CHAPTER 16

Applications of Derivatives

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. The larger of cos (ln θ) and ln (cos θ) if $e^{-\pi/2} < \theta < \frac{\pi}{2}$ is .... *(1983 - 1 Mark)*
2. The function $y = 2x^2 - \ln|x|$ is monotonically increasing for values of $x (\neq 0)$ satisfying the inequalities .... and monotonically decreasing for values of x satisfying the inequalities .... *(1983 - 2 Marks)*
3. The set of all x for which $ln(1+x) \le x$ is equal to .... *(1987 - 2 Marks)*
4. Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2 then the maximum value of A is .... *(1994 - 2 Marks)*
5 Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical then H=........... and V= .... *(1994 - 2 Marks)*

B True / False

1. If $x - r$ is a factor of the polynomial $f(x) = a_n x^4 + ... + a_0$, repeated m times $(1 < m \le n)$, then r is a root of f′(x) = 0 repeated m times. *(1983 - 1 Mark)*
2. For $0 < a < x$, the minimum value of the function $\log_a x + \log_x a$ is 2. *(1984 - 1 Mark)*

C MCQs with One Correct Answer

1. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has *(1983 - 1 Mark)*
   (a) at least one root in [0, 1]
   (b) one root in [2, 3] and the other in [–2, –1]
   (c) imaginary roots
   (d) none of these
2. AB is a diameter of a circle and C is any point on the circumference of the circle. Then *(1983 - 1 Mark)*
   (a) the area of ΔABC is maximum when it is isosceles
   (b) the area of ΔABC is minimum when it is isosceles
   (c) the perimeter of ΔABC is minimum when it is isosceles
   (d) none of these
3. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point ‘θ’ is such that *(1983 - 1 Mark)*
   (a) it makes a constant angle with the x– axis
   (b) it passes through the origin
   (c) it is at a constant distance from the origin
   (d) none of these
4. If $y = a\ln x + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then *(1983 - 1 Mark)*
   (a) $a = 2, b = -1$ (b) $a = 2, b = -\frac{1}{2}$
   (c) $a = -2, b = \frac{1}{2}$ (d) none of these
5. Which one of the following curves cut the parabola $y^2 = 4ax$ at right angles? *(1994)*
   (a) $x^2 + y^2 = a^2$ (b) $y = e^{-x/2a}$
   (c) $y = ax$ (d) $x^2 = 4ay$
6. The function defined by $f(x) = (x + 2)\, e^{-x}$ is *(1994)*
   (a) decreasing for all x
   (b) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
   (c) increasing for all x
   (d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$
7. The function $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$ is *(1995S)*
   (a) increasing on $(0, \infty)$
   (b) decreasing on $(0, \infty)$
   (c) increasing on $(0, \pi/e)$, decreasing on $(\pi/e, \infty)$
   (d) decreasing on $0, \pi/e)$, increasing on $(\pi/e, \infty)$
8. On the interval [0, 1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point *(1995S)*
   (a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$
9. The slope of the tangent to a curve $y = f(x)$ at $[x, f(x)]$ is $2x + 1$. If the curve passes through the point (1, 2), then the area bounded by the curve, the x-axis and the line $x = 1$ is *(1995S)*
   (a) $\frac{5}{6}$ (b) $\frac{6}{5}$ (c) $\frac{1}{6}$ (d) 6

10. If $f(x)=\dfrac{x}{\sin x}$ and $g(x)=\dfrac{x}{\tan x}$, where $0<x\le 1$, then in this interval *(1997 - 2 Marks)*
(a) both $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function.

