Matgeo Q.9.2.26

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Problem Statement

Find the area of the region included between $y^2 = 9x$ and $y = x$.

Variables Used

Variable	Description
e	Eccentricity of conic
$\mathbf{F}$	Focus of conic
$\mathbf{I}$	Identity matrix
$\mathbf{n}^\top \mathbf{x} = c$	Equation of directrix
$\mathbf{n}$	Slope of normal to directrix
$\mathbf{V}$	A symmetric matrix given by eigenvalue decomposition

Table: Variables

General equation of a Conic in Matrix Form

The general equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, Focus $\mathbf{F}$ and eccentricity e is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.1}$$

where,

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{3.2}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{3.4}$$

Parameters for given parabola

For the parabola $y = 9x^2$, and,

$$\text{directrix is} \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = \frac{9}{4} \tag{3.5}$$

$$\text{Focus } \mathbf{F} = \begin{pmatrix} \frac{9}{4} \\ 0 \end{pmatrix} \tag{3.6}$$

$$\text{and, eccentricity } e = 1. \tag{3.7}$$

Matrix Parameters of given Parabola

We can now find $\mathbf{V}, \mathbf{u},$ and f to represent given parabola in the matrix equation form given in 3.1.
From 3.2, we have

$$\mathbf{V} = \mathbf{I} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \end{pmatrix} \implies \mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.8}$$

$$\mathbf{u} = \frac{9}{4} \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{9}{4} \\ 0 \end{pmatrix} \implies \mathbf{u} = \begin{pmatrix} -\frac{9}{2} \\ 0 \end{pmatrix} \tag{3.9}$$

$$f = \left(\frac{9}{4}\right)^2 - \left(\frac{9}{4}\right)^2 \implies f = 0 \tag{3.10}$$

Line Parameters

The given line $y = x$ can be represented in matrix form

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \tag{3.11}$$

where

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.12}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.13}$$

Points of Intersection

The points of intersection of line and conic are given by

$$\mathbf{x} = \mathbf{h} + \kappa_i \mathbf{m} \tag{3.14}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \pm \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \right]^2 - g\left(\mathbf{h} \right) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right)} \right) \tag{3.15}$$

On putting values in 3.15, we get

$$\kappa_i = 0, 9 \tag{3.16}$$

Using 3.16 in 3.14, we get points of intersection as $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 9 \end{pmatrix}$

Calculating Area

The area between the line and the parabola is given by

$$\int_0^9 3\sqrt{x}\,dx - \int_0^9 x\,dx = \left(2(9)^{3/2} - 2(0)^{3/2}\right) - \left(\frac{(9)^2}{2} - \frac{(0)^2}{2}\right)$$
$$= (2 \cdot 27 - 0) - \left(\frac{81}{2} - 0\right)$$
$$= \frac{27}{2} \qquad (3.17)$$

So, the area between the given parabola $y^2 = 9x$ and the line $y = x$ is $\frac{27}{2}$.