What is fuzzy?

- Unclear

Basic concepts

- Degree of membership [0,1]
- Better reflect the way intelligent people think

Fuzzy representation

$A = \{ (x, \mu_A(x)) | x \in X \}$

where X represents the universe of discourse and $\mu_A(x)$ assumes values in the range from 1 to 0.

Example

Let the values of temperature in °C under consideration be

$T = \{0, 5, 10, 15, 20, 25, 30, 35, 40\}$.

Then, the term *hot* can be defined by a fuzzy set as follows

HOT = {(0,0), (5,0.1), (10,0.3), (15,0.5), (20,0.6), (25,0.7), (30,0.8), (35,0.9), (40,1.0)}.

This fuzzy set reflects the point of view that 0 °C is not hot at all, 5, 10, and 15 °C are somewhat hot, and 40 °C is indeed hot. Another person could have defined the set differently.

Representation

$A = \mu_1/x_1 + \mu_2/x_2 + \mu_3/x_3 + \ldots$

The symbol / here does not denote division, nor does the symbol + denote summation. The summation symbol is used to connect the terms and thus it means a union of single-term subsets.

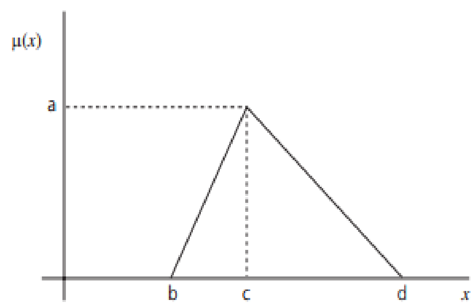
Example

Let $X = \{ x_1, x_2, x_3, x_4 \}$. One can define a fuzzy set as:

$A = 0.8/x_1 + 0.4/x_2 + 0.1/x_3 + 0.9/x_4$.

Representation as a continues function

$$\mu(x) = a\,(b-x)/(b-c)\;;\quad b \ge x \le c$$
$$= a\,(d-x)/(d-c)\;;\quad c \ge x \le d$$
$$= 0\;;\quad \text{otherwise}$$



Example

Figure 2.9 shows three sets defined graphically to represent the fuzzy sets SLOW, MEDIUM, and FAST to reflect a way of thinking about values of speed in the range of 0 to 40 km/hr.

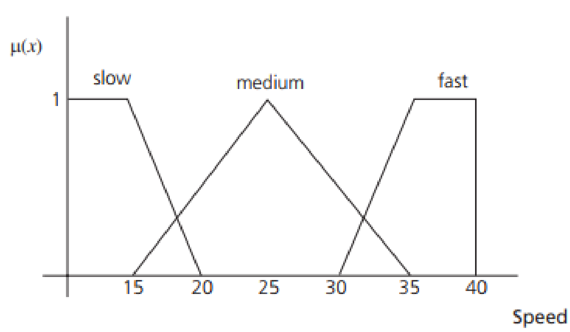


Figure 2.9: Possible graphical representation of fuzzy sets.

Fuzzy set Properties

Empty fuzzy set

A fuzzy set is referred to as empty if and only if the value of the membership function is zero for all possible members under consideration. In a short hand form this statement would read

$A = \varnothing$ iff $\mu_A(x) = 0\ \forall x \in X$. (iff and $\forall$ are short hand forms for *if and only if* and *for all values of*, respectively).

Equal fuzzy set

Two fuzzy sets A and B are said to be equal iff $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

Universal fuzzy set

A fuzzy set is universal if and only if the value of the membership function is one for all members under consideration.

α -cuts

A fuzzy set may be completely characterized by its α-cuts, defined as follows

strong α-cuts: $A_\alpha = \{ x | \mu_A(x) > \alpha \};\ \alpha \in [0,1]$

weak α-cuts: $A_{\bar{\alpha}} = \{ x | \mu_A(x) \geq \alpha \};\ \alpha \in [0,1]$

Thus, an α-cut is a crisp set that consists of all the elements of a fuzzy set whose membership functions have values greater than a specified value α, or greater than or equal to a specified value; the first condition leads to strong α-cuts and the second to weak α-cuts. All the cuts of a fuzzy set form a family of crisp sets.

Let $A = 0.2/1 + 0.5/2 + 0.6/3 + 1/4 + 0.7/5 + 0.3/6 + 0.1/7$.

