Bansilal Ramnath Agarwal Charitable Trust's

VISHWAKARMA INSTITUTE OF INFORMATION TECHNOLOGY

Department of Engineering & Applied Sciences

F.Y.B.Tech

Course material (A brief reference version for students)

Course: Engineering Mathematics - I

Unit 3: Infinite Series andFourier Series

Infinite Series

An Expression of the Form

$$u_1 + u_2 + u_3 + \dots\dots + u_n + \dots\dots \quad ....(1)$$

where each term is followed by another term following some definite rule is called a series. When the series contains an unlimited number of terms it is called an infinite series. When the number of terms is finite, it is called a finite series.

An infinite series given by (1) is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

The study of finite series presents no problem. We can find out all the terms and find their sum if necessary. The behaviour of an infinite series is however a matter of interesting study and the problem is to find whether the series is convergent and possesses a finite sum or whether it is divergent. The sum of an infinite series is to be defined first and then we have to find whether it possesses a sum according to the accepted definition.

Sequence of Partial Sums

Sum of first n terms of the series $\sum_{n=1}^{\infty} u_n$ is denoted by

$S_n = u_1 + u_2 + u_3 + \dots + u_n$, called as n^{th} partial sum of $\sum_{n=1}^{\infty} u_n$.

i.e. if $S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

$\vdots \qquad \vdots$

$S_n = u_1 + u_2 + u_3 + \dots + u_n$ and so on

Then $\{S_1, S_2, S_3, \dots\dots.., S_n \dots\dots\}$ is called as the sequence of partial sums

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CONVERGENT, DIVERGENT AND OSCILLATORY SERIES

Let $\sum_{n=1}^{\infty} u_n$ be a given infinite series of real numbers and $\{S_1, S_2, S_3, \dots\dots.., S_n \dots\dots\}$ be a sequence of its partial sums. Then series $\sum_{n=1}^{\infty} u_n$ is said to be:

1. Convergent if $\lim\limits_{n\to\infty} S_n$ is finite.
2. Divergent if $\lim\limits_{n\to\infty} S_n$ is $\pm\infty$.
3. Oscillatory if $\lim\limits_{n\to\infty} S_n$ is not unique i.e limit does not exist i.e. sequence $<S_n>$ of its partial sums is oscillatory.

Example 1:

Consider the series, $\sum_{n=1}^{\infty} u_n$ where $u_n = \dfrac{1}{n(n+1)}$. Here the corresponding sequence (S_n) of the partial sums can be obtained as follows.

$S_n = u_1 + u_2 + u_3 + \dots + u_n$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \frac{1}{n+1}\right)$$

Clearly, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$

Sequence of partial sums (S_n) converges to 1. Therefore the given series is convergent to the sum 1.

Example 2:

Consider the series 1 + 3 + 5 + ...

We have $S_n = 1 + 3 + 5 + ... + (2n - 1) = n^2$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} = \infty$$

$$\therefore S = (S_n) \text{ is divergent}$$

The given 1 + 3 + 5 + ... is divergent

Example 3:

Let $\sum_{n=1}^{\infty} u_n$, where $u_n = (-1)^n$

We observe that,

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = 1 - 1 + 1 - 1 + \cdots + (-1)^n$$

$$S_n = \begin{cases} 0, & \text{if n is even} \\ -1, & \text{if n is odd} \end{cases}$$

$$\therefore \lim_{n\to\infty} S_n = \begin{cases} 0, & \text{if n is even} \\ -1, & \text{if n is odd} \end{cases}$$

Sequence of partial sums (S_n) of given series is oscillatory.

Therefore $\sum_{n=1}^{\infty} u_n$ is oscillatory.

Properties of Infinite Series

1.The removal or addition of a finite number of terms of at the beginning of an infinite series does not affect the convergence or divergence of a series.

2.The multiplication all the terms of a series by fixed nonzero real number has no effect on its convergence or divergence or oscillation.

3.If the terms of a convergent series are grouped in parenthesis, in any manner to form new terms new terms, then the resulting series will convergence to the same sum.

GEOMETRIC SERIES

A series of the form $\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \ldots + a^{n-1} + \cdots$ is called as geometric series where a is common ratio.

Geometric series $1 + a + a^2 + a^3 + \ldots + a^{n-1} + \cdots$ (a is common ratio)

(i) converges if $|a| < 1$

(ii) diverges if $a > 1$

(iii) oscillates finitely when $a = -1$

(iv) oscillates infinitely when $a < -1$.

Proof: (i) When $|a| < 1$.

In this case we have, $S_n = 1 + a + a^2 + a^3 + \ldots + a^{n-1} = \frac{1-a^n}{1-a}$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-a^n}{1-a}$$

$$= \lim_{n\to\infty} \frac{1}{1-a} - \lim_{n\to\infty} \frac{a^n}{1-a}$$

$$= \frac{1}{1-a} \qquad (\because |a| < 1 \Rightarrow \lim_{n\to\infty} a^n = 0)$$

In this case (S_n) is convergent and therefore, the geometric series is convergent.

(ii) When $a > 1$.

In this case $S_n = \frac{a^n - 1}{a-1}$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a^n - 1}{a-1} = \infty$$

$(\because |a| > 1 \Rightarrow \lim_{n\to\infty} a^n = \infty)$

$\therefore$ when $a > 1$, (S_n) is divergent. Therefore the geometric series is divergent when $a > 1$.

When $a = 1$

In this case : $S_n = 1 + 1 + 1 + \ldots + 1$ (n terms)

$\Rightarrow \quad S_n = n$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

$\therefore$ (S_n) is divergent when $a = 1$ and the geometric series divergent when $a = 1$.

Therefore, the geometric series is divergent when $a \geq 1$

(iii) When $a = -1$, We have

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = 1 - 1 + 1 - 1 + \cdots + (-1)^n$$

$$S_n = \begin{cases} 0, & \text{if n is even} \\ -1, & \text{if n is odd} \end{cases} \qquad \therefore \lim_{n\to\infty} S_n = \begin{cases} 0, & \text{if n is even} \\ -1, & \text{if n is odd} \end{cases}$$

Sequence of partial sums (S_n) of given series is oscillatory.

Therefore $\sum_{n=1}^{\infty} u_n$ is oscillatory.

