Partial Differentiation

Partial Differential Coefficients or Partial Derivatives

Definition : First order partial Derivative:

The ordinerary derivatives of z with respect to x, treating y as constant is called the parital derivatives of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, f_x or $f_x(x, y)$.

Thus, $\frac{\partial z}{\partial x} = \lim\limits_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$, the derivative of z with respect to y keeping x as constant is $\frac{\partial z}{\partial y}$.

If z is a function of three or more variable $x_1, x_2, \ldots\ldots x_n$ then partial derivatives of z with respect to x_1 is obtained by differentiating z with espect to x_1 keeping all other variables constant and is expressed by $\frac{\partial z}{\partial x_1}$.

$\frac{\partial z}{\partial x_1}$ is the first order partial derivative of z with respect to x_1.

Second order and higher order Partial Differentiations :

Let $\qquad z = f(x, y)$

then $\qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy} \qquad\qquad \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{xy}$

in which we first differentiate z partially with respect to y considering x as constant and the result to obtained is then differentiated partially with respect to x, considering y as a constant.

In general $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$ that is the order of differentiation is commutative.

Illsutrative Examples

Type 1 - Direct partial derivatives

Example 1 :

If $u = \log(x^2 + y^2 + z^2)$ show that $x \frac{\partial^2 u}{\partial y \partial z} = \frac{y\, \partial^2 u}{\partial z\, \partial x} = z \frac{\partial^2 u}{\partial x\, \partial y}$

Solution :

Similarly we can define thrid and higher order derivatives

$$\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial t}{\partial y^2}\right] \text{etc.}$$

Error!

We have, $\qquad u = \log(x^2 + y^2 + z^2)$

$$\therefore \qquad \frac{\partial u}{\partial z} = \frac{2z}{x^2 + y^2 + z^2} \qquad \text{[treating x and y as constant]}$$

$$\text{and} \qquad \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right] = -\frac{2z\,(2y)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{-4yz}{(x^2 + y^2 + z^2)^2}$$

$$\text{Now,} \qquad x\frac{\partial u^2}{\partial y\, \partial z} = \frac{-4xyz}{(x^2 + y^2 + z^2)^2}$$

$$\text{By symmetry,} \qquad y\frac{\partial^2 u}{\partial z\, \partial x} = z\frac{\partial^2}{\partial x\, \partial y}$$

$$= \frac{4\,xyz}{(x^2 + y^2 + z^2)^2}$$

$$\text{Hence,} \qquad x\frac{\partial^2 u}{\partial y\, \partial z} = y\frac{\partial^2 u}{\partial z\, \partial x} = z\frac{\partial^2 u}{\partial x\, \partial y}$$

Example 2 :

If $v = (1 - 2xy + y^2)^{\frac{1}{2}}$ prove that

(i) $x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = y^2\, v^3$

(ii) $\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial v}{\partial y}\right] = 0$

Solution :

(i) Given $\qquad v = (1 - 2xy + y^2)^{-\frac{1}{2}}$

$$\therefore \qquad \frac{\partial v}{\partial x} = -\frac{1}{2}\,(1 - 2xy + y^2)^{-\frac{3}{2}}\,(-2y)$$

$$yv^3 = y\left[(1 - 2xy + y^2)^{-\frac{1}{2}}\right]^3 \qquad \text{...(1)}$$

$$\text{and} \qquad \frac{\partial v}{\partial y} = -\frac{1}{2}\,(1 - 2xy + y^2)^{-\frac{3}{2}}\,(-2x + 2y)$$

$$= (x - y)\,v^3 \qquad \text{...(2)}$$

$$\therefore\ x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = xyv^3 - y\,(x - y)\,v^3$$

$$= xyv^3 - yxv^3 + y^2\,v^3$$

$$= y^2\,v^3$$

$$\text{(ii) We have,} \quad \frac{\partial v}{\partial x} = yv^3 \qquad \text{...(from (1)}$$

$$\therefore \qquad (1 - x^2)\frac{\partial v}{\partial x} = y\,(1 - x^2)\,v^3$$

$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial v}{\partial x}\right] = \frac{\partial}{\partial x}\,[y\,(1 - x^2)\,v^3]$$

$$= y\left[(1-x^2)\ 3v^2\frac{\partial v}{\partial x} - 2xy^3\right]$$

$$= y\ [3\ (1-x^2)\ v^2\ yv^3 - 2xv^3]$$

$$= yv^3\ [3y\ (1-x^2)\ v^2 - 2x] \qquad \ldots(3)$$

and from (2)

$$\frac{\partial}{\partial y}\left[y^2\frac{\partial v}{\partial y}\right] = \frac{\partial}{\partial y}\ [\,y^2\ (x-y)\ v^3]$$

$$= \frac{\partial}{\partial y}\ [(xy^2 - y^3)\ v^3]$$

$$= (xy^2 - y^3)\ 3v^2\ \frac{\partial v}{\partial y} + (2xy - 3y^2)\ v^3$$

$$= (xy^2 - y^3)\ 3y^2\ (x-y)\ v^3 + (2xy - 3y^2)\ v^3 \qquad \ldots\text{From (2)}$$

$$= yv^3\ [3y\ (x-y)^2\ v^2 + 2x - 2y] \qquad \ldots(4)$$

Adding (3) and (4), we get,

$$\frac{\partial}{\partial x}\left[(1-x^2)\ \frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^3\frac{\partial v}{\partial y}\right] = yv^3\ [3y\ (1-x^2)\ v^3 - 2x + 3y\ (x-y)^2\ v^2 + 2x -$$

3y] 2x

$$= 3y^2\ v^3\ [(1-x^2)\ v^2 + (x-y)^2\ v^2 - 1]$$

$$= 3y^2\ v^3\ [(1-x^2)\ v^2 + (x^2 - 2xy + y^2)\ v^2 - 1]$$

$$= 3y^2\ v^3\ [v^2\ (1 - 2xy + y^2) - 1]$$

$$= 3y^2\ v^3\ [v^2 \cdot v^{-2} - 1]$$

$$= 3y^2\ v^3\ [1 - 1]$$

$$= 0$$

Hence,

$$\frac{\partial}{\partial x}\left[(1-x)^2\frac{\partial y}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^3\frac{\partial v}{\partial y}\right] = 0$$

Example 3 :

If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, prove that

(i) $x\ \dfrac{\partial y}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{\partial y}{\partial z} = -u$

(ii) $\dfrac{\partial^2 u}{x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 y}{\partial z^2} = 0$

Solution :

(i)

$$u = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{-1}{2\ (x^2 + y^2 + z^2)^{3/2}} \cdot 2x$$

$$= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -\frac{x^2}{(x^2+y^2+z^2)^{3/2}} \quad ...(1)$$

Similarly, $$y\frac{\partial u}{\partial y} = -\frac{y^2}{(x^2+y^2+z^2)^{3/2}} \quad ...(2)$$

and $$z\frac{\partial u}{\partial z} = -\frac{z^2}{(x^2+y^2+z^2)^{3/2}} \quad ...(3)$$

Adding (1) (2) and (3),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{-1}{(x^2+y^2+z^2)^{\frac{1}{2}}} = -u$$

Now, differentiating $\frac{\partial u}{\partial x}$ w.r.t. x partially,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \left[-\frac{(x^2+y^2+z^2)^{3/2}(1) - x\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2x}{(x^2+y^2+z^2)^3}\right]$$

$$= -\frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}[x^2+y^2+z^2-3x^2]$$

$$= -\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \quad ...(4)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = -\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \quad ...(5)$$

and $$\frac{\partial^2 u}{\partial y^2} = -\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \quad ...(6)$$

Adding (3) , (5) and (6), we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{-2x^2+y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= 0$$

Example 5 :

If $z(x+y) = x^2+y^2$ show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Solution :

$$z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)\,2x - (x^2+y^2)\,1}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

$$\therefore \quad \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x^2 - y^2)}{(x+y)^2}$$

$$= \frac{2(x-y)}{(x+y)}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \qquad ...(1)$$

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 1 - \frac{4xy}{(x+y)^2}$$

$$= \frac{(x+y)^2 - 4xy}{(x+y)^2} = \frac{x + 2xy + y^2 - 4xy}{(x+y)^2}$$

$$= \frac{(x-y)^2}{(x+y)^2} \qquad ...(2)$$

From (1) and (2),

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2}$$

$$= 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Example 7 :

If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solution :

$$\text{LHS} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) v$$

$$\left[\text{where } v = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\right]$$

$$= \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \qquad ...(1)$$

Given $$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

Now, $$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 - z^3 - 3xyz}(3x^2 - 3yz)$$

$$= \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{3(x^2 - xt)}{x^3 + y^3 + z^3 - 3xyz}$$

and $$\frac{\partial u}{\partial z} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

$\therefore$ $$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - xz)}$$

$$= \frac{3}{(x + y + z)}$$

$\therefore$ $$\frac{\partial v}{\partial x} = \frac{-3}{(x + y + z)^2}$$

Similarly $$\frac{\partial v}{\partial x} = \frac{-3}{(x + y + z)^2}$$

$$\frac{\partial v}{\partial z} = \frac{-3}{(x + y + z)^2}$$

Substituting in equation (1),

$$\text{LHS} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 = \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2}$$

$$= \frac{-9}{(x + y + z)^2} = \text{RHS}$$

Example 9 :

If $u = \log(\tan x + \tan y + \tan z)$ show that

$\sin 2x\, u_x + \sin\, 2y\, u_y + \sin 2z\, u_z = 2$

Solution :

Given $$u = \log(\tan x + \tan y + \tan z)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y + \tan z)} \sec^2 x$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{(\tan x + \tan y + \tan z)} \sec^2 y$$

$$u_z = \frac{\partial u}{\partial z} = \frac{1}{(\tan x + \tan y + \tan z)} \sec^2 z$$

$\therefore$ $$\text{LHS} = \sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\ u_z$$

$$= \frac{\sin 2x \sec^2 + \sin 2y \sec^2 y + \sin 2z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \sin x \cos x \dfrac{1}{\cos^2 x} + 2 \sin y \cos y \dfrac{1}{\cos^2 y} + 2 \sin z \cos z - \dfrac{1}{\cos^2 z}}{(\tan x + \tan y + \tan z)}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

Example 10 :

Using log if $x^x y^y z^z = c$ show that at $x = y = z$ $z_{xy} = -[x \log (ex)]^{-1}$

Solution :

From the given claim, we can regard z as a function of two independent variables x and y.

Given $x^x y^y z^z = c.$ ...(1)

Taking log of both side of equation (1) we get,

$x \log x + y \log y + z \log z = \log c$ ...(2)

differentiate (2) w.r.t. partially

$$\frac{x}{x} + \log x + 0 + \frac{z}{z}\frac{\partial z}{\partial x} + \log z \frac{\partial z}{\partial x} = 0$$

$$1 + \log x + \frac{\partial z}{\partial x}(1 + \log z) = 0$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \qquad ...(3)$$

Differentiate (2) w.r.t. y partially,

$$0 + \frac{y}{y} + \log y + \frac{z}{z}\frac{\partial z}{\partial y} + \log z \frac{\partial z}{\partial y} = 0$$

$$1 + \log y + 1\frac{\partial z}{\partial y} + \log z \frac{\partial z}{\partial y} = 0$$

$$\therefore \quad \frac{\partial z}{\partial y} = -\frac{1 + \log y}{1 + \log z}$$

$$z_{xy} = \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[-\left(\frac{1 + \log y}{1 + \log z}\right)\right]$$

$$= -(1 + \log y)\left[-(1 + \log z)^{-2}\frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$= \frac{1 + \log y}{z(1 + \log z)^2}\left[-\frac{(1 + \log x)}{1 + \log z}\right] \qquad ...\text{From (3)}$$

Given $x = y = z$ we get,

$$z_{xy} = -\frac{(1 + \log x)^2}{x(1 + \log x)^3} = -\frac{1}{x(1 + \log x)}$$

$$= -\frac{1}{x(\log e + \log x)}$$

$$= -\frac{1}{x \log ex}$$

$$= -[x \log (ex)]^{-1}$$

Type 2 : To show $f_{xy} = f_{yz}$ i.e. $\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$

Example 11 :

$$f = \log\left(\frac{x^2+y^2}{xy}\right)$$

Differentiating wr.t. x partially,

$$f_x = \frac{1}{\frac{x^2+y^2}{xy}} \quad \frac{(xy)\ (2x)\ - (x^2+y^2)\ y}{(xy)^2}$$

$$= y\ \frac{(2x^2 - x^2 - \ y^2)}{(x^2+y^2)\ xy} \quad = \frac{x^2-y^2}{x\,(x^2+y^2)}$$

Differentiating f_x w.r.t. y partially,

$$f_{yx} = \frac{\partial}{\partial y}[f_x] \quad = \frac{1}{x}\left[\frac{(x^2+y^2)\ (-2y)\ -(x^2-y^2)\ 2y}{(x^2+y^2)^2}\right]$$

$$= \frac{1}{x}\left[\frac{-2y\,(2x^2)}{(x^2+y^2)^2}\right]$$

$$= \frac{-4xy}{(x^2+y^2)^2} \qquad \text{...(1)}$$

Similarly by symmetry

$$f_y = \frac{y^2-x^2}{y\,(x^2+y^2)}$$

$$\therefore \quad f_{yx} = \frac{\partial}{\partial y}[f_y] \quad = \frac{1}{y}\left[\frac{(x^2+y^2)\ (-2x)\ -(y^2-x^2)\ (2x)}{(x^2\ y^2)^2}\right]$$

$$= \frac{-4xy}{(x^2+y^2)^2} \qquad \text{...(2)}$$

From (1) and (2),

$$f_{xy} = f_{yx} = \frac{-4xy}{(x^2+y^2)^2}$$

Example 12 :

$$f = a\ \tan^{-1}\left(\frac{x}{y}\right)$$

Solution :

$$\frac{\partial f}{\partial x} = a\ \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} \ = \frac{ay}{x^2+y^2}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[\frac{ay}{x^2+y^2}\right]$$

$$= a\left[\frac{1\,(x^2+y^2) - y2y}{(x^2+y^2)^2}\right] = \frac{a(x^2-y^2)}{(x^2+y^2)^2} \qquad \text{...(1)}$$

From given $\quad f = a\tan^{-1}\left(\frac{x}{y}\right)$

Differentiating w.r.t. y, partially,

$$f_y = \frac{\partial f}{\partial y} = a\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right) = \frac{-ax}{x^2+y^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$= -a\left[\frac{1\,(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2}\right]$$

$$= -a\frac{y^2-x^2}{(x^2+y^2)^2} = \frac{a\,(x^2-y^2)}{(x^2+y^2)^2} \quad \text{...(2)}$$

From (1) and (2),

Thus, $f_{xy} = f_{yx}$

Example 13 :

$$f = x^y + y^x$$

Solution :

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\,[\,yx^{y-1} + y^x \log y]$$

$$= yx^{y-1}\log x + x^{y-1} + xy^{x-1}\log y + \frac{y^x}{y}$$

$$= x^{y-1} + yx^{y-1}\log x + xy^{x-1}\log y + y^{x-1}$$

$$= x^{y-1}(1 + y\log x) + y^{x-1}(1 + x\log y) \quad \text{...(1)}$$

Given $f = x^y + y^x$

$$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\,\partial y} = yx^{y-1}\log x + x^y\frac{1}{x} + xy^{x-1}\log y + y^{x-1}$$

$$= yx^{y-1}\log x + x^{y-1} + xy^{x-1}\log y + y^{x-1}$$

$$= x^{y-1}(1 + y\log x) + y^{x-1}(1 + x\log y) \quad \text{...(2)}$$

Thus, From (1) and (2)

$$f_{xy} = f_{yx}$$

Example 14 :

$f = e^{ax} \sin by$

Solution :

$$f_x = \frac{\partial t}{\partial x} = e^{ax} \sin by$$

$$\therefore \quad f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\,[a^{ax} \sin by] = a\,be^{ax} \cos by \quad \text{...(1)}$$

$$\frac{\partial f}{\partial y} = be^{ax} \quad \text{casby}$$

$$\therefore \quad f_{xy} = \frac{\partial^2 f}{\partial x\, \partial y}$$

$$= abe^{ax} \cos by \quad \ldots(2)$$

Thus from (1) and (2), $f_{xy} = f_{yx}$.

Type 3 : to find n, a, b, etc.

Example 15 :

If $u(x, t) = Ae^{-gx} \sin(nt - gx)$ satisfies the equation $\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$ where A, g, n and k are constants, show that $n = 2k^2 g^2$

Solution :

$$u(x, t) = Ae^{-gx} \sin(nt - gx)$$

$$\frac{\partial u}{\partial t} = A\,n\,e^{-gx} \cos(nt - gx) \quad \ldots(1)$$

$$\frac{\partial y}{\partial x} = A\,e^{-gx}[-g\sin(nt - gx) - g\cos(nt - gx)]$$

$$= -A\,g\,e^{-gx}[\sin(nt - gx) + \cos(nt - gx)]$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = -Age^{-gx}\{-g[\sin(nt - gx) + \cos(nt - gx))] + [\cos(nt - gx)$$

$$- \sin(nt - gx))(-g)]\}$$

$$= Ag^2 e^{-gx}[2\cos(nt - gx)] \quad \ldots(2)$$

But, $$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \quad \ldots\text{(given)}$$

$\therefore$ From (1) and (2),

$$Ane^{-gx}\cos(nt - gx) = Ag^2 e^{-gx} k^2 [2\cos(nt - gx)]$$

$$\therefore \quad n = 2g^2 k^2$$

Example 16 :

$z = u(x, y)\, e^{ax + by}$ where $u(x, y)$ is such that $\frac{\partial^2 u}{\partial x\, \partial y} = 0$. If $\frac{\partial^2 z}{\partial x\, \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$, then find the constants a and b.

