Numerical Analysis of 2 Body Problem

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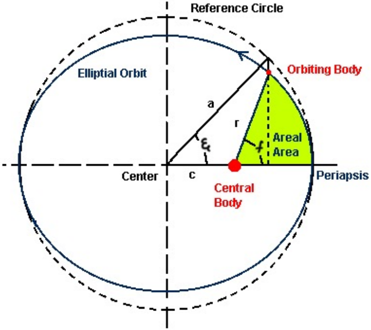
Introduction

- N - Body gravitational problem involves predicting the motion of N celestial objects due to gravitational force exerted on each object.
- Application: planning paths for space expedition, orbital motion of celestial bodies, etc.
- This project solves the basic 2 - Body problem and solves it numerically by Newton's method and Laguerre's Iterative Method.
- The results and algorithms are compared in the end to analyse the performance of the two methods.


Reference Circle
Elliptial Orbit
Orbiting Body
a
r
Areal
Area
Center
c
Central
Body
Periapsis

Mathematical Formulation

2.3 Kepler's Equation for 2-body Problem

The potential energy of a single particle moving through an external field depends only on the distance r from a fixed point (origin),

$$\mathbf{F} = -\frac{\partial V}{\partial \mathbf{r}}$$

In terms of Cartesian coordinates,

$$F_x = -\frac{\partial V}{\partial x} = -\frac{dV}{dr}\frac{\partial r}{\partial x}$$

Now,

$$\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2+y^2+z^2}}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Therefore the force along x becomes,

$$F_x = -\frac{dV}{dr}\frac{x}{r}$$

Taking the center of the field as origin, the angular momentum can be expressed as,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Also,

$$\dot{\mathbf{r}} \times \mathbf{p} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$$

and

$$\mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \mathbf{F} = 0$$

Therefore the rate of change of angular momentum becomes,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = 0$$

Thus angular momentum L is conserved and is orthogonal to r and thus r stays in the same plane. Motion is planar.

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r).$$

E-L equation for coordinate $\phi \Longrightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\mathrm{d}\left(mr^2\dot{\phi}\right)}{\mathrm{d}t} = 0$$

where

$$L_z = L = mr^2\dot{\phi} = \text{ const}$$

The total energy E can be written as,

$$E = 2T - \mathcal{L} = T + V = \text{ const}$$

$$E = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r) = \tfrac{1}{2}m\dot{r}^2 + \tfrac{L^2}{2mr^2} + V(r),$$

$$\tfrac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\tfrac{2}{m}\left[E - V_{\text{eff}}(r)\right]}$$

$$\mathrm{d}t = \tfrac{\mathrm{d}r}{\sqrt{\frac{2}{m}\left[E - V_{\text{eff}}(r)\right]}}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}$$

Trajectory $\phi = \phi(r)$ (angle ϕ as a function of r),
We know that the angular momentum can be written as $mr^2\ \mathrm{d}\phi/L = \mathrm{d}t$,

$$\mathrm{d}\phi = \frac{L\ \mathrm{d}r}{r^2\sqrt{2m\left[E - V_{\text{eff}}\left(r\right)\right]}}$$

Now for a two-body problem the Lagrangian can be expressed as,

$$\mathcal{L} = \frac{1}{2}m_1\left|\dot{\mathbf{r}}_1\right|^2 + \frac{1}{2}m_2\left|\dot{\mathbf{r}}_2\right|^2 - V\left(\left|\mathbf{r}_1 - \mathbf{r}_2\right|\right)$$

Taking origin to be the center of mass, so $\mathbf{r}_1 m_1 + \mathbf{r}_2 m_2 = 0$, and define $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, so

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}$$
$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}$$

Substituting the values of r1 and r2 from the above equation we get,

$$\mathcal{L} = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(r)$$

where

$$m \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

Here m is the reduced mass and $M \equiv m_1 + m_2$ is the total mass.

$$V\left(|\mathbf{r}_1 - \mathbf{r}_2|\right) = -\frac{G m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

$$V(r) = -\frac{G m_1 m_2}{r} = -\frac{G m M}{r}$$

In celestial mechanics μ, the gravitational mass is taken as GM or $G\left(m_1 + m_2\right)$, Note that m and M have units of mass, while μ has units of length^3/time^2.

