Fuzzy Sets Operations

We will discuss the following terms and operations:

- Min, Max operators
- Empty Fuzzy Set
- Normal Fuzzy Set
- Equality of Fuzzy Set
- Union of Two Fuzzy Sets
- Intersection of Two Fuzzy Sets
- Complement of a Fuzzy Set
- Product of Two Fuzzy Sets
- Power of a Fuzzy Set
- Linguistic Variable

?

Basic Terms & Operations

Min and Max Operators

- Many fuzzy set operations are defined through the min ($\wedge$) and max ($\vee$) operators.
- Min and max are analogous to product (.) and sum (+) in algebra).

Basic Terms & Operations

Min and Max Operators

-Min operator-

- Min operator may be used to select the minimum of two elements.
- The minimum of two elements $\mu 1$ and $\mu 2$ denoted either as min($\mu 1$, $\mu 2$), $\vee$($\mu 1$, $\mu 2$) or $\mu 1 \vee \mu 2$ is defined as

$$\mu_1 \wedge \mu_2 = min(\mu_1, \mu_2) \equiv \begin{cases} \mu_1, & \text{if} \quad \mu_1 \leq \mu_2 \\ \mu_2 & \text{if} \quad \mu_1 > \mu_2 \end{cases}$$

Example $2 \wedge 3 = 2$

Basic Terms & Operations

Min and Max Operators

-Max operator-

- Max operator may be used to select the maximum of two elements.
- The maximum of two elements $\mu 1$ and $\mu 2$ denoted either as max($\mu 1$, $\mu 2$), $\wedge$($\mu 1$, $\mu 2$) or $\mu 1 \wedge \mu 2$ is defined as

$$\mu_1 \vee \mu_2 = \max \mu_1, \mu_2) \equiv \begin{cases} \mu_1, & \text{if} \quad \mu_1 \geq \mu_2 \\ \mu_2 & \text{if} \quad \mu_1 < \mu_2 \end{cases}$$

Example $2 \vee 3 = 3$

Basic Terms & Operations

Empty Fuzzy Set

A fuzzy set A is called empty (denoted by A = ∅) if its membership function is zero everywhere in its universe of discourse X.

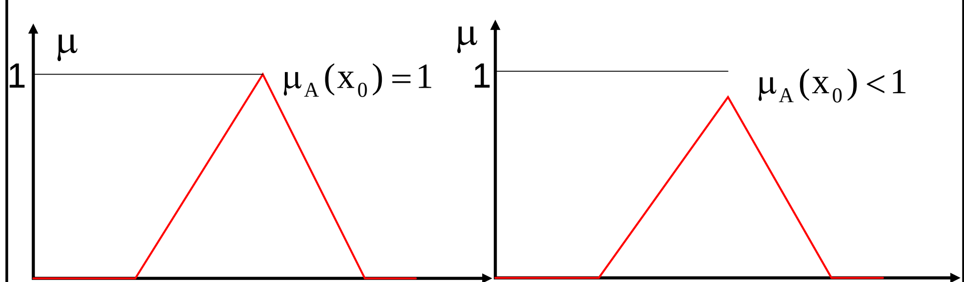
$$A \equiv \phi \quad \text{if} \quad \mu_A(x) = 0, \forall x \in X$$

$\forall x \in X$: "for any element x in X"

Basic Terms & Operations

Normal Fuzzy Set

A normal fuzzy set is one whose membership function has at least one element x_0 in the universe where its membership function equals to one.



A normal fuzzy

Subnormal fuzzy

Basic Terms & Operations

Equality of Fuzzy Sets

Two fuzzy sets are said to be equal if their membership functions are equal everywhere in the universe of discourse, that is

$$A \equiv B \quad \text{if} \quad \mu_A(x) = \mu_B(x)$$

Basic Terms & Operations

Union of Two Fuzzy Sets

The union of two fuzzy sets A and B defined over the same universe of discourse X is a new fuzzy set $A \cup B$ also on X with membership function which is the maximum of the grades of membership function of every x to A and B:

$$\mu_{A \cup B}(x) \equiv \mu_A(x) \vee \mu_B(x) \equiv \max(\mu_A(x), \mu_B(x))$$

Basic Terms & Operations

Union of Two Fuzzy Sets

What is the meaning of union in fuzzy sets?