Solution :

Given u is a function of x and y.

and $$z = u(x, y)\, e^{ax + by} \quad \ldots(1)$$

Differentiating (1) w.r.t. x, and partially, we have,

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \cdot e^{ax + by} + u \cdot e^{ax + by} \quad \ldots(2)$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \cdot e^{ax + by} + u \cdot be^{ax + by} \quad \ldots(3)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\, e^{ax+by} + b \cdot u \cdot e^{ax+by}\right]$$

$$= \frac{\partial^2 u}{\partial x\,\partial y}\, e^{ax \;+\; by} \;+\; ae^{ax \;+\; by} \;\cdot\; \frac{\partial u}{\partial y} + b$$

$$\left[\frac{\partial u}{\partial x} \cdot e^{ax+by} + a \cdot u \cdot e^{ax+by}\right]$$

$$= e^{ax+by}\left[\frac{\partial^2 u}{\partial x\,\partial y} + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} + abu\right] \qquad \ldots(4)$$

From (1) (2) (3 and (4) and given that

$$\frac{\partial^2 z}{\partial x\,\partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$$

$$\frac{\partial^2 z}{\partial x\,\partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = e^{ax+by}$$

$$\left[\frac{\partial^2 u}{\partial x\,\partial y} + a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} + abu - \frac{\partial u}{\partial x} - au - \frac{\partial u}{\partial y} - bu + u\right]$$

$$= 0 \qquad \text{Given } \frac{\partial^2 u}{\partial x\,\partial y} = 0 \qquad \ldots(5)$$

From (5),

$$e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + u\,(a-1) - u\,(b-1)\right] = 0$$

Since $\quad e^{ax+by} \neq 0 \quad$ and $\quad \frac{\partial u}{\partial y} \neq 0 \qquad \frac{\partial u}{\partial x} \neq 0$

Thus, $\quad a - 1 = 0, \qquad b - 1 = 0$

$\therefore \quad a = 1 \quad$ and $\quad b = 1$

Example 17 :

Find the values of n, so that $v = r^n\,(3\cos^2\theta - 1)$ satisfy the equation $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0$

Solution :

$$v = r^n\,(3\cos^2\theta - 1)$$

$$\frac{\partial v}{\partial r} = n\,r^{n-1}\,(3\cos^2\theta - 1)$$

$$\therefore \quad r^2\frac{\partial v}{\partial r} = nr^{n+1}\,(3\cos^2\theta - 1)$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = n\,(n+1)\,v \qquad \ldots(1)$$

$$\frac{\partial v}{\partial\theta} = r^n\,(-6\cos\theta\sin\theta)$$

$$= r^n\,(-3\sin 2\theta)$$

$$\sin\theta \frac{\partial v}{\partial \theta} = r^n(-3\sin\theta\sin 2\theta)$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = r^n[-3\cos\theta\sin 2\theta - 6\sin\theta\cos 2\theta]$$

$$= r^n[-6\cos^2\theta\sin\theta - 6\sin\theta(2\cos^2\theta - 1)]$$

$$= r^n \sin\theta\ [-6\cos^2\theta - 12\cos^2\theta + 6]$$

$$= -6r^n \sin[3\cos^2\theta - 1]$$

$$= -6r^n[3\cos^2\theta - 1]\sin\theta$$

$$= -6\,v\sin\theta \qquad \ldots(2)$$

$\therefore$ From (1) and (2),

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial r}\right) = [n(n+1) - 6]\ v$$

$$= 0$$

$\therefore$ $$[n(n+1)-6]\,v = 0$$

$\therefore$ $$n^2+n-6 = 0$$

$$(n+3)(n-2) = 0$$

$\therefore$ $$n = -3$$

or $$n = 2$$

Example 18 :

If $\theta = t^n e^{\frac{-r^2}{4t}}$ find the value of n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$

Solution :

$$\frac{\partial\theta}{\partial r} = t^n e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)$$

$$= -\frac{r}{2t}\cdot t^n e^{\frac{-r^2}{4t}}$$

$$= -\frac{r\theta}{2t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-r^3\theta}{2t}\right)$$

$$= -\frac{1}{2tr^2}\frac{\partial}{\partial r}(r^3\theta)$$

$$= -\frac{1}{2tr^2}\left[r^3\frac{\partial\theta}{\partial r} + 3r^2\theta\right]$$

$$= -\frac{1}{2tr^2}\left[\frac{-r^4\theta}{2t} + 3r^2\theta\right]$$

$$= \left[\frac{r^2}{4t^2} - \frac{3}{2t}\right]\theta \qquad \ldots(1)$$

Also, $$\frac{\partial\theta}{\partial t} = t^n e^{\frac{-r^2}{4t}} \frac{r^2}{4t^2} + nt^{n-1} e^{\frac{-r^2}{4t}}$$

$$= \frac{r^2\theta}{4\,t^2} + \frac{n\theta}{t} = \left[\frac{r^2}{4\,t^2} + \frac{n}{t}\right]\theta \qquad ...(2)$$

Given that,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\,\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$$

$\therefore$ From (1) and (2),

$$\left(\frac{r^2}{4t^2} - \frac{3}{2t}\right)\theta = \left(\frac{r^2}{4t^2} + \frac{n}{t}\right)\theta$$

$\therefore$ $$n = -\frac{3}{2}$$

Exercise 5.1

Type 1

1. If $z^3 - zx - y = 0$, prove that $\dfrac{\partial^2 z}{\partial x\,\partial y} = -\dfrac{3z^2 + x}{3z^2 - x}$
2. If $u\,(x + y) + x^2 + y^2$, prove that
$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$
3. If $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$ prove that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$
4. If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$
5. If $u = 2\,(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$
6. If $u = \tan^{-1}\left(\dfrac{\partial y}{\sqrt{1 + x^2 + y^2}}\right)$ prove that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{1}{(1 + x^2 + y^2)^{\frac{3}{2}}}.$
7. If $u = e^{x - kt}\cos(x - kt)$, prove that $\dfrac{\partial^2 u}{\partial t^2} = k^2\dfrac{\partial^2 u}{\partial x^2}$
8. If $u^2 = x^2 + y^2 + z^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2}{u}$
9. If $e^u = \tan x + \tan y$, prove that $\sin 2x\,\dfrac{\partial u}{\partial x} + \sin 2y\,\dfrac{\partial u}{\partial y} = 2$
10. If $u = x^y$, prove that $\dfrac{\partial^3 u}{\partial x^2\,\partial y} = \dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$
11. If $u = x^y$ prove that $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \dfrac{\partial^3 u}{\partial x^2\,\partial y}$
12. If $u = e^{xyz}$ prove that $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1 + 3xyz + x^2 y^2 z^2)\,u$

13. If $u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 2u$

14. If $u = e^x (x \cos y - y \sin y)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

15. If $u = e^{ax} \tan by \log z$, prove that $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \dfrac{\partial^3 u}{\partial z\, \partial y\, \partial x}$

16. If $u = x^3 - 3xy^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

17. If $u = x^p y^q$, prove that $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial x} = \dfrac{\partial^3 u}{\partial y\, \partial x^2}$

Type 2 :

Prove that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$ or $u_{xy} = u_{yx}$ it.

19. $u = e^{ax} \sin by$

20. $u = x^3 + y^3 - 3$

21. $u = \dfrac{1}{\sqrt{y}}\, e^{\frac{-(x+a)^2}{4y}}$

22. $u = x^y + y^x$

23. $u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$

24. $u = \dfrac{x}{x^2 + y^2}$

25. $u = \sin^{-1}\left(\dfrac{x}{2y}\right)$

26. $u = \log\left(\dfrac{x^3 + y^3}{xy}\right)$

27. $u = \cos^{-1}\left(\dfrac{x}{x + y}\right)$

Type 3 :

29. Find the value of n for which $z = pt^{-\frac{1}{2}}\, e^{\frac{-x^2}{na^2 t}}$ satisfies the equation $\dfrac{\partial z}{\partial t} = a^2 \dfrac{\partial^2 z}{\partial x^2}$

Ans.: $n = 4$.

Partial differentiation of function of a function (Composite function) (First order Partial derivative)

If $Z = f(u)$ and $u = \phi(x, y)$, i.e. Z is a function of u and u itself is a function of two independent vriables x and y. The two relations define Z as a fucntion of x and y. Thus Z is called a function of a function of x and y.

1. If $z = f(u)$ is differentiable function of u and $u = \phi(x, y)$ possess first order partial derivatives. i.e. $z \to u \to x, y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x}$$ treating y as constant.

$$= f'(u)\,\frac{\partial u}{\partial x} \qquad \left\{\because \frac{\partial z}{\partial u} = \frac{\partial z}{\partial u} = f'(a) \text{ z is a function of single variable u}\right\}$$

Similarly $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} \qquad \text{treating x as constant.}$$

$$= f'(u)\,\frac{\partial u}{\partial y}$$

2. If $v = f(u)$ is differentiable function of u and $u = \phi(x, 8y, z)$ possess first order partial derivatives, i.e. $v \to u \to x, y, z$, then,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial x} = f'(u)\,\frac{\partial u}{\partial x} \qquad \text{treating y, z constants.}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial y} = f'(u)\,\frac{\partial u}{\partial y} \qquad \text{treating y, z constants.}$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial z} = f'(u)\,\frac{\partial u}{\partial z} \qquad \text{treating y, z constants.}$$

Illustrative Exampels

Type 1 :

Example 1 :

If $u = f\left(\frac{x^2}{y}\right)$ prove that $x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = 0.$

Solution :

$$\frac{\partial u}{\partial x} = f'\left(\frac{x^2}{y}\right)\frac{2x}{y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\left(\frac{-x^2}{y^2}\right)$$

$$x\cdot\frac{\partial u}{\partial x} + 2u\left(\frac{\partial u}{\partial y}\right) = \frac{2x^2}{y}\,f'\left(\frac{x^2}{y}\right) - \frac{2x^2}{y}\,f'\left(\frac{x^2}{y}\right) = 0$$

Example 2 :

If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + r^2}$ prove that

$$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{1}{r}\,f'(r)$$

Solution : $$r^2 = x^2 + y^2 + r^2$$

Differentiating w.r.t. partially,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left\{f'(r)\frac{1}{r}x\right\}$$

$$= f'(r)\frac{1}{r} + f'(r)\left\{-\frac{1}{r_2}\right\}\frac{\partial r}{\partial x}\cdot x + f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r}$$

$$= \frac{f'(r)}{r} - f'(r)\frac{x^2}{r^3} + f''(r)\frac{x^2}{r^2} \quad ...(1)$$

Similarly, $u_{yy}\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{f'(r)}{r} - f'(r)\dfrac{y^2}{r^3} + f''(r)\dfrac{y^2}{r^2}$ ...(2)

and $\dfrac{\partial^2 u}{\partial z^2} = \dfrac{f'(r)}{r} - f'(r)\dfrac{z^2}{r^3} + f''(r)\dfrac{z^2}{r^2}$ ...(3)

(1) + (2) + (3) gives,

$$\text{LHS} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{3f'(r)}{r} - \frac{f'(r)}{r^3}(x^2 + y^2 + z^2) + \frac{f''(r)}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{3f'(r)}{r} - \frac{f^1(r)}{r^3}r^2 + \frac{f''(r)}{r^2}r^2$$

$$= \frac{2f'(r)}{r} - \frac{f'(r)}{r} + f''(r)$$

$$= f''(r) + \frac{f'(r)}{r} = \text{RHS}$$

Example 3 :

If $u = x\log(x + r) - r$ where $r^2 = x^2 + y^2$ find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

Solution :

Given $r^2 = x^2 + y^2$

$\therefore$ $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \ \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Given $u = x\log(x + r) - r,$

$$\frac{\partial u}{\partial x} = \log(x + r) + \frac{x}{x + r}\left(1 + \frac{\partial r}{\partial x}\right) - \frac{\partial r}{\partial x}$$

$$= \log(x + r) + \frac{x}{x + r}\left(1 + \frac{x}{r}\right) - \frac{x}{r}$$

$$= \log(x + r) + \frac{x}{x + r}\left(\frac{r + x}{r}\right) - \frac{x}{r}$$

$$= \log(x + r)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x + r}\left(1 + \frac{\partial r}{\partial x}\right) = \frac{1}{x + r}\left(1 + \frac{x}{r}\right)$$

$$= \frac{1}{x+r} \quad \frac{(r+x)}{r} \quad = \frac{1}{r} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = x \, \frac{1}{x+r} \, \frac{\partial r}{\partial y} - \frac{\partial r}{\partial y}$$

$$= \frac{x}{x+r} \, \frac{y}{r} - \frac{y}{r}$$

$$= \frac{y}{r}\left(\frac{x}{x+r} - 1\right)$$

$$= \frac{y}{r} \, \frac{[-r]}{(x+r)} = \frac{-y}{(x+r)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x+r} + y \, \frac{1}{(x+r)^2} \, \frac{\partial r}{\partial y}$$

$$= -\frac{1}{x+r} + \frac{y}{(x+r)^2} \cdot \frac{y}{r}$$

$$= -\frac{1}{x+r} + \frac{y^2}{(x+r)^2 \, r}$$

$$= -\frac{1}{x+r} + \frac{r^2 - x^2}{r\,(x+r)^2} \qquad (\because r^2 = x^2 + y^2)$$

$$= -\frac{1}{x+r} + \frac{(r-x)\,(r+x)}{r\,(x+r)^2}$$

$$= -\frac{1}{x+r} + \frac{r-x}{r\,(x+r)}$$

$$= -\frac{1}{x+r} + \frac{r}{r\,(x+r)} - \frac{x}{r\,(x+r)}$$

$$= -\frac{x}{r\,(x+r)} \qquad \ldots(2)$$

Adding (1) and (2), w eget,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r} - \frac{x}{r\,(x+r)}$$

$$= \frac{x + r - x}{r\,(x+r)} = \frac{1}{x+r}$$

Example 4 :

If $v = (x^2 - y^2) \; f(xy)$, show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (x^4 - y^4) \; f''(xy)$

Solution : $v = (x^2 - y^2) \; f(l)$ where $l = xy$

$$\therefore \quad \frac{\partial l}{\partial x} = y \text{ and } \frac{\partial l}{\partial y} = x$$

$$\frac{\partial v}{\partial x} = (x^2 - y^2) \; f'(e) \; y + 2x \, f'(e)$$

$$\frac{\partial^2 v}{\partial x^2} = y \, [(x^2 - y^2) \; f''(l) \, y + 2x \, f'(l)\,] + 2\,[f'(1) \; yx + f(1)\,]$$

$$= y^2 \, (x^2 - y^2) \; f''(1) + 4xy \, f'(1) + 2f(1) \qquad \ldots(1)$$

and $\quad v = (x^2 - y^2)\ f(l),$

$\therefore \quad \frac{\partial v}{\partial y} = (x^2 - y^2)\ f'(e)\ \frac{\partial l}{\partial y} + f(l)\ (-2y)$

$= (x^2 - y^2)\ f'(l)\ x - 2y\, f(l)$

$\frac{\partial^2 v}{\partial y^2} = x\ [(x^2 - y^2)\ f''(l)\ x - 2y\, f'(l)\,] - 2\,[y f'(l)\, x + f(l\,]$

$= (x^2 - y^2)\ x^2 f''(l) - 4xy\, f'(l) - 2f(l)$ ...(2)

Adding (1) and (2)

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = y^2 (x^2 - y^2)\ f''(l) + (x^2 - y^2)\ x^2 f''(l)$

$= (y^2 + x^2)\ (x^2 - y^2)\ x^2 f''(l)$

$= (x^4 - y^4)\ f''(l)$ Hence proven.

Example 5 :

If $u = f(r)$ and $x = r\cos\theta,\ y = r\sin\theta$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}\ f'(r)$

Solution :

Given $\quad u = f(r)$ and $r = \sqrt{x^2 + y^2}$

$\frac{\partial u}{\partial r} = f'(r)$ and $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$

$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\ \frac{dr}{\partial x}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\ \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)$

$= \frac{\partial}{\partial x}\ [f'(r)\,]\ \frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\ \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)$

$= \frac{\partial}{\partial r}\ f'(r)\ \frac{\partial r}{\partial x}\ \frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\ \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)$

$= f''(r)\ \frac{x^2}{r^2} + f'(r)\ \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$

$= f''(r)\ \frac{x^2}{r^2} + f'(r)\left[\frac{r \cdot 1 - x\,\frac{\partial r}{\partial x}}{r^2}\right]$

$= f''(r)\ \frac{x^2}{r^2} + f'(r)\ \frac{(r^2 - x^2)}{r^3}$ ...(1)

Similarly, $\quad \frac{\partial^2 u}{\partial y^2} = f''(r)\ \frac{y^2}{r^2} + f'(r)\ \frac{(r^2 - y^2)}{r^3}$ ...(2)

$\therefore$ From Equation (1) and (2),

L. H. S. $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\ \frac{[x^2 + y^2]}{r^2} + f'(r)\left[\frac{2r^2 - (x^2 + y^2)}{r^3}\right]$

$$= f''(r) + \frac{1}{r} f'(r) = \text{R. H. S.}$$

Example 6 :

If $z = e^{ax+by} f(ax - by)$ prove that, $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2\,abz.$

Solution :

$$z = e^{ax+by} f(ax - by)$$

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by)\; a + ae^{ax+by} f(ax - by) \text{ and}$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by)\;(-b) + b \cdot e^{ax+by} f(ax - by)$$

Now, LHS $= b\frac{\partial t}{\partial x} + a\frac{\partial t}{\partial y}$

$$= abe^{ax+by} f'(ax - by) + abe^{ax+by} f(ax - by)$$

$$- abe^{ax+by} f'(ax - by) + abe^{ax+by} f(ax - by)$$

$$= 2\,abe^{ax+by} f(ax - by) = 2abz = \text{RHS}$$

Example 7 :

If $x = r\cos\theta$, $y = r\sin\theta$ prove that,

1. $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

2. $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \quad (x \neq 0,\; y \neq 0)$

Solution :

1. Given $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore \quad x^2 + y^2 = r^2$ ...(1)

$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial r}{\partial y} = \frac{y}{r}$ ...(2)

Differentiating (2) partially w.r.t. x

$$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x\frac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}$$

Similarly from (3)

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{x^2+y^2}{r^3} = \frac{1}{r} \qquad ...(4)$$

Also, $$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} = 1 \qquad ...(5)$$

$\therefore$ From equation (4) and (5),

$$\text{L. H. S.} = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \text{R. H. S.}$$

