9-9.2-26

AI24BTECH11015 - Harshvardhan Patidar

Question:
Find the area of the region included between $y^2 = 9x$ and $y = x$.
Solution:

Variable	Description
$\mathbf{e}$	Eccentricity of conic
$\mathbf{F}$	Focus of conic
$\mathbf{I}$	Identity matrix
$\mathbf{n}^\top\mathbf{x} = c$	Equation of directrix
$\mathbf{n}$	Slope of normal to directrix
f	$\|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2$
$\mathbf{V}$	A symmetric matrix given by eigenvalue decomposition
$\mathbf{u}$	Vertex of conic with same directrix

TABLE 0: Variables Used

The general equation of a parabola with directrix $\mathbf{n}^\top\mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{0.1}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \tag{0.2}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{0.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{0.4}$$

For the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix}-1 & 0\end{pmatrix}\mathbf{x} = \frac{9}{4}$

$$\mathbf{V} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{0.5}$$

$$\mathbf{u} = \begin{pmatrix}-\frac{9}{2}\\0\end{pmatrix} \tag{0.6}$$

$$f = 0 \tag{0.7}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{m} = \begin{pmatrix}1\\1\end{pmatrix} \tag{0.8}$$

The point of intersection of a Line and a conic are given by

$$\mathbf{x} = \mathbf{h} + \kappa_i\mathbf{m} \tag{0.9}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right]^2 - g(\mathbf{h})(\mathbf{m}^\top \mathbf{V}\mathbf{m})} \right) \tag{0.10}$$

On solving we get the points of intersection to be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 9 \end{pmatrix}$. Area between the line and the parabola is,

$$\int_0^9 3\sqrt{x}dx - \int_0^9 xdx = \frac{27}{2} \tag{0.11}$$

So, the area between the parabola $y^2 = 9x$ and line $y = x$ is $\frac{27}{2}$

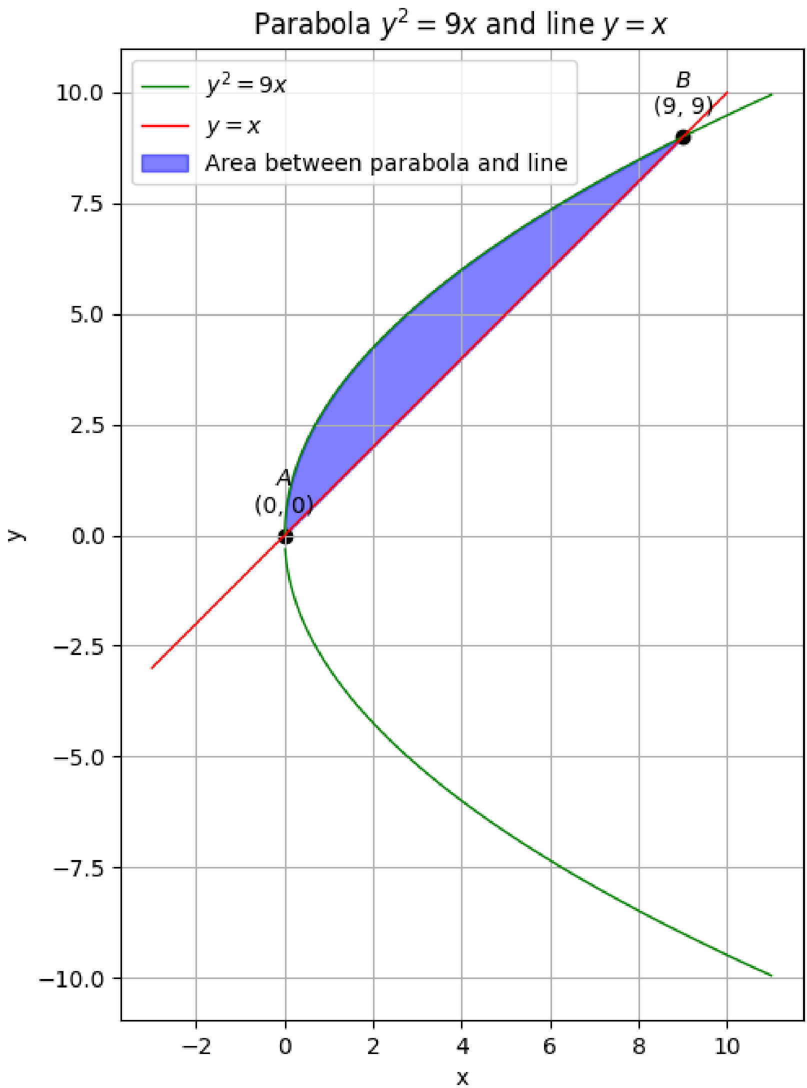

Fig. 0.1: Parabola and the line