Basic Terms & Operations

Union of Two Fuzzy Sets

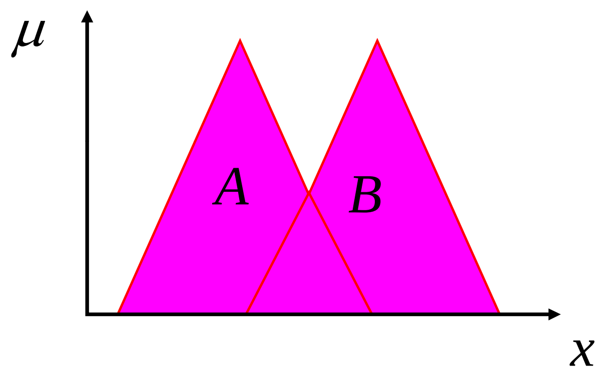
Crisp sets: Which element belongs to either set?

Fuzzy sets: How much of the element is in either set?

Basic Terms & Operations

Union of Two Fuzzy Sets

$$\mu_{A \cup B}(x) \equiv \mu_A(x) \vee \mu_B(x)$$



Basic Terms & Operations

Union of Two Fuzzy Sets

Example

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Find: $A \cup B$

Basic Terms & Operations

Union of Two Fuzzy Sets

Exercise

$$D_1 = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

$$D_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

Find: $D_1 \cup D_2$

Basic Terms & Operations

Intersection of Two Fuzzy Sets

The intersection of two fuzzy sets A and B is a new fuzzy set $A \cap B$ also on X with membership function which is the minimum of the grades of membership function of every x in X to the sets A and B:

$$\mu_{A \cap B}(x) \equiv \mu_A(x) \wedge \mu_B(x) \equiv \min(\mu_A(x), \mu_B(x))$$

Basic Terms & Operations

Intersection of Two Fuzzy Sets

What is the meaning of the intersection in fuzzy sets?

Basic Terms & Operations

Intersection of Two Fuzzy Sets

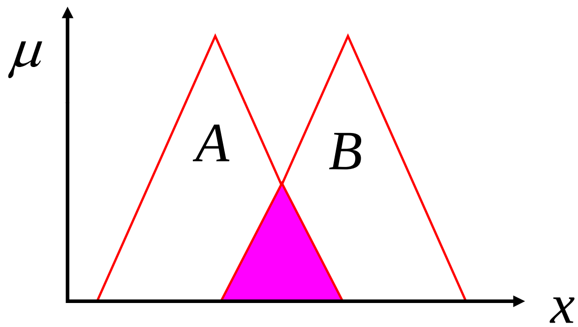
Crisp sets: Which element belongs to both sets?

Fuzzy sets: How much of the element is in both sets?

Basic Terms & Operations

Intersection of Two Fuzzy Sets

$$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x)$$



Basic Terms & Operations

Intersection of Two Fuzzy Sets

Example

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Find: $A \cap B$

Basic Terms & Operations

Intersection of Two Fuzzy Sets

Exercise

$$D_1 = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

$$D_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

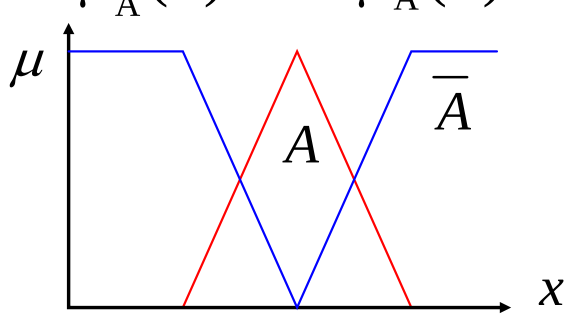
Find: $D_1 \cap D_2$

Basic Terms & Operations

Complement of a Fuzzy Set

The complement of a fuzzy set A is a new fuzzy set A also on X with membership function:

$$\mu_{\overline{A}}(x) \equiv 1 - \mu_A(x)$$



Basic Terms & Operations

Complement of a Fuzzy Set

What is the meaning of complement in fuzzy sets?

Basic Terms & Operations

Complement of a Fuzzy Set

Crisp sets: Which element does not belongs to the set?

Fuzzy sets: How much do elements not belong to the set?