11. The function $f(x)=\sin^4 x+\cos^4 x$ increases if *(1999 - 2 Marks)*
(a) $0<x<\pi/8$ (b) $\pi/4<x<3\pi/8$
(c) $3\pi/8<x<5\pi/8$ (d) $5\pi/8<x<3\pi/4$

12. Consider the following statments in S and R *(2000S)*
S : Both $\sin x$ and $\cos x$ are decreasing functions in the interval $\left(\frac{\pi}{2},\pi\right)$
R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).
Which of the following is true ?
(a) Both S and R are wrong
(b) Both S and R are correct, but R is not the correct explanation of S
(c) S is correct and R is the correct explanation for S
(d) S is correct and R is wrong

13. Let $f(x)=\int e^x(x-1)(x-2)dx$. Then f decreases in the interval *(2000S)*
(a) $(-\infty,-2)$ (b) $(-2,-1)$
(c) $(1,2)$ (d) $(2,+\infty)$

14. If the normal to the curve $y=f(x)$ at the point (3,4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3)=$ *(2000S)*
(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

15. Let $f(x)=\begin{cases}|x|, & \text{for } 0<|x|\le 2\\ 1, & \text{for } x=0\end{cases}$ then at $x=0$, f has *(2000S)*
(a) a local maximum (b) no local maximum
(c) a local minimum (d) no extremum

16. For all $x\in(0,1)$ *(2000S)*
(a) $e^x<1+x$ (b) $\log_e(1+x)<x$
(c) $\sin x>x$ (d) $\log_e x>x$

17. If $f(x)=xe^{x(1-x)}$, then $f(x)$ is *(2001S)*
(a) increasing on $[-1/2, 1]$ (b) decreasing on R
(c) increasing on R (d) decreasing on $[-1/2, 1]$

18. The triangle formed by the tangent to the curve $f(x)=x^2+bx-b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is *(2001S)*
(a) -1 (b) 3 (c) -3 (d) 1

19. Let $f(x)=(1+b^2)x^2+2bx+1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is *(2001S)*
(a) $[0, 1]$ (b) $(0, 1/2]$ (c) $[1/2, 1]$ (d) $(0, 1]$

20. The length of a longest interval in which the function $3\sin x-4\sin^3 x$ is increasing, is *(2002S)*
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

21. The point(s) on the curve $y^3+3x^2=12y$ where the tangent is vertical, is (are) *(2002S)*
(a) $\left(\pm\frac{4}{\sqrt{3}},-2\right)$ (b) $\left(\pm\sqrt{\frac{11}{3}},1\right)$
(c) $(0,0)$ (d) $\left(\pm\frac{4}{\sqrt{3}},2\right)$

22. In [0,1] Lagranges Mean Value theorem is NOT applicable to *(2003S)*
(a) $f(x)=\begin{cases}\frac{1}{2}-x & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2 & x\ge\frac{1}{2}\end{cases}$
(b) $f(x)=\begin{cases}\frac{\sin x}{x}, & x\ne 0\\ 1, & x=0\end{cases}$
(c) $f(x)=x|x|$
(d) $f(x)=|x|$

23. Tangent is drawn to ellipse
$\frac{x^2}{27}+y^2=1$ at $\left(3\sqrt{3}\cos\theta,\sin\theta\right)$ (where $\theta\in(0,\pi/2)$).
Then the value of θ such that sum of intercepts on axes made by this tangent is minimum, is *(2003S)*
(a) $\pi/3$ (b) $\pi/6$ (c) $\pi/8$ (d) $\pi/4$

24. If $f(x)=x^3+bx^2+cx+d$ and $0<b^2<c$, then in $(-\infty,\infty)$ *(2004S)*
(a) $f(x)$ is a strictly increasing function
(b) $f(x)$ has a local maxima
(c) $f(x)$ is a strictly decreasing function
(d) $f(x)$ is bounded

25. If $f(x)=x^\alpha\log x$ and $f(0)=0$, then the value of α for which Rolle's theorem can be applied in [0, 1] is *(2004S)*
(a) -2 (b) -1 (c) 0 (d) 1/2

26. If $P(x)$ is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(0)=0$, $P(1)=1$ and $P'(x)>0\ \forall x\in[0,1]$, then *(2005S)*
(a) $S=\phi$
(b) $S=ax+(1-a)x^2\ \forall\ a\in(0,2)$
(c) $S=ax+(1-a)x^2\ \forall\ a\in(0,\infty)$
(d) $S=ax+(1-a)x^2\ \forall\ a\in(0,1)$

