CHAPTER 5

Mathematical Induction and Binomial Theorem

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. The larger of $99^{50} + 100^{50}$ and 101^{50} is .... *(1982 - 2 Marks)*
2. The sum of the coefficients of the plynomial $(1+x-3x^2)^{2163}$ is .... *(1982 - 2 Marks)*
3. If $(1+ax)^n = 1 + 8x + 24x^2 + ...$ then $a = ...$ and $n = ...$ *(1983 - 2 Marks)*
4. Let n be positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1+x)^n$ are in A.P., then the value of n is .... *(1994 - 2 Marks)*
5. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is .... *(1997 - 2 Marks)*

C MCQs with One Correct Answer

1. Given positive integers $r > 1$, $n > 2$ and that the coefficient of $(3r)$th and $(r+2)$th terms in the binomial expansion of $(1+x)^{2n}$ are equal . Then *(1983 - 1 Mark)*
(a) $n = 2r$ (c) $n = 2r+1$
(c) $n = 3r$ (d) none of these
2. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is *(1983 - 1 Mark)*
(a) $\frac{405}{256}$ (b) $\frac{504}{259}$
(c) $\frac{450}{263}$ (d) none of these
3. The expression $\left(x + (x^3-1)^{\frac{1}{2}}\right)^5 + \left(x - (x^3-1)^{\frac{1}{2}}\right)^5$ is a polynomial of degree *(1992 - 2 Marks)*
(a) 5 (b) 6 (c) 7 (d) 8
4. If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively, then m is *(1999 - 2 Marks)*
(a) 6 (b) 9 (c) 12 (d) 24
5. For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$ *(2000S)*
(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$ (c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$
6. In the binomial expansion of $(a-b)^n$, $n \ge 5$, the sum of the 5th and 6th terms is zero. Then a/b equals *(2001S)*
(a) $(n-5)/6$ (b) $(n-4)/5$
(c) $5/(n-4)$ (d) $6/(n-5)$
7. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if p < q) is maximum when m is *(2002S)*
(a) 5 (b) 10 (c) 15 (d) 20
8. Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is *(2003S)*
(a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 2$
9. If ${}^{n-1}C_r = (k^2-3)\,{}^{n}C_{r+1}$, then $k \in$ *(2004S)*
(a) $(-\infty, -2]$ (b) $[2, \infty)$ (c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$
10. The value of
$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} ..... + \binom{30}{20}\binom{30}{30}$$ is where $\binom{n}{r} = {}^nC_r$ *(2005S)*
(a) $\binom{30}{10}$ (b) $\binom{30}{15}$ (c) $\binom{60}{30}$ (d) $\binom{31}{10}$
11. For r = 0, 1, …, 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to *(2010)*
(a) $B_{10} - C_{10}$ (b) $A_{10}(B^2_{10}C_{10}A_{10})$
(c) 0 (d) $C_{10} - B_{10}$
12. Coefficient of x^{11} in the expansion of $(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$ is *(JEE Adv. 2014)*
(a) 1051 (b) 1106 (c) 1113 (d) 1120

D MCQs with One or More than One Correct

1. If C_r stands for ${}^nC_r$, then the sum of the series
$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - .... + (-1)^n(n+1)C_n^2],$$
where n is an even positive integer, is equal to *(1986 - 2 Marks)*

(a) 0 (b) $(-1)^{n/2}(n+1)$

(c) $(-1)^{n/2}(n+2)$ (d) $(-1)^n n$

(e) none of these.

2. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals *(1998 - 2 Marks)*

(a) $(n-1)a_n$ (b) na_n

(c) $\frac{1}{2}na_n$ (d) None of the above

E Subjective Problems

1. Given that *(1979)*
$C_1 + 2C_2x + 3C_3x^2 + ... + 2n\,C_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$
where $C_r = \frac{(2n)!}{r!(2n-r)!}$ $r = 0, 1, 2, ..., 2n$
Prove that
$C_1^2 - 2C_2^2 + 3C_3^2 - .... - 2nC_{2n}^2 = (-1)^n n\,C_n$.

2. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for any natural number n. *(1982 - 5 Marks)*

3. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time, represented by $\sum_{0 \le i < j \le n}\sum C_iC_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$ *(1983 - 3 Marks)*

4. Use mathematical Induction to prove : If n is any odd positive integer, then $n(n^2 - 1)$ is divisible by 24. *(1983 - 2 Marks)*

5. If p be a natural number then prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n. *(1984 - 4 Marks)*

6. Given $s_n = 1 + q + q^2 + ... + q^n$;
$$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n, q \neq 1$$ Prove that
$${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + ... + {}^{n+1}C_n s_n = 2^n S_n$$ *(1984 - 4 Marks)*

7. Use method of mathematical induction $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n > 0$ *(1985 - 5 Marks)*

8. Prove by mathematical induction that– *(1987 - 3 Marks)*
$$\frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{(3n+1)^{1/2}}$$ for all positive Integers n.

9. Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+4}$. *(1988 - 5 Marks)*

10. Using mathematical induction, prove that *(1989 - 3 Marks)*
$${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + ... {}^mC_k\,{}^nC_0 = {}^{(m+n)}C_k,$$
where m, n, k are positive integers, and ${}^pC_q = 0$ for $p < q$.

11. Prove that *(1989 - 5 Marks)*
$C_0 - 2^2C_1 + 3^2C_2 - .... + (-1)^n(n+1)^2C_n = 0$,
$n > 2$, where $C_r = {}^nC_r$.

12. Prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every positive integer n. *(1990 - 2 Marks)*

13. Using induction or otherwise, prove that for any non-negative integers m, n, r and k, *(1991 - 4 Marks)*
$$\sum_{m=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

14. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{2n+1}C_{n+1}$ *(1992 - 6 Marks)*

15. Let $p \ge 3$ be an integer and α, β be the roots of $x^2 - (p+1)x + 1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$.
(i) is an integer and (ii) is not divisible by p *(1992 - 6 Marks)*

16. Using mathematical induction, prove that
$$\tan^{-1}(1/3) + \tan^{-1}(1/7) + ... \tan^{-1}\{1/(n^2+n+1)\}$$
$$= \tan^{-1}\{n/(n+2)\}$$ *(1993 - 5 Marks)*

17. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. *(1993 - 5 Marks)*

18. If x is not an integral multiple of 2π use mathematical induction to prove that : *(1994 - 4 Marks)*
$$\cos x + \cos 2x + ... + \cos nx = \cos\frac{n+1}{2}x \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}$$

19. Let n be a positive integer and *(1994 - 5 Marks)*
$(1+x+x^2)^n = a_0 + a_1x + ... + a_{2n}x^{2n}$
Show that $a_0^2 - a_1^2 + a_2^2 .... + a_{2n}^2 = a_n$

20. Using mathematical induction prove that for every integer $n \ge 1$, $(3^{2n} - 1)$ is divisible by 2^{n+2} but not by 2^{n+3}. *(1996 - 3 Marks)*

21. Let $0 < A_i < \pi$ for $i = 1, 2 ...., n$. Use mathematical induction to prove that
$$\sin A_1 + \sin A_2 ... + \sin A_n \le n \sin\left(\frac{A_1 + A_2 + ... + A_n}{n}\right)$$
where ≥ 1 is a natural number.
{You may use the fact that
$p\sin x + (1-p)\sin y \le \sin[px + (1-p)y]$,
where $0 \le p \le 1$ and $0 \le x, y \le \pi$.} *(1997 - 5 Marks)*

22. Let p be a prime and m a positive integer. By mathematical induction on m, or otherwise, prove that whenever r is an integer such that p does not divide r, p divides ${}^{mp}C_r$, *(1998 - 8 Marks)*
[Hint: You may use the fact that $(1+x)^{(m+1)p} = (1+x)^p(1+x)^{mp}$]

23. Let n be any positive integer. Prove that **(1999 - 10 Marks)**

$$\sum_{k=0}^{m}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\cdot\frac{(2n-4k+1)}{(2n-2k+1)}2^{n-2k}=\frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}2^{n-2m}$$

for each non-be gatuve integer $m \le n$. $\left\{\text{Here}\binom{p}{q}={}^{p}C_{q}\right\}$.

24. For any positive integer m, n (with $n \ge m$), let $\binom{n}{m}={}^{n}C_{m}$.

Prove that $\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+.....+\binom{m}{m}=\binom{n+1}{m+2}$.