$Sn = 1 - 1 + 1 - 1 + \ldots$ to n terms

$$\Rightarrow S_n = \begin{cases} 1 \text{ when n is odd} \\ 0 \text{ when n is} \end{cases}$$

$\therefore$ $\lim S_n = 1$ or 0

$\therefore$ (S_n) oscillates finitely.

$\therefore$ the geometric series is divergent when $a = -1$.

(iv) when $a < -1$. Let $r = -a$ then $a < -1 \Rightarrow -a > 1 \Rightarrow r > 1$

$\therefore$ $S_n = 1 + a + a^2 + \ldots + a^{n-1}$

$$= \frac{1-a^n}{1-a} = \frac{1-(-r)^n}{1-(-r)} \qquad (\because a = -r)$$

$\therefore$ $S_n = \dfrac{1-r^n}{1+r}$ when n is even

$= \dfrac{1+r^n}{1-r}$ when n is odd

Now r $>$ 1 $\Rightarrow$ $\lim\limits_{n\to\infty} r^n = \infty$

$$\lim_{n\to\infty} S_n = \begin{cases} -\infty & \text{when n is even} \\ \infty & \text{when n is odd} \end{cases}$$

Thus (S_n) oscillates infinitely.

$\therefore$ the Geometric series given oscillates infinitely when a < -1.

Example 1:

Consider the series $\sum \dfrac{1}{2^n}$. This is a geometric series with common ratio $\dfrac{1}{2} < 1$.

$\therefore$ $\sum \dfrac{1}{2^n}$ is convergent

Example 2: The geometric series $1-\dfrac{1}{3}+\dfrac{1}{3^2}-\dfrac{1}{3^3}+\ldots$ is convergent.

Since the common ratio a $= -\dfrac{1}{3} \Rightarrow |\,a\,| = \dfrac{1}{3} < 1$

$\therefore$ $\sum \dfrac{1}{3}$ is convergent.

P-Series

A series of the form $\sum \dfrac{1}{n^p} = \dfrac{1}{1^p}+\dfrac{1}{2^p}+\dfrac{1}{3^p}+\ldots\ldots+\dfrac{1}{n^p}+\ldots$ is called as p-series.

p-series (i) is convergent if p > 1 (ii) divergent if p ≤ 1.

Example 1: Examine the convergene of the series

$$1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\ldots$$

Solution : We have

$$\sum u_n = 1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\ldots$$

$$= 1+\frac{1}{(2^2)^{2/3}}+\frac{1}{(3^2)^{2/3}}+\frac{1}{(4^2)^{2/3}}+\ldots$$

$$= \frac{1}{1^{4/3}}+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{2/3}}+\ldots$$

$$= \sum_{1}^{\infty}\frac{1}{n^{4/3}} \text{ is of the form } \sum\frac{1}{n^p} \text{ where } p = \frac{4}{3}.$$

$\therefore$ Since p > 1 , the given series is convergent.

Example 2: Test the convergence of the series $\sum \frac{1}{\sqrt{n}}$

Solution :

We have $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^p}$ where p = $\frac{1}{2} < 1$

$\therefore \quad \sum \frac{1}{\sqrt{n}}$ is divergent.

Note: The Harmonic series $\sum \frac{1}{n}$ is divergent.

Zero Test (NECESSARY CONDITION FOR CONVERGENCE OF A SERIES)

If series of positive terms $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim\limits_{n\to\infty} u_n = 0$.

Note : $\lim\limits_{n\to\infty} u_n \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ is always divergent.

Example:

Consider the series

$$\sum_{n=1}^{\infty} u_n = \sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \ldots + \sqrt{\frac{n}{2(n+1)}} + \ldots$$

We have

$$u_n = \sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}} \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\left(\frac{n+1}{n}\right)^{1/2}} = \frac{1}{\sqrt{2}} \frac{1}{\left(1+\frac{1}{n}\right)^{1/2}}$$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2}} \left[\frac{1}{\left(1+\frac{1}{n}\right)^{1/2}}\right] = \frac{1}{\sqrt{2}} \neq 0$$

$\therefore \sum_{n=1}^{\infty} u_n$ is divergent.

Comparison Test:

If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two series of positive terms such that $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is finite and nonzero, then both the series behave alike i.e. both are either convergent or divergent.

Note: (i) Use this test when u_n contains powers of n (which may be positive or negative, integral or fractional).

Choose $v_n = \frac{\text{highest degree term in numerator of } u_n}{\text{highest degree term in denominator of } u_n}$.

(ii) Also use this test when u_n can be expanded in ascending powers of $\frac{1}{n}$.

In that case, choose $v_n =$ lowest power of $\frac{1}{n}$

Method:

(1) Suppose $\sum_{n=1}^{\infty} u_n$ be a given series.

(2) Choose $v_n = \frac{\text{highest degree term in numerator of } u_n}{\text{highest degree term in denominator of } u_n}$.

(3) Find $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$. This limit must be finite and non-zero.

$\therefore \sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converges together or diverges together.

(4) Compare $\sum_{n=1}^{\infty} v_n$ with p series to decide it's convergence or divergence and identify p.

$\therefore \sum_{n=1}^{\infty} v_n$ convergent $\Rightarrow \sum_{n=1}^{\infty} u_n$ convergent

and $\sum_{n=1}^{\infty} v_n$ divergent $\Rightarrow \sum_{n=1}^{\infty} u_n$ divergent.

Example 1: Test for the convegence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Solution :

We have $u_n = \frac{\sqrt{n}}{n^2+1}$

Let $v_n = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}$

$$\therefore \frac{u_n}{v_n} = \frac{\left(\frac{\sqrt{n}}{n^2+1}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \frac{n^{3/2}\sqrt{n}}{n^2+1} = \frac{n^2}{n^2+1}$$

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1 \neq 0 \text{ (a finite number)}$$

$\therefore$ By comparision test , $\sum u_n$ and $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{n^{3/2}}$ is of the form $\sum \frac{1}{n^p}$ where p = $\frac{3}{2} > 1$

$\Rightarrow \sum v_n$ is convergent

$\Rightarrow \sum u_n$ is convergent.

