Estimating water vapour using specific humidity

Team members: Rakshith M
Harsh Mishra
Arnav Bhavsar

Problem Statement

- In physics, there are many variables which we cannot measure directly. They have to be measured/estimated through indirect ways.
- One such example in Atmospheric physics is the amount of water vapour in a column of atmosphere over a particular region.
- Water vapour can be directly measured using radiosonde (RS) and GPS measurements but data is limited [1].
- Directly measurements are not done over many regions hence we need ways to estimate the water vapour over such regions using other procedures.
- This can be done by integrating specific humidity.

Data source

ECMWF

EUROPEAN CENTRE FOR MEDIUM-RANGE WEATHER FORECASTS

Specific humidity data for year 2010	Units: None											
Month/ Pressure level	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
1000	0.009524256	0.011294778	0.013060852	0.015932903	0.018136103	0.019374363	0.019037252	0.018479876	0.018075582	0.016073568	0.012334549	0.010006663
925	0.006350666	0.0058545643	0.006986905	0.01032913	0.01279582	0.015855752	0.016620865	0.016227478	0.015997427	0.01349739	0.00854233	0.0070368447
850	0.004637116	0.004421936	0.0049597444	0.007547912	0.009880751	0.013746691	0.014378573	0.013965327	0.013508816	0.01096466	0.0066001834	0.0055549294
700	0.002094466	0.0024839877	0.003456871	0.005227988	0.0062489607	0.009251846	0.009609579	0.009416727	0.008656302	0.0066710077	0.0030677668	0.002634405
600	0.0010376371	0.0009751456	0.0018980281	0.003262456	0.004146213	0.00638546	0.0069375895	0.006768338	0.00600489	0.0044482527	0.0011631423	0.001324333
500	0.00055216486	0.00040379507	0.00059105095	0.0014489787	0.0019063712	0.0037418308	0.004303301	0.0040712636	0.0033555229	0.002547167	0.0005130085	0.00064408954
400	0.00038411387	0.00022260968	0.0002437803	0.0006192389	0.00075801974	0.0016736919	0.002175074	0.002008108	0.0016175203	0.0012446054	0.00030785514	0.0003301656
300	0.00019449701	9.52E-05	0.00010070934	0.0002265306	0.0002696438	0.0005577052	0.0007652089	0.00067584176	0.00055521395	0.000416073	0.0001387235	0.00015589416
250	9.32E-05	4.82E-05	4.68E-05	9.70E-05	0.00012177222	0.00024969966	0.00033221874	0.0002882557	0.00023211613	0.00018028729	6.91E-05	8.01E-05
200	3.38E-05	2.17E-05	1.61E-05	2.94E-05	3.94E-05	7.53E-05	9.75E-05	8.33E-05	6.62E-05	5.37E-05	2.56E-05	2.90E-05
150	6.32E-06	5.71E-06	4.40E-06	5.78E-06	7.62E-06	1.29E-05	1.71E-05	1.42E-05	1.12E-05	9.33E-06	5.06E-06	5.60E-06
100	1.56E-06	1.69E-06	1.57E-06	1.42E-06	1.22E-06	1.43E-06	1.81E-06	1.80E-06	1.72E-06	1.43E-06	1.49E-06	1.54E-06
70	1.70E-06	1.57E-06	1.48E-06	1.35E-06	1.58E-06	1.69E-06	1.81E-06	1.91E-06	1.96E-06	2.01E-06	1.86E-06	1.73E-06
50	1.89E-06	1.87E-06	1.83E-06	1.75E-06	1.76E-06	1.71E-06	1.70E-06	1.71E-06	1.72E-06	1.73E-06	1.79E-06	1.81E-06
30	1.76E-06	1.88E-06	1.86E-06	1.89E-06	1.89E-06	1.89E-06	1.87E-06	1.85E-06	1.82E-06	1.80E-06	1.83E-06	1.87E-06
20	1.84E-06	1.86E-06	1.85E-06	1.89E-06	1.92E-06	1.93E-06	1.93E-06	1.91E-06	1.90E-06	1.89E-06	1.93E-06	2.01E-06
10	2.36E-06	2.37E-06	2.55E-06	2.32E-06	2.20E-06	2.14E-06	2.11E-06	2.07E-06	2.03E-06	2.05E-06	2.09E-06	2.20E-06
5	2.72E-06	2.72E-06	2.69E-06	2.62E-06	2.62E-06	2.64E-06	2.47E-06	2.38E-06	2.29E-06	2.28E-06	2.32E-06	2.46E-06
1	3.23E-06	3.18E-06	3.18E-06	3.16E-06	3.11E-06	3.09E-06	3.13E-06	3.17E-06	3.23E-06	3.33E-06	3.38E-06	3.37E-06
Water vapour value	16.57525	16.431093	20.330452	30.512035	37.40615	53.023975	56.718407	55.122498	51.529587	41.500546	22.237293	19.136929
Units: kg/m2												