2. Given $x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore \quad \theta = \tan^{-1}\frac{y}{x}$$

i.e. $\theta \to x, y$

$$\therefore \quad \frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{1}{x^2}\right) = \frac{-y}{(x^2+y^2)}$$

$$\therefore \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \qquad ...(1)$$

and $$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\,\frac{1}{x}$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \qquad ...(2)$$

$\therefore$ From equation (1) and (2)

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

Example 8 :

If $u = f(x+ay) + \phi(x-ay)$, prove that $\frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}$

Solution :

Given $$u = f(x+ay) + \phi(x-ay)$$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(x+ay) + \phi'(x-ay)$$

And $$\frac{\partial^2 u}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \qquad ...(1)$$

From given u

$$\frac{\partial u}{\partial y} = f'(x+ay)\, a + \phi''(x-ay)\,(a)$$

$$\therefore \quad \frac{\partial^2 u}{\partial^2 y} = a^2 f''(x+ay) + a^2\phi''(x-ay) \qquad ...(2)$$

From Equation (1) and (2),

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2}$$

Example 9 :

If $u = f\left(\frac{x}{y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solution :

Given $\quad u = f\left(\frac{x}{y}\right)$

$\therefore \quad \frac{\partial u}{\partial x} = f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} \quad$ and $\quad \frac{\partial u}{\partial y} = f'\left(\frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right)$

$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{y} f'\left(\frac{x}{y}\right) \quad$ ...(1)

And $\quad y\frac{\partial u}{y} = \frac{-x}{y} f'\left(\frac{x}{y}\right) \quad$ ...(2)

Adding equation (1) and (2), we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Example 10 :

If $u = xf(x+y) + y\,\phi(x+y)$ show that $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

Solution :

$$\frac{\partial u}{\partial x} = xf'(x+y) + f(x+y) + y\phi'(x+y)$$

$$\frac{\partial^2 u}{\partial x^2} = x f''(x+y) + f'(x+y) + f'(x+y) + y\phi''(x+y)$$

$$= xf''(x+y) + 2f'(x+y) + y\phi''(x+y) \quad \text{...(1)}$$

$$\frac{\partial y}{\partial y} = xf'(x+y) + y\phi'(x+y) + \phi(x+y)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = xf''(x+y) + f'(x+y) + y\phi''(x+y) + \phi'(x+y) \quad \text{...(2)}$$

$$\frac{\partial^2 u}{\partial y^2} = xf''(x+y) + y\phi''(x+y) + 2\phi'(x+y) \quad \text{...(3)}$$

From (1) , (2) and (3) we get,

$$\text{L. H. S.} = \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$= xf''(x+y) + 2f'(x+y) + y\phi''(x+y)$$

$$- 2x f''(x+y) - 2f'(x+y) - 2y\phi''(x+y)$$

$$- 2\phi'(x+y) + xf''(x+y) + y\phi''(x+y) + 2\phi'(x+y)$$

$$= 0 = \text{R. H. S.}$$

Example 11 :

If $u = \log r$, where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{r^2}$

Solution :

We have, $$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. x, we have,

$$2r\frac{\partial r}{\partial x} = 2(x-a) \quad \text{or} \quad \frac{\partial r}{\partial x} = \left(\frac{x-a}{r}\right) \qquad \text{...(2)}$$

Now, $$u = \log r$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\left(\frac{x-a}{r}\right) \qquad \text{... From (2)}$$

Thus, $$\frac{\partial u}{\partial x} = \frac{(x-a)}{r^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right)$$

$$= \frac{r^2(1) - (x-a)\cdot 2r\left(\frac{\partial r}{\partial x}\right)}{r^4}$$

$$= \frac{r^2 - 2(x-a)^2}{r^4} \qquad \text{...}\left[\because \text{ from (2) } \frac{\partial r}{\partial x} = \frac{x-a}{r}\right]$$

Similarly, by symmetry,

$$\frac{\partial^2 u}{\partial y^2} = \frac{r^2 - 2(y-b)^2}{r^4}$$

and $$\frac{\partial^2 u}{\partial z^2} = \frac{r^2 - 2(z-c)^2}{r^4}$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3r^2 - 2\{(x-a)^2 + (y-b)^2 + (z-c)^2\}}{r^4}$$

$$= \frac{3r^2 - 2r^2}{r^4} \qquad \text{...(Using (1))}$$

$$= \frac{r^2}{r^4} = \frac{1}{r^2} \qquad \text{Hence Proven}$$

Example 12 :

If $u = x\,\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$ prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$

Solution :

We have, $$u = x\,\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right) \qquad \text{...(1)}$$

Differentiating (1) partial w.r.t. x and y, we get,

$$\frac{\partial u}{\partial x} = x\left\{\phi'\left(\frac{y}{x}\right)\right\}\cdot\left(-\frac{y}{x^2}\right) + \phi\left(\frac{y}{x}\right) + \left\{\psi'\left(\frac{y}{x}\right)\right\}\cdot\left(-\frac{y}{x^2}\right)$$

and $$\frac{\partial u}{\partial y} = x\left\{\phi'\left(\frac{y}{x}\right)\right\}\cdot\left(\frac{1}{x}\right) + \left\{\psi'\left(\frac{y}{x}\right)\right\}\cdot\left(\frac{1}{x}\right)$$

$\therefore$ $$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = x\,\phi\left(\frac{y}{x}\right) \quad \text{...(2)}$$

Now differentiating (2) partially w.r.t. x and y respectively, we get,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = x\left\{\phi\left(\frac{y}{x}\right)\right\}\left(-\frac{y}{x^2}\right) + \phi\left(\frac{y}{x}\right)$$

$$= \frac{-y}{x}\phi'\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)$$

and $$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = x\left\{\phi'\left(\frac{y}{x}\right)\right\}\cdot\frac{1}{x} = \phi\left(\frac{y}{x}\right)$$

Multiplying these equations by x and y respectively and adding we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x\phi\left(\frac{y}{x}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{... (from (2))}$$

Example 13 :

If $\log_e \theta = r - x$ where $r^2 = x^2 + y^2$, show that $\dfrac{\partial^2\theta}{\partial y^2} = \dfrac{\theta\,(x^2 + ry^2)}{r^3}$

Solution :

$\theta \rightarrow r \rightarrow x, y$

Given $$\log_e \theta = r - x$$

$$r^2 = x^2 + y^2$$

$$\theta = e^{r-x} \qquad 2r\frac{\partial r}{\partial y} = 2y \qquad \therefore \frac{\partial r}{\partial y} = \frac{y}{r}$$

and $$\frac{\partial \theta}{\partial y} = e^{r-x}\frac{\partial r}{\partial y} = e^{r-x}\frac{y}{r}$$

$\therefore$ $$\frac{\partial^2\theta}{\partial y^2} = e^{r-x}\cdot\frac{\partial r}{\partial y}\cdot\frac{y}{r} + e^{r-x}\left[\frac{r\cdot 1 - y\cdot\frac{\partial r}{\partial y}}{r^2}\right]$$

$$= e^{r-x}\left[\frac{y}{r}\cdot\frac{y}{r} + \frac{r - y\cdot\frac{y}{r}}{r^2}\right]$$

$$= e^{r-x}\left[\frac{y^2}{r^2} + \frac{r^2 - y^2}{r^3}\right]$$

$$= e^{r-x}\left[\frac{ry^2 + r^2 - y^2}{r^3}\right]$$

$$= e^{r-x}\left[\frac{ry^2 + x^2 + y^2 - y^2}{r^3}\right]$$

$$= e^{r-x}\left[\frac{ry^2 + x^2}{r^3}\right]$$

$$\frac{\partial^2\theta}{\partial y^2} = \theta\left[\frac{ry^2 + x^2}{r^3}\right] = \theta\left[\frac{x^2 + ry^2}{r^3}\right]$$

Type 2 - Composite Function :

(a) Let $z = f(x, y)$ and $x = \phi(t)$ $y = \psi(t)$, so that z is a function of x, y which are functions of third variable t.

Thus, z is a function of t. In such cases z is called as a composite funciton of t. i.e. $z \rightarrow x,\ y \rightarrow t$.

Partial Differentiation of composite function: Let $z = f(x, y)$ possess continuous first order partial derivatives and $x = \phi(t)$, $y = \psi(t)$ posses continuous first order derivatives then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \qquad (z \rightarrow x,\ y \rightarrow t)$$

(b) Let $z = f(u, v)$ possess continuous first order partial derivatives and Let $u = \phi(x, y)$, $v = \psi(x, y)$ posses continuous first order partial derivatives.

i.e. $z \rightarrow u, v \rightarrow 1\,xy$ then,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \text{...(treating y constant)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \qquad \text{...(treating x constant)}$$

(c) Let $Z = f(u, v, w)$ and $u = \phi(x, y, z)$, $v = \psi(x, y, z)$ and $w = \xi(x,y, z)$. So that z is function of u, v, w and u, v, w are themselves functions of x, y, z. These relations define z as a function of x, y, z. i.e. $z \rightarrow u, v, w \rightarrow x, y, z$. Then z is called a composite function of x, y and z. i.e. $z \rightarrow u, v, w \rightarrow x, y, z$.

Let $z = f(u, v, w)$ possess continuous first order partial derivatives and let $u = \phi(x, y, z)$, $v = \psi(x, y, z)$ and $w = \zeta(x, y, z)$ possess continuous first order partial derivatives, then

$$\frac{\partial Z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x} \qquad \text{(Treating y, z constants)}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y} \qquad \text{(Treating x, z constants)}$$

$$\frac{\partial Z}{\partial z} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial z} \qquad \text{(Treating x, y constants)}$$

(∂) Similarly if $z = f(u_1, u_2, \dots\dots\dots, u_n)$ where $u_1, u_2, \dots\dots, u_n$ are functions of $x_1, x_2, x_3, \dots\dots, x_n$. Then the partial differential coefficient of z with respect to x_1, $\frac{\partial z}{\partial x_1}$ is given by

$$\frac{\partial z}{\partial x_1} = \frac{\partial z}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial z}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \dots\dots\dots + \frac{\partial z}{\partial u_n}\frac{\partial u_n}{\partial x_1} \text{(treating } (x_2, x_3, \dots\dots x_n) \text{) constant}$$

Similarly

$$\frac{\partial z}{\partial x_2} = \frac{\partial z}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial z}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \dots\dots\dots + \frac{\partial z}{\partial u_n}\frac{\partial u_n}{\partial x_2} \text{(treating } (x_1, x_3, \dots x_n) \text{) constant}$$

...and

$$\frac{\partial z}{\partial x_n} = \frac{\partial z}{\partial u_1}\frac{\partial u_1}{\partial x_n} + \frac{\partial z}{\partial u_2}\frac{\partial u_2}{\partial x_n} + \dots\dots\dots + \frac{\partial z}{\partial u_n}\frac{\partial u_n}{\partial x_n}$$

(treating $(x_1, x_2, x_3, \dots x_{n-1})$) constants

Illustrative Examples

Example 1

If $f(xy^2, z - x) = 0$ prove that $x\frac{\partial z}{\partial x} - \frac{1}{2}y\frac{\partial z}{\partial y} = 2x$

Solution :

Let $f(l, m) = 0$

where $l = xy^2$

$m = z - 2x,$

i.e. $z \to l, m \to x,y$

and derivatives of z w. r. t. x and y involved.

Differentiating w.r.t. x,

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial f}{\partial m}\frac{\partial m}{\partial x}$$

$$= \frac{\partial f}{\partial l}\; y^2 + \frac{\partial f}{\partial m}\left\{\frac{\partial z}{\partial x} - 2\right\} \qquad \dots(1)$$

Differentiating w.r.t. y,

$$\frac{\partial t}{\partial y} = 0 = \frac{\partial f}{fl}\frac{\partial l}{\partial y} + \frac{\partial f}{\partial m}\frac{\partial m}{\partial y} = \frac{\partial f}{\partial l}\; 2xy + \frac{\partial f}{\partial m}\left\{\frac{\partial z}{\partial y}\right\} \qquad \dots(2)$$

From (1), $$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}} = -\frac{\frac{\partial z}{\partial x} - 2}{y^2} \qquad \dots(3)$$

from (2) $$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}} = -\frac{\frac{\partial z}{\partial y}}{2xy} \qquad \dots(4)$$

From (3) and (4)

$$\frac{\frac{\partial z}{\partial x} - 2}{y^2} = \frac{\frac{\partial z}{\partial y}}{2xy}$$

$$\therefore \quad 2xy\left\{\frac{\partial z}{\partial x} - 2\right\} = \frac{\partial z}{\partial y}\, y^2$$

$$2x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 4x$$

$$\therefore x\frac{\partial z}{\partial x} - \frac{y}{2}\frac{\partial z}{\partial y} = 2x$$

Hence proven

Example 2 :

If $z = f(x, y)$ $x = r\cos\theta$, $y = r\sin\theta$, prove

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Solution :

We have, $z \to x$, $y \to r, \theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\cdot\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}\cdot(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\therefore \qquad \text{R. H. S.} = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)$$

$$= \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + 2\cos\theta\sin\theta\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta$$

$$+ \sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2 - 2\cos\theta\sin\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = \text{L. H. S.}$$

Example 3 :

If $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ prove that,

$$x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} + z\frac{\partial \Phi}{\partial z} = u\frac{\partial \Phi}{\partial u} + v\frac{\partial \Phi}{\partial v} + w\frac{\partial \Phi}{\partial w}$$

where ϕ is a function x, y, z.

Solution :

$\phi \to x, y, z \to u, v, w.$

Since Φ is a function of x, y, z themselves are functions of u, v, w Thus Φ is composite function of v, u, w.

$$\therefore \qquad \frac{\partial \Phi}{\partial u} = \frac{\partial \Phi}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial \Phi}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial \Phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{\partial \Phi}{\partial x}\cdot 0 + \frac{\partial \Phi}{\partial y}\cdot\sqrt{w}\cdot\frac{1}{2\sqrt{u}} + \frac{\partial \Phi}{\partial z}\cdot\sqrt{v}\cdot\frac{1}{2\sqrt{u}}$$

$$\therefore \qquad u\frac{\partial \Phi}{\partial x} = \frac{1}{2}\sqrt{uw}\,\frac{\partial \Phi}{\partial y} + \frac{1}{2}\sqrt{uv}\,\frac{\partial \Phi}{\partial z}$$

$$= \frac{1}{2}\left(y\frac{\partial \Phi}{\partial y} + z\frac{\partial \Phi}{\partial z}\right) \qquad ...(1)$$

Similarly,
$$v\frac{\partial \Phi}{\partial v} = \frac{1}{2}\left(z\frac{\partial \Phi}{\partial z} + x\frac{\partial \Phi}{\partial x}\right) \qquad ...(2)$$

and
$$w\frac{\partial \Phi}{\partial w} = \frac{1}{2}\left(x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y}\right) \qquad ...(3)$$

adding the three results, we get,

$$u\frac{\partial \Phi}{\partial u} + v\frac{\partial \Phi}{\partial v} + w\frac{\partial \Phi}{\partial w} = x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} + z\frac{\partial \Phi}{\partial z}$$

Example 4 :

If $z = f(u, v)$ where $u = x\cos\theta - y\sin\theta$

$v = x\sin\theta + y\cos\theta$

show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$, θ being constant.

Solution :

Here $z = f(u, v)$ where $u, v,$ themselves are functions of x and y amd thus be a composite function of $x \cdot y$.

Then
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\therefore \qquad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cos\theta + \frac{\partial z}{\partial v}\sin\theta \qquad ...(1)$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\therefore \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(-\sin\theta) + \frac{\partial z}{\partial v}\cos\theta \qquad ...(2)$$

Multiply (1) by x and (2) by y, then adding,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial u}\cos\theta + x\frac{\partial z}{\partial v}\sin\theta - y\frac{\partial z}{\partial u}\sin\theta + y\frac{\partial z}{\partial v}\cos\theta$$

$$= \frac{\partial z}{\partial u}(x\cos\theta - y\sin\theta) + \frac{\partial z}{\partial v}(y\cos\theta + x\sin\theta).$$

$$\therefore \qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

Hence proved

Example 5 :

If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Solution :

Let $\frac{x}{y} = l$, $\frac{y}{z} = m$, $\frac{z}{x} = n$ then, $u \to l, m, n \to x, y, z$.

u = f (l, m, n) where l, m, n themselves are function of x, y, z and hence u be a composite functions of x, y, z.

Thus,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}\left(-\frac{z}{x^2}\right) \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l} \cdot \left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial m}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial n}(0) \qquad ...(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l} \cdot (0) + \frac{\partial u}{\partial m}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial n}\left(\frac{1}{x}\right) \qquad ...(3)$$

Multiply (1) by x, (2) by y and (3) by z, then adding,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial l} - \frac{z}{x}\frac{\partial u}{\partial l} + \frac{y}{z}\frac{\partial u}{\partial m} + \frac{z}{x}\frac{\partial u}{\partial n}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$ Hence proved

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Example 6 :

If z = f (x, y) where x = $e^u \cos v$, y = $e^u \sin v$, show that,

(i) $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$

(ii) $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{fy}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$

Solution : Given z → x,y → u, v

(i) Here z be a composite function of u, v.

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial u}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v)$$

$$\therefore \quad \frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \qquad ...(1)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(- e^u \sin y) + \frac{\partial z}{\partial y}(e^u \cos v)$$

$$\therefore \quad \frac{\partial z}{\partial v} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \qquad ...(2)$$

Multiply (1) by y and (2) by x and adding then

$$\text{L. H. S. } y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(yx - xy) + \frac{\partial z}{\partial y}(y^2 + x^2)$$

$$= (x^2 + y^2)\frac{\partial z}{\partial y} = e^{2u}\frac{\partial z}{vy} = \text{R. H. S.} \qquad (\because\ x^2 + y^2 = e^{2u})$$

(ii) We have,

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial u} \quad , \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial v}$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$$

Then we have from (1) and (2),

$$\frac{\partial f}{\partial u} = e^u \cos v\frac{\partial f}{\partial x} + e^u \sin v\frac{\partial f}{\partial y} = x\frac{\partial t}{\partial x} + y\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = -e^u \sin v\frac{\partial f}{\partial x} + e^u \cos v\frac{\partial f}{\partial y} = y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$$

Squarring and adding, we get,

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = e^{2u}\left(\cos v\frac{\partial f}{\partial x} + \sin v\frac{\partial f}{\partial y}\right)^2 + e^{2u}\left(\cos v\frac{\partial f}{\partial y} - \sin v\frac{\partial f}{\partial x}\right)^2$$

$$= e^{2u}\left[\left(\frac{\partial f}{\partial x}\right)^2 (\cos^2 v + \sin^2 v) + \left(\frac{\partial f}{\partial y}\right)^2 (\sin^2 v + \cos^2 v)\right.$$

$$\left. + 2\sin v\cos v\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) - 2\sin v\cos v\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\right]$$

$$= e^{2u}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$$

$$\therefore \quad \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$$

Example 9 :

If $u = z\sin\frac{y}{x}$ where$x = 3r^2 + 2s$,

$y = 4r - 2s^3$ $\qquad z = 2r^2 - 3s^2$

Find that $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial s}$

Solution :

$u \to x, y, z \to r, s.$

Here u be a function of x, y and z where x, y, z themselves are functions of r and s.