Example 2: Test for the convergence of the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ... + \frac{2n+1}{n(n+1)(n+2)} + ...$

Solution :

We have $u_n = \frac{2n+1}{n(n+1)(n+2)}$

Let $v_n = \frac{n}{n^3} = \frac{1}{n^2}$

Then $\frac{u_n}{v_n} = \frac{(2n+1)/n(n+1)(n+2)}{1/n^2} = \frac{n^2(2n+1)}{n(n+1)(n+2)}$

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\frac{n^2(2n+1)}{n(n+1)(n+2)} = \lim_{n\to\infty}\frac{2+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2 \neq 0$$

$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is a finite and non-zero quantity.

$\therefore$ By comparision test , $\sum u_n$ and $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{n^2}$ is in the form $\sum \frac{1}{n^p}$, where p = 2 > 1

$\therefore$ $\sum v_n$ is convergent. Therefore, $\sum u_n$ is also convergent

Example 3: Test the convergence of $\sum u_n$ where $u_n = \sqrt{\frac{n^2+n-1}{n^3-2}}$

Solution : We have $u_n = \sqrt{\frac{n^2+n-1}{n^3-2}}$

Let $$v_n = \frac{(n^2)^{1/2}}{(n^3)^{1/2}} = \frac{1}{n^{1/2}} = \frac{1}{\sqrt{n}}$$

$$\therefore \frac{u_n}{v_n} = \frac{\sqrt{\frac{n^2+n-1}{n^3-2}}}{\frac{1}{\sqrt{n}}} = \sqrt{\frac{n(n^2+n-1)}{n^3-2}} = \sqrt{\frac{n^3+n^2-n}{n^3-2}} = \left(\frac{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}}{\sqrt{1-\frac{2}{n^3}}}\right)$$

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}}{\sqrt{1-\frac{2}{n^3}}}\right) = 1,$$ finite and non-zero quantity.

$\therefore$ By comparision test , $\sum u_n$ and $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ is divergent.

$\left(\therefore \sum \frac{1}{n^p} = \sum \frac{1}{n^{1/2}} \right.$ is divergent for p = $\left. \frac{1}{2} < 1 \right)$

$\therefore \sum u_n = \sqrt{\frac{n^2+n-1}{n^3-2}}$ is also divergent.

Example 4: Examine the convegence or divergence of the series

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + ...$$

Solution : We have

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Let $v_n = \frac{1}{\sqrt{n}}$

$$\therefore \frac{u_n}{v_n} = \frac{\left(\frac{1}{\sqrt{n}+\sqrt{n+1}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$\therefore \lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2} \neq 0$

$\therefore \lim_{n\to\infty} \left(\frac{u_n}{v_n}\right)$ is a finite and non-zero number.

$\therefore$ By comparision test , $\sum u_n$ and $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ is of the form $\sum \frac{1}{n^p}$ where p = $\frac{1}{2}$ < 1

From p -series test, $\sum v_n$ is divergent.

$\therefore$ The given series $\sum u_n$ is divergent.

Example 5: Test for the convergence of the series $\sum_{n=1}^{\infty} \tan \frac{1}{n}$

Solution :

Let $u_n = \tan\left(\frac{1}{n}\right)$ then

$\therefore u_n = \tan\left(\frac{1}{n}\right) = \frac{1}{n} + \frac{1}{3}.\frac{1}{n^3} + \frac{2}{15}.\frac{1}{n^5} + ...$

$= \frac{1}{n}\left[1 + \frac{1}{3n^2} + \frac{2}{15n^4} + ...\right]$

Let $v_n = \frac{1}{n}$ then

$\therefore \lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n}\left[1 + \frac{1}{3n^2} + \frac{2}{15n^4} + ...\right]}{\left(\frac{1}{n}\right)}$

$= \lim_{n\to\infty} \left[1 + \frac{1}{3n^2} + \frac{2}{15n_4} + ...\right] = 1 \neq 0$

$\lim_{n\to\infty} \left(\frac{u_n}{v_n}\right)$ is a finite, non-zero real number.

$\therefore$ By comparision test , $\sum u_n$ and $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{n}$ (Harmonic series) is divergent.

$\therefore \sum u_n$ is also divergent

Example 6: Test the convergence of the series $\sum_{n=1}^{\infty} \left(\sqrt[3]{n^3+1} - n\right)$

Solution : We have $u_n = \left(\sqrt[3]{n^3+1} - n\right) = (n^3+1)^{1/3} - n$

$$= (n^3)^{1/3}\left(1+\frac{1}{n}\right)^{1/3} - n$$

$$= n\left[\left(1+\frac{1}{n}\right)^{1/3} - 1\right]$$

$$= n\left[\left(1+\frac{1}{3n^3}-\frac{1}{9n^6}+....\right)-1\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3}-\frac{1}{9n^3}+...\right]$$

$$= \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9n^3}+...\right]$$

Let $v_n = \frac{1}{n^2}$ then

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\frac{\frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9n^2}+...\right]}{\left(\frac{1}{n^2}\right)} = \frac{1}{3} \neq 0$$

$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is finile and non-zero.

$\therefore$ By comparision test , $\sum u_n$ and $\sum v_n$ behave alike.

But $\sum u_n = \sum \frac{1}{n^2}$ is convergent by p-series test.

$\therefore \sum u_n$ is also convergent.

EXERCISE

Q.Test for the convergence of the following series

1. $\sum_{n=1}^{\infty}\frac{n+2}{(n+1)\sqrt{n}}$ $\quad(hint: Let\ v_n = \frac{1}{\sqrt{n}}.\ \ Ans: Divergent)$

2. $\sum_{n=1}^{\infty}\frac{2n^3+5}{4n^5+1}$ $\quad(hint: Let\ v_n = \frac{1}{n^2}.\ \ Ans: convergent)$

3. $\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+1}$ $\quad(hint: Let\ v_n = \frac{1}{n^2}.\ \ Ans: convergent)$

4. $\frac{2}{1}+\frac{3}{8}+\frac{4}{27}+\frac{5}{64}+...+\frac{n+1}{n^3}+....$ $\quad(hint: Let\ v_n = \frac{1}{n^2}.\ \ Ans: convergent)$

5. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+....$ $\quad(hint: Here\ u_n = \frac{2n-1}{n(n+1)(n+2)}.Let\ v_n = \frac{1}{n^2}.\ \ Ans: convergent)$

6. $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+---------$ $\quad(hint: Here\ u_n = \frac{n}{(2n-1)(2n+1)}.Let\ v_n = \frac{1}{n}.\ Ans: divergent)$