Then, the weak α-cuts for $\alpha \in [0,1]$ from 0.1 to 1 with step width of 0.1 are as follows

$A_{0.\bar{1}} = \{1, 2, 3, 4, 5, 6, 7\}$

$A_{0.\bar{2}} = A_{0.\bar{3}} = \{1, 2, 3, 4, 5, 6\}$

$A_{0.\bar{4}} = A_{0.\bar{5}} = \{2, 3, 4, 5\}$

$A_{0.\bar{6}} = \{3, 4, 5\}$

$A_{0.\bar{7}} = \{4, 5\}$

$A_{0.\bar{8}} = A_{0.\bar{9}} = A_{1.\bar{0}} = \{4\}$

Support

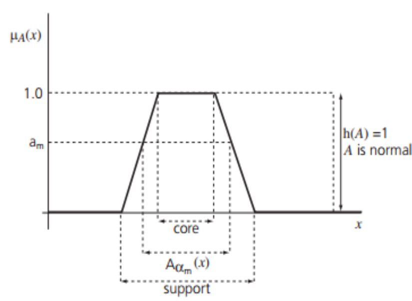
The support of a fuzzy set A is a crisp set supp(A) of all $x \in X$ such that $\mu_A(x) > 0$. It is a strong α-cut for $\alpha = 0$. The element $x \in X$ at which $\mu_A(x) = 0.5$ is referred to as the *cross-over point*. A fuzzy set whose support is a single element in X with $\mu_A(x) = 1$ is referred to as a *fuzzy singleton*.

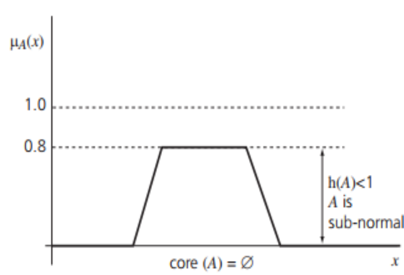
Core

The core of a fuzzy set A is a crisp set core(A) of all $x \in X$ such that $\mu_A(x) = 1$. The core of a fuzzy set may be an empty set.

Height

The height, h(A) of a fuzzy set A is the largest value of μ_A for which the α-cut is not empty. In other words, it is the largest value of the membership function attained by an element in the set. A fuzzy set with h(A) = 1 is referred to as *normal*, otherwise it will be referred to as *sub-normal*.





Operations on fuzzy set

Logical operations

COMPLEMENT

The absolute complement of a fuzzy set A is denoted by $\overline{A}$ and its membership function is defined by

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x) \text{ for all } x \in X$$

Example

Let $A = 0.7/1 + 1/1 + 0.6/3 + 0.2/4 + 0/5$.

Then, $\overline{A} = 0.3/1 + 0/2 + 0.4/3 + 0.8/4 + 1/5$.

Conti..

UNION

The union of two fuzzy sets A and B is a fuzzy set whose membership function is defined by

$$\mu_{A \cup B}(x) = \max\left[\mu_A(x), \mu_B(x)\right]$$

Example

Let $A = 0.3/1 + 0/2 + 0.4/3 + 0.8/4 + 1/5$, and

$B = 0.2/1 + 0.3/2 + 0.1/3 + 0.2/4 + 0.4/5$.

Then,

$$A \cup B = 0.3/1 + 0.3/2 + 0.4/3 + 0.8/4 + 1/5$$

Conti..

INTERSECTION

The intersection of two fuzzy sets A and B is a fuzzy set whose membership function is defined by

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)].$$

Example

Let $A = 0.3/1 + 0/2 + 0.4/3 + 0.8/4 + 1/5$, and

$B = 0.2/1 + 0.3/2 + 0.1/3 + 0.2/4 + 0.4/5$

Then,

$$A \cap B = 0.2/1 + 0/2 + 0.1/3 + 0.2/4 + 0.4/5$$

Some properties of Operations

Law of contradiction	$A \cap \bar{A} \neq \varnothing$
Law of excluded middle	$A \cup \bar{A} \neq I$
De Morgan's laws	$(A \cap B)' = A' \cup B'$ $(A \cup B)' = A' \cap B'$
Involution (Double negation)	$\bar{\bar{A}} = A$
Commutative	$A \cap B = B \cap A$ $A \cup B = B \cup A$
Associative	$A \cap (A \cap B) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Algebraic Operations

Cartesian multiplication

The Cartesian multiplication of two sets A and B is a fuzzy set C such that

$$C = A \times B$$
$$= \{\mu_C(x)/(a,b) \mid a \in A, b \in B, \mu_C(c) = \min[\mu_A(a), \mu_B(b)]\}$$

Example

Let $A = 0.2/3 + 1/5 + 0.5/7$, and
$B = 0.8/2 + 0.3/6$.

