HW -5 : Today (2-09)

Quiz - 4 :Today (2-09)

Class - 6

Exam-1:
Covers Chapters 1-5,
No electronic devices except a calculator,
One double-sided hand written sheet is allowed.

Extra Office Hours:
R : 2 pm - 5 pm,
F : 2 pm - 5 pm.

Need to Know:
How to read a computer output for SLR model, ANOVA table, etc.
How to find table values for Standard normal, t , and F-distributions.

Chapter – 5: Matrix approach to Simple Linear Regression

SLR model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \quad , \quad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$$

$$i = 1, 2, \ldots n.$$

That is

$$Y_1 = \beta_0 + \beta_1 X_1 + \varepsilon_1$$

$$Y_2 = \beta_0 + \beta_1 X_2 + \varepsilon_2$$

$$\vdots$$

$$\underbrace{Y_n}_{Y_{n\times1}} = \underbrace{\beta_0 + \beta_1 X_n}_{X_{n\times2} * \beta_{2\times1}} + \underbrace{\varepsilon_n}_{\varepsilon_{n\times1}}$$

So SLR model in matrix form:

$$Y_{n\times1} = X_{n\times2} \cdot \beta_{2\times1} + \varepsilon_{n\times1} \, ,$$

where

$$Y_{n\times1} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times1}, \quad X_{n\times2} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & X_3 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}_{2\times1}, \quad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times1}.$$

Note:

* $$X\beta = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}_{n\times 2} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}_{2\times 1} = \begin{bmatrix} \beta_0 + \beta_1 X_1 \\ \beta_0 + \beta_1 X_2 \\ \vdots \\ \beta_0 + \beta_1 X_n \end{bmatrix}$$

* **Expectation of error term**

$$E(\underline{\varepsilon}) = \begin{bmatrix} E(\varepsilon_1) \\ E(\varepsilon_2) \\ \vdots \\ E(\varepsilon_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} = \underline{0}_{n\times 1} \longleftarrow \text{zero vector.}$$

* Variance co-variance matrix of $\underline{\varepsilon}$

$$\mathrm{Var}(\underline{\varepsilon}) = ?$$

$\mathrm{cov}(X,X) = \mathrm{var}(X)$

We know that

$$\text{co-variance}(\varepsilon_i, \varepsilon_j) = \sigma\{\varepsilon_i, \varepsilon_j\} = \begin{cases} \sigma\{\varepsilon_i, \varepsilon_i\} = \sigma^2\{\varepsilon_i\} = \sigma^2 & : i=j \\ \sigma\{\varepsilon_i, \varepsilon_j\} = 0 & : i \neq j \end{cases}$$

(since independent).

$$\therefore \sigma^2\{\underline{\varepsilon}_{n\times 1}\} = \begin{bmatrix} \sigma^2\{\varepsilon_1\} & \sigma\{\varepsilon_1, \varepsilon_2\} & \sigma\{\varepsilon_1, \varepsilon_3\} & \cdots & \sigma\{\varepsilon_1, \varepsilon_n\} \\ \sigma\{\varepsilon_2, \varepsilon_1\} & \sigma^2\{\varepsilon_2\} & \cdots & \cdots & \sigma\{\varepsilon_2, \varepsilon_n\} \\ \vdots & & & & \\ \sigma\{\varepsilon_n, \varepsilon_1\} & \sigma\{\varepsilon_n, \varepsilon_2\} & \cdots & \cdots & \sigma^2\{\varepsilon_n\} \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}_{n\times n}$$

$$= \sigma^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \sigma^2 \underbrace{I_{n \times n}}_{\text{identity matrix.}}$$

$$\therefore \quad \sigma^2\{\underline{\varepsilon}\} = \sigma^2 I_{n \times n}.$$

* We also can show that

$$\underline{\varepsilon}_{n \times 1} \sim \underbrace{N(\underline{0}, \sigma^2 I_{n \times n})}$$

Multivariate normal distribution with n random variables.

