Maths 2 : Combinatorics Basics

AGENDA

- → Addition & Multiplication Rule
- → Permutation basics
- → Combination basics & properties
- → Pascal Triangle
- → Find N-th column title.

Example 1:- Given 10 girls & 7 boys. How many different pairs can be formed?

Note :- pair = 1 boy + 1 girl.

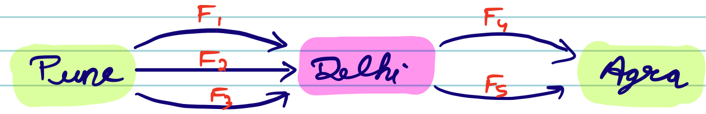
Ex:-

Girls	Boy
G_1	B_1
G_2	B_2
G_3	B_3
⋮	⋮
G_{10}	B_7

Total pairs $= 7 * 10$

$= 70$ pairs Ans

Example?

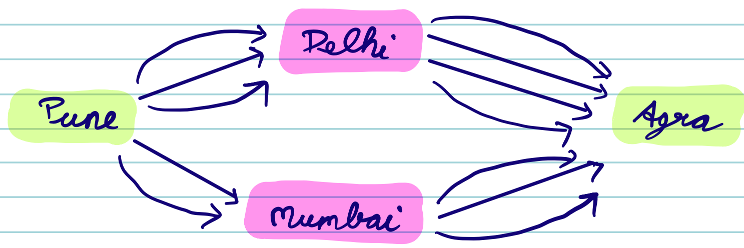


No. of ways to reach Agra from Pune via Delhi

APPROACH :- way to Reach Delhi from Pune * Way to reach Agra from Delhi

$\Rightarrow$ 3 * 2 $\Rightarrow$ 6 ways

Quiz! No. of ways of reaching Agra from Pune?



APPROACH :- Way to Reach Agra from Pune via Delhi + Way to reach Agra from Pune via Mumbai

$\Rightarrow$ (3 * 4) (2 * 3)

$\Rightarrow$ 12 + 6 $\Rightarrow$ 18 Ans

CONCLUSION

$\Rightarrow$ multiplication (*) = AND: used when counting possibilities that occur together in sequence.

$\Rightarrow$ Addition (+) = OR: used when counting possiblities that occur in separat way

PERMUTATION

↳ Permutation is defined as the arrangement of objects.

↳ In Permutation, order matters

i.e. → $(i, j) \neq (j, i)$

Eg :- i) AB, BA

ii) RGB, BGR, RBG, ...

Example 1 :- Given 3 distinct characters, in how many ways we can arrange them?

Ex:- S = "a b c"

Ans $\underline{3}\ \underline{2}\ \underline{1} \Rightarrow 6\,[3\times2\times1]$ Arrangement

↳ $3!$

a [b → c, c → b]

b [a → c, c → a]

c [a → b, b → a]

$\Rightarrow$ S = " $\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}$ "

$\Rightarrow 4 * 3 * 2 * 1 = 24\ (4!)$ Ans

Quiz 2 In how many ways n distinct characters can be arranged?

$\hookrightarrow \quad n * (n-1) * (n-2) * \ldots\ldots * 1$

$\Rightarrow \boxed{n!\ \text{Ans}}$

Ques :- In how many way you can arrange 2 out of 4 characters

a b c d $\Rightarrow$ $\underset{(a,b,c,d)}{\underline{4}}$ $\underline{3}$ $\Rightarrow$ $4*3$

$\Rightarrow 12$

(except character selected for first pos)

$\Rightarrow$ Generalizing -

Ques :- In how many way you can arrange 2 out of 3 characters

a b c $\rightarrow$ $\underline{3}$ $\underline{2}$ $\Rightarrow$ $3*2$ $\Rightarrow 6$