Then,

$$A \times B = \min[0.2,0.8]/(3,2) + \min[0.2,0.3]/(3,6) + \min[1,0.8]/(5,2) + \min[1,0.3]/(5,6) + \min[0.5,0.8]/(7,2) + \min[0.5,0.3]/(7,6)$$
$$= 0.2/(3,2) + 0.2/(3,6) + 0.8/(5,2) + 0.3/(5,6) + 0.5/(7,2) + 0.3/(7,6).$$

Algebraic multiplication

The algebraic product of two fuzzy sets A and B leads to a fuzzy set C such that

$$AB = \{\mu_A(a)\mu_B(b)/x \mid x \in A, x \in B\}.$$

Example

Let $A = 0.2/3 + 1/5 + 0.5/7$, and

$B = 0.1/3 + 0.3/7 + 0.2/8$.

Then, $AB = (0.2)(0.1)/3 + (1)(0)/5 + (0.5)(0.3)/7 + (0)(0.2)/8$
$= 0.02/3 + 0.15/7$

Exponent

Raising a set A to the power of α is a special case of algebraic multiplication. It is defined by

$$A^{\alpha} = \left\{ \left(\mu_A(x)\right)^{\alpha} / x \middle| x \in A \right\}.$$

Example

Let $A = 0.5/4 + 1/5 + 0.4/6$.

Then, $A^2 = (0.25)^2/4 + (1)^2/5 + (0.4)^2/6$

$= 0.25/4 + 1/5 + 0.16/6$, and

$A^{0.5} = (0.5)^{0.5}/4 + (1)^{0.5}/5 + (0.4)^{0.5}/6$

$= 0.7/4 + 1/5 + 0.63/6$.

Example

Let the range of temperature under consideration be 5, 10, 20, 30, and 40°C. Let also the set A represent the concept "hot" from a given point of view:

$A = 0.1/5 + 0.2/10 + 0.4/20 + 0.6/30 + 0.7/40$.

Then, *very hot* could be expressed by

$A^2 = 0.01/5 + 0.04/10 + 0.16/20 + 0.36/30 + 0.49/40$.

It follows that *very very hot* can be expressed by $(A^2)^2 = A^4$.

Concentration and Dilation

These two operations are unique to fuzzy sets; they do not have counterparts in classical sets. The concentration operation is defined as:

$\text{CON}(A) = A^2$

The dilation process is defined as:

$\text{DIL}(A) = A^{0.5}$

Example

Let A = 0.5/4 + 1/5 + 0.4/6.

Then, $\text{CON}(A) = 0.25/4 + 1/5 + 0.16/6$, and

$\text{DIL}(A) = 0.7/4 + 1/5 + 0.63/6$.

Conti..

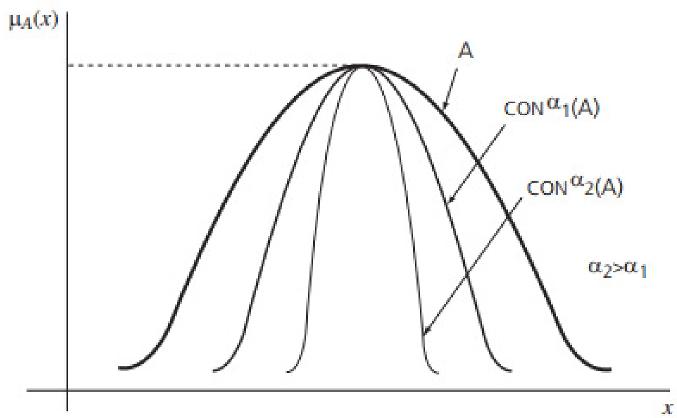
The operations concentration and dilation could be composed with themselves, eg, $\text{CON}^2(A) = A^4$,

and $\text{DIL}^2(A) = A^{\frac{1}{4}}$. In general one can write

$\text{CON}^{\alpha}(A) = A^{2\alpha}$, and

$\text{DIL}^{\alpha}(A) = A^{0.5\alpha}$.

with α being an integer ≥ 2.



Algebraic sum

The algebraic sum of two sets A and B is a fuzzy set with a membership function given by:

$$\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

where the symbol + here denotes algebraic summation.

Example

Let C be the algebraic summation of two fuzzy sets A and B expressed by:

$C = A + B$,

$A = 0.2/1 + 1/2 + 0.4/3$, and

$B = 0.3/1 + 0.4/2 + 0.2/7$.

$$\begin{aligned}
\mu_C(x_1) &= 0.2 + 0.3 - (0.2)(0.3) \\
&= 0.44 \\
\mu_C(x_2) &= 1 + 0.4 - (1)(0.4) \\
&= 1 \\
\mu_C(x_3) &= 0.4 + 0 + (0)(0.4) \\
&= 0.4 \\
\mu_C(x_4) &= 0.2 + 0 + (0)(0.2) \\
&= 0.2
\end{aligned}$$

Accordingly, C = 0.44/1 + 1/2 + 0.4/3 + 0.2/7.