Basic Terms & Operations

Complement of a Fuzzy Set

Example

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Find: $\overline{A}, \overline{B}$

Basic Terms & Operations

Complement of a Fuzzy Set

Exercise

$$D_1 = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

$$D_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

Find: $\overline{D_1}, \overline{D_2}$

Basic Terms & Operations

Algebraic Product of Two Fuzzy Sets

Product of two fuzzy sets A and B defined on the same universe of discourse X is a new fuzzy set A·B, with membership function that equals to the algebraic product of the membership function of A and B:

$$\mu_{A\cdot B}(x) \equiv \mu_A(x) \cdot \mu_B(x)$$

Basic Terms & Operations

Product of Two Fuzzy Sets

Example

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Find: $A \cdot B$

Basic Terms & Operations

Product of Two Fuzzy Sets

Exercise

$$D_1 = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

$$D_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

Find: $\overline{\mathrm{D}}_1 \cdot \overline{\mathrm{D}}_2$

Basic Terms & Operations

Multiplying a Fuzzy Set by a Crisp Number

We can multiply the membership function of a fuzzy set by the crisp number a to obtain a new fuzzy set called a·A. Its membership function is

$$\mu_{a \cdot A}(x) \equiv a \cdot \mu_A(x)$$

Basic Terms & Operations

Algebraic Sum of Two Fuzzy Sets

Summing of two fuzzy (a.k.a Probabilistic sum) sets A and B defined on the same universe of discourse X is a new fuzzy set (A+B) -A·B, with membership function that equals to the algebraic sum of the membership function of A and B:

$$\mu_{A+B}(x) \equiv \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Basic Terms & Operations

Sum of Two Fuzzy Sets

Example

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Find: $A + B$

Basic Terms & Operations

Sum of Two Fuzzy Sets

Exercise

$$D_1 = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

$$D_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

Find: $\overline{\mathrm{D}}_1 + \overline{\mathrm{D}}_2$

Basic Terms & Operations

Exercise

Consider the fuzzy sets A and B where:

A={(x_1, 0.2), (x_2, 0.4), (x_3, 1), (x_4, 0.3), (x_5, 0.1)}

B={(x_1, 0.1), (x_2, 0.3), (x_3, 0.6), (x_4, 0.4)}

Solve:

- algebraic product
- algebraic sum.

Basic Terms & Operations

Bounded Product

The bounded product of two fuzzy sets A and B is given by:

$$\mu_{A\Theta B}(x) \equiv \max\left(0, \mu_A(x) + \mu_B(x) - 1\right)$$

Bounded Sum

The bounded sum of two fuzzy sets A and B is by:

$$\mu_{A\oplus B}(x) \equiv \min\left\{1, \mu_A(x) + \mu_B(x)\right\}$$

Basic Terms & Operations

Drastic Product

The drastic product of two fuzzy sets A and B is given by:

$$\mu_{A\otimes B}(x) \equiv \begin{cases} \mu_A(x) & \text{for } \mu_B(x) = 1 \\ \mu_B(x) & \text{for } \mu_A(x) = 1 \\ 0 & \text{for } \mu_A(x), \mu_B(x) < 1 \end{cases}$$

Basic Terms & Operations

Exercise

Consider the fuzzy sets A and B where:

A={(x_1, 0.2), (x_2, 0.3), (x_3, 1), (x_4, 0.5), (x_5, 0.1), }
B={(x_1, 0.0), (x_2, 0.3), (x_3, 0.6), (x_4, 0.7), (x_5, 0.9)}

Perform the following on A and B:
(1). Algebraic product
(2). Algebraic sum
(3). Bounded product
(4). Bounded sum
(5). Drastic product

Basic Terms & Operations

Power of a Fuzzy Set

The α power of A is a new fuzzy set A , with membership function

$$\mu_{A^\alpha}(x) \equiv [\mu_A(x)]^\alpha$$

Basic Terms & Operations

Power of a Fuzzy Set

Example

$$A=\left\{\frac{1}{2}+\frac{0.5}{3}+\frac{0.3}{4}+\frac{0.2}{5}\right\}$$

$$B=\left\{\frac{0.5}{2}+\frac{0.7}{3}+\frac{0.2}{4}+\frac{0.4}{5}\right\}$$

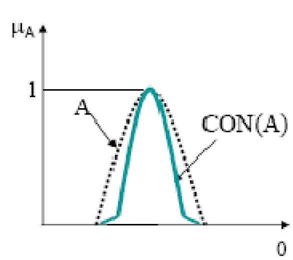
Find: A^2 $B^{1/2}$

Basic Terms & Operations

Power of a Fuzzy Set

CON (concentration)

$$\mu_{A^2}(x) \equiv \mu_{CON(A)}(x) = [\mu_A(x)]^2$$

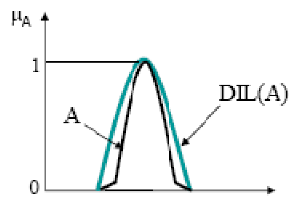


Basic Terms & Operations

Power of a Fuzzy Set

DIL (dilatation)

$$\mu_{A^{1/2}}(x) \equiv \mu_{DIL(A)}(x) = \sqrt{\mu_A(x)}$$

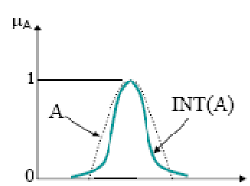


Basic Terms & Operations

Power of a Fuzzy Set

INT (Contrast Intensification)

$$\mu_{INT(A)}(x) \equiv \begin{cases} 2[\mu_A(x)]^2, & 0 \le \mu_A(x) \le 0.5 \\ 1-2[1-\mu_A(x)]^2, & 0.5 \le \mu_A(x) \le 1 \end{cases}$$