27. The tangent to the curve $y=e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ *(2007 -3 marks)*
(a) on the left of $x=c$ (b) on the right of $x=c$
(c) at no point (d) at all points

28. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then, ***(2008)***
(a) C_1 and C_2 touch each other only at one point.
(b) C_1 and C_2 touch each other exactly at two points
(c) C_1 and C_2 intersect (but do not touch) at exactly two points
(d) C_1 and C_2 neither intersect nor touch each other

29. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is ***(2008)***
(a) 0 (b) 1 (c) 2 (d) 3

30. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is ***(2008)***
(a) even and is strictly increasing in $(0, \infty)$
(b) odd and is strictly decreasing in $(-\infty, \infty)$
(c) odd and is strictly increasing in $(-\infty, \infty)$
(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

31. The least value of $a \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \ge 1$, for all $x > 0$, is ***(JEE Adv. 2016)***
(a) $\frac{1}{64}$ (b) $\frac{1}{32}$ (c) $\frac{1}{27}$ (d) $\frac{1}{25}$

D MCQs with One or More than One Correct

1. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \ldots.. < a_n$. The function $P(x)$ has
(a) neither a maximum nor a minimum ***(1986 - 2 Marks)***
(b) only one maximum
(c) only one minimum
(d) only one maximum and only one minimum
(e) none of these.

2. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then ***(1986 - 2 Marks)***
(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$
(e) none of these.

3. The smallest positive root of the equation, $\tan x - x = 0$ lies in ***(1987 - 2 Marks)***
(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \pi\right)$
(c) $\left(\pi, \frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2}, 2\pi\right)$
(e) None of these

4. Let f and g be increasing and decreasing functions, respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) - h(1)$ is ***(1987 - 2 Marks)***
(a) always zero (b) always negative
(c) always positive (d) strictly increasing
(e) None of these.

5. If $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x & 2 < x \le 3 \end{cases}$ then: ***(1993 - 2 Marks)***
(a) $f(x)$ is increasing on $[-1, 2]$
(b) $f(x)$ is continues on $[-1, 3]$
(c) $f'(2)$ does not exist
(d) $f(x)$ has the maximum value at $x = 2$

6. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then ***(1998 - 2 Marks)***
(a) h is increasing whenever f is increasing
(b) h is increasing whenever f is decreasing
(c) h is decreasing whenever f is decreasing
(d) nothing can be said in general.

7. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f ***(1998 - 2 Marks)***
(a) does not exist because f is unbounded
(b) is not attained even though f is bounded
(c) is equal to 1
(d) is equal to –1

8. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is ***(1998 - 2 Marks)***
(a) 0 (b) 1
(c) 2 (d) infinite

9. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\, dt$ has a local minimum at $x =$ ***(1999 - 3 Marks)***
(a) 0 (b) 1 (c) 2 (d) 3

10. $f(x)$ is cubic polynomial with $f(2) = 18$ and $f(1) = -1$. Also $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$, then ***(2006 - 5M, –1)***
(a) the distance between $(-1, 2)$ and $(a, f(a))$, where $x = a$ is the point of local minima is $2\sqrt{5}$
(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$
(c) $f(x)$ has local minima at $x = 1$
(d) the value of $f(0) = 15$

11. Let $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\, dt$, $x \in [1, 3]$ then $g(x)$ has ***(2006 - 5M, –1)***
(a) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$
(b) local maxima at $x = 1$ and local minima at $x = 2$
(c) no loeal maxima
(d) no local minima

12. For the function

$$f(x) = x \cos \frac{1}{x},\ x \geq 1, \qquad \textit{(2009)}$$

(a) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$

(b) $\lim\limits_{x \to \infty} f'(x) = 1$

(c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

13. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then *(2012)*

(a) f has a local maximum at $x = 2$

(b) f is decreasing on $(2, 3)$

(c) there exists some $c \in (0, \infty)$, such that $f''(c) = 0$

(d) f has a local minimum at $x = 3$

14. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are *(JEE Adv. 2013)*

(a) 24 (b) 32 (c) 45 (d) 60

15. Let $f: (0, \infty) \to R$ be given by $f(x) \int_{\frac{1}{x}}^{x} e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t}$. Then *(JEE Adv. 2014)*