7. $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+----------$

$(hint: Here\ u_n=\frac{1}{\sqrt{n(n+1)}}. Let\ v_n=\frac{1}{n}.\ \ Ans: divergent)$

8. $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+----------$ $(hint: Here\ u_n=\frac{\sqrt{n}}{2n+3}. Let\ v_n=\frac{1}{\sqrt{n}}.\ \ Ans: divergent)$

9. $\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\cdots\cdots$ $(hint: Here\ u_n=\sqrt{\frac{n}{(n+1)^3}}. Let\ v_n=\frac{1}{n}.\ \ Ans: divergent)$

10. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+----------$ $(hint: Here\ u_n=\frac{1}{n(n+3)}.\ Let\ v_n=\frac{1}{n^2}.\ \ Ans: Convergent)$

11. $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ (Refer solved example 5. Ans: divergent)

Cauchy's n^{th} Root test:

If $\sum_{n=1}^{\infty} u_n$ is series of positive terms such that $\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=l$, then

(i) $\sum_{n=1}^{\infty} u_n$ is convergent if $l<1$

(ii) $\sum_{n=1}^{\infty} u_n$ is divergent if $l>1$

(iii) Test fails if $l=1$.

Note: Use this test when u_n is in the form $(f(n))^{\text{powers of } n}$ or $(\text{constant})^n$

Example 1: Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

Solution :

In this case we have $u_n=\left(\frac{n}{n+1}\right)^{n^2}$

$$\Rightarrow u_n^{1/n}=\left|\left(\frac{n}{n+1}\right)^{n^2}\right|^{1/n}=\left(\frac{n}{n+1}\right)^n$$

$$=\frac{1}{\left(\frac{n+1}{n}\right)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{1}{\left(1+\frac{1}{n}\right)^n}\right| = \frac{1}{e} < 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is convergent.

Example 2: Test for convergence $\sum_{1}^{\infty} \frac{2^n}{n^3}$

Solution : Let $u_n = \frac{2^n}{n^3}$

$$(u_n)^{\frac{1}{n}} = \left(\frac{2^n}{n^3}\right)^{1/n} = \frac{2}{n^{3/n}}$$

$$\therefore \lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{2}{n^{3/n}} = 2\lim_{n\to\infty}\frac{1}{n^{3/n}} = 2\lim_{n\to\infty}\frac{1}{(n^{1/n})^3}$$

But $\lim_{n\to\infty} n^{1/n} = 1.$

$$\therefore \lim_{n\to\infty}(u_n)^{\frac{1}{n}} = 2 > 1$$

Hence, by Cauchy's root test. $\sum u_n$ is divergent.

Example 3: Test for convergence $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$

Solution : $u_n = \frac{1}{(\log n)^n}$

$$(u_n)^{\frac{1}{n}} = \left[\frac{1}{(\log n)^n}\right]^{1/n} = \frac{1}{\log n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\log n} = 0 < 1.$$

Therefore by Cauchy's root test, $\sum u_n$ is convergent

EXERCISE

Q. Test for convergence

(i) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$ (Ans: Convergent) (ii) $\sum_{n=1}^{\infty}\left(\frac{n^3}{3^n}\right)$ (Ans:Convergent)

(iii) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n}$ (Ans:Root Test Fails. But by Zero test, series is divergent)

(iv) $\sum\left(\frac{n}{n+1}\right)^{n^2}$ (Ans: Convegent) (v) $\sum\left(\frac{n+1}{3^n}\right)^{n}$ (Ans: Divergent)

(vi) $\sum n.\left(\frac{3}{4}\right)^{n}$ Ans: Convergent

D'Alembert's Ratio Test :

If $\sum_{n=1}^{\infty} u_n$ is series of positive terms such that $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = l$, then

(i) $\sum_{n=1}^{\infty} u_n$ is convergent if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ is divergent if $l > 1$

(iii) Test fails if $l = 1$.

Note : Use this test when u_n contains powers of n or u_n contains factorials.

Example 1: Test for convergent the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....$

Solution:

We have $\sum_{n=1}^{\infty} u_n = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....$

then $u_n = \frac{1}{(n-1)!} \Rightarrow u_{n+1} = \frac{1}{n!} = \frac{1}{n.(n-1)!}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{1}{n}$

$\therefore \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\frac{1}{n} = 0 < 1$

Hence by D' Alemberts ratio test, $\sum_{n=1}^{\infty} u_n$ is convergent.

Example 2: Test for congence the series $1+\frac{1}{2^2}+\frac{1}{3^2}+.....+\frac{1}{n^n}$

Solution :

We have $u_n = \frac{1}{n^n}$

$$\Rightarrow u_{n+1} = \frac{1}{(n+1)^{n+1}}$$

$$\therefore \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left[\frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}}\right] = \lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\frac{n^n}{\left(\frac{n+1}{n}\right)^n}\right|$$

$$= \lim_{n\to\infty}\frac{1}{n+1}\cdot\lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= 0 \times \frac{1}{e} = 0 < 1$$

$\therefore$ By D Alemberts ratio test,the given series is convergent

Example 3: Test for the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n.n!}{n^n}$

Solution :

Let $u_n = \frac{2^n.n!}{n^n}$ then $u_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}$

$$\therefore \lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left(\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\right)\left(\frac{n^n}{2^n n!}\right) = \lim_{n\to\infty} 2\frac{n^n}{(n+1)^n} = \lim_{n\to\infty} 2\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$$

$\therefore$ Hence by D' Alemberts ratio lest, $\sum_{n=1}^{\infty} \frac{2^n.n!}{n^n}$ is convergent.

Example 4: Test for convergence $\sum_{n=1}^{\infty} \frac{n^3 + a}{2^n + a}$ (a > 0)

Solution :

We have $u_n = \frac{n^3+a}{2^n+a}$ $\therefore u_{n+1} = \frac{(n+1)^3+a}{2^{n+1}+a}$

$$\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left|\frac{(n+1)^3+a}{2^{n+1}+a} \cdot \frac{2^n+a}{n^3+a}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n^3\left[\left(1+\frac{1}{n}\right)^3 + \frac{a}{n^3}\right].2^n\left(1+\frac{a}{2^n}\right)}{2^{n+1}\left[1+\frac{a}{2^{n+1}}\right]n^3\left(1+\frac{a}{n^3}\right)}\right|$$

$$= \frac{1}{2} \lim_{n\to\infty} \left[\frac{\left[\left(1+\frac{1}{n}\right)^3 + \frac{a}{n^3}\right]\left[1+\frac{a}{n^3}\right]}{\left(1+\frac{a}{2^{n+1}}\right)\left(1+\frac{a}{n^3}\right)}\right]$$

$$= \frac{1}{2} < 1$$

$\therefore$ By D' Alemberts ratio test , $\sum_{n=1}^{\infty} \frac{n^3 + a}{2^n + a}$ converges.