So the normal error regression model in matrix form is

$$Y_{n \times 1} = X_{n \times 2} \, \beta_{2 \times 1} + \varepsilon_{n \times 1}, \text{ where } \varepsilon_{n \times 1} \sim N(\underline{0}, \sigma^2 I_{n \times n}).$$

* Least Square Estimation

The normal Equations:

$$n b_0 + b_1 \Sigma X_i = \Sigma Y_i$$

$$\underbrace{b_0 \Sigma X_i + b_1 \Sigma X_i^2} = \Sigma X_i Y_i$$

$A' = A^T$ — transpose of A.

$$\underbrace{\begin{bmatrix} n & \Sigma X_i \\ \Sigma X_i & \Sigma X_i^2 \end{bmatrix}}_{X'X} \underbrace{\begin{bmatrix} b_0 \\ b_1 \end{bmatrix}}_{b_{2 \times 1}} = \underbrace{\begin{bmatrix} \Sigma Y_i \\ \Sigma X_i Y_i \end{bmatrix}}_{X'Y}$$

Let $\underline{b}_{2 \times 1} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}_{2 \times 1}$, then the normal Equations in matrix form,

$$(X'X)_{2\times 2}\, \underset{\sim}{b}_{2\times 1} = (X'Y)_{2\times 1}$$

$AX = Y$

$\underbrace{A^{-1}A}X = A^{-1}Y$

$\Rightarrow X = A^{-1}Y$

* Estimators of β_0 and β_1

$$\underbrace{(X'X)^{-1}(X'X)}\, b = (X'X)^{-1}(X'Y)$$

$$\Rightarrow \boxed{b_{2\times 1} = (X'X)^{-1}(X'Y)} \Rightarrow \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} \square \\ \square \end{pmatrix}$$

Note:

$$* \quad X'X = \begin{bmatrix} 1 & \cdots & 1 \\ X_1 & \cdots & X_n \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}_{2\times 2} = \begin{bmatrix} n & \Sigma X_i \\ \Sigma X_i & \Sigma X_i^2 \end{bmatrix}_{2\times 2}.$$

* Fitted Values

$$\hat{Y}_1 = b_0 + b_1 X_1$$

$$\hat{Y}_2 = b_0 + b_1 X_2$$

$$\vdots$$

$$\underbrace{\hat{Y}_n}_{\hat{Y}_{n\times 1}} = \underbrace{b_0 + b_1 X_n}_{X_{n\times 2}\, b_{2\times 1}}$$

So Let $\hat{Y} = \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \vdots \\ \hat{Y}_n \end{bmatrix}_{n\times 1}$, then the fitted values in matrix form is

$$\hat{Y}_{n\times 1} = X_{n\times 2} \cdot \underline{b}_{2\times 1}.$$

But $\quad X'X\underline{b} = X'Y$

$$\Rightarrow \quad b = (X'X)^{-1}X'Y$$

Now $\quad \hat{Y}_{n\times 1} = \underbrace{X\cdot(X'X)^{-1}X'}_{H}Y$

hat matrix.

Let $H = X\cdot(X'X)^{-1}X'$, then $\hat{Y} = HY$.

Note:

* matrix H is called the hat matrix
* H is Symmetric (ie $H' = H$).
* H is idempotent (ie $H^2 = H'H = H$).
* H depends only on X (ie H – constant).

* Residuals

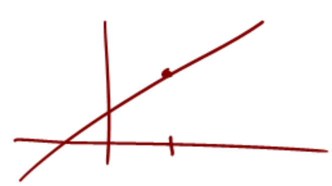
Let $\quad \underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}_{n\times 1}$ then,

$$e_{n\times 1} = Y_{n\times 1} - \hat{Y}_{n\times 1}$$

$$= \underline{Y}_{n\times 1} - H\,Y_{n\times 1}$$

$$= (I-H)Y$$

$$\Rightarrow \boxed{\underline{e}_{n\times 1} = (I-H)_{n\times n}Y_{n\times 1}}$$



* Expectation of $\underline{e}$

$E(Y) = E(\hat{Y})$

$$E(\underline{e}) = E(\underbrace{(I-H)}Y)$$

$$= (I-H)E(Y) = (I-H)X\underline{\beta}$$

* Estimator: $\hat{E}(\underline{e}) = (I-H)Xb$.

* Variance of $\underline{e}$ (Variance co-variance of $\underline{e}$)

$$\boxed{Var(aX) = a^2 Var(X) \quad a - \text{constant}}$$

$$\sigma^2\{\underline{e}\} = \sigma^2\{(I-H)Y\}$$
$$= (I-H)' \underbrace{\sigma^2\{Y\}}(I-H)$$
$$= (I-H)(\sigma^2 I_{n\times n})(I-H) \quad (\because (I-H) \text{ is Symmetric}).$$
$$= \sigma^2 \cdot (I-H)\cdot(I-H)$$
$$= \sigma^2(I-H) \quad (\because (I-H) \text{ is idempotent}).$$

$$\boxed{\sigma^2\{e\} = \sigma^2(I-H)}$$

* Estimator $\longrightarrow$ $\hat{\sigma}^2\{e\} = MSE(I-H)$.