(a) $f(x)$ is monotonically increasing on $[1, \infty)$

(b) $f(x)$ is monotonically decreasing on $(0, 1)$

(c) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(d) $f(2^x)$ is an odd function of x on R

16. Let f, g: $[-1, 2] \to \mathbb{R}$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are) *(JEE Adv. 2015)*

(a) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(b) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(c) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(d) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

17. Let $f: \mathbb{R} \to (0, \infty)$ and $g: \mathbb{R} \to \mathbb{R}$ be twice differentiable functions such that f'' and g'' are continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim\limits_{x \to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then *(JEE Adv. 2016)*

(a) f has a local minimum at $x=2$

(b) f has a local maximum at $x=2$

(c) $f''(2) > f(2)$

(d) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

E Subjective Problems

1. Prove that the minimum value of $\frac{(a+x)(b+x)}{(c+x)}$, $a, b > c, x > -c$ is $(\sqrt{a-c} + \sqrt{b-c})^2$. *(1979)*

2. Let x and y be two real variables such that $x > 0$ and $xy = 1$. Find the minimum value of $x+y$. *(1981 - 2 Marks)*

3. For all x in $[0, 1]$, let the second derivative $f''(x)$ of a function $f(x)$ exist and satisfy $|f''(x)| < 1$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all x in $[0, 1]$. *(1981 - 4 Marks)*

4. Use the function $f(x) = x^{1/x}, x > 0$. to determine the bigger of the two numbers e^π and π^e *(1981 - 4 Marks)*

5. If $f(x)$ and $g(x)$ are differentiable function for $0 \leq x \leq 1$ such that $f(0) = 2, g(0) = 0, f(1) = 6; g(1) = 2$, then show that there exist c satisfying $0 < c < 1$ and $f'(c) = 2g'(c)$. *(1982 - 2 Marks)*

6. Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$ where $0 \leq c \leq 5$. *(1982 - 2 Marks)*

7. If $ax^2 + \frac{b}{x} \geq c$ for all positive x where $a > 0$ and $b > 0$ show that $27ab^2 \geq 4c^3$. *(1982 - 2 Marks)*

8. Show that $1 + x \ln(x + \sqrt{x^2+1}) \geq \sqrt{1+x^2}$ for all $x \geq 0$ *(1983 - 2 Marks)*

9. Find the coordinates of the point on the curve $y = \frac{x}{1+x^2}$ where the tangent to the curve has the greatest slope. *(1984 - 4 Marks)*

10. Find all the tangents to the curve $y = \cos(x+y)$, $-2\pi \leq x \leq 2\pi$, that are parallel to the line $x + 2y = 0$. *(1985 - 5 Marks)*

11. Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum. *(1985 - 5 Marks)*

12. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$. *(1987 - 4 Marks)*

13. Investigate for maxima and minima the function *(1988 - 5 Marks)*

$$f(x)=\int_1^x [2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]dt$$

14. Find all maxima and minima of the function

$y=x(x-1)^2, 0 \le x \le 2$ *(1989 - 5 Marks)*

Also determine the area bounded by the curve $y=x\,(x-1)^2$, the y-axis and the line $y=2$.

15. Show that $2\sin x+\tan x \ge 3x$ where $0 \le x < \dfrac{\pi}{2}$. *(1990 - 4 Marks)*

16. A point P is given on the circumference of a circle of radius r. Chord QR is parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. *(1990 - 4 Marks)*

17. A window of perimeter P (including the base of the arch) is in the form of a rectangle surmounded by a semi circle. The semi- circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass transmits three times as much light per square meter as the coloured glass does.

