}\phi = \frac{L \ \mathrm{d}r}{r^2\sqrt{2m\left[E + \frac{m\mu}{r}\right] - \frac{L^2}{r^2}}} = \frac{L \ \mathrm{d}r}{r^2\sqrt{2\left[\tilde{E} + \frac{\mu}{r}\right] - \frac{L^2}{r^2}}}$$

where $\tilde{L} \equiv L/m$ is the modulus of the angular momentum per unit mass and $\tilde{E} = E/m$ is the total energy per unit mass. After integrating the above equation we get:

$$\phi = \arccos \frac{(L/r) - \left(m^2\mu/L\right)}{\sqrt{2mE + \frac{m^3b^2}{L^2}}} + \phi_0 = \arccos \frac{\frac{L^2}{m^2\mu r} - 1}{\sqrt{1 + \frac{2EL^2}{m^3\mu^2}}} + \phi_0 = \arccos \frac{\frac{L^2}{\mu^2} - 1}{\sqrt{1 + \frac{2\tilde{E}\tilde{L}^2}{\mu^2}}} + \phi_0$$

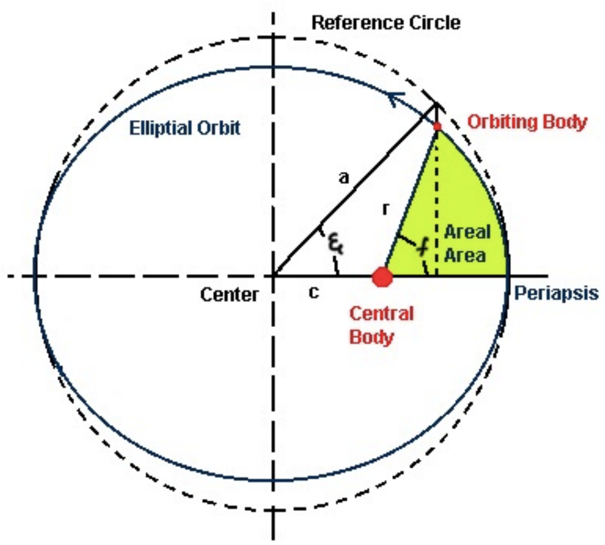
with ϕ_0 constant (verified by differentiation). Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

Defining $\ell \equiv L^2/m^2\mu = L^2/\mu$ and $e \equiv \sqrt{1 + (2EL^2/m^3\mu^2)} = \sqrt{1 + (2EL^2/\mu^2)}$ we get

$$\frac{\ell}{r} = 1 + e \cos f$$

Here $f = \phi - \phi_0$ is the true anomaly. This is the equation of a conic section where ℓ is the semi-latus rectum and e is the eccentricity. r is the distance from one focus. ϕ_0 is such that $\phi = \phi_0$ at the pericentre (perihelion).



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \ell/\left(1 - e^2\right) \quad (\text{ Semi } - \text{ major axis })$$

$$b = \ell/\sqrt{1 - e^2} \quad (\text{ Semi } - \text{ minor axis }),$$

$$b = a\sqrt{1 - e^2}$$

From the relations among e, ℓ, L and E, we get

$$a=\ell/\left(1-e^{2}\right)=\mu/2|\vec{E}|$$

$$b=\ell/\sqrt{1-e^{2}}=\bar{L}/\sqrt{2|\tilde{E}|}$$

Pericentre and apocentre. We remember that the distance r from one of the foci is given by the equation

$$r=\frac{\ell}{1+e\cos\phi}=\frac{a\left(1-e^{2}\right)}{1+e\cos\phi}$$

where we have assumed $\phi_0=0$, so $f=\phi$. Therefore the apocentre $(\cos\phi=-1)$ is $r_{\text{apo}}=a(1+e)$ and the

$$\text{pericentre } (\cos\phi=1) \text{ is } r_{\text{peri}}=a(1-e)$$

Also for motion in a central field the sectorial velocity $\mathrm{d}A/\mathrm{d}t$ is constant (from Kepler's second law).