Analysis of Variance Results

Sum of Squares

$$SSTO = \Sigma(Y_i - \bar{Y})^2$$
$$= \Sigma(Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2)$$
$$= \Sigma Y_i^2 - 2\bar{Y}(\Sigma Y_i) + n\bar{Y}^2$$
$$= \Sigma Y_i^2 - 2n\bar{Y}^2 + n\bar{Y}^2$$
$$= \Sigma Y_i^2 - n\bar{Y}^2$$
$$= \Sigma Y_i^2 - n\left(\frac{\Sigma Y_i}{n}\right)^2$$
$$= \underbrace{\Sigma Y_i^2}_{Y'Y} - \underbrace{\frac{1}{n}(\Sigma Y_i)^2}_{Y'JY}$$

$\bar{Y} = \frac{\Sigma Y_i}{n} \Rightarrow \Sigma Y_i = n\bar{Y}$

$$[Y_1, \dots Y_n]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

Let $J = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n}$ then, $\boxed{SSTO = Y'Y - Y'JY}$

$$SSE = \underbrace{\sum e_i^2}_{e'e} = \underline{e}'\underline{e}$$

* $A \cdot B \neq B \cdot A$
* $(A\underline{B})' = B' \cdot A'$

$$= (y - xb)'(y - xb)$$

$$= y'y - (xb)'y - y'(xb) + (xb)'(x\underline{b})$$

$$= y'y - b'x'y - y'xb + b'\underbrace{x'x(x'x)^{-1}}_{I}\underline{x'y}$$

$$= y'y - \cancel{b'x'y} - y'xb + \cancel{b'x'}y$$

$$= y'y - y'xb$$

$$= y'y - b'x'y$$

$$\therefore \boxed{SSE = y'y - b'x'y}$$

$$\therefore SSR = SSTo - SSE$$

$$= y'y - \left(\tfrac{1}{n}\right)y'Jy - (y'y - b'x'y)$$

$$\boxed{SSR = b'x'y - (1/n)y'Jy}$$

Regression coefficients

The variance co-variance matrix of b.

$$\sigma^2\{b\} = \begin{bmatrix} \sigma^2\{b_0\} & \sigma\{b_0, b_1\} \\ \sigma\{b_0, b_1\} & \sigma^2\{b_1\} \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} \dfrac{\sigma^2}{n} + \dfrac{\sigma^2 \bar{x}^2}{\sum(x_i - \bar{x})^2} & \dfrac{-\bar{x}\sigma^2}{\sum(x_i - \bar{x})^2} \\ \dfrac{-\bar{x}\sigma^2}{\sum(x_i - \bar{x})^2} & \dfrac{\sigma^2}{\sum(x_i - \bar{x})^2} \end{bmatrix}_{2\times 2}$$

$$= (X'X)^{-1}$$

$$\therefore \boxed{\sigma^2\{b_{2\times 2}\} = \sigma^2 (X'X)^{-1}_{2\times 2}}$$

* Estimated variance co-variance matrix

$$S^2\{b\} = MSE\,(X'X)^{-1}$$

* Mean Response (at $X = X_n$)

Let $X_{n_{2\times1}} = [1 \quad X_n]'_{1\times 2}$, then the fitted value,

$$\hat{Y}_n = X_n\, b$$

* Variance-covariance matrix of $\hat{Y}_n$:

$$\sigma^2\{\hat{Y}_n\} = \sigma^2\{X_n b\} = X_n' \underbrace{\sigma^2\{b\}}_{\sigma^2 (X'X)^{-1}} X_n$$

$$= \sigma^2 X_n' (X'X)^{-1} X_n .$$

* Estimated variance co-variance of $\hat{Y}_n$

$$S^2\{\hat{Y}_n\} = MSE\; X_n' (X'X)^{-1} X_n .$$

* Prediction of New observation.

$$\sigma^2\{pred\} = \sigma^2\{Y_{n(new)}\} + \sigma^2\{\hat{Y}_n\}$$

$$= \sigma^2 I + \sigma^2 X_n' (X'X)^{-1} X_n$$

$$= \sigma^2 [I + X_n' (X'X)^{-1} X_n]$$

* Estimated Variance

$$S^2\{Pred\} = MSE\left[I + X_n' (X'X) X_n \right]_6$$