What is the ratio for the sides of the rectangle so that the window transmits the maximum light? *(1991 - 4 Marks)*

18. A cubic $f(x)$ vanishes at $x=2$ and has relative minimum / maximum at $x=-1$ and $x=\dfrac{1}{3}$ if $\displaystyle\int_{-1}^{1} f\,dx=\frac{14}{3}$, find the cubic $f(x)$. *(1992 - 4 Marks)*

19. What normal to the curve $y=x^2$ forms the shortest chord? *(1992 - 6 Marks)*

20. Find the equation of the normal to the curve

$y=(1+x)^y+\sin^{-1}(\sin^2 x)$ at $x=0$ *(1993 - 3 Marks)*

21. Let $f(x)=\begin{cases} -x^3+\dfrac{(b^3-b^2+b-1)}{(b^2+3b+2)}, & 0 \le x < 1 \\ 2x-3, & 1 \le x \le 3 \end{cases}$ *(1993 - 5 Marks)*

Find all possible real values of b such that $f(x)$ has the smallest value at $x=1$.

22. The curve $y=ax^3+bx^2+cx+5$, touches the x-axis at $P(-2, 0)$ and cuts the y axis at a point Q, where its gradient is 3. Find a, b, c. *(1994 - 5 Marks)*

23. The circle $x^2+y^2=1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR. *(1994 - 5 Marks)*

24. Let (h, k) be a fixed point, where $h>0, k>0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin. *(1995 - 5 Marks)*

25. A curve $y=f(x)$ passes through the point $P(1,1)$. The normal to the curve at P is $a(y-1)+(x-1)=0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P. *(1996 - 5 Marks)*

26. Determine the points of maxima and minima of the function

$f(x)=\dfrac{1}{8}\ln x-bx+x^2, x>0$, where $b \ge 0$ is a constant. *(1996 - 5 Marks)*

27. Let $f(x)=\begin{cases} xe^{ax}, & x \le 0 \\ x+ax^2-x^3, & x>0 \end{cases}$ *(1996 - 3 Marks)*

Where a is a positive constant. Find the interval in which $f'(x)$ is increasing.

28. Let $a+b=4$, where $a<2$, and let $g(x)$ be a differentiable function.

If $\dfrac{dg}{dx}>0$ for all x, prove that $\int_0^a g(x)\,dx+\int_0^b g(x)dx$ increases as $(b-a)$ increases. *(1997 - 5 Marks)*

29. Suppose $f(x)$ is a function satisfying the following conditions
(a) $f(0)=2, f(1)=1$, *(1998 - 8 Marks)*
(b) f has a minimum value at $x=5/2$, and
(c) for all x,

$$f'(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

where a, b are some constants. Determine the constants a, b and the function $f(x)$.

30. A curve C has the property that if the tangent drawn at any point P on C meets the co-ordinate axes at A and B, then P is the mid-point of AB. The curve passes through the point $(1, 1)$. Determine the equation of the curve. *(1998 - 8 Marks)*

31. Suppose $p(x)=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n$. If $|p(x)| \le |e^{x-1}-1|$ for all $x \ge 0$, prove that

$|a_1+2a_2+\ldots..+na_n| \le 1$. *(2000 - 5 Marks)*

32. Let $-1 \le p \le 1$. Show that the equation $4x^3-3x-p=0$ has a unique root in the interval $[1/2, 1]$ and identify it. *(2001 - 5 Marks)*

33. Find a point on the curve $x^2+2y^2=6$ whose distance from the line $x+y=7$, is minimum. *(2003 - 2 Marks)*

34. Using the relation $2(1-\cos x)<x^2, x \ne 0$ or otherwise, prove that $\sin(\tan x) \ge x, \forall\, x \in \left[0, \dfrac{\pi}{4}\right]$ *(2003 - 4 Marks)*

35. If the function $f : [0,4] \to R$ is differentiable then show that

(i) For $a, b \in (0,4)$, $(f(4))^2 - (f(0))^2 = 8f'(a)f(b)$

(ii) $\int_0^4 f(t)dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)] \forall\ 0 < \alpha, \beta < 2$