Thus u be a composite functions of r and s.

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= z\cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)(6r) + z\cos\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)(4) + \sin\frac{y}{x}(4r)$$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{z}{x}\cos\left(\frac{y}{x}\right)\left(4 - \frac{6yr}{x}\right) + 4r\sin\frac{y}{x}$$

$$\text{And} \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= z\cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)(2) + z\cos\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)(-6s^2) + \sin\frac{y}{x}(-6s)$$

$$\therefore \quad \frac{\partial u}{\partial s} = -2\frac{z}{x}\cos\left(\frac{y}{x}\right)\left(\frac{y}{x} + 3s^2\right) - 6s\sin\left(\frac{y}{x}\right)$$

Example 10 :

If $z = f(u, v)$, $u = x^2 - 2xy - y^2$, $v = y$ show that $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ is equivalent to $\frac{\partial z}{\partial x} = 0$.

Solution : Given $z \to u, v \to x, y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \quad \text{...(1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \quad \text{...(2)}$$

Since $u = x^2 - 2xy - y^2,\ v = y$

$$\frac{\partial u}{\partial x} = 2x - 2y, \quad , \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = -2x - 2y \quad , \quad \frac{\partial v}{\partial y} = 1$$

Substitute $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ in (1) we get,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\,2(x-y) + \frac{\partial z}{\partial u}\cdot 0$$

$$= 2(x-y)\frac{\partial z}{\partial u}$$

Multiply both sides by $(x + y)$

$$(x+y)\frac{\partial z}{\partial x} = 2(x^2 - y^2)\frac{\partial z}{\partial u} \quad \text{...(3)}$$

and substitute $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ in (2), we get,

$$\frac{\partial z}{\partial y} = -2(x+y)\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\cdot 1$$

$$= -2(x+y)\frac{\partial z}{\partial u} + 0 \quad \left(\because \frac{\partial z}{\partial v} = 0\right)$$

Multiplied both sides by $(x - y)$

$$(x-y)\frac{\partial z}{\partial y} = -2(x^2 - y^2)\frac{\partial z}{\partial u} \quad \text{...(4)}$$

Adding (3) and (4), we get,

$$\therefore (x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial u} = 0$$

Hence Proven

Example 11 :

If $V = f(r, \theta)$, where $r^2 = x^2 + y^2 + z^2$ and $z \tan\theta = \sqrt{x^2 + y^2}$ prove that,

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = r \cdot \frac{\partial V}{\partial r}$$

Solution : $V \rightarrow r,\ \theta \rightarrow x, y, z.$

$$V = f(r, \theta)$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} \qquad ...(1)$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial V}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} \qquad ...(2)$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial V}{\partial \theta} \cdot \frac{\partial \theta}{\partial z} \qquad ...(3)$$

$$r^2 = x^2 + y^2 + z^2 \quad ; \quad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad ; \quad \sec^2\theta\frac{\partial \theta}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} \cdot \frac{1}{z}$$

$$\frac{\partial \theta}{\partial x} = \frac{x}{z\sqrt{x^2 + y^2}} \cdot \frac{1}{\sec^2\theta}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \& \quad \frac{\partial \theta}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}} \cdot \frac{1}{\sec^2\theta} \cdot \frac{1}{z}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \qquad \frac{\partial \theta}{\partial z} = -\frac{\sqrt{x^2 + y^2}}{z^2 \sec^2\theta}$$

Substitute $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ in (1), we get,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r} \cdot \frac{x}{r} + \frac{\partial V}{\partial \theta} \cdot \frac{x}{z\sqrt{x^2 + y^2}} \cdot \frac{1}{\sec^2\theta}$$

Multiplying by x

$$x\frac{\partial V}{\partial x} = \frac{x^2}{r}\frac{\partial V}{\partial r} + \frac{x^2}{z\sqrt{x^2 + y^2}} \cdot \frac{1}{\sec^2\theta}\frac{\partial V}{\partial \theta} \qquad ...(4)$$

Substitute $\frac{\partial r}{\partial y}$ and $\frac{\partial \theta}{\partial y}$ in (2), we get,

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r} \cdot \frac{y}{r} + \frac{y}{z\sqrt{x^2 + y^2}} \cdot \frac{1}{\sec^2\theta}\frac{\partial V}{\partial \theta}$$

Multiplying by y

$$y\frac{\partial V}{\partial y} = \frac{y^2}{r}\frac{\partial V}{\partial r} + \frac{y^2}{z\sqrt{x^2 + y^2}} \cdot \frac{1}{\sec^2\theta}\frac{\partial V}{\partial \theta} \qquad ...(5)$$

Substitute $\frac{\partial r}{\partial z}$ and $\frac{\partial \theta}{\partial z}$ in (3), we get,

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial r} \cdot \left(\frac{z}{r}\right) + \frac{\partial V}{\partial \theta} \cdot \left(\frac{-\sqrt{x^2 + y^2}}{z^2 \sec^2\theta}\right)$$

Multiplying by z

$$z\frac{\partial v}{\partial z} = \frac{z^2}{r}\frac{\partial V}{\partial r} - \frac{\sqrt{x^2+y^2}}{z\sec^2\theta}\frac{\partial V}{\partial \theta} \quad ...(6)$$

Adding (4), (5) and (6) we get,

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = \frac{x^2}{r}\frac{\partial V}{\partial r} + \frac{x^2}{z\sqrt{x^2+y^2}\sec^2\theta}\frac{\partial V}{\partial \theta}$$

$$\frac{y^2}{r}\frac{\partial V}{\partial r} + \frac{y^2}{z\sqrt{x^2+y^2}\sec^2\theta}\frac{\partial V}{\partial \theta} + \frac{z^2}{r}\frac{\partial V}{\partial r} - \frac{\sqrt{x^2+y^2}}{z\sec^2\theta}\frac{\partial V}{\partial \theta}$$

$$= \frac{(x^2+y^2+z^2)}{r}\frac{\partial V}{\partial r} + \frac{(x^2+y^2-x^2-y^2)}{z\sec^2\theta\sqrt{x^2+y^2}}\frac{\partial V}{\partial \theta}$$

$$= r\frac{\partial V}{\partial r} + \frac{0}{z\sec^2\theta\sqrt{x^2+y^2}}\cdot\frac{\partial V}{\partial \theta} = r\cdot\frac{\partial V}{\partial r}$$

$$\therefore x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = r\cdot\frac{\partial V}{\partial r}$$

Type 3 :

Example 1 :

If $f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$, prove that

$$(mz - ny) + (lz - nx)\frac{\partial x}{\partial y} + (mx - ly)\frac{\partial x}{\partial z} = 0$$

Solution :

If $f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$

Let $u = lx + my + nz$

and $v = x^2 + y^2 + z^2$

$\therefore \quad f(u, v) = 0 \quad ...(1)$

Since implicit relation between u, v, i.e. x, y, z indicates that one of the three variables x, y, z can be expressed as function of remaining two variables which are independent. The result to be proved indicated that x is a function of y, z.

Thus, $f \to u, v \to x, y$ and $x \to y, z$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \quad ...(2)$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} = 0 \quad ...(3)$$

From (2), $$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{-\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} ;$$

From (3), $$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial z}} ;$$

$$\therefore \qquad \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial z}}$$

$$\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} \qquad \text{...(4)}$$

$$u = lx + my + nz$$

Differentiate partially w.r.t. y and noting that x is a function of y, z we have,

$$\frac{\partial u}{\partial y} = l\frac{\partial x}{\partial y} + m \qquad \text{z is constant}$$

Differentiating partialy w.r.t. z keeping y constant.

$$\frac{\partial u}{\partial z} = l\frac{\partial x}{\partial z} + n$$

But $\qquad v = x^2 + y^2 + z^2$

$$\therefore \qquad \frac{\partial v}{\partial y} = 2x\frac{\partial x}{\partial y} + 2y$$

$$\frac{\partial v}{\partial z} = 2x\frac{\partial x}{\partial z} + 2z$$

Substituting In (4),

$$\left(l\frac{\partial x}{\partial t} + n\right)\left(2x\frac{\partial x}{\partial y} + 2y\right) = \left(l\frac{\partial x}{\partial y} + m\right)\left(2x\frac{\partial x}{\partial z} + 2z\right)$$

$$lx\frac{\partial x}{\partial z}\frac{\partial x}{\partial y} + ly\frac{\partial x}{\partial z} + nx\frac{\partial x}{\partial y} + ny = lx\frac{\partial x}{\partial y}\frac{\partial x}{\partial z} + l\,z\frac{\partial x}{\partial z} + mx\frac{\partial x}{\partial z} + z$$

$$(mz - ny) + (lz - nx)\frac{\partial x}{\partial y} + (mx - ly)\frac{\partial x}{\partial z} = 0$$

Hence Proved

Example 2 :

If $\qquad x^2 = au + bv,\ y^2 = au - bv$

and $\qquad V = f(x, y)$ then show that

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2\left(u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v}\right)$$

Solution : Given $V \to x, y \to$?? v

We have, $$\frac{\partial V}{\partial u} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial V}{\partial x}\left(\frac{a}{2x}\right) + \frac{\partial V}{\partial y}\left(\frac{a}{2y}\right) \qquad \text{...(1)}$$

$$\because \qquad x^2 = au + bv$$

$$\therefore \qquad 2x\frac{\partial x}{\partial u} = a \qquad \therefore \frac{\partial u}{\partial x} = \frac{a}{2x}]$$

Similarly, $$\frac{\partial x}{\partial v} = \frac{b}{2x}, \quad \frac{\partial y}{\partial v} = -\frac{b}{2y}$$

Again, $$\frac{\partial V}{\partial v} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial V}{\partial x}\left(\frac{b}{2x}\right) + \frac{\partial V}{\partial y}\left(\frac{-b}{2y}\right) \qquad \ldots(2)$$

Multiplying equation (1) by u and (2) by v, and adding we get,

$$u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v} = \left(\frac{au+bv}{2x}\right)\frac{\partial V}{\partial x} + \left(\frac{au-bv}{2y}\right)\frac{\partial V}{\partial y} + \frac{x^2}{2x}\frac{\partial V}{\partial x} + \frac{y^2}{2y}\frac{\partial V}{\partial y}$$

$$\therefore \quad x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2\left(u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v}\right)$$

Example 3 :

If $x = r\cos\theta$, $y = r\sin\theta$ where r and θ are functions of t, prove that $x\frac{\partial y}{\partial t} - y\frac{\partial x}{\partial t} = r^2\frac{\partial \theta}{\partial t}$

Solution : Given $x, y \to r, \theta \to t$

Here x and y are composite functions of t.

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial x}{\partial \theta} \cdot \frac{\partial \theta}{\partial t}$$

$$= \cos\theta\frac{\partial r}{\partial t} - r\sin\theta\frac{\partial \theta}{\partial t}$$

Also, $$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial r} \cdot \frac{\partial y}{\partial t} + \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= \sin\theta\frac{\partial r}{\partial t} + r\cos\theta\frac{\partial \theta}{\partial t}$$

$$\therefore \quad \text{L. H. S. } x\frac{\partial y}{dt} - y\frac{yx}{dt}$$

$$= r\cos\theta\left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{dq}{dt}\right) - r\sin\theta\left(\cos\theta\frac{\partial r}{\partial t} - r\sin\theta\frac{\partial \theta}{\partial t}\right)$$

$$= r^2(\cos^2\theta + \sin^2\theta)\frac{\partial \theta}{\partial t}$$

$$= r^2\frac{\partial \theta}{\partial t} = \text{R. H. S.}$$

Type 4 : Examples on variable to be treated as constant.

Let $Z = f(u, v)$ and $u = \phi(x, y)$, $v = \psi(x, y)$.

i.e. $z \to u$, $v \to x, y$

Then $\left(\frac{\partial u}{\partial x}\right)_y$ means the partial derivative of u w.r.t. x treatign y constant.

To get $\left(\frac{\partial x}{\partial u}\right)_v$ from the given three relations we first express x in terms of u and v.

i.e. $x = f_1(u, v)$ then we can find partial derivative of x w.r.t. u treating v constant.

To get $\left(\frac{\partial v}{\partial y}\right)_u$ express v in terms of y and u.

i.e. $v = f_2(y, u)$ then we can find partial derivative of v w.r.t. y treating u constant.

Example 1 :

If $x = r\cos\theta$, $y = r\sin\theta$, show that,

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y \quad \text{and} \quad \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = r\left(\frac{\partial \theta}{\partial x}\right)_y$$

Solution :

(i) Since $x = r\cos\theta$

$$\therefore \quad \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$$

Also, $r^2 = x^2 + y^2$

$$\therefore \quad 2r\left(\frac{\partial r}{\partial x}\right)_y = 2x$$

$$\therefore \quad \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \cos\theta$$

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$$

(ii) Since $x = r\cos\theta$

$$\therefore \quad \left(\frac{\partial x}{\partial \theta}\right)_r = -r\sin\theta$$

$$\frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = -\sin\theta$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$= -\frac{y}{r^2}$$

$$\therefore \quad r\left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{y}{r^2} = -\frac{r\sin\theta}{r} = -\sin\theta$$

$$\therefore \quad \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = r\left(\frac{\partial \theta}{\partial x}\right)_y$$

Example 2 :

If $\quad u = ax + by \quad\quad v = bx - ay,$

find the value of $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$ **(May 1998)**

Solution :

$$u = ax + by$$

Differentiating u w.r.t. x treating y constant.

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = a \quad \text{...(1)}$$

$$v = bx - ay$$

$$\therefore \quad 1 = -a\left(\frac{\partial y}{\partial v}\right)_x$$

$$\left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{a} \quad \text{...(2)}$$

Now,

$$u = ax + by$$

$$v = bx - ay$$

$$\therefore \quad au + bv = (a^2 + b^2)\, x$$

$$x = \frac{au + bv}{a^2 + b^2}$$

Similarly

$$bu - av = (b^2 + a^2)\, y$$

$$\therefore \quad y = \frac{bu - av}{a^2 + b^2}$$

$$x = \frac{au + bv}{a^2 + b^2}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{a^2 + b^2} \quad \text{...(3)}$$

$$y = \frac{bu - av}{a^2 + b^2}$$

$$-1 = \frac{-a}{a^2 + b^2}\left(\frac{\partial v}{\partial y}\right)_u \quad \text{...(4)}$$

From 1, 2, 3 and 4

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = a \cdot \frac{a}{a^2 + b^2} \cdot \left(\frac{-1}{a}\right)\left(-\frac{a^2 + b^2}{a}\right)$$

$$= 1$$

Example 3 :

If $ux + vy = 0$, $\frac{u}{x} + \frac{v}{y} = 1$, Prove that,

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2 + y^2}{y^2 - x^2}$$

Solution :

$$ux + vy + 0 = 0$$

$$\frac{1}{x}u + \frac{1}{y}v - 1 = 0 \qquad \text{...(By determinant)}$$

$$\frac{u}{-y} = \frac{v}{x} = \frac{1}{\frac{x}{y} - \frac{y}{x}} = \frac{xy}{x^2 - y^2}$$

$$\therefore \qquad u = \frac{-xy^2}{x^2 - y^2} \qquad \text{and} \qquad v = \frac{x^2 y}{x^2 - y^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = -y^2 \frac{(x^2 - y^2) - x(2x)}{(x^2 - y^2)^2}$$

$$= -y^2 \frac{(-x^2 - y^2)}{(x^2 - y^2)^2}$$

$$= \frac{(x^2 + y^2)\, y^2}{(x^2 - y^2)^2} \qquad \text{...(1)}$$

$$\left(\frac{\partial v}{\partial y}\right)_x = x^2 \frac{x^2 - y^2 - y(-2y)}{(x^2 - y^2)^2}$$

$$= \frac{x^2}{(x^2 - y^2)^2} (y^2 + x^2) \qquad \text{...(2)}$$

Subtracting (2) from (1), we get,

$$\text{LHS} = \frac{(x^2 + y^2)}{(x^2 - y^2)^2} (y^2 - x^2)$$

$$= -\frac{(x^2 + y^2)(x^2 - y^2)}{(x^2 - y^2)^2}$$

$$= -\frac{(x^2 + y^2)}{x^2 - y^2}$$

$$= \frac{x^2 + y^2}{y^2 - x^2}$$

$$= \text{RHS.}$$

Example 4 :

If $x^2 = au + bv$, $y = au - bv$ prove that

$$\left(\frac{\partial u}{\partial x}\right)_x \left(\frac{\partial x}{\partial u}\right)_v = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

Solution :

$$x^2 = au + bv$$

Differentiating w.r.t. u treating v constant

$$2x \left(\frac{\partial x}{\partial u}\right)_v = a$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x} \qquad \text{...(1)}$$

$$y = au - bv$$

Differentiating y w.r.t. v treating u constant

$$\left(\frac{\partial y}{\partial v}\right)_u = -b \qquad ...(2)$$

$$x^2 = au + bv$$

$$y = au - bv$$

Adding $\qquad x^2 + y = 2au$

Subtracting, $\qquad x^2 - y = 2\,bv$

$$\frac{x^2 + y}{2a} = u$$

Differentiating w.r.t. x, treating y constant, we get,

$$\frac{1}{2a} \cdot 2x = \left(\frac{\partial u}{\partial x}\right)_y$$

$$\therefore \qquad \left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a} \qquad ...(3)$$

$$\frac{x^2 - y}{2b} = v$$

Differentiating w.r.t. y treating x constant, we get,

$$-\frac{1}{2b} = \left(\frac{\partial v}{\partial y}\right)_x$$

$$\therefore \qquad \left(\frac{\partial v}{\partial y}\right)_x = \frac{-1}{2b} \qquad ...(4)$$