Example 5 : Test for the convergence or divergence of the series $\sum \frac{x^n}{3^n.n^2}.(x>0)$

Solution :

Here we have $u_n = \frac{x^n}{3^n n^2}$ $\therefore u_{n+1} = \frac{x^{n+1}}{3^{n+1}(n+1)^2}$

Now $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left| \frac{x^n.x}{3^n.3(n+1)^2} \cdot \frac{3^n.n^2}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x}{3}\left(\frac{n}{n+1}\right)^2 \right|$

$= \frac{|x|}{3} \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = \frac{|x|}{3} = \frac{x}{3}$

By D' Alemberts test,

(1) $\sum_{n=1}^{\infty} u_n$ converges if $\frac{x}{3} < 1$ i.e., $\frac{x}{3} < 1$ converges if x < 3.

(2) $\sum_{n=1}^{\infty} u_n$ diverges when $\frac{x}{3} > 1$ i.e. $\sum_{n=1}^{\infty} u_n$ diverges when x > 3.

(3) When $\frac{x}{3} = 1$ i.e. if x = 3 the test fails.

When x = 3 we have

$\sum_{n=1}^{\infty} u_n = \sum \frac{3^n}{3^n.n^2} = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ where p = 2 > 1

By p-series test, $\sum \frac{1}{n^p}$ is convergent.

$\therefore$ The given series $\sum_{n=1}^{\infty} u_n$ is convergent when x $\leq$ 3 and divergent when x > 3.

EXERCISE

1.Test for convergence

(a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (Ans: Convergent)

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ (Ans:Divergent)

(c) $\sum \frac{n^2(n+1)^2}{n!}$ (Ans: Convergent)

(d) $\sum_{n=1}^{\infty} \frac{5^n+a}{3^n+b}, a>0, b>0$ (Ans: Divergent)

(e) $\sum_{n=1}^{\infty} \frac{5^{n-1}}{n!}$ (Ans: Convergent)

2. Test for the convergence of the series.

(a) $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \ldots$ Ans: Divergent

(b) $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+...$ Ans: Convergent

(c) $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+...$ Ans: Divergent

(d) $1+\frac{1}{2}+\frac{1}{2.4}+\frac{1}{2.4.6}+....$ Ans: Convergent

(e) $\frac{2}{1}+\frac{2.5}{1.5}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+....$ Ans: Convergent

3. Examine the convergence of the series

(i) $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...(x>0)$

(ii) $\frac{2x}{1}+\frac{3x^2}{8}+\frac{4x^3}{27}+...+\frac{(n+1)x^n}{n^3}+...$

(iii) $x+\frac{3}{5}x^2+\frac{8}{10}x^3+...+\frac{(n^2-1)}{(n^2+1)}x^n+...$

Answers : (i) convergent, (ii) convergent if $x \leq 1$, divergent if $x > 1$, (iii) convergent if $x < 1$, divergent if $x \geq 1$.

4. Test for the convergence of the series

(a) $\sum \frac{x^n}{n^2}$ (b) $\sum \frac{(n+1)^n}{n^{n+1}}.x^n$ (c) $\sum \frac{n!x^n}{n^n}$

Answers : (a) Converges when $x \leq 1$ and diverges when $x > 1$.

(b) Converges when $x < 1$ and diverges when $x \geq 1$.

(c) Converges when $x < e$ and diverges when $x \geq e$.

Raabe's Test (Higher Ratio Test)

If $\sum_{n=1}^{\infty} u_n$ is series of positive terms such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l$ then

(i) $\sum_{n=1}^{\infty} u_n$ is convergent if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ is divergent if $l < 1$

(iii) Test fails if $l = 1$.

Note : When Ratio test fails use Raabe's Test

Example 1: Determine the nature of the series $\sum \frac{1.5.9...(4n-3)}{3.7.11....(4n-1)}$

Solution :

Let $u_n = \frac{1.5.9...(4n-3)}{3.7.11....(4n-1)}$

Then $u_{n+1} = \frac{1.5.9...(4n-3)(4n+1)}{3.7.11....(4n-1)(4n+3)}$

We note that lim $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left|\frac{4n+1}{4n+3}\right| = \lim_{n\to\infty} \left|\frac{4+\frac{1}{n}}{4+\frac{3}{n}}\right| = 1$

Therefore, the nature of the series is not determined by the D'alemberts ratio test.

Now consider: $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n+3}{4n+1} - 1\right) = \lim_{n\to\infty} n\left(\frac{2}{4n+1}\right) = \frac{1}{2} < 1$

By Raabe's test, the given series is divergent.

EXERCISE

Q. Test the convergence

(a) $\sum \frac{1^2.4^2.7^2.....(3n-2)^2}{3^2.6^2.9^2...(3n)^2}$ (b) $\frac{2}{3.5} + \frac{2.4}{3.5.7} + \frac{2.4.6}{3.5.7.9} + ...$

(c) $\sum_{n=1}^{\infty} \frac{1^2 5^2 9^2 ...(4n-3)^2}{4^2 8^2 12^2 ...(4n)^2}$ (d) $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + ...$

Answers: (a) convergent, (b) convergent, (c) convergent (d) divergent

Alternating Series :

An infinite series with alternate positive and negative terms is called an alternating series.

Alternating series is given by

$\sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 \;......+ (-1)^{n-1} u_n + \;......$

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

Leibnitz's Test:

An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent if

(i)Each term is numerically less than it's preceding term, i.e. $u_{n+1} < u_n$ for all n.

(ii) $\lim_{n\to\infty} u_n = 0$

Example 1: Test for convergence of the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + ...$

Solution : We have $u_n = \frac{1}{\sqrt{n}}$ $\therefore\; u_{n+1} = \frac{1}{\sqrt{n+1}}$

Now $\sqrt{n+1} > \sqrt{n}$ for all n ∈ N.