The physics!

CWV = Column water vapour

S = specific humidity

z = height

$$CWV = \int_{h_0}^{h_{top}} s \cdot dz$$

Hydrostatic equation $\Rightarrow$ $\frac{dp}{dz} = -\rho \cdot g$

ρ = density, g = acceleration due to gravity.

$$\Rightarrow CWV = \frac{-1}{g} \int_{p_0}^{p_{top}} \left(\frac{s}{\rho}\right) \cdot dp$$

$$\boxed{CWV = \frac{-1}{g} \int_{p_0}^{p_{top}} (sh) \cdot dp}$$

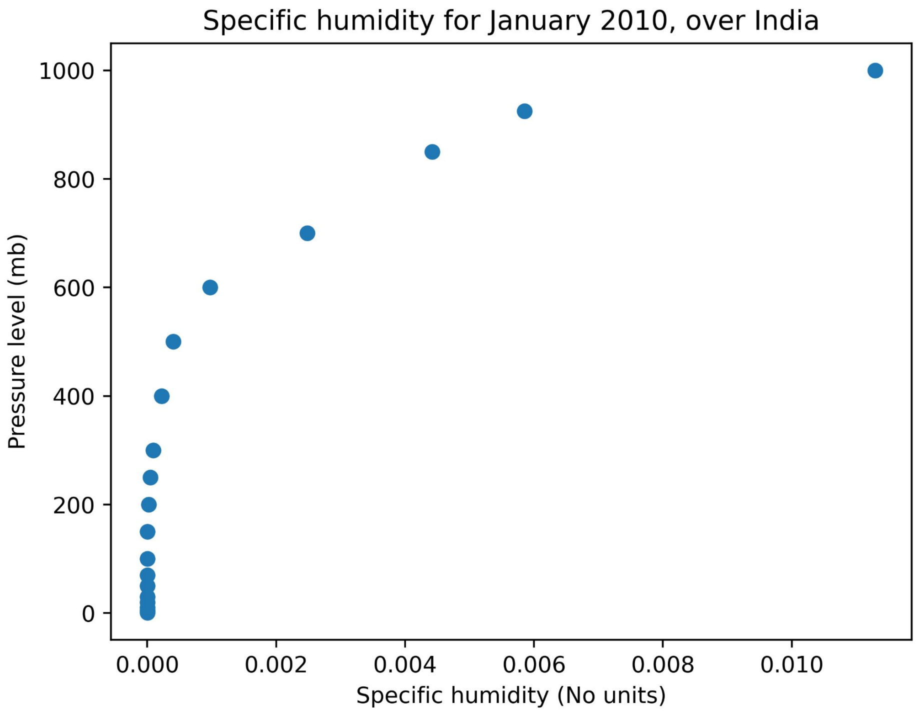
Dimensions of s = kg/m^3

Given data = sh = dimensionless

Here p is pressure. The Hydrostatic equation gives the relationship between atmospheric height and pressure. The given data for specific humidity is in pressure coordinates which is specific humidity per unit density of air so it is unitless.