Arrangements

Ques :- In how many way you can arrange 2 out of 5 characters

a b c d e $\Rightarrow$ $\underline{5}$ $\underline{4}$ $\Rightarrow$ $5*4$ $\Rightarrow 20$

Arrangement

Ques :- In how many way you can arrange 3 out of N characters

$(N, (N-1), (N-2) \ldots\ldots 1)$ $\Rightarrow$ $\underline{N}$ $\underline{(N-1)}$ $\underline{(N-2)}$

$\Rightarrow N * (N-1) * (N-2)$

Ques :- In how many way you can arrange r out of N characters

$$\underbrace{\underline{N}\quad \underline{(N-1)}\quad \underline{(N-2)}\quad \underline{(N-3)}\ \ldots\ldots\ \underline{(N-(r-1))}}_{r\ \text{places}}$$

$\Rightarrow N * (N-1) * (N-2) * (N-3) \ldots\ldots * (N-r+1)$

Multiply & Divide the above expression with

$$\left[(N-r) * (N-r-1) * (N-r-2) \ldots\ldots * 1\right]$$

$$\Rightarrow \frac{N * (N-1) * (N-2) * \ldots\ldots * (N-r+1) * (N-r) * (N-r-1) * \ldots\ldots * 1}{(N-r) * (N-r-1) * (N-r-2) * \ldots\ldots * 1}$$

$$\Rightarrow \boxed{\frac{N!}{(N-r)!} = {}^{N}P_{r}}$$

Format for writing No. of ways to arrange r places from n distinct characters.

COMBINATIONS

- Combination is defined as the no. of ways to select something.
- In combination, order of selection doesn't matter

i.e. $(i, j) == (j, i)$

Eg:- RB = BR
RGB = BGR

Example 1:- Given 4 player, count the no. of ways of selecting 3 players.

Ex:- {P_1 P_2 P_3 P_4}

Ways to select ⇒

P_1	P_2	P_3
P_1	P_2	P_4
P_1	P_3	P_4
P_2	P_3	P_4

4 ways

Example 2:- Given 4 players, write the no. of ways to arrange players in 3 slots.

Ways to select ⇒

P_1	P_2	P_3
P_1	P_2	P_4
P_1	P_3	P_4
P_2	P_3	P_4

Ways to arrange the above 4 Selection

$P_1 P_2 P_3$	$P_1 P_2 P_4$	$P_1 P_3 P_4$	$P_2 P_3 P_4$
$P_1 P_3 P_2$	$P_1 P_4 P_2$	$P_1 P_4 P_3$	$P_2 P_4 P_3$
$P_2 P_1 P_3$	$P_2 P_1 P_4$	$P_3 P_1 P_4$	$P_3 P_2 P_4$
$P_2 P_3 P_1$	$P_2 P_4 P_1$	$P_3 P_4 P_1$	$P_3 P_4 P_2$
$P_3 P_1 P_2$	$P_4 P_1 P_2$	$P_4 P_1 P_3$	$P_4 P_2 P_3$
$P_3 P_2 P_1$	$P_4 P_2 P_1$	$P_4 P_3 P_1$	$P_4 P_3 P_2$

Observations

① First select 3 elements & then arranging will give us Permutation.

i.e. Permutation = Combinations + arrangement.

② For every selection = 6 arrangements

⇒ Total No. of Selections * No. of arrangement of each Selection = Total No. of arrangement

Example 3 :- Given N distinct elements, in how many ways we can select r elements such that

$$0 <= r <= n$$

Given n distinct element, arrange r element

$$^nP_r = \frac{n!}{(n-r)!}$$

if r → arrange r elements $= r!$

& No of selection for r distinct element $= 1$

∴ By using Unitary Math

selection of $r!$ arrangement $= 1$

" " 1 " $= \frac{1}{r!}$

" " ^nP_r " $= \frac{1}{r!} * {^nP_r}$

$$= \frac{1}{r!} * \frac{n!}{(n-r)!}$$

Selecting r out of N $= \frac{n!}{r!(n-r)!}$

$$\Rightarrow \boxed{^nC_r = \frac{n!}{r!(n-r)!}}$$

Imp Observation

$$\boxed{{}^{n}C_{r} * r! = {}^{n}P_{r}}$$

arrangement

Selection

⇒ Formula for Selecting 2 element out of N (Pair)

