Unit II : Differential Calculus

(Mean Value Theorems, Exapansions of functions and Indeterminate forms)

Mean Value Theorems

Closed interval: An interval of the form $a \le x \le b$, that includes every point between a and b and also the end points, is called a closed interval and is denoted by $[a,b]$.

Open Interval: An interval of the form $a < x < b$, that includes every point between a and b but not the end points, is called an open interval and is denoted by (a,b)

Continuity: A real valued function $f(x)$ is said to be continuous at a point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

The function $f(x)$ is said to be continuous in an interval if it is continuous at every point in the interval.
Roughly speaking, if we can draw a curve without lifting the pen, then it is a continuous curve otherwise it is discontinuous, having discontinuities at those points at which the curve will have breaks or jumps.
We note that all elementary functions such as algebraic, exponential, trigonometric, logarithmic, hyperbolic functions are continuous functions. Also the sum, difference, product of continuous functions is continuous. The quotient of continuous functions is continuous at all those points at which the denominator does not become zero.

Differentiability: A real valued function $f(x)$ is said to be differentiable at point x_0 if

$\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists uniquely and it is denoted by $f'(x_0)$.

A real valued function f(x) is said to be differentiable in an interval if it is differentiable at every point in the interval or if $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists uniquely. This is denoted by $f'(x)$. We say that either $f'(x)$ exists or f(x) is differentiable.

Geometrically, it means that the curve is a smooth curve. In other words a curve is said to be smooth if there exists a unique tangent to the curve at every point on it. For example a circle is a smooth curve. Triangle, rectangle, square etc are not smooth, since we can draw more number of tangents at every corner point.

We note that if a function is differentiable in an interval then it is necessarily continuous in that interval. The converse of this need not be true. That means a function is continuous need not imply that it is differentiable.

I) Rolle's Theorem:

The Rolle's theorem is named in the honor of mathematician Michael Rolle (1652-1719). It is an important theorem in calculus, having wide applications. It forms the base for the other mean value theorems which are the Lagrange's mean value theorem and the Cauchy's mean value theorem.

Statement:

Suppose a function $f(x)$ satisfies the following three conditions:

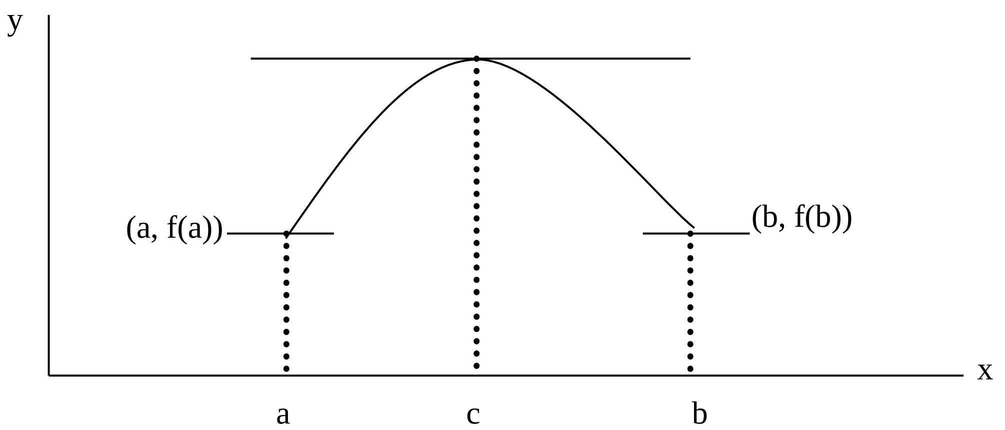
(i) $f(x)$ is continuous in a closed interval $[a,b]$

(ii) $f(x)$ is differentiable in the open interval (a,b)

(iii) $f(a) = f(b)$

Then there exists at least one point c in the open interval (a,b) such that $f'(c) = 0$

Geometrical Meaning of Rolle's Theorem: Consider a curve $f(x)$ that satisfies the conditions of the Rolle's Theorem as shown in figure:



As we see the curve $f(x)$ is continuous in the closed interval $[a,b]$, the curve is smooth i.e. there can be a unique tangent to the curve at any point in the open interval (a,b) and also $f(a) = f(b)$. Hence by Rolle's Theorem there exist at least one point c belonging to (a,b) such that $f'(c) = 0$. In other words there exists at least one point at which the tangent drawn to the curve will have its slope zero or lies parallel to x-axis.