Bounded sum

The symbol ⊕ denotes the bounded sum of two fuzzy sets. The operation leads to a fuzzy set with a membership function defined by:

$$\mu_{A\oplus B}(x) = \min\left[1, \left(\mu_A(x) + \mu_B(x)\right)\right]$$

Example

Let $A = 0.5/3 + 1/4 + 0.8/5$, and

$B = 0.2/3 + 0.4/5$.

Then, $A \oplus B = 0.7/3 + 1/4 + 1/5$.

Bounded difference

The symbol $\ominus$ denotes the bounded difference of two fuzzy sets. The operation leads to a fuzzy set with a membership defined by:

$$\mu_{A\ominus B}(x) = \min\left[1, \left(\mu_A(x) + \mu_B(x)\right)\right]$$

Example

Let $A = 0.5/3 + 0.3/5 + 0.7/8$, and

$B = 0.4/3 + 0.1/5$.

Then, $A \ominus B = \mu(x_1)/3 + \mu(x_2)/5 + \mu(x_3)/8$

where $\mu(x_1) = \min[1,(0.5 - 0.4)] = 0.1$,

$\mu(x_2) = \min[1,(0.3 - 0.1)] = 0.2$, and

$\mu(x_3) = \min[1,(0.7 - 0.0)] = 0.7$.

hence, $A \ominus B = 0.1/3 + 0.2/5 + 0.6/8$.

Identities relating sets and algebraic operations.

$$\mu_A(x) \wedge \mu_B(x) = \mu_B(x) \ominus \left(\mu_B(x) \ominus \mu_A(x)\right)$$

$$\mu_A(x) \vee \mu_B(x) = \mu_A(x) + \left(\mu_B(x) \ominus \mu_A(x)\right)$$

$$\left|\mu_A(x) - \mu_B(x)\right| = \left(\mu_A(x) \ominus \mu_B(x)\right) + \left(\mu_B(x) \ominus \mu_A(x)\right)$$

fuzzification

- Is an operation of transforming CRISP set to a fuzzy set OR fuzzy set to a fuzzier set.

Fuzzy Set Relations

Presentation of relations

A relation can be defined by listing all the member pairs $(a,b) \in R$. It can also be expressed using a coordinate diagram, a matrix, a mapping diagram, or an arrow diagram.

Example

Let A = {desk, bag, book}, and B = {wood, plastic, paper}.

The Cartesian product of these two sets leads to
$A \times B$ = {(wood,desk), (wood,bag), (wood,book), (plastic,desk), (plastic,bag), (plastic,book), (paper,desk), (paper,bag), (paper,book)}.
From this set one may select a subset such that
R = {(wood,desk), (plastic,bag), ((paper,bag), (paper,book)}.

The relation could equivalently be represented using a matrix as shown below.

R	wood	plastic	paper
desk	1	0	0
bag	0	1	1
book	0	0	1

Fundamentals of Fuzzy Relations

Fuzzy relations are fuzzy sets defined on universal sets which are Cartesian products. They capture the strength of association among elements of two or more sets, not just whether such an association exists or not.

Example

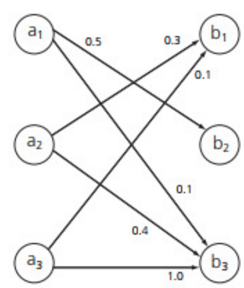
Let $A = \{a_1, a_2, a_2\}$ and $b = \{b_1, b_2, b_3\}$.

Let R be a relation from A to B given by

$$R = 0.1/(a_1,b_3) + 0.5/(a_1,b_2) + 0.3/(a_2,b_1) + 0.4/(A_2,B_3) + 1.0/(A_3,B_3) + 0.1/(A_3,B_1)$$

The corresponding fuzzy matrix would be

$$M_R = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0 & 0 \\ 0.1 & 0.4 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array}$$



Operations on Binary Fuzzy Relations

Inverse of a fuzzy relation

The inverse of a fuzzy relation R on $A \times B$ is denoted by R^{-1}. It is a relation on $B \times A$ defined by $R^{-1}(b,a) = R(a,b)$ for all pairs $(b,a) \in B \times A$.