Therefore $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all n ∈ N.

$\Rightarrow \qquad u_{n+1} < u_n$ for all n ∈ N.

Further $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Therefore by Leibnitz's test, the given series is convergent.

Example 2 : Test for convergence $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) + \ldots$

Solution :

We have $$u_n = \frac{1+2+3+\ldots\ldots+n}{(n+1)^3} = \left(\frac{n(n+1)}{2}\right).\frac{1}{(n+1)^3} = \frac{n}{2(n+1)^2}$$

and $u_{n+1} = \frac{n+1}{2(n+2)^2}$

$$\therefore u_{n+1} - u_n = \frac{(n+1)}{2(n+2)^2} - \frac{n}{2(n+1)^2} = \frac{1}{2}\frac{(n+1)^3 - n(n+2)^2}{(n+1)^2(n+2)^2}$$

$$= -\frac{n^2+n-1}{2(n+2)^2(n+1)^2} < 0$$

$\therefore u_{n+1} - u_n < 0$

$\Rightarrow \qquad u_{n+1} < u_n$ for all n ∈ N.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2(n+1)^2} = 0$

Both the conditions of Leibnitz's test are satisfied.
Therefore, the given series is convergent.

EXERCISE

Test for convergence of the series

1. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$
2. $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots$
3. $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + ---------$
4. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$
5. $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}}$
6. $\log\left(\frac{1}{2}\right) - \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) - \log\left(\frac{4}{5}\right) + \ldots.$

Fourier Series

Periodic function: A function f(x) is said to be periodic with period T if f(x+T) = f(x) for all real x and if there is some positive number T such that f(x+T) = f(x) then such smallest positive number is called a fundamental period of f(x).

Example: f(x) =sinx periodic of period 2π

Fourier series: If f(x) is a periodic function of period 2L, defined in the interval $c \le x \le c+2L$ and satisfies the Dirichlet's conditions, then f(x) can be represented by trigonometric series (series of sine and cosine) as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{.....................................(1)}$$

$$where \quad a_0 = \frac{1}{L}\int_c^{c+2L} f(x)dx$$

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

a_0, a_n & b_n are called as Fourier coefficients.

Important Formulae

$$\int uv dx = uv_1 - u'v_2 + u''v_3 \text{ ....}$$

$$\int x^2 \cos nx dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)$$

$$\int e^{ax}\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$\int e^{ax}\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$$

$$\int e^{-x}\cos(nx)dx = \frac{e^{-x}}{1+n^2}\left[-\cos nx + n\sin nx\right]$$

$$\int e^{\alpha x}\sin(nx)dx = \frac{e^{\alpha x}}{d^2+n^2}\left[\alpha\sin nx - n\cos nx\right]$$

Type I (c=0 and L= π)

If f(x) is periodic function with period 2π in the interval $0 \le x \le 2\pi$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx, \qquad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

Example 1: Expand following function as Fourier series

$f(x) = x^2 \quad where \quad 0 \le x \le 2\pi.$

Solution :

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{2\pi} = \frac{1}{\pi} \cdot \frac{8}{3} \pi^3 = \frac{8}{3} \pi^2.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx$$

$$= \frac{1}{\pi} \left[\left[\frac{x^2 \sin nx}{n} \right]_0^{2\pi} + \left[2x \left(\frac{-\cos nx}{n^2} \right) \right]_0^{2\pi} + 2 \left[\left(\frac{-\sin nx}{n^3} \right) \right]_0^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n} \left[x^2 \sin nx \right]_0^{2\pi} + \pi \frac{2}{n^2} \left[x \cos nx \right]_0^{2\pi} - \frac{2}{n^3} \left[\sin nx \right]_0^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[0 + \frac{2}{n^2} .2\pi .1 \right]$$

$$= \frac{1}{\pi} \frac{4\pi}{n^2}$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx dx$$

$$= \frac{1}{\pi} \left\{ \left[2x \left(\frac{-\cos nx}{n^2} \right) \right]_0^{2\pi} - \left[2x \left(\frac{-\sin nx}{n^2} \right) \right]_0^{2\pi} + \left[2 \frac{\cos nx}{n^3} \right]_0^{2\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{-1}{n} \left[x^2 \cos nx \right]_0^{2\pi} + \frac{2}{n^2} \left[x \sin nx \right]_0^{2\pi} + \frac{2}{n^3} \left[\cos nx \right]_0^{2\pi} \right\}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left[4\pi^2 - 0\right] + \frac{2}{n^2}\left[0-0\right] + \frac{2}{n^3}\left[1-1\right] \ \right]$$

$$= -\frac{4\pi^2}{\pi n}$$

$$= -\frac{4\pi}{n}$$

$$\therefore \ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\alpha}\frac{4}{n^2}\cos nx + \sum_{n=1}^{\alpha}\frac{-4\pi}{n}\sin nx$$

Example 2. Find a Fourier series of the function f(x) = $\left(\frac{\pi - x}{2}\right)^2$ when $0 \le x \le 2\pi$.

Hence deduce that i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$ $\quad$ $ii)\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ... = \frac{\pi^2}{12}$

$iii)\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$ $\quad$ $Ans: \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$

Solution :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{4\pi}\int_0^{2\pi}\left(\pi - x\right)^2 dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2 - 2x\pi + x^2\right)dx = \frac{1}{4\pi}\left[\pi^2 x - x^2\pi + \frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3}\right] = \frac{1}{4\pi}\left[-2\pi^3 + \frac{8\pi^3}{3}\right] = \frac{1}{4\pi}.\frac{2\pi^3}{3}$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_o^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_o^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nx dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left(\pi - x\right)^2 \cos nx dx$$

$$= \frac{1}{4\pi}\left\{\left[\left(\pi - x\right)^2 \frac{\sin nx}{n}\right]_0^{2\pi} - \left[-2\left(\pi - x\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi} + \left[2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}\right\}$$

$$= \frac{1}{4\pi}\left\{\frac{1}{n}\left[(\pi - x)^2 \sin nx\right]_0^{2\pi} - \frac{2}{n^2}\left[(\pi - x)\cos nx\right]_0^{2\pi} - \frac{2}{n^3}\left[\sin nx\right]_0^{2\pi}\right\}$$

