MA - 2021

AI24BTECH11015 - Harshvardhan Patidar

1) Let $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an inner product on the vector space $\mathbb{R}^n$ over $\mathbb{R}$. Consider the following statements:
P: $|\langle u,v\rangle| \leq \frac{1}{2}\left(\langle u,u\rangle + \langle v,v\rangle\right)$ for all $u,v \in \mathbb{R}^n$.
Q: If $\langle u,v\rangle = \langle 2u,-v\rangle$ for all $v \in \mathbb{R}^n$, then $u = 0$.
Then
   a) both P and Q are TRUE
   b) P is TRUE and Q is FALSE
   c) P is FALSE and Q is TRUE
   d) both P and Q is FALSE

Q.2 - Q.13 Numerical Answer Type (NAT), carry TWO mark each (no negative marks).

2) Let G be a group of order 5^4 with center having 5^2 elements. Then the number of conjugacy classes in G is _______.
3) Let F be a finite field and $F^\times$ be the group of all nonzero elements of F under multiplication. If $F^\times$ has a subgroup of order 17, then the smallest possible order of the field F is _______.
4) Let $R = \{z = x + iy \in \mathbb{C} : 0 < x < 1\}$ and $-11\pi < y < 11\pi$ and Γ be the positively oriented boundary of R. Then the value of the integral
$$\frac{1}{2\pi i}\int_\Gamma \frac{e^z dz}{e^z - 2}$$
is _______.
5) Let $D = \{z \in \mathbb{C} : |z| < 2\pi\}$ and $f : D \to \mathbb{C}$ be the function defined by
$$f(z) = \begin{cases} \frac{3z^2}{(1-\cos z)} & \text{if } z \neq 0 \\ 6 & \text{if } z = 0 \end{cases}$$
If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in D$, then $6a_2 =$_______.
6) The number of zeroes (counting multiplicity) of $P(z) = 3z^5 + 2iz^2 + 7iz + 1$ in the annular region $\{z \in \mathbb{C} : 1 < |z| < 7\}$ is _______.
7) Let A be a square matrix such that $\det(xI - A) = x^4(x-1)^2(x-2)^3$, where $\det(M)$ denotes the determinant of a square matrix M.
If $\operatorname{rank}\left(A^2\right) < \operatorname{rank}\left(A^3\right) = \operatorname{rank}\left(A^4\right)$, then the geometric multiplicity of the eigenvalue 0 of A is _______.
8) If $y = \sum_{k=0}^{\infty} a_x x^k, (a_0 \neq 0)$ is the power series solution of the differential equation $\frac{d^2y}{dx^2} - 24x^2y = 0$, then $\frac{a_4}{a_0}$ = _______.

9) If $u(x,t) = Ae^{-t}\sin x$ solves the following initial boundary value problem

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, 0 < x < \pi, \qquad && t > 0,\\
u(0,t) &= u(\pi,t) = 0, && t > 0,
\end{aligned}$$

$$u(x,0) = \begin{cases} 60, & 0 < x \le \frac{pi}{2}, \\ 40, & \frac{\pi}{2} < x < \pi, \end{cases}$$

then $\pi A =$ ____.

10) Let $V = \left\{p : p(x) = a_0 + a_1 x + a_2 x^2, a_0, a_1, a_2 \in \mathbb{R}\right\}$ be the vector space of all polynomials of degree at most 2 over the real field $\mathbb{R}$. Let $T : V \to V$ be the linear operator given by

$$T(p) = (p(0) - p(1)) + (p(0) + p(1))x + p(0)x^2.$$

Then the sum of the eignevalues of T is ____.

11) The quadrature formula

$$\int_0^2 xf(x)\,dx \approx \alpha f(0) + \beta f(1) + \gamma f(2)$$

is exact for all polynomials of degree ≤ 2. Then $2\beta - \gamma =$ ____.

12) For each $x \in (0,1]$, consider the decimal representation $x = \cdot d_1 d_2 d_3 \cdots d_n \cdots$. Define $f : (0,1] \to \mathbb{R}$ by $f(x) = 0$ if x is rational and $f(x) = 18n$ if x is irrational, where n is the number of zeroes immediately after the decimal point up to the first nonzero digit in the decimal representation of x. Then the Lebesgue integral $\int_0^1 f(x)\,dx =$ ____

13) Let $\tilde{x} = \begin{pmatrix} 11/3 \\ 2/3 \\ 0 \end{pmatrix}$ be an optimal solution of the following Linear Programming Problem P :

$$\text{Maximize } 4x_1 + x_2 - 3x_3$$

subject to

$$\begin{aligned}
2x_1 + 4x_2 + ax_3 &\le 10,\\
x_1 - x_2 + bx_3 &\le 3,\\
2x_1 + 3x_2 + 3x_3 &\le 11,\\
x_1 \ge 0, x_2 \ge 0 \text{ and } x_3 \ge 0,&\ \text{ where } a, b \text{ are real numbers.}
\end{aligned}$$

If $\tilde{y} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is an optimal solution of the dual of P, then $p + q + r =$ ____. (round off to two decimal places.)