From (1) and (3)

$$\text{LHS} = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \cdot \frac{a}{2x} = \frac{1}{2}$$

From (2) and (4),

$$\text{RHS} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = -\frac{1}{2b}\,(-b) = \frac{1}{2}$$

$$\therefore \qquad \text{LHS} = \text{RHS}$$

Example 5 :

If $\qquad ux + uy = 0$

$$\frac{u}{x} + \frac{u}{y} = 1, \text{ prove that}$$

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_0 = 0$$

Solution :

Given $\quad ux + vy = 0 \qquad \therefore \qquad y = -\frac{ux}{v}$

and $\quad \frac{u}{x} + \frac{v}{y} = 1 \qquad \therefore \quad \frac{u}{x} + v\left(\frac{-v}{ux}\right) = 1$

i.e. $$\frac{u^2 - v^2}{ux} = 1$$

$$x = \frac{u^2 - v^2}{u} = u - \frac{v^2}{u}$$

Differentiating w.r.t. u treating v constant.

$$\left(\frac{\partial x}{\partial u}\right)_v = 1 + \frac{v^2}{u^2} \quad ...(1)$$

$$x = \frac{vy}{u}$$

$$u\left(\frac{-u}{vy}\right) + \frac{v}{y} = 1$$

Differentiating w.r.t. v treating u constant.

$$\therefore \quad y = \frac{v^2 - u^2}{v} = v - \frac{u^2}{v}$$

$$\left(\frac{\partial y}{\partial v}\right)_u = 1 + \frac{u^2}{v^2} \quad ...(2)$$

From (1) and (2),

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = \frac{u}{x}\left(1 + \frac{v^2}{u^2}\right) + \frac{v}{y}\left(1 + \frac{u^2}{v^2}\right)$$

$$= \frac{u^2 + v^2}{xu} + \frac{v^2 + u^2}{vy}$$

$$(u^2 + v^2)\left(\frac{1}{ux} + \frac{1}{vy}\right) = (u^2 + v^2)\left(\frac{vy + ux}{xu\,yv}\right)$$

$$= 0 \quad (\because\ ux + vy = 0)$$

Example 6 :

If $x = \frac{\cos\theta}{u}$, $y = \frac{\sin\theta}{u}$, prove that

$$\left(\frac{\partial x}{\partial u}\right)_0 \cdot \left(\frac{\partial u}{\partial x}\right)_v + \left(\frac{\partial v}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial y}\right)_x = 1$$

Solution :

The given equations are , $$x = \frac{\cos\theta}{u} \quad ...(1)$$

$$y = \frac{\sin\theta}{u} \quad ...(2)$$

Differentiating partially w.r.t. u we have,

$$\left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2} \quad ...(3)$$

$$\left(\frac{\partial y}{\partial u}\right)_\theta = -\frac{\sin\theta}{u^2} \quad ...(4)$$

To find $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial u}{\partial y}\right)_x$, we shall solve the given equations (1) and (2) for u in terms of x and y.

Thus, we have, $$x^2 + y^2 = \frac{\cos^2\theta}{u^2} + \frac{\sin^2\theta}{u^2} = \frac{1}{u^2}$$

$$u = \frac{1}{\sqrt{x^2+y^2}} \qquad \text{...(5)}$$

Differentiating (5) partially, w.r.t.. x and y respectively, we have,

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{2}(x^2+y^2)^{-3/2}(2x) = \frac{-x}{(x^2+y^2)^{-3/2}}$$

$$\left(\frac{\partial u}{\partial y}\right)_x = -\frac{1}{2}(x^2+y^2)^{-3/2}(2y) = \frac{-y}{(x^2+y^2)^{-3/2}}$$

Then, $$\left(\frac{\partial x}{\partial u}\right)_\theta\left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta\left(\frac{\partial u}{\partial y}\right)_y = \left(-\frac{\cos\theta}{u^2}\right)\left[-\frac{x}{(x^2+y^2)^{3/2}}\right] + \left(-\frac{\sin\theta}{u^2}\right)\left[\frac{-y}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{1}{u^2(x^2+y^2)^{3/2}}[x\cos\theta + y\sin\theta]$$

$$= \frac{1}{u^2(1/u^{2)3/2}}\left[\frac{\cos\theta}{u}\cdot\cos\theta + \frac{\sin\theta}{u}\cdot\sin\theta\right]$$

$$= \frac{u(\cos^2\theta+\sin^2\theta)}{u} = 1$$

Example 7 :

If $x = u\tan v$, $y = u\sec v$, prove that,

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x\left(\frac{\partial v}{\partial y}\right)_x$$

Solution :

Given $\quad x = u\tan v$ and $y = u\sec v$

$\therefore \quad y^2 - x^2 = u^2(\sec^2 v - \tan^2 v) = u^2 \qquad$ ...(1)

Differentiating (1), w.r.t. x,

$$-2x = 2u\left(\frac{\partial u}{\partial x}\right)_y$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = -\frac{x}{u}$$

Differentiating (1) w.r.t. y,

$$2y = 2u\left(\frac{\partial u}{\partial y}\right)_x$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{y}{u}$$

$$\frac{x}{y} = \frac{u \tan v}{u \sec v} = \sin v \qquad \ldots(2)$$

Differentiating (2) w.r.t. x,

$$\frac{1}{y} = \cos v \left(\frac{\partial v}{\partial x}\right)_y$$

$$\left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{y \cos v} = \frac{\sec v}{y} = \frac{\sec v}{u \sec v} = \frac{1}{u}$$

Differentiating (2) w.r.t. y,

$$x\left(-\frac{1}{y^2}\right) = \cos v \left(\frac{\partial v}{\partial y}\right)_x$$

$$\left(\frac{\partial v}{\partial y}\right)_x = -\frac{x}{y^2 \cos v} = -\frac{x}{y^2} \sec v$$

$$= \frac{-x}{y^2}\frac{y}{u} = \frac{-x}{uy} \qquad [\because \sec v = \frac{y}{u}]$$

$$\text{LHS} = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{u}\left(-\frac{x}{u}\right) = -\frac{x}{u^2}$$

and $$\text{RHS} \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = \frac{y}{u}\left(-\frac{1}{uy}\right) = -\frac{x}{u^2}$$

$\therefore$ LHS = RHS.

Exercise 6.2

Type 1 :

1. If $u = f(r)$, where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} + (r)$

2. If $u = f(r)$, where $r = \frac{x^2}{y}$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 3xy \frac{\partial^2}{\partial x \partial y} + 2y^2 \frac{\partial^2 u}{\partial y^2} = 0$.

3. If $u = \frac{1}{r}$, where $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

4. If u r^m, where $r^2 = x^2 + y^2 + z^2$, prove that, $u_{xx} + u_{yy} + u_z = m(m+1)r^{m-2}$

5. If $u = f(r)$ were $r = x^2y$, prove that $x\frac{\partial u}{\partial x} = 2y\frac{\partial u}{\partial y}$

6. If $u = \log r$ where $r^2 = x^2 + y^2 + z^2$, prove that

$$(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$$

7. If $u = \log r$ where $r = x^3 + y^3 - x^2y - xy^2$ prove that,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \frac{4}{(x + y^2)}$$

Type 2 :

8. If $u = f(v, s)$ where $r = x + y$, $s = x - y$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$

9. If $u = f(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Hint: Take $x - y = 1$, $y - z$ m and $z - x = n$)

10. If $u = f(u, v) = x^2 + y^2$ and $r = 2xy$, prove that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(u^2 - r^2)\frac{\partial z}{\partial u}$$

11. If $u = (e^{x-y}, e^{y-z}, e^{z-x})$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Hint: $l = e^{x-y}$, $m = e^{y-z}$, $n = e^{z-x}$

12. If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$ prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

13. If $z = f(x, y)$, $x = eu + e^{-v}$, $y = e^{-u} - e^{v}$, prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Type 3 :

14. If $f(x + y + z, x^2 + y^2 + z^2) = 0$, prove that, $(z - y) + (z - x)\frac{\partial y}{\partial x} + (x - y)\frac{\partial x}{\partial z} = 0$

15. If $f(x + y + z, x^2 + y^2 + z^2) = 0$, prove that, $(z - x) + (z - y)\frac{\partial y}{\partial x} + (y - x)\frac{\partial x}{\partial z} = 0$

16. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - y)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$$

Hint : $x^2 + 2yz = l$, $y^2 + 2zx = m$

17. If $f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$, prove that

$$(lz - nx) + (mz - ny)\frac{\partial y}{\partial x} + (ly - mn)\frac{\partial y}{\partial z} = 0.$$

18. If $z = f(x, y)$ and $x = u \cosh v$, $y = u \sin y$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial x}\right)^2$$

19. If $z = f(u, v)$, $u = x^2 - 2xy - y^2$, $v = y$ prove that

$$(x + y)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y} = (x - y)\frac{\partial z}{\partial x}$$

20. If $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$, prove that $u\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial \theta} = (y - x)\dfrac{\partial z}{\partial x} - (y + x)\dfrac{\partial z}{\partial y}$

Hint Let $z = f(x, y)$ and $x, y \to u, \theta$

Type 4 :

21. If $x = \dfrac{r}{2}(e^{\theta} + e^{-\theta})$, $y = \dfrac{r}{2}(e^{\theta} - e^{-\theta})$, thenprove that $\left(\dfrac{\partial x}{\partial r}\right)_{\theta} = \left(\dfrac{\partial r}{\partial x}\right)_{y}$

Hint $\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$, $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

22. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left[x\left(\dfrac{\partial x}{\partial r}\right)_{\theta} + y\left(\dfrac{\partial y}{\partial r}\right)_{\theta}\right]^2 = x^2 + y^2$

23. If $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$, prove that $\left(\dfrac{\partial x}{\partial u}\right)_{\theta}\left(\dfrac{\partial u}{\partial x}\right)_{y} = \cos^2\theta$

24. If $\tan^2\theta = \dfrac{y}{x}$, $\sec^2\phi = x + y$ and $u = f(\theta, \phi)$, prove that $\left(\dfrac{\partial \theta}{\partial x}\right)_{y}\left(\dfrac{\partial \theta}{\partial y}\right)_{x} = -\dfrac{1}{4(x + y)^2}$

Homogeneous Function

Definition : An $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} y^2 + \dots\dots\dots + a_n y^n$, where all the terms are of the same degree, is called a Homogeneous expression in x and y of the n^{th} degree.

The above expression can be written as,

$$\begin{aligned} z &= f(x, y) \\ &= a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} y^2 + \dots\dots\dots\dots + a_n y^n \\ &= x^n\left[a_0\left(\frac{y}{x}\right)^0 + a_1\left(\frac{y}{x}\right)^1 + a_2\left(\frac{y}{x}\right)^2 \dots\dots\dots + a_n\left(\frac{y}{x}\right)^n\right] \end{aligned}$$

$$= x^n \phi\left(\frac{y}{x}\right)$$

OR $$z = f(x, y)$$

$$= y^n \left[\left(\frac{x}{y}\right)^n + a_1 \left(\frac{x}{y}\right)^{n-1} + a_2 \left(\frac{x}{y}\right)^{n-3} + \ldots\ldots\ldots + a_n\right]$$

$$= y^n \phi\left(\frac{x}{y}\right)$$

Thus, $z = x^n \phi\left(\frac{y}{x}\right)$ or $z = y^n \phi\left(\frac{x}{y}\right)$ is defined as a homogeneous expression of n^{th} degree of the variables x and y.

e.g.

1. $z = x^4 y - 4x^3 y^2 + 3xy^4 + 4xy^4 + y^5$ is a hom ogeneous function of degree five.

$$= x^5 \left[\frac{y}{x} - 4\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^5\right]$$

$$= x^5 \phi\left(\frac{y}{x}\right)$$

2. The function $\log\left(\frac{y}{x}\right)$ is an homogeneous function of degree zero.

We can write, $\log\left(\frac{y}{x}\right) = x^0 \log\frac{y}{x}$

3. The function $\cos\left(\frac{x^3 + y^3}{x + y}\right)$ is not a homogeneous function because it cannot be express in the form

$$x^n f\left(\frac{y}{x}\right) \text{ or } y^n f\left(\frac{y}{x}\right)$$

Euler's theorem on Homogeneous Function :

If u is a homogeneous function of degree n is x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

Proof :

Given u is homogeneous function of degree n in x and y.

i.e. $$u = x^n \left(\frac{y}{x}\right) \qquad \ldots(1)$$

Differentiating (1) partially w.r.t. x and y respectively, we have,

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$= -y \cdot x^{n-2} f'\left(\frac{y}{x}\right) + nx^{n-1} f\left(\frac{y}{x}\right) \qquad \ldots(2)$$

and $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$= x^{n-1} f'\left(\frac{y}{x}\right) \qquad \ldots(3)$$

Multiplying the equation (2) by x and the equation (3) by y and adding we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -y\,x^{n-1} f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{u}{x}\right) + y\,x^{n-1} f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right)$$

$$= n \cdot u \qquad \text{...(from (1))}$$

Hence proved.

Euler's theorem for more than two variable on homogeneous functions :

Statement : If u is a homogeneous function of $x_1, x_2, \ldots\ldots\ldots, x_n$ of degree n then

$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + \ldots\ldots\ldots + x_n\frac{\partial u}{\partial x_n} = nu$$

Deductions from Euler's theorem

1. **If z is a homogeneous function of two variables x, y of degree n, then**

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)\,z$$

Proof :

z is a homogeneous function of x, y of degree n, we have, by Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \text{...(1)}$$

Differentiating equation (1) partially with respect to x,

$$\left(x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\cdot 1\right) + y\frac{\partial^2 z}{\partial x\,\partial y} = n\frac{\partial z}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} = (n-1)\frac{\partial z}{\partial x} \qquad \text{...(2)}$$

Differentiating equation (1) partialy with respect to y, we get,

$$\Rightarrow \quad y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial y\,\partial x} = (n-1)\frac{\partial z}{\partial y} \qquad \text{...(3)}$$

Since, $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial x\,\partial y}$

Multiplying equation (2) by x, equation (3) by y, and adding we get,

$$\left(x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x\,\partial y}\right) + \left(y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial x\,\partial y}\right)$$

$$= (n-1)\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)\,nz = n(n-1)\,z$$

2. If z is hom ogeneous function of x, y of degree n and $z = f(u)$.

then, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)}$

and $\dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = g(u)\ [g'(u) - 1]$

where $g(u) = n \dfrac{f(u)}{f'(u)}$

Proof : z is homogeneous function of x, y of degree n, we have, by Euler's theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \qquad ...(1)$$

Given : $z = f(u)$,

$$\therefore \quad \frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y}$$

Substituting in equation (1), we get,

$$x \cdot f'(u) \frac{\partial u}{\partial x} + y \cdot f'(u) \frac{\partial u}{\partial y} = n f \cdot (u)$$

i.e.
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)} = g(u) \qquad ...(2)$$

$$\left[\because \frac{n f(u)}{f'(u)} = g(u) \right]$$

Differentiating equation (2), partially w.r.t. x, we get,

$$\left(x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot 1 \right) + y \frac{\partial^2 u}{\partial x \partial y} = g'(u) \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = [g'(u) - 1] \frac{\partial u}{\partial x} \qquad ...(3)$$

Differentiating equation (2) partially w.r.t. y, we get,

$$\Rightarrow \quad y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial^2 u}{\partial x \partial y} = [g'(u) - 1] \frac{\partial u}{\partial y} \qquad ...(4)$$

Multiplying equation (3) by x and equation (4) by y and adding, we get,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = [g'(u) - 1] \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

$$= g(u)\ [g; (u) - 1] \qquad ...\text{(from (1))}$$

Illustrative Examples

Type 1 : Verification

Verify Euler's theorom for the following functions.

Example 1 :

Verity Euler's theorem, when $t = ax^2 + 2hxy + by^2$

Solution :

$$t = ax^2 + 2hxy + by^2 = x^2 \left[a + 2h \frac{y}{x} + b \left(\frac{y^2}{x} \right) \right]$$

which is the homogeneous equation of x and y of order 2.

$\therefore$ For Euler's theorem

To prove $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = n \cdot f = 2f$

Given $$f = ax^2 + 2hxy + by^2$$

$$\therefore \quad \frac{\partial f}{\partial x} = 2\,ax + 2hy \quad \text{...(1)}$$

and $$\frac{\partial f}{\partial y} = 2hx + 2by \quad \text{...(2)}$$

Now, multiplying (1) and by x and (2) by y and adding we get,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2ax^2 + 2hxy + 2hyx + 2by^2$$

$$= 2\,(ax^2 + 2hxy + 2by^2)$$

$$= 2f$$

Example 2 :

$$f(x, y) = \frac{\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)}{\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)} \quad \text{...(1)}$$

Solution :

Given $$t(x, y) = x^{\frac{1}{3}} \frac{\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{3}}\right]}{x^{\frac{1}{4}}\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}$$

$$= x^{\frac{1}{3}-\frac{1}{4}} \frac{\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{3}}\right]}{\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}$$

$$= x^{\frac{1}{12}} \frac{\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{3}}\right]}{\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}$$

which is the homogeneous equation of x and y of order $\frac{1}{12}$.