Example

If $M_R = \begin{bmatrix} 0.3 & 1.0 & 0 \\ 1.0 & 0.5 & 0.3 \\ 0.0 & 0.0 & 0.8 \end{bmatrix}$, then

$$M_{R^{-1}} = \begin{bmatrix} 0.3 & 1.0 & 0.0 \\ 1.0 & 0.5 & 0.0 \\ 0.0 & 0.3 & 0.8 \end{bmatrix}.$$

Composition of fuzzy relations

As explained in Section 3.2, the composition of two crisp binary relations P and Q requires that the relations be compatible—i.e., $A \times B$ and $B \times C$ share the set B. The composition $R = P \circ Q$ consists of pairs (a,b) from $A \times C$ that are connected through at least one element in B. In fuzzy relations, the connections have degrees of strength stemming from the fact P and Q are fuzzy sets.

There are several types of fuzzy relations compositions. The most common in engineering applications is the *max-min composition*. In this composition scheme the strength of the connection is determined by the smaller strength connection of the two in a chain that connects a to c. Among the chains that connect the two elements, the one with the largest strength is the one that is selected to characterize the relation. In mathematical shorthand, we can write

$$P \circ Q = \{\max \min[\mu_P(a,b), \mu_Q(b,c)]/(a,c) \mid a \in A, b \in B, c \in C\}$$

The properties of crisp relations given in Table 3.1 are valid for fuzzy relations composition as well.

Example

Let $A = \{a_1, a_2\}$,

$B = \{b_1, b_2\}$, and

$C = \{c_1, c_2\}$.

Let P be a relation from A to B defined by:

$$\begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ M_P = & \begin{bmatrix} 0.4 & 0.2 \\ 0.8 & 0.3 \end{bmatrix} \begin{array}{c} a_1 \\ a_2 \end{array} \end{array}$$

Let Q be a relation from B to C defined by:

$$\begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ M_Q = & \begin{bmatrix} 0.3 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{array}{c} b_1 \\ b_2 \end{array} \end{array}$$

Then,

$$P \circ Q = \begin{bmatrix} 0.4 & 0.2 \\ 0.8 & 0.3 \end{bmatrix} \circ \begin{bmatrix} 0.3 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$$
$$= \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$$

where r_1, r_2, r_3, and r_3 are calculated as follows:

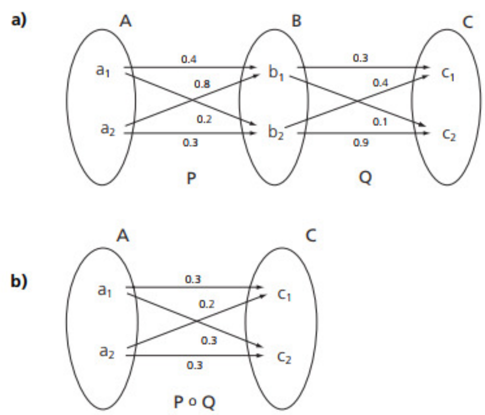
r_1 = max[min(0.4,0.3), min(0.2,0.4)]
= max[0.3,0.2]
= 0.3

r_2 = max[min(0.4,0.1), min(0.2,0.9)]
= max[0.1,0.2]
= 0.2

r_3 = max[min(0.8,0.3), min(0.3,0.4)]
= max[0.3,0.3]
= 0.3

r_4 = max[min(0.8,0.1), min(0.3,0.9)]
= max[0.1,0.3]
= 0.3

Hence, $M_{P \circ Q} = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}$



Types of Fuzzy Relations

Reflexive	$\mu_R(x,x) = 1$ and $\mu_R(x,y) < 1$, $x \neq y$
Anti-reflexive	$\mu_R(x,x) = 0$ and $\mu_R(x,y) < 1$, $x \neq y$
Symmetric	$\mu_R(x,y) = \mu_R(y,x)$
Anti-symmetric	$\mu_R(x,y) \neq \mu_R(y,x)$
max-min transitive	$\mu_R(x,z) \geq \max \min[\mu(x,y), \mu(x,z)]$
Identity relation	Diagonal elements of M_R are all ones
Zero relation	All elements of M_R are zeros
Universal	All elements of M_R are ones

Fuzzy Reasoning

A fuzzy algorithm has several rules, such as

Rule 1: IF x is A_1, THEN y is B_1

Rule 2: IF x is A_2, THEN y is B_2

.
.
.

Rule n: IF x is A_n, THEN y is B_n

The rules can be also written as

$A_1 \rightarrow B_1$

$A_2 \rightarrow B_2$

.
.
.

$A_n \rightarrow B_n$