

# Understanding Backpropagation with an Example

## and the “Math” behind it...

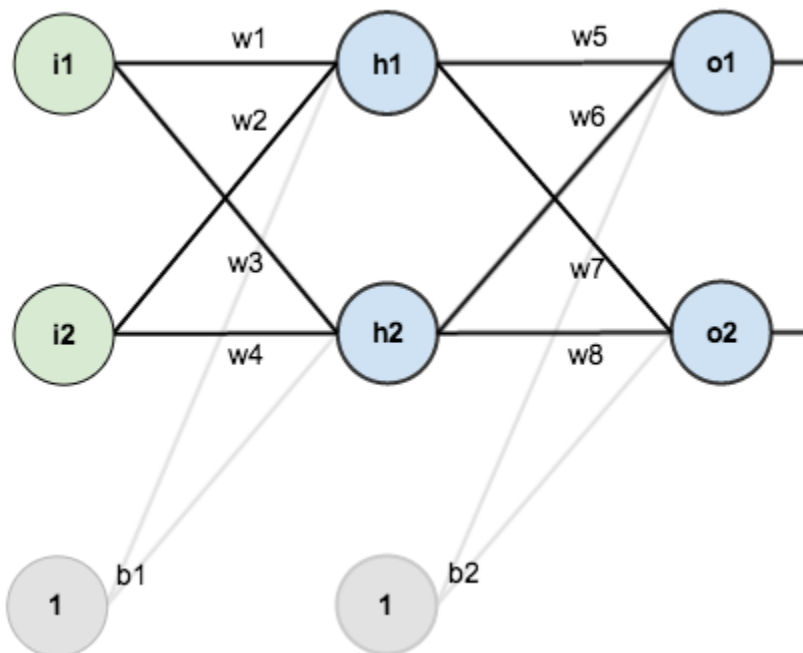
### Background

Backpropagation is a common method for training a neural network. In this document, we explain how it works with a concrete example that we can compare with own calculations in order to ensure we understand backpropagation correctly.

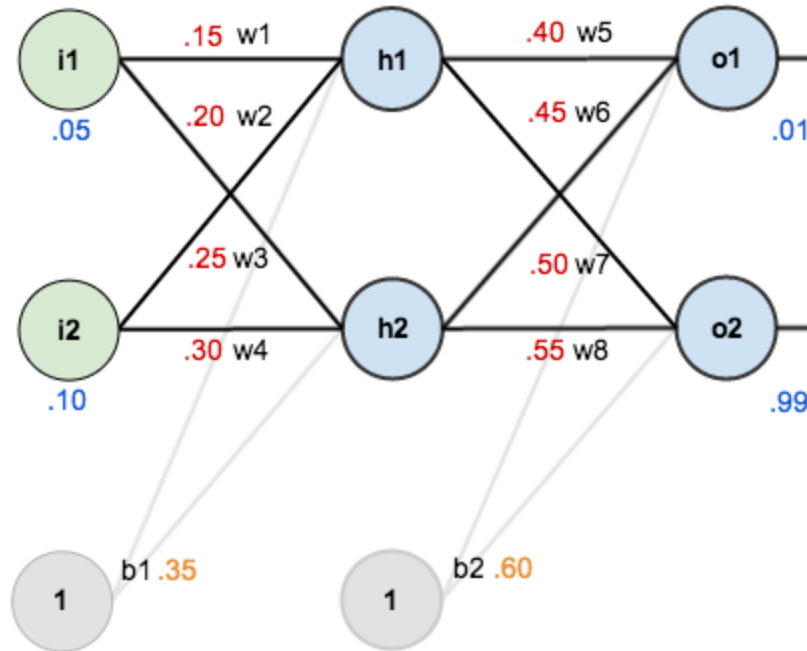
### Overview

- Use a neural network with **two inputs, two hidden neurons, two output neurons**.
- Additionally, the hidden and output neurons will include a bias.

Here's the basic structure of a Neural Network what we have mentioned:



In order to have some numbers to work with, here are the **initial weights**, the **biases**, and **training inputs/outputs**:



**The goal of backpropagation is to optimize the weights** so that the neural network can learn how to correctly map arbitrary inputs to outputs.

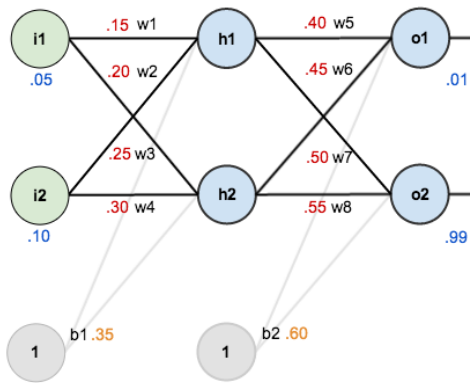
We're going to work with:

- a single training set: given **inputs** 0.05 and 0.10,
- we want the neural network to **output** 0.01 and 0.99.