Example 1: Verify Rolle's Theorem for $f(x) = x^2$ in $[-1, 1]$

Solution : First we check whether the conditions of Rolle's theorem hold good for the given function:

(i) $f(x) = x^2$ is an elementary algebraic function, hence it is continuous every where and so also in $[-1, 1]$.

(ii) $f'(x) = 2x$ exists in the interval $(-1, 1)$ i.e. the function is differentiable in (-1, 1).

(iii) Also we see that $f(-1)=(-1)^2=1$ and $f(1)=1^2=1$ i.e., $f(-1)=f(1)$

Hence the three conditions of the Rolle's Theorem hold good.

$\therefore$ By Rolle's Theorem, there exists point c in $(-1, 1)$ such that $f'(c)=0$.

That means 2c = 0 $\Rightarrow$ c= 0 $\in(-1,1)$

Hence Rolle's Theorem is verified.

Example 2: Verify Rolle's Theorem for $f(x)=x(x+3)e^{-x/2}$ in $[-3, 0]$

Solution: (i) $f(x)$ is a product of elementary algebraic and exponential functions which are continuous and hence it is continuous in $[-3,0]$.

(ii) $f'(x)=(2x+3)e^{-x/2}-\frac{1}{2}(x^2+3x)e^{-x/2}$

$=[(2x+3)-\frac{1}{2}(x^2+3x)]e^{-x/2}$

$=-\frac{1}{2}[x^2-x-6]e^{-x/2}=-\frac{1}{2}(x-3)(x+2)e^{-x/2}$ exists in (-3, 0).

$\therefore$ $f(x)$ is differentiable in (-3, 0).

(iii) $f(-3)=0$ and also $f(0)=0$

$\therefore f(-3)=f(0)$

That is, the three conditions of the Rolle's Theorem hold good.

$\therefore$ By Rolle's Theorem, there exists point c in (-3, 0) such that $f'(c)=0$.

$\therefore -\frac{1}{2}(c-3)(c+2)e^{-c/2}=0 \Rightarrow c=3,-2,\infty$

Out of these values of c, since $-2\in(-3,0)$, the Rolle's Theorem is verified.

Example 3: Verify Rolle's Theorem for $f(x)=(x-a)^m(x-b)^n$ in [a, b] where a < b and a, b>0.

Solution:

(i) $f(x)$ is a product of elementary algebraic functions which are continuous and hence it is continuous in [a, b]

(ii) $f'(x)=m(x-a)^{m-1}(x-b)^n+n(x-a)^m(x-b)^{n-1}$

$=[m(x-b)+n(x-a)](x-a)^{m-1}(x-b)^{n-1}$

$=[x(m+n)-(mb+na)](x-a)^{m-1}(x-b)^{n-1}$ exists in (a,b).

$\therefore$ $f(x)$ is differentiable in (a, b).

(iii) $f(a)=0$ and also $f(b)=0$

$\therefore f(a)=f(b)$

Hence the three conditions of the Rolle's Theorem hold good.

$\therefore$ By Rolle's Theorem, there exists point c in (a,b) such that $f'(c)=0$.

$$\therefore [c(m+n)-(mb+na)](c-a)^{m-1}(c-b)^{n-1}=0 \Rightarrow c=\frac{mb+na}{m+n},\ a,\ b$$

Out of these values of c, since $c=\dfrac{mb+na}{m+n}\in(a,b)$, the Rolle's Theorem is verified.

Example 4: Verify Rolle's Theorem for $f(x)=\log\dfrac{x^2+ab}{x(a+b)}$ in [a, b]

Solution:

(i) $f(x)=\log(x^2+ab)-\log x-\log(a+b)$ is the sum of elementary logarithmic functions which are continuous and hence it is continuous in [a, b]

(ii) $f'(x)=\dfrac{2x}{x^2+ab}-\dfrac{1}{x}$ exists in (a,b).

$\therefore$ $f(x)$ is differentiable in (a, b).

(iii) $f(a)=\log\dfrac{a^2+ab}{a(a+b)}=\log\dfrac{a^2+ab}{a^2+ab}=\log 1=0$

and similarly $f(b)=\log\dfrac{b^2+ab}{b(a+b)}=\log\dfrac{b^2+ab}{b^2+ab}=\log 1=0$

Hence the three conditions of the Rolle's Theorem hold good.