(2003 - 4 Marks)

36. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \geq 1$ then prove that $P(x) > 0$ for all $x > 1$. *(2003 - 4 Marks)*

37. Using Rolle's theorem, prove that there is at least one root in $(45^{1/100}, 46)$ of the polynomial $P(x) = 51x^{101} - 2323(x)^{100} - 45x + 1035$. *(2004 - 2 Marks)*

38. Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \geq \frac{3x(x+1)}{\pi}$. Explain the identity if any used in the proof. *(2004 - 4 Marks)*

39. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the cuve $y = f(x)$ at the point (1, 2). *(2005 - 2 Marks)*

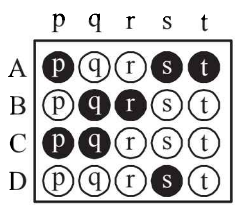
40. If $p(x)$ be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$ and $p(x)$ has maxima at $x = -1$ and $p'(x)$ has minima at $x = 1$. Find the distance between the local maxima and local minima of the curve. *(2005 - 4 Marks)*

41. For a twice differentiable function $f(x)$, $g(x)$ is defined as $g(x) = (f'(x)^2 + f''(x))\, f(x)$ on $[a, e]$. If for $a < b < c < d < e$, $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2$, $f(e) = 0$ then find the minimum number of zeros of $g(x)$. *(2006 - 6M)*

F Match the Following

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



1. In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book.

Let the functions defined in column I have domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *(1992 - 2 Marks)*

Column I	Column II
(A) $x + \sin x$	(p) increasing
(B) $\sec x$	(q) decreasing
	(r) neither increasing nor decreasing

G Comprehension Based Questions

PASSAGE - 1

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

1. The line $y = x$ meets $y = ke^x$ for $k \leq 0$ at *(2007 - 4 marks)*

(a) no point (b) one point

(c) two points (d) more than two points

2. The positive value of k for which $ke^x - x = 0$ has only one root is *(2007 - 4 marks)*

(a) $\frac{1}{e}$ (b) 1

(c) e (d) $\log_e 2$

3. For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is *(2007 - 4 marks)*

(a) $\left(0, \frac{1}{e}\right)$ (b) $\left(\frac{1}{e}, 1\right)$

(c) $\left(\frac{1}{e}, \infty\right)$ (d) $(0, 1)$

PASSAGE - 2

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in IR$ and let

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt \text{ for all } x \in (1, \infty). \quad (2012)$$

4. Consider the statements:
P : There exists some $x \in R$ such that $f(x) + 2x = 2(1 + x^2)$
Q : There exists some $x \in R$ such that $2f(x) + 1 = 2x(1 + x)$
Then
(a) both P and Q are true
(b) P is true and Q is false
(c) P is false and Q is true
(d) both P and Q are false

5. Which of the following is true?
(a) g is increasing on $(1, \infty)$
(b) g is decreasing on $(1, \infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

PASSAGE - 3

Let $f : [0, 1] \to \mathbb{R}$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x, x \in [0, 1]$.

6. Which of the following is true for $0 < x < 1$? ***(JEE Adv. 2013)***
(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$
(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

7. If the function $e^{-x} f(x)$ assumes its minimum in the interval $[0, 1]$ at $x = \frac{1}{4}$, which of the following is true?
(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ ***(JEE Adv. 2013)***
(b) $f'(x) > f(x), 0 < x < \frac{1}{4}$
(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$
(d) $f'(x) < f(x), \frac{3}{4} < x < 1$

I Integer Value Correct Type

1. The maximum value of the function
$f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set
A= $\{x \mid x^2 + 20 \leq 9x\}$ is ***(2009)***

2. Let $p(x)$ be a polynomial of degree 4 having extremum at
$x = 1, 2$ and $\lim\limits_{x \to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2.$
Then the value of p (2) is ***(2009)***