Eg:- (a, b, c, d)

$$\to {}^{n}C_{2} = \frac{n!}{2!(n-2)!}$$

$$= \frac{n * (n-1) * \cancel{(n-2)!}}{2! * \cancel{(n-2)!}}$$

$$= \frac{n * (n-1)}{2}$$

Break :- 10:14 — 10:24

PROPERTIES OF COMBINATIONS

① Property 1

↳ The No. of ways of selecting 0 items from N items, i.e. no. of ways to not select anything, will always be 1.

$$^nC_0 = \frac{N!}{(N-0)!\,0!} = 1$$

② Property 2

↳ The No. of ways of selecting N items from N items, i.e. no. of ways to select everything, will always be 1.

$$^nC_n = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\,0!} = 1$$

③ Property 3

↳ No. of ways of selecting $(n-r)$ items from n:

$$^nC_{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!}$$

$$= \frac{n!}{(n-r)!\,(n-n+r)!} = \frac{n!}{(n-r)!\,r!}$$

$$= {}^nC_r$$

So, $\boxed{{}^nC_{\underbrace{n-r}_{\text{Not Selecting } r}} = {}^nC_r}$

Eg:- Select 3 Boy out of 4.

Eg:- {B_1 B_2 B_3 B_4}

Selection ⇒ Selecting 3	Not Selecting
B_1 B_2 B_3	B_4
⇒ B_1 B_2 B_4	B_3
⇒ B_1 B_3 B_4	B_2
B_2 B_3 B_4	B_1

★★★

(4) Property 4 [special property]

Given n district element, select r out of them

For each element, I have 2 option, either to select or, skip (Not select)

n item

_ _ _ _ _ _ _ _ _ _ _ _

r places

Select → $^{n-1}C_{r-1}$

Not select → $^{n-1}C_{r}$

$$\boxed{^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}}$$

Eg:- B_1 B_2 B_3 B_4 B_5

_ _ _

select or not select

$$\boxed{{}^5C_3 = {}^4C_2 + {}^4C_3}$$

Proof

$$\boxed{{}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r}$$

RHS

$$\frac{(n-1)!}{(r-1)!\,(n-1-(r-1))!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r+1)!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r+1)*(n-1-r)!} + \frac{(n-1)!}{r*(r-1)!\,(n-1-r)!}$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r)!}\left[\frac{1}{n-\cancel{1}-r+\cancel{1}} + \frac{1}{r}\right]$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r)!}\left[\frac{\cancel{r} + (n-\cancel{r})}{(n-r)*r}\right]$$

$$\Rightarrow \frac{n*(n-1)!}{[r*(r-1)!]*[(n-r)*(n-r-1)!]}$$

$$\Rightarrow \frac{n!}{r!\,(n-r)!} = {}^nC_r = L.H.S$$

$\therefore$ HP

Question : **PASCAL TRIANGLE**

Generate Pascal's triangle for given value of n.

Ex: $n = 4$

${}^0C_0$						1				
${}^1C_0$	${}^1C_1$					1	1			
${}^2C_0$	${}^2C_1$	${}^2C_2$			$\Rightarrow$	1	2	1		
${}^3C_0$	${}^3C_1$	${}^3C_2$	${}^3C_3$			1	3	3	1	
${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$		1	4	6	4	1

Brute force

↳ 2 loops for going over rows & columns. Then one nested inner loop to calculate ${}^nC_r$

$$\boxed{\begin{array}{l} TC - O(N^3) \\ SC - O(1) \end{array}}$$

↳ **OPTIMIZATION**

IDEA (1) Can we use somehow use special property :-

i.e. ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

Eg:- ${}^3C_2 = {}^2C_1 + {}^2C_2$

This is upper left

this is upper ele.