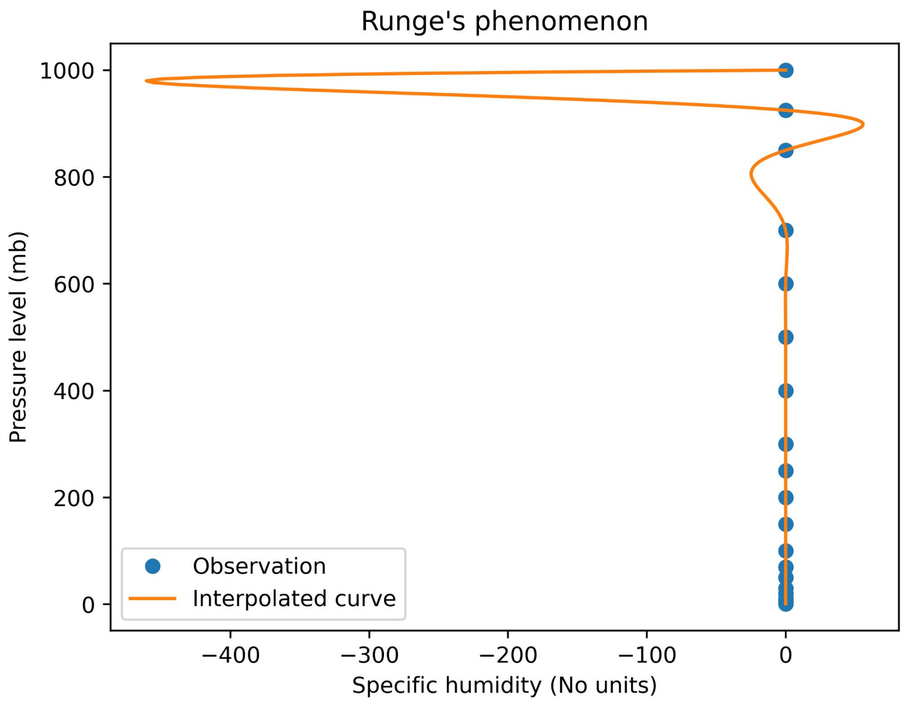
- There is a problem however.
- To integrate, we would need the values of specific humidity at all heights.
- But data is available only for certain heights.

Solution: Interpolation!



Approach 1: Lagrangian Interpolation

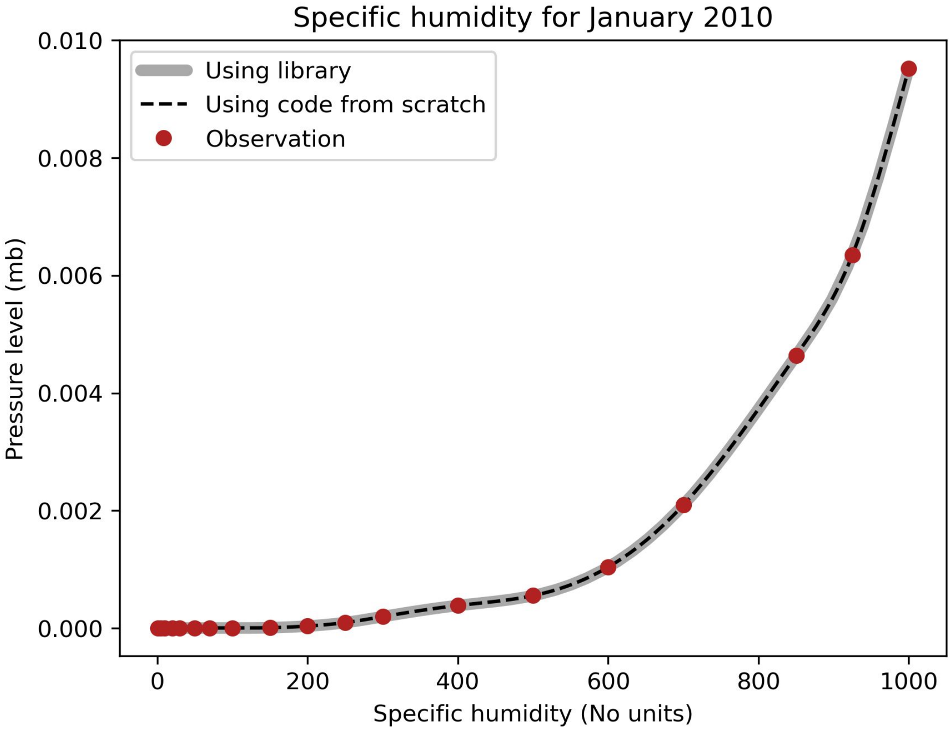
- Here we run into a problem. Interpolation does not work well at the boundary.
- This is called Runge’s phenomenon [3].
- It is a problem of oscillation at the edges of an interval that occurs when the interpolated polynomials are of high degree.