3. Let f be a real-valued differentiable function on **R** (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then find the value of $f(-3)$ ***(2010)***

4. Let f be a function defined on **R** (the set of all real numbers) such that $f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$ for all $x \in$ **R**.
If g is a function defined on **R** with values in the interval $(0, \infty)$ such that
$$f(x) = ln(g(x)), \text{ for all } x \in \boldsymbol{R}$$
then the number of points in R at which g has a local maximum is ***(2010)***

5. let $f : IR \to IR$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is ***(2012)***

6. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is ***(2012)***

7. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2 \leq h \leq 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \leq h \leq 1} \Delta(h)$,
then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ ***(JEE Adv. 2013)***

8. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1, 3)$ is ***(JEE Adv. 2014)***

9. A cylindrical container is to be made from certain solid material with the following constraints: It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.
If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is ***(JEE Adv. 2015)***

Section-B JEE Main / AIEEE

1. The maximum distance from origin of a point on the curve $x = a\sin t - b\sin\left(\frac{at}{b}\right)$
$y = a\cos t - b\cos\left(\frac{at}{b}\right)$, both $a, b > 0$ is [2002]
(a) $a-b$ (b) $a+b$ (c) $\sqrt{a^2+b^2}$ (d) $\sqrt{a^2-b^2}$

2. If $2a+3b+6c=0, (a,b,c \in R)$ then the quadratic equation $ax^2+bx+c=0$ has [2002]
(a) at least one root in [0, 1] (b) at least one root in [2, 3]
(c) at least one root in [4, 5] (d) none of these

3. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals [2003]
(a) $\frac{1}{2}$ (b) 3 (c) 1 (d) 2

4. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is [2004]
(a) $\left(\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2,-4)$ (c) $\left(\frac{-9}{8}, \frac{9}{2}\right)$ (d) $(2, 4)$

5. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x-1)$. If its graph passes through the point (2,1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is [2004]
(a) $(x+1)^2$ (b) $(x-1)^3$ (c) $(x+1)^3$ (d) $(x-1)^2$

6. The normal to the curve $x = a(1+\cos\theta), y = a\sin\theta$ at 'θ' always passes through the fixed point [2004]
(a) (a, a) (b) $(0, a)$ (c) $(0,0)$ (d) $(a, 0)$

7. If $2a+3b+6c=0$, then at least one root of the equation $ax^2+bx+c=0$ lies in the interval [2004]
(a) $(1,3)$ (b) $(1,2)$ (c) $(2,3)$ (d) $(0,1)$

8. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is [2005]
(a) $2ab$ (b) ab (c) $\sqrt{ab}$ (d) $\frac{a}{b}$

9. The normal to the curve [2005]
$x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ at any point θ' is such that
(a) it passes through the origin
(b) it makes an angle $\frac{\pi}{2}+\theta$ with the x- axis
(c) it passes through $\left(a\frac{\pi}{2}, -a\right)$
(d) it is at a constant distance from the origin

10. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm,then the rate at which the thickness of ice decreases is [2005]
(a) $\frac{1}{36\pi}$ cm/min. (b) $\frac{1}{18\pi}$ cm/min.
(c) $\frac{1}{54\pi}$ cm/min. (d) $\frac{5}{6\pi}$ cm/min

11. If the equation $a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots\ldots + a_1x = 0$
$a_1 \neq 0, n \geq 2$, has a positive root $x = \alpha$, then the equation $na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots\ldots\ldots + a_1 = 0$ has a positive root, which is [2005]
(a) greater than α
(b) smaller than α
(c) greater than or equal to α
(d) equal to α

12. The function $f(x) = \frac{x}{2}+\frac{2}{x}$ has a local minimum at [2006]
(a) $x=2$ (b) $x=-2$
(c) $x=0$ (d) $x=1$

13. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is [2006]
(a) $\frac{3}{2}x^2$ (b) $\sqrt{\frac{x^3}{8}}$ (c) $\frac{1}{2}x^2$ (d) πx^2

14. A value of c for which conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is [2007]
(a) $\log_3 e$ (b) $\log_e 3$ (c) $2\log_3 e$ (d) $\frac{1}{2}\log_3 e$

15. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in **[2007]**

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

16. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p+q)$ is **[2007]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) 2.

17. Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds? **[2008]**

(a) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(b) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(c) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(d) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

18. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? **[2008]**

(a) 7 (b) 1 (c) 3 (d) 5

19. Let $f(x) = x|x|$ and $g(x) = \sin x$.
Statement-1 : gof is differentiable at $x = 0$ and its derivative is continuous at that point.
Statement-2 : gof is twice differentiable at $x = 0$. **[2009]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

20. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$: **[2009]**

(a) P(–1) is not minimum but P(1) is the maximum of P
(b) P(–1) is the minimum but P(1) is not the maximum of P
(c) Neither P(–1) is the minimum nor P(1) is the maximum of P
(d) P(–1) is the minimum and P(1) is the maximum of P

21. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is **[2010[**

(a) $y = 1$ (b) $y = 2$ (c) $y = 3$ (d) $y = 0$

22. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} k - 2x, \text{ if } x \le -1 \\ 2x + 3, \text{ if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is **[2010]**

(a) 0 (b) $-\frac{1}{2}$ (c) –1 (d) 1

23. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function defined by

$$f(x) = \frac{1}{e^x + 2e^{-x}}$$ **[2010]**

Statement -1 : $f(c) = \frac{1}{3}$, for some $c \in \mathbf{R}$.

Statement -2 : $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in \mathbf{R}$

(a) Statement -1 is true, Statement -2 is true ; Statement -2 is **not** a correct explanation for Statement -1.
(b) Statement -1 is true, Statement -2 is false.
(c) Statement -1 is false, Statement -2 is true .
(d) Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

24. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is **[2011]**

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

25. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then f has **[2011]**

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local maximum at π and 2π

26. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[2012]**

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{9}{2}$

27. Let $a, b \in R$ be such that the function f given by $f(x) = ln|x| + bx^2 + ax, x \ne 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1 : f has local maximum at $x = -1$ and *at* $x = 2$.

Statement-2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$ **[2012]**

(a) Statement-1 is false, Statement-2 is true.
(b) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(d) Statement-1 is true, statement-2 is false.

28. A line is drawn through the point (1,2) to meet the coordinate axes at *P* and *Q* such that it forms a triangle *OPQ*, where O is the origin. If the area of the triangle *OPQ* is least, then the slope of the line *PQ* is : **[2012]**

(a) $-\frac{1}{4}$ (b) -4 (c) -2 (d) $-\frac{1}{2}$

29. The intercepts on *x*-axis made by tangents to the curve, $y = \int_0^x |t|\,dt,\ x \in \mathbb{R}$, which are parallel to the line $y = 2x$, are equal to : **[JEE M 2013]**

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

30. If *f* and *g* are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1), g(0) = 0$ and $f(1) = 6$, then for some $c \in]0,1[$ **[JEE M 2014]**

(a) $f'(c) = g'(c)$ (b) $f'(c) = 2g'(c)$
(c) $2f'(c) = g'(c)$ (d) $2f'(c) = 3g'(c)$

31. Let f(x) be a polynomial of degree four having extreme values at x = 1 and x = 2. If $\lim_{x \to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$, then f(2) is equal to : **[JEE M 2015]**

(a) 0 (b) 4 (c) -8 (d) -4

32. Consider :

$$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right).$$

A normal to y = f(x) at $x = \frac{\pi}{6}$ also passes through the point : **[JEE M 2016]**

(a) $\left(\frac{\pi}{6}, 0\right)$ (b) $\left(\frac{\pi}{4}, 0\right)$ (c) $(0, 0)$ (d) $\left(0, \frac{2\pi}{3}\right)$

33. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then: **[JEE M 2016]**

(a) $x = 2r$ (b) $2x = r$
(c) $2x = (\pi + 4)r$ (d) $(4 - \pi)x = \pi r$