For $V = -\alpha/r$ with constant $\alpha > 0$. We are describing the motion of a particle m moving in a central potential $V(r) = -\alpha/r$.

For the gravitational two-body problem with reduced mass $m, \alpha = G\left(m_1 + m_2\right) m = \mu m$. So we will take $\alpha = m\mu$.

We saw that radial motion in a central field is similar to 1-D motion with effective potential energy.

$$V_{\text{eff}}\left(r\right) = -\frac{m\mu}{r} + \frac{L^2}{2mr^2},$$

Minimum of V_{eff} at $r = L^2/m^2\mu . V_{\text{eff ,min}} = -\mu m^2/2L^2$. When $E = V_{\text{eff ,min}}$ the orbit is circular.

Motion is possible only when $E > V_{\text{eff.}}$ If $E < 0$ motion is finite. If $E > 0$ motion is infinite.

Path: $\phi = \phi(r)$. Take

$$\mathrm{d}\phi = \frac{L\ \mathrm{d}r}{r^2\sqrt{2m\left[E - V_{\text{eff}}(r)\right]}} = \frac{L\ \mathrm{d}r}{r^2\sqrt{2m[E - V(r)] - \frac{L^2}{r^2}}}$$

and substitute $V = -m\mu/r$. We get

$$\mathrm{d}\phi = \frac{L\ \mathrm{d}r}{r^2\sqrt{2m\left[E + \frac{m\mu}{r}\right] - \frac{L^2}{r^2}}} = \frac{L\ \mathrm{d}r}{r^2\sqrt{2\left[\tilde{E} + \frac{\mu}{r}\right] - \frac{L^2}{r^2}}}$$

where $\tilde{L} \equiv L/m$ is the modulus of the angular momentum per unit mass and $\tilde{E} = E/m$ is the total energy per unit mass. After integrating the above equation we get:

$$\phi = \arccos \frac{(L/r) - \left(m^2\mu/L\right)}{\sqrt{2mE + \frac{m^3 b^2}{L^2}}} + \phi_0 = \arccos \frac{\frac{L^2}{m^2\mu r} - 1}{\sqrt{1 + \frac{2EL^2}{m^3\mu^2}}} + \phi_0 = \arccos \frac{\frac{L^2}{\mu^2} - 1}{\sqrt{1 + \frac{2\tilde{E}\tilde{L}^2}{\mu^2}}} + \phi_0$$

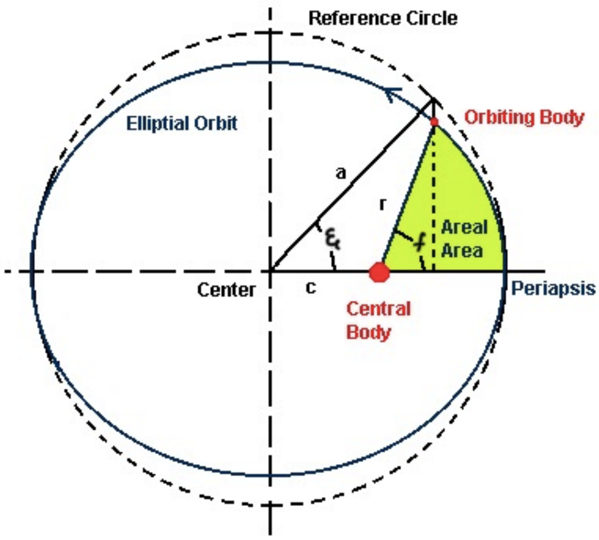
with ϕ_0 constant (verified by differentiation). Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

Defining $\ell \equiv L^2/m^2\mu = L^2/\mu$ and $e \equiv \sqrt{1 + (2EL^2/m^3\mu^2)} = \sqrt{1 + (2EL^2/\mu^2)}$ we get

$$\frac{\ell}{r} = 1 + e\cos f$$

Here $f = \phi - \phi_0$ is the true anomaly. This is the equation of a conic section where ℓ is the semi-latus rectum and e is the eccentricity. r is the distance from one focus. ϕ_0 is such that $\phi = \phi_0$ at the pericentre (perihelion).