Approach 2: Cubic Spline Interpolation

- It is used very often to avoid the problem of Runge's phenomenon.
- It gives a smoother interpolating polynomial and has smaller error than some other interpolating polynomials such as Lagrange polynomial and Newton polynomial.
- We assume that the points (xi,yi) and (xi+1,yi+1) are joined by a cubic polynomial
- $S_i(x)=a_ix^3+b_ix^2+c_ix+d_i$ that is valid for $x_i \leq x \leq x_{i+1}$ for i=1,…,n−1 [4]

AIM: To determine coefficients a,b,c,d



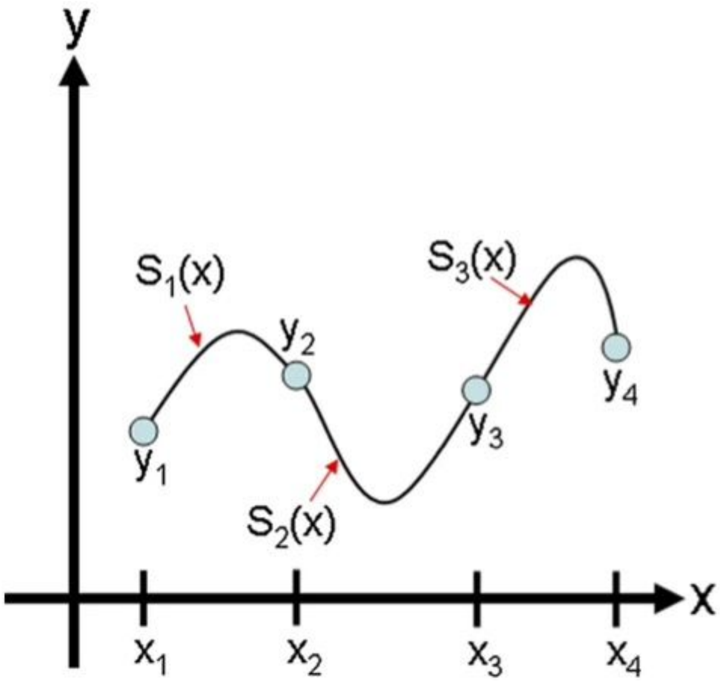
Cubic functions must intersect the data the points on the left and the right

$$S_i(x_i) = y_i, \quad i = 1, \ldots, n-1,$$
$$S_i(x_{i+1}) = y_{i+1}, \quad i = 1, \ldots, n-1$$

To achieve a smooth interpolation, we further require the first and second derivatives of S(x) to be continuous. This results in the additional constraints

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}), \quad i = 1, \ldots, n-2,$$
$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}), \quad i = 1, \ldots, n-2,$$

4n coefficients and number of constraints is **2n** + **2(n – 1)** = **4n** – **2**



ALGORITHM

Compute changes

$$h_i = x_i - x_{i+1} \text{ for } i = 0..n-2$$

Creating Diagonal matrix

$$a_i = \frac{h_i}{h_i + h_{i+1}} \text{ for } i = 0..n-3$$

$$b_i = 2 \text{ for } i = 0..n-1$$

$$c_i = \frac{h_i}{h_{i-1} + h_i} \text{ for } i = 1..n-2$$

Setting the target i.e RHS of equation

$$d_i = 6\frac{\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}}{h_i + h_{i+1}} \text{ for } i = 1..n-2$$