Let's start by defining an infinitesimal sector bounded by the path as follows:

$$\mathrm{d}A=\frac{1}{2}r^{2}\ \mathrm{d}\phi$$

$\mathrm{d}A/\mathrm{d}t=r^{2}(\ \mathrm{d}\phi/\mathrm{d}t)/2=L/(2m)=$ const is the sectorial velocity $\Longrightarrow$ the particle's position vector sweeps equal areas in equal times (Kepler's second law).

$$\bar{L}=r^{2}\dot{\phi}=2\frac{\mathrm{d}A}{\mathrm{d}t}=\text{const}$$

we get period T for elliptic orbit:

$$\tilde{L}\ \mathrm{d}t=2\ \mathrm{d}A\Longrightarrow T\bar{L}=2A=2ab\pi$$

where $A=\pi ab$ is the area of the ellipse.

$$T=2\pi a^{3/2}/\sqrt{\mu}=\pi\mu/\sqrt{2|\bar{E}|^{3}}$$

which is Kepler's third law $T\propto a^{3/2}$. Keep in mind that the period is solely determined by the amount of energy available. We have used definitions of a, b as functions of $\bar{L}$ and $\bar{E}: a=\mu/2|\bar{E}|, b=\bar{L}/\sqrt{2|\bar{E}|}$ Mean motion. Kepler's third law can be written as

$$T^{2}\mu=4\pi^{2}a^{3}\qquad\text{or}\qquad\mu=n^{2}a^{3}$$

with $n \equiv 2\pi/T$ is the mean motion (i.e. the mean angular velocity).

Kepler's first law: the orbit of each planet is an ellipse with the Sun in one of its foci. It is just a special case of the general result that the orbits are ellipses for $E < 0$. Using the expression for the orbital energy we can relate the velocity modulus v to r and a as follows:

$$\bar{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

so

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

or

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1}$$

Kepler's equation $\rightarrow$ We have seen that for motion in a central field the time dependence of the coordinates is given by

$$\mathrm{d}t = \frac{\sqrt{m}\ \mathrm{d}r}{\sqrt{2[E - V(r)] - \frac{L^2}{mr^2}}}$$

For Kepler's potential

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{2\left[\tilde{E} + \frac{\mu}{r}\right] - \frac{\tilde{L}^2}{r^2}}},$$

can be integrated analytically (see below, for instance for elliptic orbits: Kepler's equation).

From the time dependence of radial coordinate (see above) we have, in the case of elliptic orbit:

$$\mathrm{d}t = \frac{r\ \mathrm{d}r}{\sqrt{2|\tilde{E}|}\sqrt{-r^2 + \mu r/|\tilde{E}| - \tilde{L}^2/2|\tilde{E}|}}$$

Note that

$$\frac{\tilde{L}^2}{2|\tilde{E}|} = b^2 = a^2\left(1 - e^2\right) = a^2 - a^2e^2$$

and

$$\mu r/|\tilde{E}| = 2ar, \quad \text{because} \quad \mu = 2a|\tilde{E}|$$

so

$$\mathrm{d}t = \frac{r\ \mathrm{d}r}{\sqrt{2|\tilde{E}|}\sqrt{a^2e^2 - (r-a)^2}}$$

Let us introduce the angular variable ξ, known as the eccentric anomaly. We substitute

$$r = a(1 - e\cos\xi)$$

$$\mathrm{d}t = \sqrt{a^2/2|\tilde{E}|}(1 - e\cos\xi)\mathrm{d}\xi$$

$$t = \sqrt{a^3/\mu}(\xi - e\sin\xi) + \text{ const}$$

where we have used $\mu = 2|\tilde{E}|a$. Note that $0 < \xi < 2\pi$: we made the calculation for $[0, \pi]$ (so $\sin\xi = \sqrt{1-\cos^2\xi}$. The calculation for $[\pi, 2\pi]$ is similar, with $\sin\xi = -\sqrt{1-\cos^2\xi}$ So, for an elliptic orbit

$$\begin{aligned} r &= a(1 - e\cos\xi) \\ t - \tau &= \sqrt{a^3/\mu}(\xi - e\sin\xi) \end{aligned}$$

where τ is the time of pericentric passage [because $\xi = 0$ when $t = \tau$, so $r = a(1-e) = r_{\text{peri}}$]. The above equation is known as Kepler's equation. Here $0 \leq \xi \leq 2\pi$ for one period. To obtain ξ (then r) as a function of t Kepler's equation must be solved numerically. Note that $\sqrt{a^3/\mu} = T/2\pi$. Often the eccentric anomaly is indicated with E, instead of ξ Mean anomaly, true anomaly and eccentric anomaly. In Kepler's equation ξ is the eccentric anomaly. Kepler's equation can be written as

$$\boxed{\mathcal{M} = \xi - e\sin\xi}$$

where $\mathcal{M} = n(t-\tau)$ is the mean anomaly, with $n = 2\pi/T$ mean motion and $T = 2\pi 3/2/\sqrt{\mu}$ is the period.