## The Forward Pass

Let's see what the neural network currently predicts given the weights and biases above and inputs of 0.05 and 0.10. To do this we'll feed those inputs forward through the network.

We figure out the *total net input* to each hidden layer neuron, *squash* the total net input using an *activation function* (here we use the *logistic function*), then repeat the process with the output layer neurons.



Here's how we calculate the total net input for  $h_1$ :

$$net_{h1} = w_1 * i_1 + w_2 * i_2 + b_1 * 1$$

$$net_{h1} = 0.15 * 0.05 + 0.2 * 0.1 + 0.35 * 1 = 0.3775$$

We then squash it using the logistic function to get the output of  $h_1$ :

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}} = \frac{1}{1+e^{-0.3775}} = 0.593269992$$

Carrying out the same process for  $h_2$  we get:

$$out_{h2} = 0.596884378$$

We repeat this process for the output layer neurons, using the output from the hidden layer neurons as inputs.

Here's the output for  $o_1$ :

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$net_{o1} = 0.4 * 0.593269992 + 0.45 * 0.596884378 + 0.6 * 1 = 1.105905967$$

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}} = \frac{1}{1+e^{-1.105905967}} = 0.75136507$$

And carrying out the same process for  $o_2$  we get:

$$out_{o2} = 0.772928465$$

## Calculating the Total Error

We can now calculate the error for each output neuron using the squared error function and sum them to get the total error:

$$E_{total} = \sum \frac{1}{2}(target - output)^2$$

The  $\frac{1}{2}$  is included so that exponent is cancelled when we differentiate later on. The result is eventually multiplied by a learning rate anyway so it doesn't matter that we introduce a constant here.

For example, the target output for  $o_1$  is 0.01 but the neural network output 0.75136507, therefore its error is:

$$E_{o1} = \frac{1}{2}(target_{o1} - out_{o1})^2 = \frac{1}{2}(0.01 - 0.75136507)^2 = 0.274811083$$

Repeating this process for  $o_2$  (remembering that the target is 0.99) we get:

$$E_{o2} = 0.023560026$$

The total error for the neural network is the sum of these errors:

$$E_{total} = E_{o1} + E_{o2} = 0.274811083 + 0.023560026 = 0.298371109$$

## The Backwards Pass (Backpropagation)

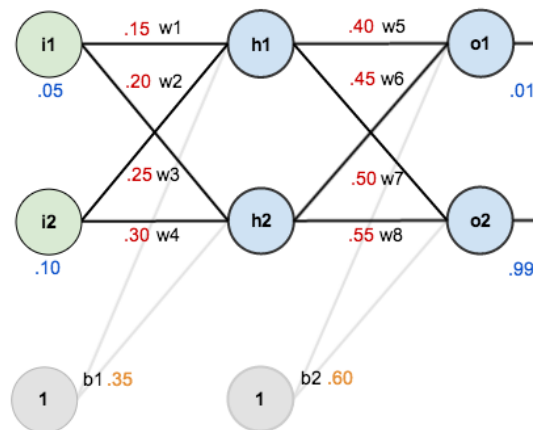
Our goal with backpropagation is to update each of the weights in the network so that they cause the actual output to be closer the target output, thereby minimizing the error for each output neuron and the network as a whole.

### Output Layer

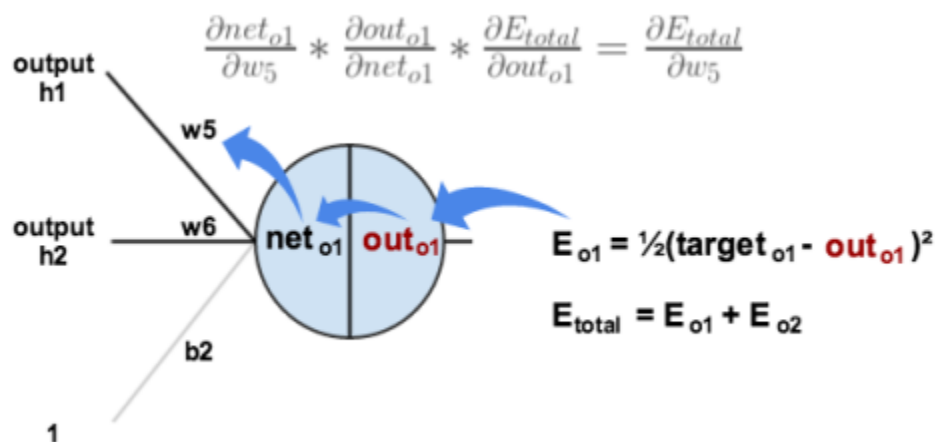
Consider  $w_5$ . We want to know how much a change in  $w_5$  affects the total error, aka  $\frac{\partial E_{total}}{\partial w_5}$ .