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \ell/\left(1 - e^2\right) \quad (\text{ Semi } - \text{ major axis })$$

$$b = \ell/\sqrt{1-e^2} \quad (\text{ Semi } - \text{ minor axis }),$$

$$b = a\sqrt{1-e^2}$$

From the relations among e, ℓ, L and E, we get

$$a=\ell/\left(1-e^2\right)=\mu/2|\vec{E}|$$

$$b=\ell/\sqrt{1-e^2}=\bar{L}/\sqrt{2|\tilde{E}|}$$

Pericentre and apocentre. We remember that the distance r from one of the foci is given by the equation

$$r=\frac{\ell}{1+e\cos\phi}=\frac{a\left(1-e^2\right)}{1+e\cos\phi}$$

where we have assumed $\phi_0=0$, so $f=\phi$. Therefore the apocentre $(\cos\phi=-1)$ is $r_{\text{apo}}\ =a(1+e)$ and the

$$\text{pericentre } (\cos\phi=1) \text{ is } r_{\text{peri}}\ =a(1-e)$$

Also for motion in a central field the sectorial velocity $\mathrm{d}A/\mathrm{d}t$ is constant (from Kepler's second law).

Let's start by defining an infinitesimal sector bounded by the path as follows:

$$\mathrm{d}A=\frac{1}{2}r^2\ \mathrm{d}\phi$$

$\mathrm{d}A/\mathrm{d}t=r^2(\ \mathrm{d}\phi/\mathrm{d}t)/2=L/(2m)=\text{const}$ is the sectorial velocity $\Longrightarrow$ the particle's position vector sweeps equal areas in equal times (Kepler's second law).

$$\bar{L}=r^2\dot{\phi}=2\frac{\mathrm{d}A}{\mathrm{d}t}=\text{const}$$

we get period T for elliptic orbit:

$$\tilde{L}\ \mathrm{d}t=2\ \mathrm{d}A\Longrightarrow T\bar{L}=2A=2ab\pi$$

where $A=\pi ab$ is the area of the ellipse.

$$T=2\pi a^{3/2}/\sqrt{\mu}=\pi\mu/\sqrt{2|\bar{E}|^3}$$

which is Kepler's third law $T\propto a^{3/2}$. Keep in mind that the period is solely determined by the amount of energy available. We have used definitions of a, b as functions of $\bar{L}$ and $\bar{E}: a=\mu/2|\bar{E}|, b=\bar{L}/\sqrt{2|\bar{E}|}$ Mean motion. Kepler's third law can be written as

$$T^2\mu=4\pi^2a^3\quad\text{ or }\quad\mu=n^2a^3$$

with $n \equiv 2\pi/T$ is the mean motion (i.e. the mean angular velocity).

Kepler's first law: the orbit of each planet is an ellipse with the Sun in one of its foci. It is just a special case of the general result that the orbits are ellipses for $E < 0$. Using the expression for the orbital energy we can relate the velocity modulus v to r and a as follows:

$$\bar{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

so

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

or

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1}$$

Kepler's equation $\rightarrow$ We have seen that for motion in a central field the time dependence of the coordinates is given by

$$\mathrm{d}t = \frac{\sqrt{m}\ \mathrm{d}r}{\sqrt{2[E - V(r)] - \frac{L^2}{mr^2}}}$$

For Kepler's potential

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{2\left[\tilde{E} + \frac{\mu}{r}\right] - \frac{\tilde{L}^2}{r^2}}},$$

can be integrated analytically (see below, for instance for elliptic orbits: Kepler's equation).

From the time dependence of radial coordinate (see above) we have, in the case of elliptic orbit:

$$\mathrm{d}t = \frac{r\ \mathrm{d}r}{\sqrt{2|\tilde{E}|}\sqrt{-r^2 + \mu r/|\tilde{E}| - \tilde{L}^2/2|\tilde{E}|}}$$

Note that

$$\frac{\tilde{L}^2}{2|\tilde{E}|} = b^2 = a^2\left(1 - e^2\right) = a^2 - a^2e^2$$

and

$$\mu r/|\tilde{E}| = 2ar, \qquad \text{because} \qquad \mu = 2a|\tilde{E}|$$

so

$$\mathrm{d}t = \frac{r\ \mathrm{d}r}{\sqrt{2|\tilde{E}|}\sqrt{a^2e^2 - (r-a)^2}}$$