② Edge Case :- ${}^{n}C_0$ & ${}^{n}C_n = 1$

Eg:- $n = 4$

${}^{0}C_0$

${}^{1}C_0$ ${}^{1}C_1$

${}^{2}C_0$ ${}^{2}C_1$ ${}^{2}C_2$

${}^{3}C_0$ ${}^{3}C_1$ ${}^{3}C_2$ ${}^{3}C_3$

${}^{4}C_0$ ${}^{4}C_1$ ${}^{4}C_2$ ${}^{4}C_3$ ${}^{4}C_4$

$\Rightarrow$

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

PSEUDO CODE

```
Pascal triangle ( n) {
        nCR [n+1] [n+1];
        for (i= 0; i<=n; i++) {
                nCR [i] [0] = 1;
                nCR [i] [i] = 1;
                for ( j =1 ; j <i ; j++) {
                        nCR[i][j]= nCR[i-1][j-1]
                                  +nCR[i-1][j];
                }
        }
}
```

$\boxed{\begin{array}{l} TC \rightarrow O(N^2) \\ SC \rightarrow O(N^2) \end{array}}$

Question :- **EXCEL COLUMN TITLE**

Find the n^{th} column title, the columns are titled as A, B, C……. & after Z, its AA, AB, AC……. & so on.

Given the column no., find the title of the column

	1	2	3		26	27	28	29		52	53	54
Ex:-	A	B	C	…..	Z	AA	AB	AC	….	AZ	BA	BB …..

$n = 3 \rightarrow C$

$n = 30 \rightarrow AD$

$n = 50 \rightarrow AX$

$n = 78 \rightarrow$?? (C.H.W.) → BZ

$(1 - 26) \rightarrow$ A to Z

$(27 - 52) \rightarrow$ AA to AZ

$(53 - 78) \rightarrow$ BA to BZ

$(79 - 104) \rightarrow$ CA to CZ

⋮

INTUTION / IDEA

Base 2 [0-1]	Base 8 [0-7]	Base 26 [1-26]
0	0	A → 1
1	1	B → 2
10	2	C → 3
11	3	⋮
100	4	Z → 26
	5	AA
	6	AB
	7	AC
	10	⋮
	11	AZ
	12	
	13	
	14	
	⋮	
	17	
	20	
	⋮	

So, this is simply Base 26 Conversion.

Decimal to Base 26

$N = 1000$

26	1000	12
26	38	12
26	1	1
	0	

$(1, 12, 12) \Rightarrow (ALL)_{26}$

Eg 2 N = 78

26	78	0 → Problem
26	3	3 → C
	0	

Way 1

↳ whenever you got remainder as 0 since it will be not possible so just take Ans as 1 less & keep remainder as 26

i.e.

26	78	26 → Z
26	2	2 → B
	0	

⇒ (BZ) Ans

Way 2

(A − Z) → (1 to 26)

or (A − Z) → (0 to 25)

Eg:- N = 78 ← this is 1 based

N = 78 − 1 ← 0 based.

26	78-1=77	25	→ Z
26	2-1=1	1	→ B
	0		

=> (BZ)

Eg:

26	1000-1= 999	11	→ L
26	38-1 = 37	11	→ L
26	1-1 = 0	0	→ A
	0		

(ALL) ✓

PSEUDO CODE

```
void    columnTitle( int n) {
        ans = " ";
        while ( n>0) {
            ans= (char)[(n-1)%26 +'A'] + ans;
                        // char + String
            n= (n-1)/26;
        }
        return ans;
```

3

$$\boxed{\begin{aligned} TC &= \log_{26}(N) \\ SC &= O(1) \end{aligned}}$$