

9-9.2-26

AI24BTECH11015 - Harshvardhan Patidar

Question:

Find the area of the region included between $y^2 = 9x$ and $y = x$.

Solution:

Variable	Description
\mathbf{e}	Eccentricity of conic
\mathbf{F}	Focus of conic
\mathbf{I}	Identity matrix
$\mathbf{n}^\top \mathbf{x} = c$	Equation of directrix
\mathbf{n}	Slope of normal to directrix
f	$\ \mathbf{n}\ ^2 \ \mathbf{F}\ ^2 - c^2 e^2$
\mathbf{V}	A symmetric matrix given by eigenvalue decomposition
\mathbf{u}	Vertex of conic with same directrix

TABLE 0: Variables Used

The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (0.1)$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \quad (0.2)$$

$$\mathbf{u} = c e^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \quad (0.3)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (0.4)$$

For the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = \frac{9}{4}$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (0.5)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{9}{2} \\ 0 \end{pmatrix} \quad (0.6)$$

$$f = 0 \quad (0.7)$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (0.8)$$

The point of intersection of a Line and a conic are given by

$$\mathbf{x} = \mathbf{h} + \kappa_i \mathbf{m} \quad (0.9)$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V} \mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{h} + \mathbf{u})]^2 - g(\mathbf{h}) (\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (0.10)$$

On solving we get the points of intersection to be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 9 \end{pmatrix}$. Area between the line and the parabola is,

$$\int_0^9 3\sqrt{x} dx - \int_0^9 x dx = \frac{27}{2} \quad (0.11)$$

So, the area between the parabola $y^2 = 9x$ and line $y = x$ is $\frac{27}{2}$

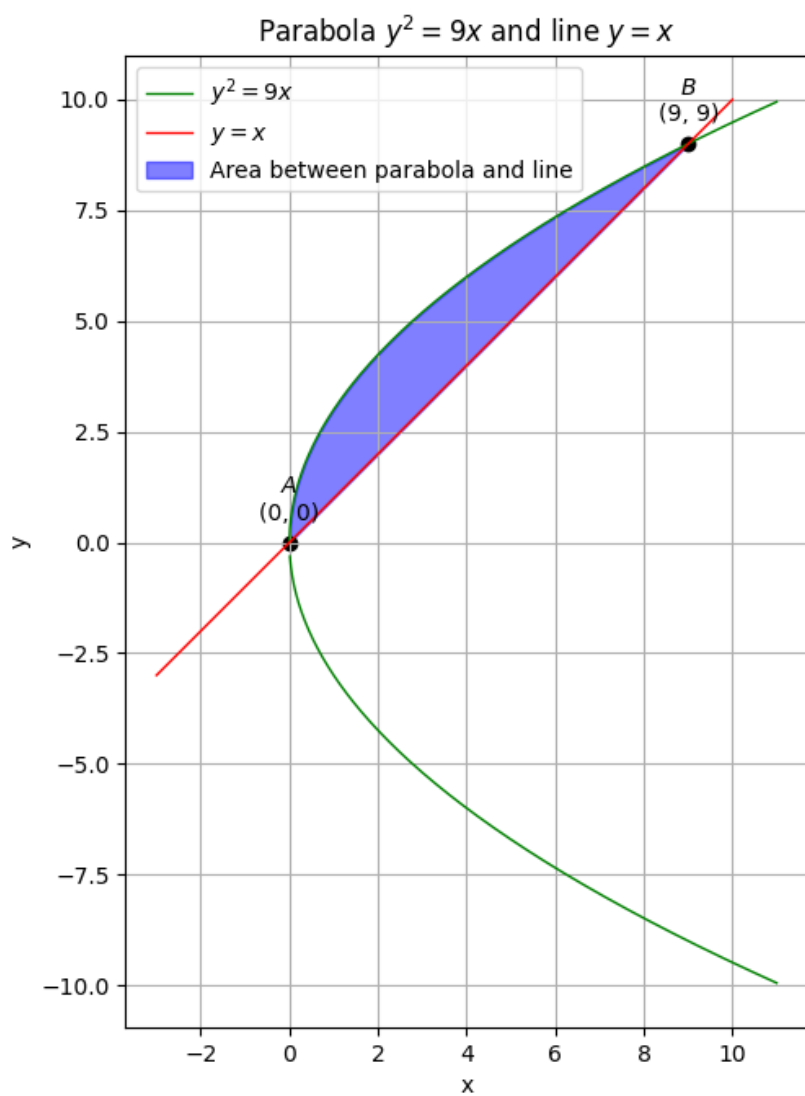


Fig. 0.1