$\frac{\partial E_{total}}{\partial w_5}$  is read as “the partial derivative of  $E_{total}$  with respect to  $w_5$ ”. You can also say “the gradient with respect to  $w_5$ ”.

By applying the chain rule we know that:



Visually, here’s what we’re doing:



Change in Error (Etotal)  $\leftarrow$  change in Out01  $\leftarrow$  Change in Net sum (Net01)  $\leftarrow$  Change in weight ( $w_5$ )

We need to figure out each piece in this equation.

First, how much does the total error change with respect to the output?

$$E_{total} = \frac{1}{2}(target_{o1} - out_{o1})^2 + \frac{1}{2}(target_{o2} - out_{o2})^2$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = 2 * \frac{1}{2}(target_{o1} - out_{o1})^{2-1} * -1 + 0$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = -(target_{o1} - out_{o1}) = -(0.01 - 0.75136507) = 0.74136507$$

$-(target - out)$  is sometimes expressed as  $out - target$

When we take the partial derivative of the total error with respect to  $out_{o1}$ , the quantity  $\frac{1}{2}(target_{o2} - out_{o2})^2$  becomes zero because  $out_{o1}$  does not affect it which means we're taking the derivative of a constant which is zero.

Next, how much does the output of  $o1$  change with respect to its total net input?

The partial derivative of the logistic function is the output multiplied by 1 minus the output:

$$out_{o1} = \frac{1}{1 + e^{-net_{o1}}}$$

$$\frac{\partial out_{o1}}{\partial net_{o1}} = out_{o1}(1 - out_{o1}) = 0.75136507(1 - 0.75136507) = 0.186815602$$

Finally, how much does the total net input of  $o1$  change with respect to  $w_5$ ?

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial w_5} = 1 * out_{h1} * w_5^{(1-1)} + 0 + 0 = out_{h1} = 0.593269992$$

**Putting it all together:**

$$\frac{\partial E_{total}}{\partial w_5} = 0.74136507 * 0.186815602 * 0.593269992 = 0.082167041$$

**To decrease the error, we then subtract this value from the current weight (optionally multiplied by some learning rate, eta, which we'll set to 0.5):** Actually,  $w_5 = 0.4$ .

$$w_5^+ = w_5 - \eta * \frac{\partial E_{total}}{\partial w_5} = 0.4 - 0.5 * 0.082167041 = 0.35891648$$

Some sources use  $\alpha$  (alpha) to represent the learning rate, others use  $\eta$  (eta), and others even use  $\epsilon$  (epsilon).

We can repeat this process to get the new weights  $w_6$ ,  $w_7$ , and  $w_8$ :