Basic Terms & Operations

Exercise

- Consider the fuzzy set A:
  A={(1,0.1), (2,0.8), (3,1), (4,0.2), (5, 0.5)}

- Solve:
  - A^3
  - CON(A)
  - DIL(A)
  - INT(A)
- Draw graphs showing the operations

Basic Terms & Operations

Exercise

- For the fuzzy set with the following membership function (take the elements of the universe of discourse to be 0, 0.5, 1, 1.5 and 2), find and draw the graphs of
  - CON(A)
  - DIL(A)

$$\mu_A(u) = \begin{cases} u & for\ 0 \le u \le 1 \\ 2-u & for\ 1 \le u \le 2 \end{cases}$$

Basic Terms & Operations

Linguistic Variables

- Many engineering rules can be formulated based on the following observations and actions:

  IF **so and so conditions** THEN **so and so action**

- Fuzzy logic allows to a certain extent **natural language** to be used in solving many engineering problems as well as other practical applications.

- Since fuzzy logic can handle some form of uncertainty and imprecision which are inherent in natural language, thus it can be used as the mathematical foundation of our natural language.

Basic Terms & Operations

Linguistic Hedges and Operators

- A fuzzy set can be regarded as corresponding to a linguistic value such as 'tall', and a linguistic variable 'height' can be regarded as ranging over such linguistic values.

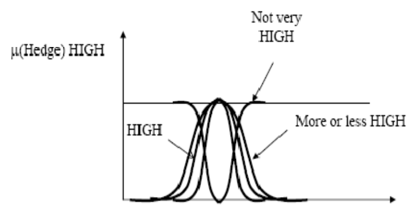
- One powerful aspect of fuzzy sets in this context is the ability to deal with linguistic quantifiers or 'hedges'.

Basic Terms & Operations

Linguistic Hedges and Operators

- Hedges such as more or "less", "very", "not very", "slightly", etc., correspond to modifications in the membership function of the fuzzy set involved.

- The fuzzy set operations such CON, DIL, INT, etc. (see the Table of Hedges and Operators) can be used to modify the fuzzy set HIGH such as shown below.



Basic Terms & Operations

Linguistic Hedges and Operators

Table of Hedges and Operators

Hedge	Operator definition
Very F	CON=F^2
More or Less F	DIL=$F^{0.5}$
Plus F	$F^{1.25}$
Not F	1-F
Not very F	1-CON(F)
Slightly F	INT [Plus F AND Not (Very F)]

Basic Terms & Operations

Linguistic Hedges and Operators

Table of Hedges and Operators

Hedge	Operator definition
Very F	CON=F^2
More or Less F	DIL=$F^{0.5}$
Plus F	$F^{1.25}$
Not F	1-F
Not very F	1-CON(F)
Slightly F	INT [Plus F AND Not (Very F)]

Basic Terms & Operations

Example

Suppose we have a universe integers, Y={1,2,3,4,5}. We define the following linguistic terms as a mapping onto Y:

$$'Small' = \left\{ \frac{1}{1} + \frac{.8}{2} + \frac{.6}{3} + \frac{.4}{4} + \frac{.2}{5} \right\}$$

$$'Large' = \left\{ \frac{.2}{1} + \frac{.4}{2} + \frac{.6}{3} + \frac{.8}{4} + \frac{1}{5} \right\}$$

Find: very small, not very small, not very small and not very large, very large

Linguistic Variables & Values

Example

Ship age is defined as follows:

$$'Old' = \left\{ \frac{0}{0} + \frac{.1}{5} + \frac{.3}{10} + \frac{.5}{15} + \frac{.7}{20} + \frac{.9}{25} + \frac{.1}{30} \right\}$$

$$'Young' = \left\{ \frac{1}{0} + \frac{.9}{5} + \frac{.7}{10} + \frac{.5}{15} + \frac{.3}{20} + \frac{.1}{25} + \frac{.0}{30} \right\}$$

Find: very old, very old or very young, not very old and more or less young