Solving tridiagonal system

$$c'_0 = \frac{c_0}{b_0}$$

$$c'_i = \frac{c_i}{b_i - c'_{i-1}a_{i-1}} \text{ for } i = 1..n-2$$

$$c'_{n-1} = 0$$

$$d'_0 = \frac{d_0}{b_0}$$

$$d'_i = \frac{d_i - d'_{i-1}a_{i-1}}{b_i - c'_{i-1}a_{i-1}} \text{ for } i = 1..n-1$$

$$x_{n-1} = d'_{n-1}$$

$$x_i = d'_i - c'_i x_{i+1} \text{ for } i = n-2..0$$

Calculate the coefficients

SPLINE FUNCTION

$$S(x) = \frac{(M_{j+1} - M_j)h_j^2}{6}z^3 + \frac{M_j h_j^2}{2}z^2 + (y_{j+1} - y_j - \frac{(M_{j+1} + 2M_j)h_j^2}{6})z + y_j$$

Numerical Integration

- Numerical integration is a technique used to approximate the definite integral of a function using numerical methods.

- Numerical integration is used when the function cannot be integrated analytically or when the integral is too complicated to solve by hand.

- There are several numerical integration methods, such as the midpoint rule, trapezoidal rule, Simpson's rule, and Gaussian quadrature

Trapezoidal Rule

- The trapezoid rule is a numerical integration method used to approximate the definite integral of a function over a given interval.
- It is a simple method that involves approximating the area under the curve of the function using trapezoids with equal width, where the height of each trapezoid is the value of the function at the endpoints of the subinterval.
- The Formula for Trapezoidal Rule is given by

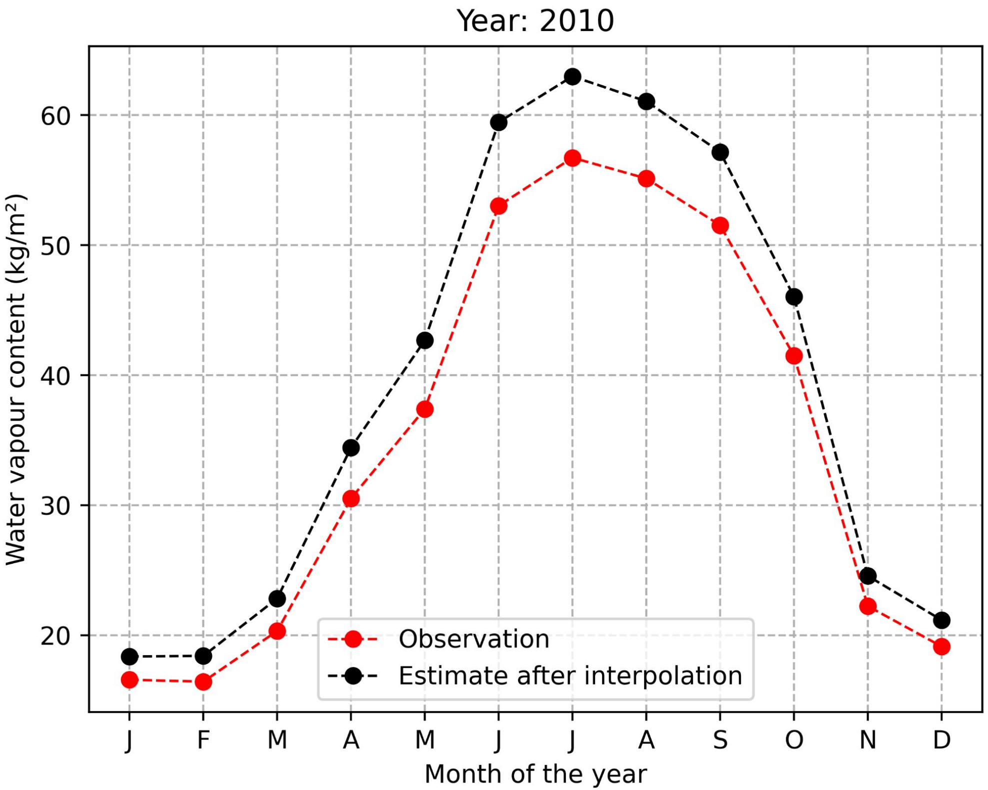
$$\int_a^b f(x)\,dx \approx (b-a)\cdot\frac{1}{2}(f(a)+f(b))$$

Results and Conclusion

- Hence, using interpolation and numerical integration we can estimate water vapour content from specific humidity reasonably well.

- We can use this method whenever observations are not available.

- The % error for our estimate is 11.5%



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