Hence to prove

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{12}\,f$$

Now, from (1),

$$\frac{\partial f}{\partial x} = \frac{(x^{\frac{1}{4}} + y^{\frac{1}{4}})\,\frac{1}{3}\,x^{\frac{-2}{3}} - (x^{\frac{1}{3}} + y^{\frac{1}{3}})\left(\frac{1}{4}\,x^{\frac{-3}{4}}\right)}{(x^{\frac{1}{4}} + y^{\frac{1}{4}})^{2}}$$

$$= \frac{1}{12}\left[4\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right) x^{\frac{-2}{3}} - 3\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) x^{\frac{-3}{4}}\right] \quad \text{...(1)}$$

Similarly, $$\frac{\partial f}{\partial y} = \frac{1}{12}\,\frac{\left[4\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right) y^{\frac{-2}{3}} - 3\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) y^{\frac{-3}{4}}\right]}{\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)^{2}} \quad \text{...(2)}$$

Multiplying (1) by x and (2) by y and adding we get,

$$x\frac{\partial f}{\partial y} + y\frac{\partial f}{\partial y}$$

$$= \frac{1}{12}\,\frac{\left[4\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) - 3\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\right]}{\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)^{2}}$$

$$= \frac{1}{12}\,\frac{\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)}{\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)^{2}}$$

$$= \frac{1}{12}\,\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}$$

$$= \frac{1}{12}\,f$$

Example 3 :

$$u = \sqrt{x} + \sqrt{y} + \sqrt{z}$$

Solution :

Given : $$u = \sqrt{x} + \sqrt{y} + \sqrt{z}$$

$$= x^{\frac{1}{2}}\left[1 + \sqrt{\frac{y}{x}} + \sqrt{\frac{z}{x}}\right]$$

$$= x^{\frac{1}{2}}\; f\left(\frac{y}{x}, \frac{z}{x}\right)$$

which is homogeneous function of degree $\frac{1}{2}$ in x, y and z.

Hence to prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}\,u$

Now, $$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}},\quad \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}},\quad \frac{\partial u}{\partial z} = \frac{1}{2\sqrt{z}}$$

$$\therefore\; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{2\sqrt{x}} + \frac{y}{2\sqrt{y}} + \frac{z}{2\sqrt{y}}$$

$$= \frac{\sqrt{x}}{2} + \frac{\sqrt{y}}{2} + \frac{\sqrt{z}}{2}$$

$$= \frac{1}{2}(\sqrt{x} + \sqrt{y} + \sqrt{z})$$

$$= n(u)$$

(which satisfies the Euler's theorem)

Type 2 : Application of Euler's theorem - First Order Partial derivative

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Example 4 :

If $\psi(x, y) = \log \frac{x^2 + y^2}{x + y}$ show that $x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} = 1$

Solution :

Given $\psi(x, y) = \log \frac{x^2 + y^2}{x + y}$

$$e^{\psi(x, y)} = \frac{x^2 + y^2}{x + y} = u \text{ ...(say)}$$

$$\therefore \quad u = \frac{x^2\left[1 + \left(\frac{y}{x}\right)^2\right]}{x\left[1 + \left(\frac{y}{x}\right)\right]}$$

$$= \frac{x\left[1 + \left(\frac{y}{x}\right)^2\right]}{1 + \frac{y}{x}}$$

$\therefore$ u is a homogeneous function of x and y of order 1.

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = 1 \cdot u \qquad \text{...(By Euler's theorem)}$$

$$\therefore \quad x\frac{\partial e^{\psi(x, y)}}{\partial x} + y\frac{\partial e^{\psi(x, y)}}{\partial y} = 1 \cdot e^{\psi(x, y)}$$

$$\therefore x\, e^{\psi(x, y)} \cdot \frac{\partial \psi}{\partial x} \quad y\, e^{\psi(x, y)} \cdot \frac{\partial \psi}{\partial y} = e^{\psi(x, y)}$$

$$\therefore \quad x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} = 1.$$

Hence Proved

Example 5 :

If $\sin u = \frac{x^2 y^2}{x + y}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3 \tan u$

Solution :

Given $\sin u = \frac{x^2 y^2}{x + y}$

$$= \frac{x^4 \left(\frac{y}{x}\right)^2}{x\left(1 + \frac{y}{x}\right)}$$

$$= \frac{x^3 \left(\frac{y}{x}\right)^2}{\left(1 + \frac{y}{x}\right)}$$

$$= z$$

(Where z is the homogeneous function of x and y of order n = 3)

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3 \cdot z \qquad \text{... (By Euler's theorem)}$$

$$\therefore \quad x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = 3 \sin u$$

$$\therefore \quad x\frac{\partial \sin u}{\partial u} \cdot \frac{\partial u}{\partial x} + y\frac{\partial \sin u}{\partial u} \cdot \frac{\partial u}{\partial y} = 3 \sin u$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\,\frac{\sin u}{\cos u}$$

$$= 3 \tan u$$

Hence proved.

Example 6 :

If $u = f\left(\frac{y}{x}\right) + \sqrt{x^2 + y^2}$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Solution :

Let $u = v + w$

where $v = f\left(\frac{y}{x}\right)$,

$w = \sqrt{x^2 + y^2}$

$v = f\left(\frac{y}{x}\right)$ is a homogeneous functions of degree 0.

$\therefore$ By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0 \qquad \text{...(1)}$$

$\omega = \sqrt{x^2 + y^2}$ is a homogeneous function of degree 1.

$$\therefore \quad x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} = \omega \qquad \text{...(2)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x\partial}{\partial x}(v + w) + y\frac{\partial}{\partial y}(v + w)$$

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y}$$

$$= 0 + w \qquad \text{...(From (1) and (2))}$$

$$= \omega = \sqrt{x^2 + y^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2 + y^2}$$ Hence proved.

Example 7 :

If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$ then prove that,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$$

Solution :

Given, $$f = x^{-2}\left[1 + \frac{1}{\frac{y}{x}} - \frac{\log\frac{y}{x}}{1 + \frac{y}{x}}\right]$$

$f(x, y)$ is a homogeneous function of degree -2 in x and y.

$\therefore$ By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f$$

$$\therefore \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$$

Type 2 : Second Order Partial derivative

Example 8 :

If $u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}}$ then show that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{1}{144}\tan u\,(13 + \tan^2 u)$$

Solution :

From given $$\operatorname{cosec} u = \sqrt{\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}}$$

$$\therefore \quad z = f(u) = \sqrt{\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}}$$

$$= x^{\frac{1}{4} - \frac{1}{6}}\sqrt{\frac{1 + \left(\frac{y}{x}\right)^{\frac{1}{2}}}{1 + \left(\frac{y}{x}\right)^{\frac{1}{3}}}}$$

Thus z is a homogeneous functions of degree $\frac{1}{12}$ in x and y.

$\therefore$ By Euler's Deduction

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)} \qquad \text{...Where } z = f(u) = \text{cosec } u \text{ and } n = \frac{1}{12}$$

$$\therefore \quad \frac{n\,f(u)}{f(u)} = G(u) = \frac{1}{12}\left(\frac{\text{cosec } u}{-\text{cosec } u \cot u}\right) = \frac{-1}{12}\tan u$$

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1]$$

$$= -\frac{1}{12}\tan u\left[-\frac{1}{12}\sec^2 u - 1\right]$$

$$= +\frac{1}{12}\tan u\left[\frac{\sec^2 u + 12}{12}\right]$$

$$= \frac{1}{144}\tan u\,[1 + \tan^2 u + 12]$$

$$= \frac{1}{144}\tan u\,[13 + \tan^2 u]$$

Hence proved

Example 9 :

If $u = \sin^{-1}\sqrt{x^2+y^2}$ then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Solution : $u = \sin^{-1}\sqrt{x^2+y^2}$

$\therefore f(u)$ is a homogeneous function of degree 1 where

$$\sqrt{x^2+y^2} = \sin u = f(u) \text{ and } n = 1.$$

$$\therefore \quad \frac{n\,f(u)}{f'(u)} = \frac{\sin u}{\cos u} = \tan u = G(u)$$

By Euler's deduction,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)} = G(u)$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

and $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= G(u)\,[G'(u) - 1]$$

$$= \tan u\,[\sec^2 u - 1]$$

$$= \tan u\,[\tan^2 u]$$

$$= \tan^3 u$$

Example 10 :

If $z = x^n f\left(\frac{y}{x}\right) + y^{-n}\phi\left(\frac{x}{y}\right)$ prove that,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

Solution :

Let $z = u + v$

where $u = x^n f\left(\frac{y}{x}\right)$

and $v = y^{-n} \phi\left(\frac{y}{x}\right)$

$u = x^n f\left(\frac{y}{x}\right)$ is a homogeneous function of deegree n in x, y.

and $v = y^{-n} \phi\left(\frac{y}{x}\right)$ is also a homogeneous function of degree – n in x and y.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = nu \qquad ...(1)$$

and by Euler's deduction,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u \qquad ...(2)$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \qquad ...(3)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = -n(-n-1)\,v$$

$$= n(n+1)\,v \qquad ...(4)$$

$$\text{LHS} = x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$= x^2\frac{\partial^2}{\partial x^2}(u+v) + 2xy\frac{\partial^2}{\partial x\,\partial y}(u+v) + y^2\frac{\partial^2}{\partial y^2}(u+v)$$

$$+ x\frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v)$$

$$= \left\{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right\} +$$

$$\left\{x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2}\right\}$$

$$+ \left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\} + \left\{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right\}$$

$$= n(n-1)\,u + n(n+1)\,v + nu - nv \qquad ...\text{(From (1), (2), (3) and (4))}$$

$$= n^2(u+v)$$

$$= n^2 z$$

$$= \text{RHS}$$

Example 11 :

If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, where u is a homogeneous function of degree n in x, y, z then show that $u_x^2 + u_y^2 + u_z^2 = 2\,nu$

Solution :

Given, u is a homogeneous function of degree n in x, y, z.

$\therefore$ By Euler's theorem

$$\left.\begin{aligned} x\,u_x + y\,u_y + z\,u_z &= nu \\ \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} & \end{aligned}\right\} \quad \ldots(1)$$

Differentiating (1) w.r.t. x,

$$\frac{1}{a^2+u}\;2x - x^2\frac{1}{(a^2+u)^2}\;u_x - y^2\frac{1}{(b^2+u)^2}\;u_x - z^2\frac{1}{(c^2+u)^2}\;u_x = 0$$

$$\therefore\quad u_x\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{2x}{a^2+u}$$

$$\therefore\qquad u_x = \frac{2x}{(a^2+u)\;k}$$

where
$$k = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$$

Similarly,
$$u_y = \frac{2y}{(b^2+u)\,k} \quad\text{and}\quad u_z = \frac{2z}{(c^2+u)\,k}$$

$$\therefore\qquad u_x^2 + u_y^2 + u_z^2 = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{4}{k^2} \quad \ldots(2)$$

$$= \frac{4}{k^2}\;[k] = \frac{4}{k}$$

From (1),

$$xu_x + yu_y + zu_z = \frac{2x^2}{k\,(a^2+u)} + \frac{2y^2}{(b^2+u)\,k} + \frac{2z^2}{(c^2+u)\,k}$$

$$= \frac{2}{k}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$= \frac{2}{k}$$

$$= nu \qquad \left[\because\ \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2} = 1\right]$$

$$\therefore\qquad k = \frac{2}{nu}$$

From (2), $u_x^2 + u_y^2 + u_z^2 = 4\dfrac{nu}{2} = 2nu$ Hence Proved.

Example 12 :

If $u = x\,\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} \quad 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Solution :

Let $u = z_1 + z_2$, where $z_1 = x\,\phi\left(\dfrac{y}{x}\right)$ and $z_2 = \psi\left(\dfrac{y}{x}\right)$.

Hence z_1 is a homogeneous function of x and y of degree 1 and z_2 is a homogeneous function of x and y of degree 0.

$$\therefore x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y} = 1 \cdot z_1 \qquad \text{and} \qquad x\frac{\partial z_2}{\partial x} + y\frac{\partial z_2}{\partial y} = 0 \quad \text{...(By Euler's theorem)}$$

Now,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}(z_1 + z_2) + y\frac{\partial}{\partial y}(z_1 + z_2)$$
$$= \left(x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y}\right) + \left(x\frac{\partial z_2}{\partial x} + y\frac{\partial z_2}{\partial y}\right)$$
$$= 1 \cdot z_1 + 0 \cdot z_2$$

Thus,
$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = z_1 \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. x and y respectively, we get,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial z_1}{\partial x} \qquad \text{...(2)}$$

and
$$x\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial z_1}{\partial y} \qquad \text{...(3)}$$

Multiplying (2) by x and (3) by y and adding, we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$
$$= x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + z_1$$
$$= 1 \cdot z_1 \qquad \left[\because x\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial u}{\partial y} = z_1 \text{ ...... By (1)}\right]$$

and
$$x\left(\frac{\partial z_1}{\partial x}\right) + y\left(\frac{\partial z_1}{\partial y}\right) = z_1$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2\,xy + \frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$
$$= 0$$

Example 13 :

If f (x, y, z) is a homogeneous function of the n^{th} degree in x, y, z prove that,

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + z^2\frac{\partial^2 f}{\partial z^2} + 2yz\frac{\partial^2 f}{\partial y\,dz} + 2zx\frac{\partial^2 f}{\partial z\,\partial x} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} = n\,(n-1)$$
f (x, y, z).

Solution :

Here f (x, y, z) is a homogeneous function of the n^{th} degee in x, y, z.

$\therefore \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are homogeneous functions of the $(n-1)^{th}$ degree in x, y, z.

So using Euler's theorem of $\frac{\partial f}{\partial x}$, we have,

$$x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = (n-1)\frac{\partial f}{\partial x}$$

$$\text{or } x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y\,\partial x} + z\frac{\partial^2 f}{\partial z\,\partial x} = (n-1)\frac{\partial f}{\partial x} \qquad \ldots(1)$$

Similarly,

$$x\frac{\partial^2 f}{\partial x\,\partial y} + y\frac{\partial^2 f}{\partial y^2} + z\frac{\partial^2 f}{\partial y\,\partial z} = (n-1)\frac{\partial f}{\partial y} \qquad \ldots(2)$$

$$\text{and } x\frac{\partial^2 f}{\partial z\,\partial x} + y\frac{\partial^2 f}{\partial y\,\partial z} + z\frac{\partial^2 f}{\partial z^2} = (n-1)\frac{\partial f}{\partial z} \qquad \ldots(3)$$

Multiplying (1) by x, (2) by y and (3) by z and adding we get,

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + z^2\frac{\partial^2 f}{\partial z^2} + 2yz\frac{\partial^2 f}{\partial y\,\partial z} + 2zx\frac{\partial^2 f}{\partial z\,\partial x} + 2xy\frac{\partial^2 f}{\partial x\,\partial y}$$

$$= (n-1)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\quad z\frac{\partial f}{\partial z}\right)$$

$$\left\{\begin{array}{l}\because \text{ Given f is a homogeneous function} \\ \text{of } n^{\text{th}} \text{ degree in x, y, z} \\ \therefore\ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf\end{array}\right\}$$

$$= (n-1)\,n\,f(x, y, z) = n(n-1)\,f(x, y, z).$$ Hence proved

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Example 14 :

If f (x, y) and ϕ (x, y) are homogeneous functions of x, y of degree p, q respectively and u = f (x, y) + ϕ (x, y) show that

$$f(x, y) = \frac{1}{p(p-q)}\left[x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{d^2 u}{\partial x\,dy} + y^2\frac{d^2 u}{dy^2}\right] - \frac{q-1}{p(p-q)}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

Solution :

Let $f(x, y) = f$ and $\phi(x, y) = \phi$

Then, $u = f + \phi$ ...(1)

Since f and ϕ are homogeneous functions of degree p and q respectively, we have,

$$x\frac{df}{dx} + y\frac{df}{dy} = p\cdot f \qquad x\frac{\partial\phi}{dx} + y\frac{\partial\phi}{dy} = q\cdot\phi$$

On Adding

$$x\left(\frac{df}{dx} + \frac{d\phi}{dx}\right) + y\left(\frac{df}{dx} + \frac{d\phi}{dy}\right) = pf + q\phi$$

i.e. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = pf + q\phi \qquad \ldots(\text{by } (1))$$

Also, $$x^2\frac{d^2 f}{dx^2} + 2xy\frac{df}{dx\,dy} + y^2\frac{d^2 f}{dy^2} = p(p-1)f$$

and $$x^2 \frac{d^2 \phi}{dx^2} + 2xy \frac{d\phi}{dx\,dy} + y^2 \frac{d^2 \phi}{dy^2} = q(q-1)\,\phi$$

So that, on ading and using equation (1), we have,

$$x^2 \frac{d^2u}{dx^2} + 2xy \frac{du}{dx\,dy} + y^2 \frac{d^2u}{dy^2} = p(p-q)\,f + q(q-1)\,\phi \qquad ...(2)$$

Hence from (1) and (2),

$$\frac{1}{p(p-q)}\left[x^2 \frac{d^2u}{dx^2} + 2xy \frac{du}{dx\,dy} + y^2 \frac{d^2u}{dy^2}\right] - \frac{q-1}{p(p-q)}\left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right]$$

$$= \frac{1}{p(p-q)}\,[p(p-1)\,f + q(q-1)\,\phi] - \frac{q-1}{p(p-q)}\,[pf + q\phi]$$

$$= \frac{1}{p(p-q)}\,[p(p-1)\,f + q(q-1)\,\phi - (q-1)\,(pf + q\phi)]$$

$$= \frac{1}{p(p-q)}\,\{\{p(p-1) - p(q-1)\}\,f + \{q(q-1)\ q(q-1)\}\,\phi\}$$

$$= \frac{1}{p(p-q)} \cdot p(p-q)\,f = f(x, y)$$

Exercise 6.3

Type 1 : Verify the Euler's theorem for the following functions

1. $u = \dfrac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{x + y + z}$

2. $u = 3x^2\,yz + 5xy^2z + 4z^4$

3. $u = \dfrac{x^2 + y^2 + z^2}{x + y + z}$

4. $u = \tan{-1}\left(\dfrac{x^2 + y^2}{y}\right)$

5. $u = \dfrac{x + y + z}{\sqrt{x} + \sqrt{y} + \sqrt{z}}$

6. $u = \log\left(\dfrac{xy + yz}{x^2 + y^2 + z^2}\right)$

Type 2 :

7. If $u = \log v$ where v is homogeneous function of degree n is x and y prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n$

8. If $u = \sin{-1}\left[\dfrac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}\right]$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{1}{20}\,\tan u$

9. If $x = eu\ \tan v$, $y = eu \sec v$ prove that $\left(x \dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right)\left(x \dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y}\right) = 0$

10. If $u = \dfrac{1}{r}\,f(\theta)$, $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(x \dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right) = -u$

Hint $u = \dfrac{1}{\sqrt{x2 + y2}}\ f\left(\tan{-1}\dfrac{y}{x} = x{-1}\ \square\ \left(\dfrac{1}{4}\right)\right)$ homogeneous function.