Hence or otherwise, prove that

$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+.....+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}$. **(2000 - 6 Marks)**

25. For every positive integer n, prove that $\sqrt{(4n+1)}<\sqrt{n}+\sqrt{n+1}<\sqrt{4n+2}$. Hence or otherwise, prove that $[\sqrt{n}+\sqrt{(n+1)}]=[\sqrt{4n+1}]$, where $[x]$ denotes the greatest integer not exceeding x. **(2000 - 6 Marks)**

26. Let a, b, c be positive real numbers such that $b^2-4ac>0$ and let $\alpha_1=c$. Prove by induction that

$\alpha_{n+1}=\dfrac{a\alpha_n^2}{(b^2-2a(\alpha_1+\alpha_2+...+\alpha_n))}$ is well – defined and

$\alpha_{n+1}<\dfrac{\alpha_n}{2}$ for all $n=1, 2, \ldots$ (Here, 'well – defined' means that the denominator in the expression for α_{n+1} is not zero.) **(2001 - 5 Marks)**

27. Use mathematical induction to show that $(25)^{n+1}-24n+5735$ is divisible by $(24)^2$ for all $n=1, 2, ...$ **(2002 - 5 Marks)**

28. Prove that **(2003 - 2 Marks)**

$$2^k\binom{n}{0}\binom{n}{k}-2^{k-1}\binom{n}{1}\binom{n-1}{k-1}+2^{k-2}\binom{n-2}{k-2}-.....(-1)^k\binom{n}{k}\binom{n-k}{0}=\binom{n}{k}.$$

29. A coin has probability p of showing head when tossed. It is tossed n times. Let p_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1=1$, $p_2=1-p^2$ and $p_n=(1-p).\ p_{n-1}+p(1-p)p_{n-2}$ for all $n \ge 3$.

Prove by induction on n, that $p_n=A\alpha^n+B\beta^n$ for all $n \ge 1$, where α and β are the roots of quadratic equation

$x^2-(1-p)x-p(1-p)=0$ and $A=\dfrac{p^2+\beta-1}{\alpha\beta-\alpha^2}$, $B=\dfrac{p^2+\alpha-1}{\alpha\beta-\beta^2}$. **(2000 - 5 Marks)**

I Integer Value Correct Type

1. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = **(JEE Adv. 2013)**

2. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+...+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is **(JEE Adv. 2016)**

Section-B JEE Main / AIEEE

1. The coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ are **[2002]**
(a) equal
(b) equal with opposite signs
(c) reciprocals of each other
(d) none of these

2. If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coeficient in the expansion is **[2002]**
(a) 1594 (b) 792 (c) 924 (d) 2924

3. The positive integer just greater than $(1+0.0001)^{10000}$ is **[2002]**
(a) 4 (b) 5 (c) 2 (d) 3

4. r and n are positive integers $r>1$, $n>2$ and coefficient of $(r+2)^{th}$ term and $3r^{th}$ term in the expansion of $(1+x)^{2n}$ are equal, then n equals **[2002]**
(a) $3r$ (b) $3r+1$ (c) $2r$ (d) $2r+1$

5. If $a_n=\sqrt{7+\sqrt{7+\sqrt{7+......}}}$ haing n radical signs then by methods of mathematical induction which is true **[2002[**
(a) $a_n>7\ \forall\ n\ge1$ (b) $a_n<7\ \forall\ n\ge1$
(c) $a_n<4\ \forall\ n\ge1$ (d) $a_n<3\ \forall\ n\ge1$

6. If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is **[2003]**
(a) 6th term (b) 7th term (c) 5th term (d) 8th term.

7. The number of integral terms in the expansion of $(\sqrt{3}+\sqrt[8]{5})^{256}$ is **[2003]**
(a) 35 (b) 32 (c) 33 (d) 34

8. Let $S(K)=1+3+5...+(2K-1)=3+K^2$. Then which of the following is true **[2004]**
(a) Principle of mathematical induction can be used to prove the formula
(b) $S(K)\Rightarrow S(K+1)$
(c) $S(K)\not\Rightarrow S(K+1)$
(d) $S(1)$ is correct

9. The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same if α equals **[2004]**
(a) $\dfrac{3}{5}$ (b) $\dfrac{10}{3}$ (c) $\dfrac{-3}{10}$ (d) $\dfrac{-5}{3}$

10. The coefficient of x^n in expansion of $(1+x)(1-x)^n$ is [2004]
(a) $(-1)^{n-1}n$ (b) $(-1)^n(1-n)$
(c) $(-1)^{n-1}(n-1)^2$ (d) $(n-1)$

11. The value of ${}^{50}C_4+\sum_{r=1}^{6}{}^{56-r}C_3$ is [2005]
(a) ${}^{55}C_4$ (b) ${}^{55}C_3$ (c) ${}^{56}C_3$ (d) ${}^{56}C_4$

12. If $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, then which one of the following holds for all $n\geq 1$, by the principle of mathematical induction [2005]
(a) $A^n=nA-(n-1)I$ (b) $A^n=2^{n-1}A-(n-1)I$
(c) $A^n=nA+(n-1)I$ (d) $A^n=2^{n-1}A+(n-1)I$