Let us introduce the angular variable ξ, known as the eccentric anomaly. We substitute

$$r = a(1 - e\cos\xi)$$

$$\mathrm{d}t = \sqrt{a^2/2|\tilde{E}|}(1 - e\cos\xi)\mathrm{d}\xi$$

$$t = \sqrt{a^3/\mu}(\xi - e\sin\xi) + \ \text{const}$$

where we have used $\mu = 2|\tilde{E}|a$. Note that $0 < \xi < 2\pi$: we made the calculation for $[0,\pi]$ (so $\sin\xi = \sqrt{1-\cos^2\xi}$. The calculation for $[\pi, 2\pi]$ is similar, with $\sin\xi = -\sqrt{1-\cos^2\xi}$ So, for an elliptic orbit

$$r = a(1 - e\cos\xi)$$
$$t - \tau = \sqrt{a^3/\mu}(\xi - e\sin\xi)$$

where τ is the time of pericentric passage [because $\xi = 0$ when $t = \tau$, so $r = a(1-e) = r_{\mathrm{peri}}$]. The above equation is known as Kepler's equation. Here $0 \leq \xi \leq 2\pi$ for one period. To obtain ξ (then r) as a function of t Kepler's equation must be solved numerically. Note that $\sqrt{a^3/\mu} = T/2\pi$. Often the eccentric anomaly is indicated with E, instead of ξ Mean anomaly, true anomaly and eccentric anomaly. In Kepler's equation ξ is the eccentric anomaly. Kepler's equation can be written as

$$\boxed{\mathcal{M} = \xi - e\sin\xi}$$

where $\mathcal{M} = n(t-\tau)$ is the mean anomaly, with $n = 2\pi/T$ mean motion and $T = 2\pi 3/2/\sqrt{\mu}$ is the period.

Newton's Method

We have to find the roots of the following function:

$$\mathcal{M} = \xi - e \sin \xi$$

1. To find the zeros, we initialised eccentric anomaly E = mean anomaly M
2. We approximated the value of E by iterating $E_{i+1} = E_i - \frac{f'(E_i)}{f(E_i)}$
3. Threshold error was set to 10^-4
4. Number of iterations required to achieve the desired accuracy was calculated to compare the performance later.

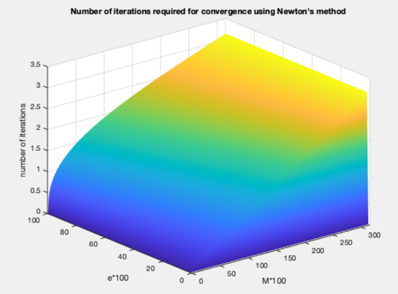
```
% Newton's Iteration Method Algorithm
clearvars;
iterations=zeros(100,315);
E_array=zeros(100,315);
for e = 0.01:0.01:0.99                     % Range of eccentricity
    j=int16(e*100)+1;                      % for matrix
    for M = 0.01:0.01:3.14                 % Range of Mean Anomaly
        k=int16(M*100)+1;
        E=M;                               % Initial Guess
        f_E = @(E) E-M-e*sin(E);           % Keplers Eqn function
        fdash = @(E) 1-e*cos(E);           % First derivative of function
        i=0;
        % Newton's Formula
        while (true)
            a=f_E(E)/fdash(E);
            E = E - a;
            if(abs(a/E)<1e-4)
                break
            end
            i=i+1;
        end
        E_array(j,k)=E;
        if i>10
            iterations(j,k)=10;
            continue
        end
        iterations(j,k)=i;
    end
end
surf(iterations)                               % 3D plot of M,e,iterations
surf(E_array)
shading interp
xlabel('M*100')
ylabel('e*100')
zlabel('number of iterations')
title("Number of iterations required for convergence using Newton's method")
```