$$w_6^+ = 0.408666186$$

$$w_7^+ = 0.511301270$$

$$w_8^+ = 0.561370121$$

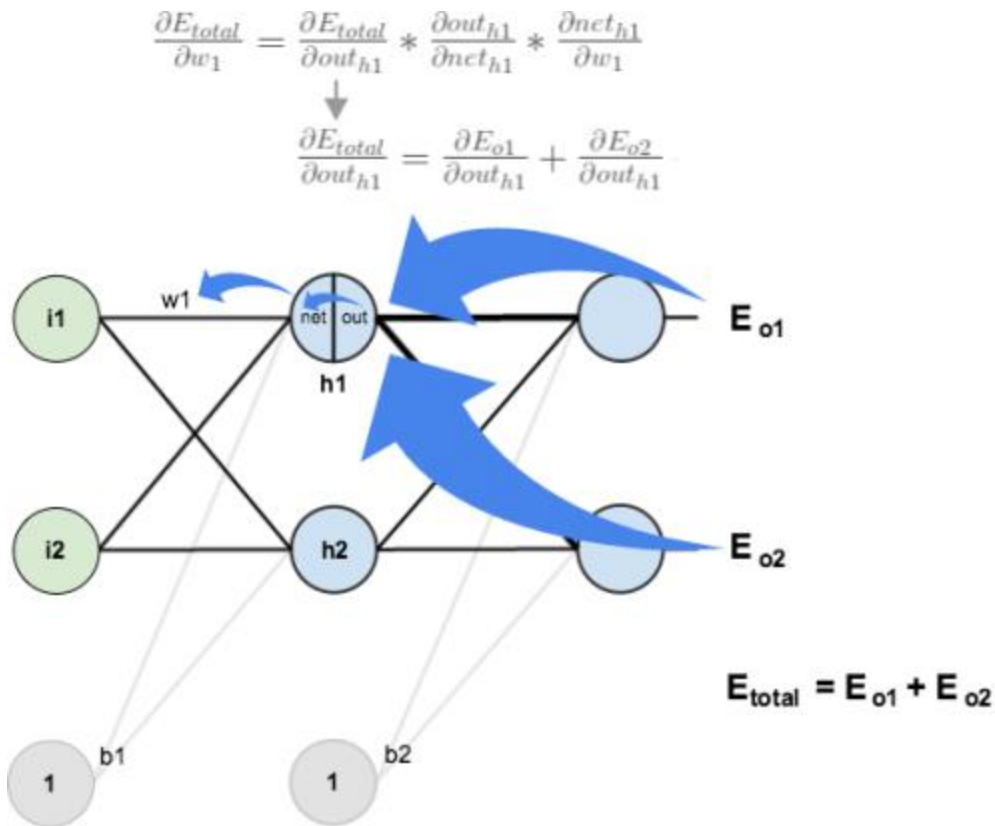
We perform the actual updates in the neural network *after* we have the new weights leading into the hidden layer neurons (ie, we use the original weights, not the updated weights, when we continue the backpropagation algorithm below).

## Hidden Layer

Next, we'll continue the backwards pass by calculating new values for  $w_1$ ,  $w_2$ ,  $w_3$ , and  $w_4$ .

Big picture, here's what we need to figure out:

Visually:



We're going to use a similar process as we did for the output layer, but slightly different to account for the fact that the output of each hidden layer neuron contributes to the output (and therefore error) of multiple output neurons. We know that  $out_{h1}$  affects both  $out_{o1}$  and  $out_{o2}$  therefore the  $\frac{\partial E_{total}}{\partial out_{h1}}$  needs to take into consideration its effect on the both output neurons:

Starting with  $\frac{\partial E_{o1}}{\partial out_{h1}}$ :

We can calculate  $\frac{\partial E_{o1}}{\partial net_{o1}}$  using values we calculated earlier:

And  $\frac{\partial net_{o1}}{\partial out_{h1}}$  is equal to  $w_5$ :

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial out_{h1}} = w_5 = 0.40$$

Plugging them in:



Following the same process for  $\frac{\partial E_{o2}}{\partial out_{h1}}$ , we get:

$$\frac{\partial E_{o2}}{\partial out_{h1}} = -0.019049119$$

Therefore:

Now that we have  $\frac{\partial E_{total}}{\partial out_{h1}}$ , we need to figure out  $\frac{\partial out_{h1}}{\partial net_{h1}}$  and then  $\frac{\partial net_{h1}}{\partial w}$  for each weight:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}}$$

$$\frac{\partial out_{h1}}{\partial net_{h1}} = out_{h1}(1 - out_{h1}) = 0.59326999(1 - 0.59326999) = 0.241300709$$

We calculate the partial derivative of the total net input to  $h_1$  with respect to  $w_1$  the same as we did for the output neuron:

$$net_{h1} = w_1 * i_1 + w_3 * i_2 + b_1 * 1$$

$$\frac{\partial net_{h1}}{\partial w_1} = i_1 = 0.05$$

Putting it all together:

$$\frac{\partial E_{total}}{\partial w_1} = 0.036350306 * 0.241300709 * 0.05 = 0.000438568$$

We can now update  $w_1$ :

$$w_1^+ = w_1 - \eta * \frac{\partial E_{total}}{\partial w_1} = 0.15 - 0.5 * 0.000438568 = 0.149780716$$

Repeating this for  $w_2$ ,  $w_3$ , and  $w_4$

$$w_2^+ = 0.19956143$$

$$w_3^+ = 0.24975114$$

$$w_4^+ = 0.29950229$$

- Finally, we've updated all of our weights!
- When we **fed forward** the 0.05 and 0.1 inputs originally, the **error** on the network was **0.298371109**.
- **After this first round of backpropagation**, the total error is now down to **0.291027924**.
- It might not seem like much, but after repeating this process 10,000 times, for example, the error plummets to 0.0000351085.
- At this point, when we feed forward 0.05 and 0.1, the two outputs neurons generate 0.015912196 (vs 0.01 target) and 0.984065734 (vs 0.99 target).