$$= \frac{1}{4\pi}\left\{0 - \frac{2}{n^2}(-\pi - \pi)\right\}$$

$$= \frac{1}{4\pi}.\frac{4\pi}{n^2}$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nxdx = \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \sin nxdx$$

$$= \frac{1}{4\pi}\left\{\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} - \left[-2(\pi - x)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} + \left[2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}\right\}$$

$$= \frac{1}{4\pi}\left\{\frac{-1}{n}\left[(\pi - x)^2 \cos nx\right]_0^{2\pi} - \frac{2}{n^2}\left[(\pi - x)\sin nx\right]_0^{2\pi} + \frac{2}{n^3}\left[\cos nx\right]_0^{2\pi}\right\}$$

$$= \frac{1}{4\pi}\left\{\frac{-1}{n}\left[\pi^2 - \pi^2\right] + \frac{2}{n^3}[0 - 0]\right\}$$

$$= 0.$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx + 0.\sin nx\right)$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\therefore \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x + \text{...........} \quad (*)$$

(i) *Put* $x = 0$ *in* $(*)$

$$\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots\ldots$$

i.e. $\quad \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots\ldots\ldots\ldots = \dfrac{\pi^2}{6}.$

(ii) *Put* $x = \pi$ *in* $(*)$

$$\left(\frac{\pi - \pi}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \frac{1}{4^2}\cos 4\pi + \ldots\ldots\ldots\ldots$$

$$0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$$

i.e. $\quad \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots\ldots\ldots = \dfrac{\pi^2}{12}.$

(iii) *Adding result of* (i) *and* (ii), *we get*

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$\therefore \frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Example 3 . Find Fourier Series for f(x) = x where o < x < π
= 2π-x where π < x< 2π

Solution:

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \left[4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \frac{4\pi^2}{2} - \frac{3\pi^2}{2}\right\}$$

$$= \frac{1}{\pi}\left[\pi^2\right]$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\left\{\int_o^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\frac{(-\sin nx)}{n}\right]_0^{\pi} - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{(-\sin nx)}{n}\right]_{\pi}^{2\pi} - \left[(-1)\frac{-\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{0 + \frac{1}{n^2}\left((-1)^n - 1\right) - \frac{1}{n^2}\left(1 - (-1)^n\right)\right\}$$

$$= \frac{1}{\pi}\frac{2}{n^2}\left[(-1)^n - 1\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{x(-\cos nx)}{n}\right]_0^{\pi} - \left[\frac{-\sin nx}{n^2}\right] + \left[\frac{(2\pi - x)(-\cos nx)}{n}\right]_{\pi}^{2\pi} - \left[\frac{(-1)\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n}(+\pi)(-1)^n + \frac{1}{n}\left[0 + \pi(-1)^n\right] - \frac{1}{n^2}(0)\right\}$$

$$= \frac{1}{\pi}(0) = 0$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2}{1^2}\cos x \frac{-2}{3^2}\cos 3x \frac{-2}{5^2}\cos sx.....\right]$$

Exercise

1. Find a Fourier series for the function $f(x) = e^{-x}$ where $0 < x < 2\pi$.

   i. Ans: $e^{-x} = -\left(\frac{e^{-2\pi} - 1}{\pi}\right) + \sum_{n=1}^{\infty} \left(\frac{\cos nx + n \sin nx}{1^2 + n^2}\right)$

2. Find a Fourier series of the function $f(x) = x^2$ when $0 \le x \le 2\pi$ and $f(x+2\pi) = f(x)$

   a. $Ans: x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx - 4\pi \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$

3. Find a Fourier series of the function $f(x) = \frac{1}{2}(\pi - x)$ when $0 \le x \le 2\pi$.

   a. $Ans: \frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$

4. Find a Fourier series of the function f(x) =xsinx in the interval $0 \le x \le 2\pi$

   $Ans: x\sin x = -1 - \frac{1}{2} + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1}\cos nx$

5. Find a Fourier series for the periodic function f(x) defined in the interval $0 \le x \le 2\pi$ as

$$f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ \\ 0 & \pi \le x \le 2\pi \end{cases} \quad and \quad f(x+2\pi) = f(x)$$

Hence deduce that $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... = \frac{1}{2}$ $\quad Ans:\ f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}\cos 2nx + \frac{1}{2}\sin x$

6. Find a Fourier series of the function $f(x) = \left(\frac{\pi - x}{2}\right)^2$ when $0 \le x \le 2\pi$.

   Hence deduce that i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$ $\quad ii)\ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ... = \frac{\pi^2}{12}$

   $iii)\ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$ $\quad Ans: \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$

7. Find a Fourier series for the function..

   f(x) = cos x, $-\pi < x < 0$.

   = sin x, $0 < x < \pi$.

   Ans:$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 1}{\pi(n^2 - 1)}\cos(nx) + \sum_{n=1}^{\infty} \frac{n[(-1)^{n+1} - 1]}{\pi(n^2 - 1)}\sin(nx)$

Type II   (c= - π   and L= π)

Fourier Series for Even & Odd functions

Properties of definite integral for odd and even function:

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \qquad \text{if f(x) is an even function}$$

$$\int_{-a}^{a} f(x)dx = 0 \qquad \text{if f(x) is an odd function}$$

This property can be applied to Fourier series:

Case I: If f(x) is even function in the interval $-\pi \le x \le \pi$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx, \qquad b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Case II: If f(x) is odd function in the interval $-\pi \le x \le \pi$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = 0, \qquad a_n = 0, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Case III: f(x) is neither even nor odd function in the interval $-\pi \le x \le \pi$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

Example. Find Fourier series for f(x) = x^2 where $-\pi \le x \le \pi$.
Solution : f(x) is an even function.

$$\therefore \; b_n = 0.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{x^2 \sin nx}{n}\right]_0^{\pi} - \left[\frac{2x(-\cos nx)}{n^2}\right]_0^{\pi} + \left[\frac{2(-\sin nx)}{n^3}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2}{n^2}\left(\pi(-1)^n\right)\right\}$$

$$= \frac{4}{n^2}(-1)^n$$

$$\therefore \; f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

Example 2. f(x) =x^2 in interval (-2,2)
Solution: L=2

Function is even.