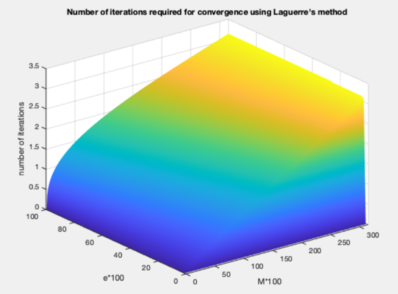
Laguerre's Method

1. Since this method is used to find zeros of polynomials, we expanded the *sin* function into polynomial form.
2. Just like in Newton's Method, we initially set E = M.
3. We approximated the value of E by Laguerre's Method:
   $G = \frac{p'(E_k)}{p(E_k)}$
   $H = G^2 - \frac{p''(E_k)}{p(E_k)}$
   $a = \frac{n}{G \pm \sqrt{(n-1)(nH-G^2)}}$
   $E_{k+1} = E_k - a$
4. Threshold error was set to 10^-4
5. Number of iterations required to achieve the desired accuracy was calculated to compare the performance later.

```
% Laguerre?s Iteration Method Algorithm
clearvars;
iterations=zeros(100,315);
E_array=zeros(100,315);
n=10;
for e = 0.01:0.01:0.99                    % Range of eccentricity
    j=int16(e*100)+1;                     % for matrix
    for M = 0.01:0.01:3.14                % Range of Mean Anomaly
        k=int16(M*100)+1;
        E=M;                              % Initial Guess
        f_E = @(E) E-M-e*sin(E);          % Keplers Eqn function
        fdash = @(E) 1-e*cos(E);          % First derivative of function
        fddash = @(E) e*sin(E);
        i=0;
        % Laguerre's Formula
        while (true)
            g=(fdash(E))/f_E(E);
            h=g^2-((fddash(E))/f_E(E));
            a=n/(g+sign(g)*sqrt((n-1)*(n*h-g^2)));
            E=E-a;
            a/E;
            if(abs(a/E)<1e-4)
                break
            end
            i=i+1;
            E_array(j,k)=E;
        end
        iterations(j,k)=i;
    end
end
surf(iterations)                          % 3D plot of M,e,iterations
surf(E_array)
shading interp
xlabel('M*100')
ylabel('e*100')
zlabel('number of iterations')
title("Number of iterations required for convergence using Laguerre's method")
```

Comparing Results and Discussions

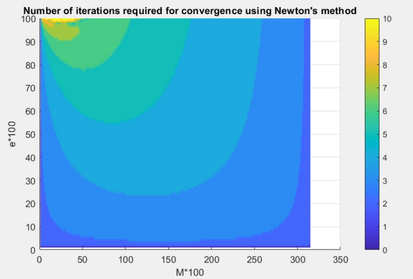


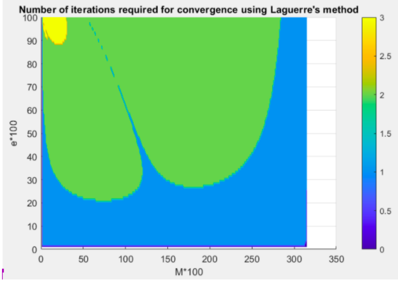


Both graphs look exactly the same.

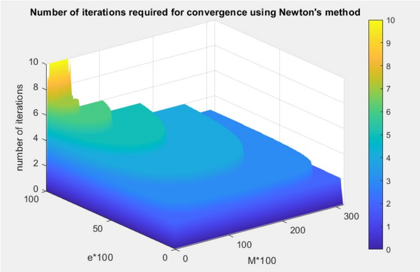
Comparing Results and Discussions

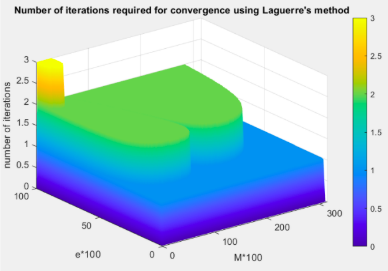
Number of iterations required for each e and M to find E.





Comparing Results and Discussions





In Newton's Method, number of iterations is in the range of 1 to 10, while in Laguerre's method, 1 or 2 iterations seem sufficient in majority cases.

Conclusion

1. Both methods show same results for a given input value of e and M. Thus, they are equally accurate.
2. Laguerre's method requires far less number of iterations to reach the result compared to Newton's method.
3. Clearly, Laguerre's Iterative Method is much more efficient than Newton's method.

Thank you