3 Numerical Analysis

3.1 Newton's Iteration Method

Newton's method is an iterative methods to find the roots of a function. if we want to find the roots of a function $f(x) = 0$, we can start with a guess that could be based on the graph. Suppose our initial guess is $x = c$ then we can find the equation of the tangent of the curve at point $x = c$. The point at which that tangent line intersects the x-axis is our new root which is closer to the actual root. Doing this through iterations, we can get closer to the actual root.

Here is the general algorithm:
1. Start with a guess value x_0.
2. Find the next approximation using the formula, $x_1 = x_0 - \frac{f'(x_0)}{f(x_0)}$.
3. Continue the iterative process using the formula, $x_{i+1} = x_i - \frac{f'(x_i)}{f(x_i)}$ until we get the desired accuracy.

3.2 Laguerre's Iterative Method

Laguerre's method is a root-finding algorithm for polynomials. It can be used to find the root of the polynomial $p(x) = 0$. This iterative methods involves the first and second derivative which can be computed either analytically or numerically.
1. Start with a guess value $x_0 = c$.
2. Find the next approximation using the following,

$G = \frac{p'(x_k)}{p(x_k)}$

$H = G^2 - \frac{p''(x_k)}{p(x_k)}$

$a = \frac{n}{G \pm \sqrt{(n-1)(nH - G^2)}}$

$x_{k+1} = x_k - a$

We can continue this until we get the desired accuracy.

Though this method is used to find the roots of an n-degree polynomial, this method can also be used in this case. The sin function in the formula can be expanded using Taylor expansion and the value of n that is used while executing the method, is where we truncate off the series since Taylor expansion of sine is an infinite series.

4 Algorithm and MATLAB Program

4.1 Newton's Iteration Method Algorithm

- Firstly what we did was to run a for loop defining the range of e (eccentricity) thereby followed by another for loop defining the range of M (mean anomaly)
- Then as an initial value, we set E (eccentric anomaly) equal to M (mean anomaly)
- Followed by this we write the function of E in terms of M and E and find its derivative.
- Now we write a while loop where we run the Newtons iterative method (as mentioned above) In which we find values of E every time the loop runs till the error is less than 1e-4.
- We will count the number of iterations and store its value in a matrix with the rows and column being the e and M.
- Now in order to compare the efficiency of both the curves we will store the value of E (eccentric anomaly) in a matrix same as before with the rows and columns being e and M.
- We will plot a 3D curve with the x-axis being M, y-axis being e and z-axis being the number of iterations.
- We will plot a 3D curve with the x-axis being M, y-axis being e and z-axis being E (eccentric anomaly).

```
% Newton's Iteration Method Algorithm
clearvars;
iterations=zeros(100,315);
E_array=zeros(100,315);
for e = 0.01:0.01:0.99                    % Range of eccentricity
    j=int16(e*100)+1;                     % for matrix
    for M = 0.01:0.01:3.14                % Range of Mean Anomaly
        k=int16(M*100)+1;
        E=M;                              % Initial Guess
        f_E = @(E) E-M-e*sin(E);          % Keplers Eqn function
        fdash = @(E) 1-e*cos(E);          % First derivative of function
        i=0;
        % Newton's Formula
        while (true)
            a=f_E(E)/fdash(E);
            E = E - a;
            if(abs(a/E)<1e-4)
                break
            end
            i=i+1;
        end
        E_array(j,k)=E;
        if i>10
            iterations(j,k)=10;
            continue
        end
        iterations(j,k)=i;
    end
end
surf(iterations)                              % 3D plot of M,e,iterations
surf(E_array)
shading interp
xlabel('M*100')
ylabel('e*100')
zlabel('number of iterations')
title("Number of iterations required for convergence using Newton's method")
```