Therefore b_n =0

$$a_n = \frac{2}{2}\int_0^{2} x^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[\frac{x^2 \sin\left(\frac{n\pi x}{2}\right)}{n\pi/2}\right]_0^2 - \left[\frac{2x\cos\left(n\pi\, x/2\right)}{n^2\,\pi^2/4}\right]_0^2 + \left[\frac{2\left(-\sin n\pi\, x/2\right)}{n^3\,\pi^3/8}\right]_0^2$$

$$= \frac{4}{n^2}\cdot\frac{4}{\pi^2}(-1)^n$$

$$= \frac{16}{n^2\pi^2}\left[(-1)^n\right]$$

$$a_o = \int_0^2 x^2 dx = \left[\frac{x3}{3}\right]_0^2 = \frac{8}{3}$$

$$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{2}\right)$$

Exercise

1) Find Fourier series to represent the function f(x)= $\pi^2 - x^2$ in the interval $-\pi \le x \le \pi$
Hence deduce that

$$i)\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...=\frac{\pi^2}{12} \qquad ii)\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...=\frac{\pi^2}{8} \qquad Ans: \pi^2-x^2=\frac{2\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

2) Find a Fourier series of the function f(x) = x when $-\pi \le x \le \pi$ and f(x+ 2π)=f(x) .

$$Ans: x=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

3) Obtain the Fourier series for the function f(x) = $x-x^2$ defined in the interval $-\pi \le x \le \pi$

Hence deduce that $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...=\frac{\pi^2}{12}$ $\qquad Ans: x-x^2=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin nx$

4) Obtain the Fourier series for the function f(x) =x^2 defined in the interval $-\pi \le x \le \pi$

Hence deduce that i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$ $\qquad ii)\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...=\frac{\pi^2}{12}$

$$iii)\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...=\frac{\pi^2}{8} \qquad Ans: x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

5) Obtain Fourier series for the function f(x) = x sin x. where $-\pi < x < \pi$,

Ans: $f(x) = 1 + (\frac{-1}{2})\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{(n^2-1)}$ cos (nx)

$$6)\ f(x)=\begin{cases}\cos x & -\pi<x<0\\ -\cos x & 0<x<\pi\end{cases} \quad and \quad f(x+2\pi)=f(x) \qquad Ans:\ f(x)=-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n}{(2n)^2-1}\sin 2nx$$

Half Range Sine and Cosine Series expansions:

In Fourier series we studied so far, range of the function is equal to period.
Thus for half range expansions, range is half the period.

1) Fourier Half Range Cosine series in the interval $0 \le x \le L$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)$$

where $\quad a_0=\frac{2}{L}\int_0^L f(x)dx, \quad a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

2) Fourier Half Range Sine series in the interval $0 \le x \le L$

f(x)= $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

3) Fourier Half Range Cosine series in the interval $0 \le x \le \pi$

f(x) $= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ & $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$

4) Fourier Half Range Sine series in the interval $0 \le x \le \pi$

f(x)= $\sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

Thus for half range expansions, range is half the period.

Examples:

1. Find half range cosine series for f(x) = x in interval (0, π)

Solution: Here L= π

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin x}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

Half range cosine series is $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$

2. Find half range sine series for f(x) = x^2 in interval (0, 2).
Solution: Here L=2

$$a_0 = 0 \ \ and \ \ a_n = 0.$$

$$b_n = \frac{2}{2}\int_0^2 x^2 \sin\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{x^2\left(-\cos n\pi \,x/2\right)}{n\pi/2} - \frac{2x\left(-\sin n\pi \,x/2\right)}{n^2\pi 2/4} + \frac{2\cos n\pi\, x/2}{n^3\pi\, 3/8}\right]_0^2$$

$$= \frac{-4(-1)^n}{n\pi/2} + \frac{2(-1)^n}{n^3\pi 3/8} - \frac{2}{n^3\pi 3/8}$$

$$b_n = \frac{-8(-1)^n}{n\pi} - \frac{16}{n^3\pi^3}\left(1-(-1)^n\right)$$

Half range sine series,

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{-8(-1)n}{n\pi} - \frac{16}{n^3\pi^3}\left[1-(-1)^n\right]\right\}\sin\frac{n\pi x}{2}$$

Exercise

1) Find the half range cosine series of

i) $f(x) = x \quad 0 \le x \le \pi$ $\qquad Ans: x = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right)$

ii) $f(x) = x^2 \quad 0 \le x \le \pi$ $\qquad Ans: x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2}\cos 2x + \frac{1}{3^2}\cos 3x - .....\right)$

iii) $f(x) = x^2 \quad 0 < x < 2$ $\qquad Ans: x^2 = \frac{4}{3} + \frac{16}{\pi^2}\left(-\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\frac{2\pi x}{2} - \frac{1}{3^2}\cos\frac{3\pi x}{2} + ...\right)$

iv) $f(x) = \sin x$ where $0 < x < p$. Also show that $p/4 = 1 - 1/3 + 1/5 - 1/7 + ...$

$$Ans: f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} cos(nx)$$

2) Find the half range sine series of

i) $f(x) = x \ , \ 0 \le x \le \pi$ $\qquad Ans: x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x.....\right)$

ii) $f(x) = x^2, \ 0 \le x \le \pi$ $\qquad Ans: x^2 = \frac{2}{\pi}\left(\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2}\right)\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x + ...\right)$

iii) f(x) = x when $0 < x < \pi/2$.
= π- x when $\pi/2 < x < \pi$.

Ans:f(x) = $\sum_{n=1}^{\infty} \dfrac{4\sin(n\pi/2)}{n^2\pi}$ sin (nx)

Fourier Series on arbitrary interval

Obtain the Fourier series for the periodic function

a)f(x)= $\begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$ $\&\,\text{Period} = 10 \quad Ans: f(x) = \dfrac{3}{2} + \dfrac{6}{\pi}\left[\sin\dfrac{\pi x}{5} + \dfrac{1}{3}\sin\dfrac{3\pi x}{5} + \dfrac{1}{5}\sin \pi x\right]$

$$b)\ f(x) = \begin{cases} \dfrac{2k}{l}x & 0 \le x \le \dfrac{l}{2} \\ \dfrac{2k}{l}(l-x) & \dfrac{l}{2} \le x \le l \end{cases}$$

Ans: $f(x) = \dfrac{8k}{\pi^2}\sum_{n=1}^{\infty}\left(\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\right)\sin\dfrac{n\pi x}{l}$