11. If $u = \tan{-1}\left(\dfrac{x3 + y3}{x - y}\right)$, prove that, $\left(x \dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right) = \sin 2u$

12. If $z = f(x, y)$ and u, v are homogeneous function of degree n in x, y then prove that

$$\left(x\frac{\partial z}{\partial x} + y\frac{dz}{dy}\right) = n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

Type 3 :

13. If $u = \log(x3 + y3 - x2\,y - xy2)$, prove that $x2\ uxx + 2xy + uxy + y2\ uyy = -3.$

14. If $u = \tan{-1}\ \dfrac{\sqrt{x3 + y3}}{\sqrt{x} + \sqrt{y}}$, prove that $x2\ uxx + 2xy\ uxy + y2\ uyy = -2\sin 3\ u$ cos u.

15. If $u = \left(\dfrac{x3 + y3}{y\sqrt{x}}\right) + \dfrac{1}{x7}\ \sin{-1}\left[\dfrac{x2 + y2}{x2 + 2xy}\right]$, prove that at the point (1, 2),

$$x2 + uxx + 2xy\ uxy + y2\ uyy + xux + yuy = \frac{81}{8} + \frac{49}{2}\ \square.$$

16. If $u = \sin{-1}\sqrt{x2 + y2}$, prove that $x2\ uxx + 2xy\ uxy + y2\ uyy = \tan 3u$(Dec. 1998)

17. If $y = x\cos u$, prove that $x2\ uxx + 2xy\ uxy + y2\ uyy = 0$

Hint $x0\ \cos^{-1}\left(\dfrac{y}{x}\right)$, homogeneous function of degree 0

18. If $u = \dfrac{x2 + y2}{x + y}$, prove that $x\ uxx + yuxy - u\ ux = 0.$

Hint u is homogeneous function

$\square$ $xux + yuy = u$, Differentiating partialy w.r.t. x

19. If $u = x\ \sin{-1}\left(\dfrac{y}{x}\right)$ prove that, $x2\ uxx + 2xy\ uxy + y2\ uyy = 0.$

20. If $u = \log\sin\left\{\dfrac{\square\ (2x2 + y2 + xz)^{\frac{1}{2}}}{2\ (x2 + xy + 2yz + z2)^{\frac{1}{3}}}\right\}$, Find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z$

$$\frac{d2u}{dz2}$$

when $x = 0,\ y = 1,\ z = 2.$ Ans.: $\dfrac{1}{12}\pi$

Hint $u = \log\sin v\ \square\ \dfrac{\partial u}{\partial x} = \cot v\ \dfrac{\partial v}{\partial x}$

$\dfrac{\partial u}{\partial y} = \cot v\ \dfrac{\partial u}{\partial y}$, $\dfrac{\partial u}{\partial z} = \cot v\ \dfrac{\partial v}{\partial z}$, is homogeneous function of degree $\dfrac{1}{3}$

Total Derivative :

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \varphi(t)$. Then we can express u as a function of t alone by substituting the value of x and y in $u = f(x, y)$. The ordinary differential coefficient $\dfrac{du}{dt}$ is the ordinary derivative of u with respect to t. This $\dfrac{du}{dt}$ is called the *total differential coeficient of u* or the *total derivative*.

Theorem :

If u be a composite function of t possesses continuous partial derivatives with respect to x and y and x, y possesses derivative with respect to t, then the total differential coefficient.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Note : 1. The total derivative of u with respect to t is $\frac{du}{dt}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \qquad ...(1)$$

Equation (1) can be written in terms of differentials as

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

This du is called total differential.

2. If u be a composite function of t given by the relation $u = f(x, y, z)$, $x = \phi(t)$, $y = \psi(t)$ and $z = \zeta(t)$, where u possess continuous partial derivatives w.r.t. x, y, and z and x, y, z posses derivatives w.r.t. to t total derivative

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Explicit / Implicit function

If y is directly expressed in terms of x,

i.e. $y = f(x)$, then y is called as *explicit* function of x.

If a function is expressed in terms of x and y

i.e. $f(x, y) = 0$, then f (x,y) is called as *imlicit* function of x and y.

Let $u = f(x, y) = 0$ then by the total derivative

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{\partial u}{\partial x} \cdot 1 + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \quad \text{or} \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{p}{q} \qquad ...(1)$$

where $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$

Differentiating w.r.t. x, we get,

$$\frac{d^2y}{dx^2} = -\left[\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}\right]$$

$$p = \frac{\partial f}{\partial x}$$

i.e. is also a function of x and y.

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial p}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{d^2 t}{dx^2} + \frac{\partial^2 f}{\partial y\, \partial x} \cdot \frac{\partial y}{\partial x}$$

$$= r + s\left(-\frac{p}{q}\right)$$

where $r = \frac{\partial^2 f}{\partial x^2}$ and $s = \frac{\partial^2 f}{\partial x\, \partial y}$

Similarly, $t = \frac{\partial f^2}{\partial y^2}$

$$\therefore \quad \frac{d^2 y}{dx^2} = -\left[\frac{q\left(r - \frac{ps}{q}\right) - p\left(s - \frac{pt}{q}\right)}{q^2}\right]$$

$$= -\left[\frac{q^2 r - 2\,pqs + p^2 t}{q^3}\right] = \frac{1}{q^3}\;[-q^2 r + 2\,pqs - p^2 t]$$

$$= \frac{1}{q^3}\begin{vmatrix} r & s & p \\ s & t & q \\ p & q & o \end{vmatrix}$$

Illustrative Examples

Example 1 :

If $x^y + y^x = a^b$, find $\frac{dy}{dx}$

Solution :

Let $f(x, y) = x^y + y^x - a^b$

Then we have, $f(x, y) = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + x\, y^{x-1}}$$

Example 2 :

If $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y)$, prove that $\frac{dy}{dx} = \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$

Solution :

It is given that

$$\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y)$$

Let $\quad f(x, y) = \dfrac{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}}{x-y} - a$

Then $\quad f(x, y) = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$= -\frac{\dfrac{\left\{\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right\}(x-y) - 1\left\{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}\right\}}{(x-y)^2}}{\dfrac{\left\{\frac{1}{2}(1-y^2)^{-1/2}(-2y)\right\}(x-y) - (-1)\left\{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}\right\}}{(x-y)^2}}$$

$$= -\frac{\dfrac{-x(x-y)}{\sqrt{(1-x^2)}} - \sqrt{(1-x^2)} - \sqrt{(1-y^2)}}{\dfrac{-y(x-y)}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)} + \sqrt{(1-y^2)}}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} \cdot \frac{-x^2 + xy - (1-x^2) - \sqrt{(1-x^2)} - \sqrt{(1-y^2)}}{-yx + y^2 + \sqrt{(1+x^2)}\sqrt{(1-y^2)} + 1 - y^2}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} \cdot \frac{xy - 1 - \sqrt{(1-x^2)}\sqrt{(1-y^2)}}{-xy + 1 + \sqrt{(1+x^2)}\sqrt{(1-y^2)}}$$

$$= \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$$

Hence proved

Example 3 :

Find $\dfrac{dy}{dx}$ if $(\cos x)^y = (\sin y)^x$

Solution :

Let $\quad f = (\cos x)^y - (\sin y)^x = 0$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$= -\frac{y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log \sin y}{(\cos x)^y \log \cos x - x(\sin y)^{x-1} \cos y}$$

$$= -\left[\frac{-y(\cos x)^{y-1}\sin x - (\cos x)^y \log \sin y}{(\sin y)^x \log \cos x - x\cos y(\sin y)^{x-1}}\right] \{\because (\cos x)^y = (\sin y)^x\}$$

$$= -\frac{(\cos x)^y\,[-y\tan x - \log \sin y]}{(\sin y)^x\,[\log \cos x - x\cot y]}$$

$$= +\frac{y\tan x + \log \sin y}{\log \cos x - x\cot y}$$

Example 4 :

If the curves $f(x, y) = 0$ and $\phi(x, y) = 0$ touch, show that at their point of contact

$$\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}$$

Solution :

$$m_1 = \text{slope at any point of the curve.}$$

$$f(x, y) = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$m_2 = \text{slope at any point of the curve.}$$

$$\phi(x, y) = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

At the pint of contact $m_1 = m_2$

$$-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

$$\therefore \quad \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial y}$$

Example 5 :

If $u = x\,e^y\,z$ where $y = \sqrt{a^2 - x^2}$ & $z = \sin^3 x$ find $\frac{du}{dx}$

Solution :

$$u \to x\ ,\ y\ ,\ z \text{ and } y, z \to x$$

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}$$

$$= e^y z + x\,e^y z \cdot \frac{(-2x)}{2\sqrt{a^2 - x^2}} + xe^y\ 3\sin^2 x \cos x$$

$$= e^y\ z\left[1 - \frac{x^2}{\sqrt{a^2 - x^2}} + 3x \cot x\right]$$

Example 6 :

If $u = x\log(x\,y)$ where $x^3 + y^3 + 3\,xy = 1$. Find $\frac{du}{dx}$ at $(1, 1)$

Solution : $u \to x,\ y$ and $y \to x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \log(x\,y) + \frac{x}{xy} \cdot y + \frac{x}{xy}\ x\left\{\frac{dy}{dx}\right\}$$

$$= \log(x\,y) + 1 + \frac{x}{y}\left\{\frac{dy}{dx}\right\}$$

Let $\quad f = x^3 + y^3 + 3\,xy - 1 = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{3x^2 + 3y}{3y^2 + 3x}$$

$$= -\frac{(x^2 + y)}{(y^2 + x)}$$

$$\Rightarrow \quad \frac{du}{dx} = \log(xy) + 1 - \frac{x}{y}\frac{(x^2 + y)}{y^2 + x} \qquad \text{...(from (1)}$$

At (1, 1) $\quad \frac{du}{dx} = 1 - 1 = 0$

Example 7 :

If $x^2 y - e^z + x \sin z = 0$ and $x^2 + y^2 + z^2 = a^2$, then evaluate $\frac{dy}{dx}$ and $\frac{dx}{dz}$

Solution :

Let $\quad f = x^2 y - e^z + x \sin z = 0$ and $g = x^2 + y^2 + z^2 - a^2 = 0$

From total differentials df and dg

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = 0$$

and

$$dg = \frac{\partial g}{\partial x}\, dx + \frac{\partial g}{\partial y}\, dy + \frac{\partial g}{\partial z}\, dz = 0$$

$$(2xy + \sin z)\, dx + x^2\, dy + (x \cos z - e^z)\, dz = 0$$

$$x\, dx + y\, dy + z\, dz = 0$$

By determinant method,

$$\frac{dx}{x^2 z - y(x \cos z - e^z)} = \frac{dy}{x(x \cos z - e^z) - z(2xy + \sin z)}$$

$$= \frac{dz}{y(2xy + \sin z) - x^3}$$

$$\frac{dy}{dx} = \frac{x^2 \cos z - xe^z - 2xyz - \sin z}{x^2 z - xy \cos z + ye^z}$$

$$\frac{dx}{dz} = \frac{x^2 z - xy \cos z + ye^z}{2xy^2 + y \sin z - x^3}$$

Example 8 :

If $u = \sin(x^2 + y^2)$, where $a^2 x^2 + b^2 y^2 = c^2$. find $\frac{du}{dx}$.

Solution :

We have, $\quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \qquad \text{...(1)}$

Now, $\quad u = \sin(x^2 + y^2)$

$\therefore \quad \dfrac{\partial u}{\partial x} = 2x \cos(x^2 + y^2)$

and $\quad \dfrac{\partial u}{\partial y} = 2y \cos(x^2 + y^2)$

Since $\quad a^2 x^2 + b^2 y^2 = c^2$

Differentiating w.r.t. x (ordinary differentiation).

$$2a^2 x + 2b^2 y \left(\frac{dy}{dx}\right) = 0$$

or $\quad \dfrac{dy}{dx} = -\dfrac{(a^2 x)}{(b^2 y)}$

$\therefore$ From (1) $\quad \dfrac{du}{dx} = 2x \cos(x^2 + y^2) - [2y \cos(x^2 + y^2)]\left[-\dfrac{(a^2 x)}{(b^2 y)}\right]$

Example 9 :

If $x^n + y^n = a^n$ then prove that $\dfrac{d^2y}{dx^2} = -(n-1)\, a^n \dfrac{x^{n-2}}{y^{2n-1}}$

Solution : Let $\quad f = x^n + y^n - a^n = 0$

$$p = \frac{\partial f}{\partial x} = nx^{n-1}$$

$$q = \frac{\partial f}{\partial y} = ny^{n-1}$$

$$r = \frac{\partial^2 f}{\partial x^2} = n(n-1)\, x^{n-2}$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 0$$

$$t = n(n-1)\, y^{n-2}$$

$$\frac{d^2y}{dx^2} = -\frac{p^2 t - 2pqs + q^2 r}{q^3}$$

$$= -\frac{n^2 x^{2n-2}\, n(n-1)\, y^{n-2} + n^2 y^{2n-2}\, n(n-1)\, x^{n-2}}{n^3 y^{3n-3}}$$

$$= -\frac{n^3 (n-1)\, x^{n-2} y^{n-2} (x^n + y^n)}{n^3 y^{3n-3}}$$

$$= -\frac{a^n (n-1)\, x^{n-2}}{y^{2n-1}}$$

Hence proved

Example 10 :

If $(\tan x)^y + y^{\cot x} = a$, find $\dfrac{dy}{dx}$.

Solution :

Let $\quad f(x, y) = (\tan x)^y + y^{\cot x} - a = 0$

Then, $\quad \left(\dfrac{\partial f}{\partial x}\right) = y\, (\tan x)^{y-1} \sec^2 x + y^{\cot x} \log y\, (-\operatorname{cosec}^2 x)$

and $\left(\frac{\partial f}{\partial y}\right) = (\tan x)^y \ \log \tan x + (\cot x)\ y^{\cot x - 1}$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{y\,(\tan x)^{y-1}\sec^2 x - y^{\cot x}\log y\ \mathrm{cosec}^{2x}}{(\tan x)^y \log \tan x + \cot x \cdot y^{\cot x - 1}}$$

Example 11 :

If $x^m + y^m = a^m$ show that $\frac{d^2y}{dx^2} = -(m-1)\ a^m \cdot \frac{x^{m-2}}{y^{m-1}}$

Solution :

Let $f(x, y) = x^m + y^m - a^m = 0$

$$p = \frac{\partial f}{\partial x} = mx^{m-1}, \qquad q = \frac{\partial f}{\partial y} = my^{m-1}$$

$$r = \frac{\partial^2 f}{\partial x^2} = m\,(m-1)\ x^{m-2}, \qquad s = \frac{\partial^2 f}{\partial x\,\partial y} = 0,$$

and $t = \frac{\partial^2 f}{\partial y^2} = m\,(m-1)\ x^{m-2}$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{q^2 r - 2\,pqs + p^2 t}{q^3}$$

$$= -\frac{m^2 y^{2m-2}\, m\,(m-1)\ x^{m-2} + m^2 x^{2m-2} \cdot m\,(m-1)\,y^{m-2}}{m^3\, y^{3m-3}}$$

$$= -\frac{m^3\,(m-1)\ x^{m-2}\, y^{m-2}\,(y^m + x^m)}{m^3\, y^{3m-3}}$$

$$\frac{d^2y}{dx^2} = -\frac{(m-1)\ x^{m-2}\ y^{m-2}\, a^m}{y^{3m-3}}$$

$$= -(m-1)\ a^m\ \frac{x^{m-2}}{y^{2m-1}}$$

Exercise 6.4

1. If $x^4 + y^4 + 4a^2 xy = 0$, prove that $(y^3 + a^2 x)\ \frac{d^2y}{dx^2} = {}^2a^2 xy\ (x^2 y^2 + {}^3a4)$

2. If $u = \sin \frac{x}{y}$ and $x = et$, $y = t^2$ verify that $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

3. If $x^3 + y^3 = {}^3ax^2$ prove that $\frac{d^2y}{dx^2} = -\frac{{}^2a^2 x^2}{y5}$

4. If $\cos y = \frac{{}^3 + 5\cos x}{5 + {}^3\cos x}$, prove that $\frac{dy}{dx} = \frac{4}{5 + {}^3\cos x}$

5. If $z = x \log xy$ and $x^3 + y^3 + {}^3xy = 0$ prove that $\frac{du}{dx} = \log xy + \frac{y^3 - x^3}{y\,(y^2 + x)}$

6. If $u = \sin(x^2 + y^2)$ where $a^2 x^2 + b^2 y^2 = c$, prove that $\frac{du}{dx}\ {}^2\left(1 - \frac{a^2}{b^2}\right)$ x cos $(x^2 + y^2)$

7. If $x^2 y^2 = -\sin y$, prove that $\frac{dy}{dx} = -\left[\frac{^2xy4}{4x^2 y^3 + \cos y}\right]$

Change of Independent Variables

If $u = f(x, y)$

where $x = f_1(r, \theta)$ and $y = f_2(r, \theta)$

then $u \Rightarrow x, y \rightarrow r, \theta$

By the chain rule differential w.r.t. r and θ partially, we get,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad ...(1)$$

or

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \quad ...(2)$$

We solve equation (1) and (2) simultaneously

We get the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in terms of $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$

Differentiating equations (1) and (2) w.r.t. r and θ partially, we get,

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right)\frac{\partial y}{\partial r}$$

and

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial y}{\partial \theta}$$

If $u = f(r, \theta)$

where $r = f_1(x, y)$ and $\theta = f_2(x, y)$

i.e. $u \rightarrow x, y \rightarrow r, \theta$

then we get, $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad ...(3)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \quad ...(4)$$

We solve equation (3) and (4) simultaneously, we get, $\frac{\partial x}{\partial r}$ and $\frac{\partial y}{\partial r}$ in terms of $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$. Similarly, we get teh higher order partial derivatives of u.

i.e. $\frac{\partial^2 u}{\partial r^2}$, $\frac{\partial^2 u}{\partial \theta^2}$ or $\frac{d^{2u}}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ etc.