4.2 Laguerre's Iterative Method Algorithm

- Firstly what we did was to run a for loop defining the range of e (eccentricity) thereby followed by another for loop defining the range of M (mean anomaly).
- Then as an initial value, we set E (eccentric anomaly) equal to M (mean anomaly).
- Followed by this we write the function of E in terms of M and E and find its derivative two times as required in Laguerre's iterative method.
- Now we write a while loop where we run Laguerre's iterative method (as mentioned above) In which we find values of E every time the loop runs till the error is less than 1e-4.
- We will count the number of iterations and store its value in a matrix with the rows and column being the e and M.
- Now in order to compare the efficiency of both the curves we will store the value of E (eccentric anomaly) in a matrix same as before with the rows and columns being e and M.
- Now we will plot a 3D curve with the x-axis being M, y-axis being e and z-axis being the number of iterations.
- We will plot a 3D curve with the x-axis being M, y-axis being e and z-axis being E (eccentric anomaly).

```
% Laguerre?s Iteration Method Algorithm
clearvars;
iterations=zeros(100,315);
E_array=zeros(100,315);
n=10;
for e = 0.01:0.01:0.99                    % Range of eccentricity
    j=int16(e*100)+1;                     % for matrix
    for M = 0.01:0.01:3.14                % Range of Mean Anomaly
        k=int16(M*100)+1;
        E=M;                              % Initial Guess
        f_E = @(E) E-M-e*sin(E);          % Keplers Eqn function
        fdash = @(E) 1-e*cos(E);          % First derivative of function
        fddash = @(E) e*sin(E);
        i=0;
        % Laguerre's Formula
        while (true)
            g=(fdash(E))/f_E(E);
            h=g^2-((fddash(E))/f_E(E));
            a=n/(g+sign(g)*sqrt((n-1)*(n*h-g^2)));
            E=E-a;
            a/E;
            if(abs(a/E)<1e-4)
                break
            end
            i=i+1;
            E_array(j,k)=E;
        end
        iterations(j,k)=i;
    end
end
surf(iterations)                          % 3D plot of M,e,iterations
surf(E_array)
shading interp
xlabel('M*100')
ylabel('e*100')
zlabel('number of iterations')
title("Number of iterations required for convergence using Laguerre's method")
```

5 Results and Discussion

From running the codes given above, we get following graphs:

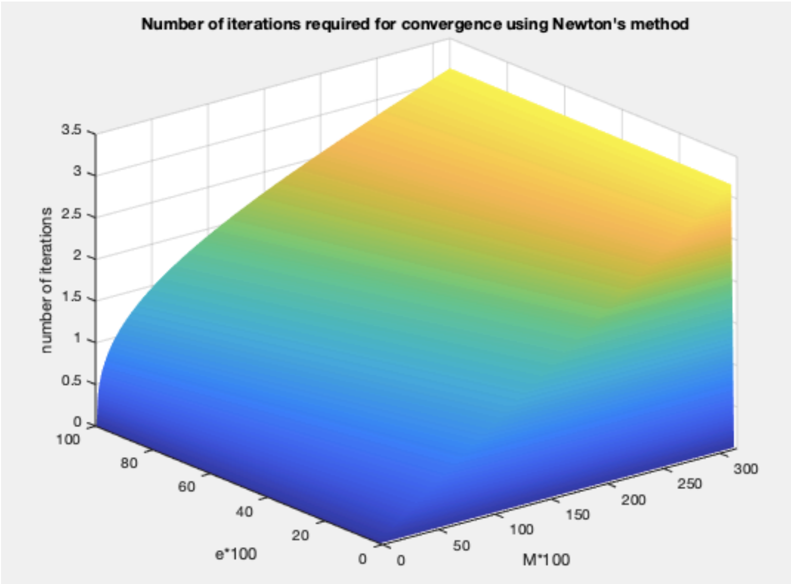


Figure 2: Plot of variation in E with e and M using Newton's method

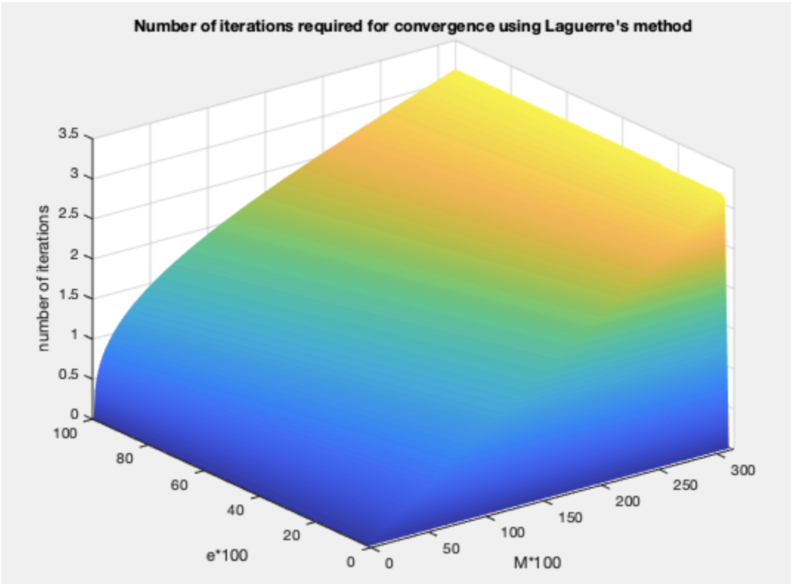


Figure 3: Plot of variation in E with e and M using Laguerre's method

As we can see, the two resulting graphs are exactly the same. Hence both the iterative methods give the same value for E at given specific e and M. Now, we can put the value of the eccentricity in the graph and we can get the value of E for a varying value of M. By doing this we can determine the motion of the body.

Here, M can be used to keep track of time from the formula, and E can be used to determine the angular position of the body. Also, we can conclude that the given code correctly calculates the required value.

Now, we need to compare the efficiency of both the codes. While calculating the value of E at each quantified values of e and M, we also calculate the number of iterations that are required to calculate the value of E. Now, we have the number of iterations required for each e and M, we plot these values for each of the method and compare these graphs.

The graphs are as follows:

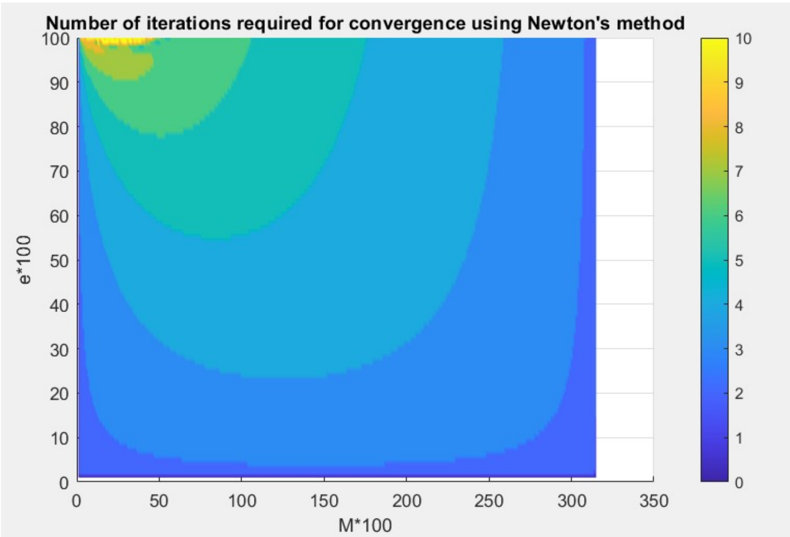


Figure 4: 2D plot of number of iteration requires using Newton's method

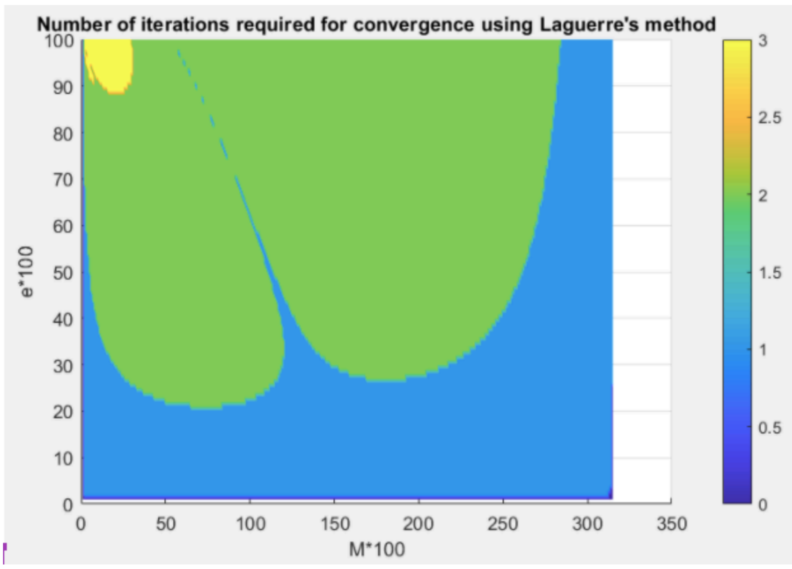


Figure 5: 2D plot of number of iteration requires using Laguerre's method

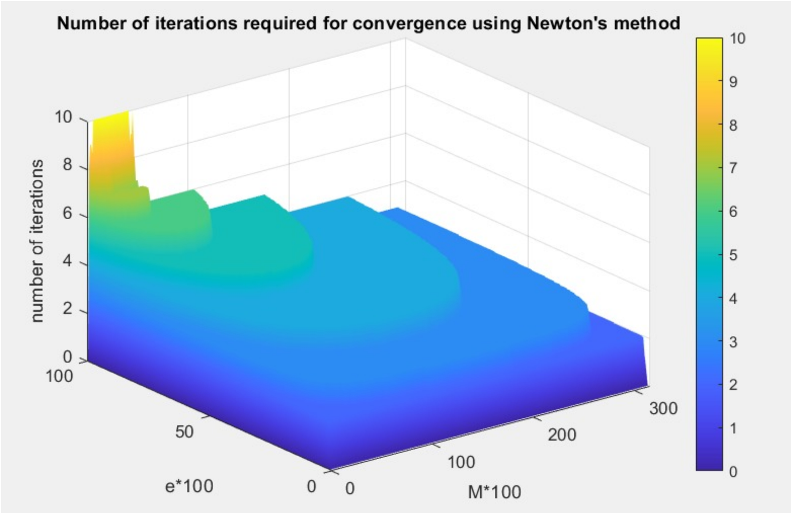


Figure 6: 3D simulation of M,e,Number of iterations using Newton's method.

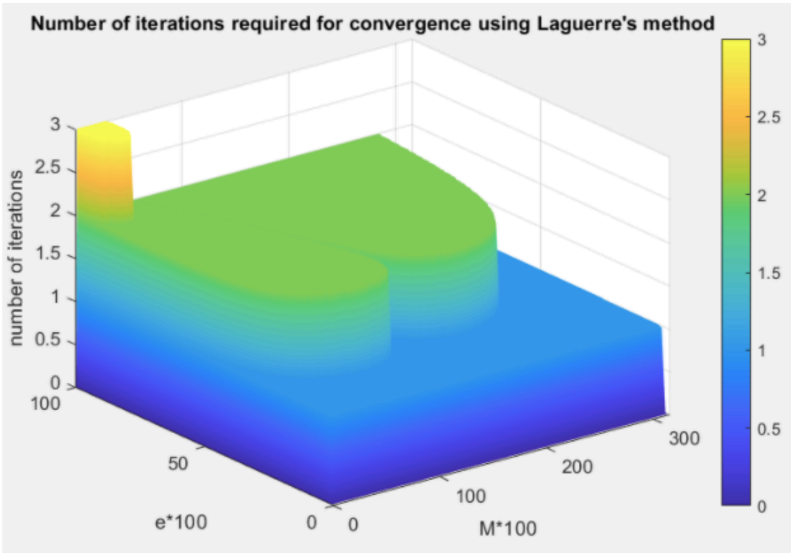


Figure 7: 3D simulation of M,e,Number of iterations using Laguerre's method.

From the graphs above, we can see that in the case of Newton's method, the number of iterations have a range of 1 to 10. Similarly, for the Laguerre's method, the range is from 1 to 3. Also, in the case of Laguerre's method, majority of the area covered in 2-D plot represents 1 or 2 iterations. While the majority of the area in the Newton's method belongs to 2 to 6 iterations.

Hence we can safely conclude that Laguerre's method is much more efficient than the Newton's method.

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