13. If the coefficient of x^7 in $\left[ax^2+\left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax-\left(\frac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation [2005]
(a) $a-b=1$ (b) $a+b=1$
(c) $\frac{a}{b}=1$ (d) $ab=1$

14. If x is so small that x^3 and higher powers of x may be neglected, then $\frac{(1+x)^{\frac{3}{2}}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{\frac{1}{2}}}$ may be approximated as [2005]
(a) $1-\frac{3}{8}x^2$ (b) $3x+\frac{3}{8}x^2$
(c) $-\frac{3}{8}x^2$ (d) $\frac{x}{2}-\frac{3}{8}x^2$

15. If the expansion in powers of x of the function $\frac{1}{(1-ax)(1-bx)}$ is $a_0+a_1x+a_2x^2+a_3x^3$...... then a_n is [2006]
(a) $\frac{b^n-a^n}{b-a}$ (b) $\frac{a^n-b^n}{b-a}$
(c) $\frac{a^{n+1}-b^{n+1}}{b-a}$ (d) $\frac{b^{n+1}-a^{n+1}}{b-a}$

16. For natural numbers m, n if $(1-y)^m(1+y)^n=1+a_1y+a_2y^2+$....... and $a_1=a_2=10$, then (m,n) is [2006]
(a) $(20,45)$ (b) $(35,20)$
(c) $(45,35)$ (d) $(35,45)$

17. In the binomial expansion of $(a-b)^n$, $n\geq 5$, the sum of 5th and 6th terms is zero, then a/b equals [2007]
(a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$ (c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$.

18. The sum of the series [2007]
${}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+.....-.....+{}^{20}C_{10}$ is
(a) 0 (b) ${}^{20}C_{10}$ (c) $-{}^{20}C_{10}$ (d) $\frac{1}{2}{}^{20}C_{10}$

19. **Statement -1 :** $\sum_{r=0}^{n}(r+1)\,{}^nC_r=(n+2)2^{n-1}$. [2008]
Statement-2 : $\sum_{r=0}^{n}(r+1)\,{}^nC_rx^r=(1+x)^n+nx(1+x)^{n-1}$.
(a) Statement -1 is false, Statement-2 is true
(b) Statement -1 is true, Statement-2 is true; Statement -2 is a correct explanation for Statement-1
(c) Statement -1 is true, Statement-2 is true; Statement -2 is not a correct explanation for Statement-1
(d) Statement -1 is true, Statement-2 is false

20. The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9 is : [2009]
(a) 2 (b) 7 (c) 8 (d) 0

21. Let $S_1=\sum_{j=1}^{10}j(j-1)\,{}^{10}C_J$, $S_2=\sum_{j=1}^{10}j\,{}^{10}C_j$ and $S_3=\sum_{j=1}^{10}j^2\,{}^{10}C_j$.
Statement-1 : $S_3=55\times 2^9$.
Statement-2: $S_1=90\times 2^8$ and $S_2=10\times 2^8$. [2010]
(a) Statement -1 is true, Statement -2 is true ; Statement -2 is **not** a correct explanation for Statement -1.
(b) Statement -1 is true, Statement -2 is false.
(c) Statement -1 is false, Statement -2 is true .
(d) Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

22. The coefficient of x^7 in the expansion of $(1-x-x^2+x^3)^6$ is [2011]
(a) –132 (b) –144 (c) 132 (d) 144

23. If n is a positive integer , then $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}-1\right)^{2n}$ is : [2012]
(a) an irrational number
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

24. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is [JEE M 2013]
(a) 4 (b) 120 (c) 210 (d) 310

25. If the coefficents of x^3 and x^4 in the expansion of $\left(1+ax+bx^2\right)(1-2x)^{18}$ in powers of x are both zero, then (a,b) is equal to: [JEE M 2014]
(a) $\left(14,\frac{272}{3}\right)$ (b) $\left(16,\frac{272}{3}\right)$ (c) $\left(16,\frac{251}{3}\right)$ (d) $\left(14,\frac{251}{3}\right)$

26. The sum of coefficients of integral power of x in the binomial expansion $\left(1-2\sqrt{x}\right)^{50}$ is : [JEE M 2015]
(a) $\frac{1}{2}\left(3^{50}-1\right)$ (b) $\frac{1}{2}\left(2^{50}+1\right)$
(c) $\frac{1}{2}\left(3^{50}+1\right)$ (d) $\frac{1}{2}\left(3^{50}\right)$

27. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x\neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is : [JEE M 2016]
(a) 243 (b) 729 (c) 64 (d) 2187