Illustrative Examples

Example 1 :

If $u = x^2 + y^2$, $v = 2xy$ and $z = f(u, v)$, show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2 - v^2}\,\frac{\partial z}{\partial u}$$

Solution :

Given $z = (u, v)$

$\therefore \quad z \to u, v \to x, y$

where $\quad u = x^2 + y^2$ and $v = 2xy$

$\therefore \quad \dfrac{\partial u}{\partial x} = 2x$ and $\dfrac{\partial u}{\partial y} = 2y.$

$\therefore \quad \dfrac{\partial v}{\partial x} = 2y$ and $\dfrac{\partial v}{\partial y} = 2x$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= 2\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$$

Multiplying it by x, we get,

$$x\frac{\partial z}{\partial u} = 2\left(x^2\frac{\partial z}{\partial u} + xy\frac{\partial z}{\partial v}\right) \qquad ...(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= 2\left(y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)$$

Multiplying it by y, we get,

$$y\frac{\partial z}{\partial y} = 2\left(y^2\frac{\partial z}{\partial u} + xy\frac{\partial z}{\partial v}\right) \qquad ...(2)$$

Substracting (2) from (1), we get,

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2 - y^2)\frac{\partial z}{\partial u} \qquad ...(3)$$

But, $\quad x^2 - y^2 = \sqrt{(x^2 - y^2)^2}$

$$= [(x^2 + y^2)^2 - 4x^2y^2]^{\frac{1}{2}}$$

$$= [u^2 - v^2]^{\frac{1}{2}}$$

$\therefore$ From (3)

$$\text{LHS} = 2\,[u^2 - v^2]^{\frac{1}{2}}\frac{\partial z}{\partial u} = \text{RHS}$$

Example 2 :

If $z = f(x,y)$, $u = e^x$ & $v = e^y$, the show that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial v \partial v}$

Solution : Given $z \to x, y \to u, v$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\, e^x + \frac{\partial z}{\partial v}(0)$$

$$= \frac{\partial z}{\partial u}\, e^x$$

$$= \frac{\partial z}{\partial u} \; u$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial u}\left(u\,\frac{\partial z}{\partial u}\right)(0) + \frac{\partial}{\partial v}\left(u\,\frac{\partial z}{\partial u}\right) e^y$$

$$= u\,\frac{\partial^2 z}{\partial v\,\partial u}\; v$$

$$= uv\,\frac{\partial^2 z}{\partial u\,\partial v}$$

Hence proved

Example 3 :

By changing the independent variables u and v to x and y by means of the relations -

$x = u\cos\alpha - v\sin\alpha,\; y = u\sin\alpha + v\cos\alpha$

show that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ transforms into $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

or if $x = a\cos\alpha - V\sin\alpha,\; y = u\sin\alpha + v\cos\alpha$

then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$

Solution :

Given x and y in terms of u and v.

Thus, z is composite function of u and v.

i.e. $z \to x, y \to u, v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \cos\alpha\,\frac{\partial z}{\partial x} + \sin\alpha\,\frac{\partial z}{\partial y}$$

Here we use equivalent operator

(i.e. if $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ then $\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(z)$

$\therefore \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$

$\frac{\partial}{\partial x}$ is called equivalent operator.)

$$\frac{\partial}{\partial u}(z) = \cos\alpha\,\frac{\partial}{\partial x}(z) + \sin\alpha\,\frac{\partial}{\partial y}(z)$$

i.e. $$\frac{\partial}{\partial u}(z) = \left[\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right](z)$$

$\therefore$ $$\frac{\partial}{\partial u} = \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y} \qquad ...(1)$$

Also, $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= -\sin\alpha \frac{\partial z}{\partial x} + \cos\alpha \frac{\partial z}{\partial y}$$

or $$\frac{\partial}{\partial y}(z) = -\sin\alpha \frac{\partial}{\partial y}(z) + \cos\alpha \frac{\partial}{\partial y}(z)$$

$\therefore$ $$\frac{\partial}{\partial y} = -\sin\alpha \frac{\partial}{\partial x} + \cos\frac{\partial}{\partial y} \qquad \text{...(2)}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$

From the equivalent operator $\frac{\partial}{\partial u}$ i.e. form equation (1)

$$\frac{\partial^2 z}{\partial u^2} = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right)$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2} \qquad \text{...(3)}$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) \qquad \text{...from equation 2)}$$

$$= \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\,\partial y} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2} \qquad \text{...(4)}$$

Adding (3) and (4), we get,

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

Hence proved.

Example 4 :

If $z = f(x, y)$ and $u = l\,x + my$; $v = lx - my$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

Solution : Given $z \to u, v \to x, y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\,l + \frac{\partial z}{\partial v}(-m)$$

$\therefore$ $$\frac{\partial}{\partial x}(z) = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)(z)$$

By the equivalent operator

$$\frac{\partial}{\partial x} = l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v} \qquad \text{...(1)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial x}\right)\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$= l\frac{\partial}{\partial u}\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right) - m\frac{\partial}{\partial v}\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$= l^2\frac{\partial^2 z}{\partial u^2} - 2lm\frac{\partial^2 z}{\partial u\,\partial v} + m^2\frac{\partial^2 z}{\partial v^2} \quad ...(2) \quad \left[\because \frac{\partial^2 z}{\partial u\,\partial v} = \frac{\partial^2 z}{\partial v\,\partial u}\right]$$

Also, $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}$$

$$= \left(m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right)(z)$$

$$\therefore \quad \frac{\partial}{\partial y} = m\frac{\partial}{\partial u} + l\frac{\partial z}{\partial v} \quad ...(3)$$

Now, $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]$$

From equation (3)

$$\frac{\partial^2 z}{\partial y^2} = \left[m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}\right]\left[m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right]$$

$$= m\frac{\partial}{\partial u}\left[m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right] + l\frac{\partial}{\partial v}\left[m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right]$$

$$= m^2\frac{\partial^2 z^2}{\partial u^2} + 2lm\frac{\partial^2 z}{\partial u\,\partial v} + l^2\frac{\partial^2 z}{\partial v^2}$$

Adding (3) and (4), we get,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y} = (l^2 + m^2)\frac{\partial^2 z}{\partial u^2} + (m^2 + l^2)\frac{\partial^2 z}{\partial v^2}$$

$$= (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Hence proved

Example 5 :

If $u = x + ay$, $v = x + ay$ change $2f_{xx} - 5f_{xy} + 3f_{yy} = 0$ into $f_{uv} = 0$. Find a and b.

Solution : Given $u = x + ay$ $\quad v = x + by$

$$\therefore \quad \frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a \quad \frac{\partial v}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = b$$

Let $f = \phi(u, v)$

$\therefore\ f \to u, v \to x, y$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial y}{\partial x}$$

$$= \frac{\partial f}{\partial u} + \frac{df}{dv}$$

$\therefore \qquad \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}$

Also, $\qquad \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y}$

$$= \frac{\partial f}{\partial u} a + \frac{\partial f}{\partial v} b$$

$$\therefore \qquad \frac{\partial}{\partial y} = a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v} \qquad ...(2)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) \qquad ...(\text{From } (1)$$

$$= \frac{\partial^2 f}{\partial u^2} + \frac{2\partial^2 f}{\partial u\, \partial v} + \frac{\partial f^2}{\partial v^2} \qquad \because \frac{\partial^2 f}{\partial u\, \partial v} = \frac{\partial^2 f}{\partial v\, \partial u}$$

$$= f_{uu} + 2 f_{uv} + f_{vv}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

From (1) and (2)

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{a\partial f}{\partial u} + \frac{b\partial f}{\partial v}\right)$$

$$= af_{ua} + (a+b)\ f_{uv} + b\, f_{vv} \qquad ...(4)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}\right)$$

$$= a^2\ f_{au} + 2\, ab\, f_{uv} + b^2 f_{vv} \qquad ...(5)$$

Multiplying equation (3) by 2, equation (4) by – 5 and equation (5) by 3 and adding , we get,

$$2\, f_{xx} - 5\, f_{xy} - 3\, f_{yy} = 2\,[\, f_{uu} + 2\, f_{uv} + f_{vv}] - 5\,[a\, f_{uu} + (a+b)\ f_{uv} + b\, f_{vv})\,]$$
$$+ 3\,[a^2 f_{uu} + 2\, abf_{uv} + b^2 f_{vy}]$$
$$= (3a^2 - 5a + 2)\ f_{uu} + (6ab - 5a - 5b + 4)\ f_{uv} + (3b^2 - 5b + 2)\ f_{uv}$$

But given that the equation

$$2\, f_{xx} - 5\, f_{xy} + 3\, f_{yy} = 0$$

transform into $\qquad f_{uv} = 0$

i.e. $(3a^2 - 5a + 2)\ f_{uu} + (6ab - 5a - 5b + 5)\ f_{uv} + (3b^2 - 5b + 2)\ f_{vv} = 0 = fuv$

Comparing the coefficient of f_{uu} , f_{uv} , f_{vv} we get,

$3a^2 - 5a + 2 = 0 \qquad$ and $\qquad 3b^2 - 4b + 2 = 0$

$\therefore (a-1)\ (3a-2) = 0 \qquad\qquad (b-1)\ (3b-2) = 0$

$\therefore \quad a = 1, \frac{2}{3} \quad$ and $\quad b = 1, \frac{2}{3}$

The values $\quad a = 1, \; b = 1,$ we get,

$u = x + y \quad$ and $\quad v = x + y$

$\therefore \quad u = v$

Similarly, $\quad$ if $a = \frac{2}{3}$, $b = \frac{2}{3}$

we get $u = v$

$\therefore$ Rejecting these pairs, we have,

$a = 1$ and $b = \frac{2}{3} \quad$ or $\quad a = \frac{2}{3}$ and $b = 1$

Example 6 :

If $z = f(x, y)$, $x = u + v$, $y = uv$ prove that $\frac{\partial^2 z}{\partial x^2} - y \frac{\partial^2 z}{\partial y^2} = \frac{1}{u-v}\left[u \frac{\partial^2 z}{\partial u^2} - v \frac{\partial^2 z}{\partial v^2}\right]$

Solution :

Given, $\quad z = f(x, y)$

$x = u + v$

$y = uv$

i.e. $z \to x, y \to u, v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{dz}{dy}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\, 1 + \frac{dz}{dy}\, v \qquad \ldots(1)$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left[\frac{\partial z}{\partial u}\right]$$

$$= \frac{\partial}{\partial u}\left[\frac{\partial z}{\partial x} + v\frac{dz}{dy}\right] \qquad \ldots\text{(From (1))}$$

$$= \frac{\partial}{\partial u}\left[\frac{\partial z}{\partial x}\right] + \frac{\partial}{\partial u}\left[v\frac{dz}{dy}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{\partial y}{\partial u} + v\left[\frac{\partial}{\partial x}\left(\frac{dz}{dy}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{dz}{dy}\right)\frac{\partial y}{\partial u}\right]$$

$$= \frac{\partial^2 z}{\partial x^2}\, 1 + \frac{\partial^2 z}{\partial y\,\partial x}\, v + v\left[\frac{\partial^2 z}{\partial x\,\partial y}\, 1 + \frac{\partial^2 z}{\partial y^2}\, v\right]$$

$$= \frac{\partial^2 z}{\partial x^2} + 2v\frac{\partial^2 z}{\partial x\,\partial y} + v^2\frac{\partial^2 z}{\partial x\,\partial y} \qquad \ldots(2)$$

Also $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{dz}{dy}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x} 1 + \frac{dz}{dy} u$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left[\frac{\partial z}{\partial x} + u\frac{dz}{dy}\right]$$

$$= \frac{\partial}{\partial v}\left[\frac{\partial z}{\partial x}\right] + u\frac{\partial}{\partial v}\left[\frac{dz}{dy}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial v}$$

$$+ u\left[\frac{\partial}{\partial x}\left(\frac{dz}{dy}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{dz}{dy}\right)\frac{\partial y}{\partial v}\right]$$

$$= \frac{\partial^2 z}{\partial x^2} 1 + \frac{\partial^2 z}{\partial y\,\partial x} u + u\left[\frac{\partial^2 z}{\partial x\,\partial y} 1 + \frac{\partial^2 z}{\partial y^2} u\right]$$

$$= \frac{\partial^2 z}{\partial x^2} + 2u\frac{\partial^2 z}{\partial x\,\partial y} + u^2\frac{\partial^2 z}{\partial y^2} \qquad \ldots(3)$$

Multiplying equation (2) by u, equation (3) by v and subtracting, we get,

$$u\frac{\partial^2 z}{\partial u^2} - v\frac{\partial^2 z}{\partial v^2} = (u-v)\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}(vu^2 - u^2 v)$$

$$= (u-v)\left[\frac{\partial^2 z}{\partial x^2} - uv\frac{\partial^2 z}{\partial y^2}\right]$$

$$= (u-v)\left[\frac{\partial^2 z}{\partial x} - y\frac{\partial^2 z}{\partial y^2}\right]$$

Divide by $(u-v)$, we get,

$$\frac{1}{(u-v)}\left[u\frac{\partial^2 z}{\partial u^2} - v\frac{\partial^2 z}{\partial v^2}\right] = \frac{\partial^2 z}{\partial u^2} - y\frac{\partial^2 z}{\partial y^2}$$

Hence proved

Exercise 6.5

1. If $x + y = 2e^{\theta}\cos\phi$, $x - y = 2ie^{\theta}\sin\phi$ prove that $\frac{\partial^2 z}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = {}^4xy\frac{\partial^2 u}{\partial x\,\partial y}$

2. Using transformation $x = r\cos\theta$, $y = r\sin\theta$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ gets transformed into $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} + \frac{\partial u}{\partial r^2}\frac{\partial^2 u}{\partial^2 \theta} = 0$

3. $x =$ ev sec u, $y =$ ev tan u and $\phi = f(x, y)$, then show that
$$\cos u\left(\frac{\partial^2 \phi}{\partial u\,\partial v} - \frac{\partial \phi}{\partial u}\right) = xy\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + (x^2 + y^2)\frac{\partial^2 \phi}{\partial x\,\partial y}$$

4. Find the values of the constants a and b such that the substitution $u = x + ay$ and $v = x + by$ transform the equation
$9\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial x\,\partial y} + 2\frac{\partial^2 f}{\partial y^2} = 0$ into $\frac{\partial^2 f}{\partial u\,\partial v} = 0$ where f is a function of x and y.

5. If $u = f(x, y)$ where $x = \xi \cos\theta - \eta \sin\theta$, $y = \xi \sin\theta + \eta \cos\theta$

then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 u}{\partial \eta^2}$

Laplace Operator

In physics a function $f(x, y, z)$ satisfies the most important partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \qquad \ldots(1)$$

This equation is called *Laplace's equation.*

In equation (1) the expression $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ is called the *Laplacian* of f and is denoted by $\nabla^2 f$ or Δf.

The differential operator $\nabla^2 = \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ (real "Nabla squared" or "delta") is called the *Laplace operator.*

Using this operator, we may write equation (1) in the form of $\nabla^2 f = 0$.

Illustrative Examples

Example 1 :

if $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$

$$\nabla^2 u = f''(r) + \frac{2}{r} f(r)$$

Solution : Given $u \to r \to x, y, z$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$

$$= f'(r)\left[\frac{r - x\dfrac{\partial r}{\partial x}}{r^2}\right] + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}.$$

$$= f'(r)\left[\frac{r^2 - x^2}{r^3}\right] + \frac{x^2}{r^2} f''(r) \qquad \ldots(1)$$

Similarly, by symmetry,

$$\frac{\partial^2 u}{\partial y^2} = f'(r)\left[\frac{r^2 - y^2}{r^3}\right] + \frac{y^2}{r^2} f''(r) \qquad \ldots(2)$$

and

$$\frac{\partial^2 u}{\partial z^2} = f'(r)\left[\frac{r^2 - z^2}{r^3}\right] + \frac{z^2}{r^2} f''(r) \qquad \ldots(3)$$

Adding (1) (2) and (3), we get,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{f'(r)}{r^3}[3r^2 - (x^2+y^2+z^2)] + \frac{f''(r)}{r^2}(x^2+y^2+z^2)$$

$$= 2\frac{f'(r)}{r} + f''(r) = f''(r) + \frac{2}{r}f'(r)$$ Hence proved.

Example 2 :

Prove that $\nabla^4 e^r = e^r + \frac{4}{r} e^r$

Solution : Let $f(r) = e^r$

$\therefore \quad f'(r) = e^r \quad$ and $\quad f''(r) = e^r$

But $$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r) \quad ...(1)$$

$$\nabla^2 f(r) = e^r + \frac{2}{r} e^r = e^r\left[1 + \frac{2}{r}\right]$$

Let $$F(r) = e^r\left[1 + \frac{2}{r}\right]$$

$$F'(r) = e^r\left[1 + \frac{2}{r}\right] + e^r\left[-\frac{2}{r^2}\right] \quad ...(2)$$

and $$F''(r) = e^r\left[1 + \frac{2}{r}\right] + e^r\left[-\frac{2}{r^2}\right] + \frac{4}{r^3}e^r - \frac{2}{r^2}e^r$$

$$= e^r\left[1 + \frac{2}{r} - \frac{2}{r^2} - \frac{2}{r^2} + \frac{4}{r^3}\right]$$

$$= e^r\left[1 + \frac{2}{r} - \frac{4}{r^2} + \frac{4}{r^3}\right] \quad ...(3)$$

From equation (2)

$$\frac{2}{r}F(r) = e^r\left[\frac{2}{r} + \frac{4}{r^2} - \frac{4}{r^3}\right] \quad ...(4)$$

$$\nabla^4 e^r = \nabla^2 \nabla^2 e^r = \nabla^2 F(r)$$

From equation (1),

$$\nabla^2 F(r) = f''(r) + \frac{2}{r} f'(r)$$

From equation (3) and (4),

$$\nabla F(r) = e^r\left[1 + \frac{2}{r} - \frac{4}{r^2} + \frac{4}{r^3} + \frac{2}{r} + \frac{4}{r^2} - \frac{4}{r^3}\right]$$

$$= e^r\left[1 + \frac{4}{r}\right]$$

$$= e^r + \frac{4}{r} e^r$$

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