$\therefore$ By Rolle's Theorem, there exists point c in (a,b) such that $f'(c)=0$.

$$f'(c)=\frac{2c}{c^2+ab}-\frac{1}{c}=0 \Rightarrow \frac{2c^2-c^2-ab}{c(c^2+ab)}=0 \Rightarrow c^2-ab=0 \Rightarrow c=\sqrt{ab}$$

Since $c=\sqrt{ab}\in(a,b)$, the Rolle's Theorem is verified.

Example 5: State whether Rolle's theorem is applicable for the function f(x)= $x-x^3$ in the interval (-1, 1) or not.

Solution : $f(x)=x-x^3 \ \ for\ -1\le x\le 1$

1.Being polynomial function, f(x) is continuous on closed interval [-1, 1] .

2.f(x) is differentiable on the open interval (-1, 1) as $f'(x)=1-3x^2$ exists.

3.f(– 1) = – 1 – (– 1) = – 1+1=0 and f(1) = 1– 1 = 0 i.e f(– 1) = f(1) = 0.

Thus all conditions of Rolle's theorem are satisfied by f(x) .

$$f'(x)=0 \Rightarrow 1-3x^2=0 \Rightarrow x^2=\frac{1}{3} \Rightarrow \ x=\pm\frac{1}{\sqrt{3}}$$

$$\therefore c=\pm\frac{1}{\sqrt{3}} \ \ \& \ -1<\pm\frac{1}{\sqrt{3}}<1$$

Example 6: By using Rolle's Theorem prove that between any two real roots of the equation $e^x \sin x = 1$, there is at least one root of $e^x \cos x + 1 = 0$.

Solution: Let 'a' and 'b' be roots of the equation $e^x \sin x = 1$, or sinx = e^{-x}.

Let f(x) = sinx - e^{-x} $\rightarrow f'(x)=\cos x + e^{-x}$
as 'a' & 'b' are roots of f(x) $\therefore$ f(a)=0 & f(b)=0
f(x) is continous on [a,b] and differntiable on (a, b)
$\therefore$ All coditions of Rolle's theorem are satisfied by f(x)
$\therefore$ By Rolle's theorem there exist at least one real number 'c'
between a & b such that $f'(x)=0 \rightarrow$ real number $'c'$ is root of $\cos x + e^{-x}=0$

Exercise 1

Q. State whether Rolle's theorem is applicable for the following functions in a given interval. If so find appropriate value of c.

a) f(x)= tanx where $0 \le x \le \pi$.

b) f(x)= $x^2(1-x^2)^2$ where $0 \le x \le 1$.

c) $f(x)=\sqrt{1-x^2}$ in the interval $[-1, 1]$

d) $f(x)=\sin x$ in the interval $[0, \pi]$

e) $f(x)=\dfrac{x^2-x-6}{x-1}$ in the interval $[-2,3]$

f) $f(x)=e^x(\sin x-\cos x)$ in $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]$,

g) $f(x)=x(x-2)e^{x/2}$ in $[0,2]$

h) $f(x)=\dfrac{\sin 2x}{e^{2x}}$ in $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]$.

II) Lagrange's Mean Value Theorem (LMVT) :

It is also known as the First Mean Value Theorem.

Statement:

Suppose a function $f(x)$ satisfies the following two conditions:

(i) $f(x)$ is continuous in a closed interval $[a,b]$

(ii) $f(x)$ is differentiable in the open interval (a,b)

Then there exists at least one point c in the open interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Proof: Consider the function $\phi(x)$ defined by $\phi(x)=f(x)-kx$ where k is a constant to be found such that $\varphi(a)=\phi(b)$.

Since $f(x)$ is continuous in the closed interval $[a,b]$, $\phi(x)$ which is a sum of continuous functions is also continuous in the closed interval $[a,b]$.

$\phi'(x)=f'(x)-kx$............(1) exists in the interval (a,b) as $f(x)$ is differentiable in (a,b).

We have $\phi(a)=f(a)-ka$ and $\phi(b)=f(b)-kb$

$\therefore \phi(a)=\phi(b)\Rightarrow f(a)-ka=f(b)-kb\Rightarrow k=\dfrac{f(b)-f(a)}{b-a}$..........(2)

This means that when k is chosen as in (2) we will have $\phi(a)=\phi(b)$

Hence the conditions of Rolle's Theorem hold good for $\phi(x)$ in $[a,b]$

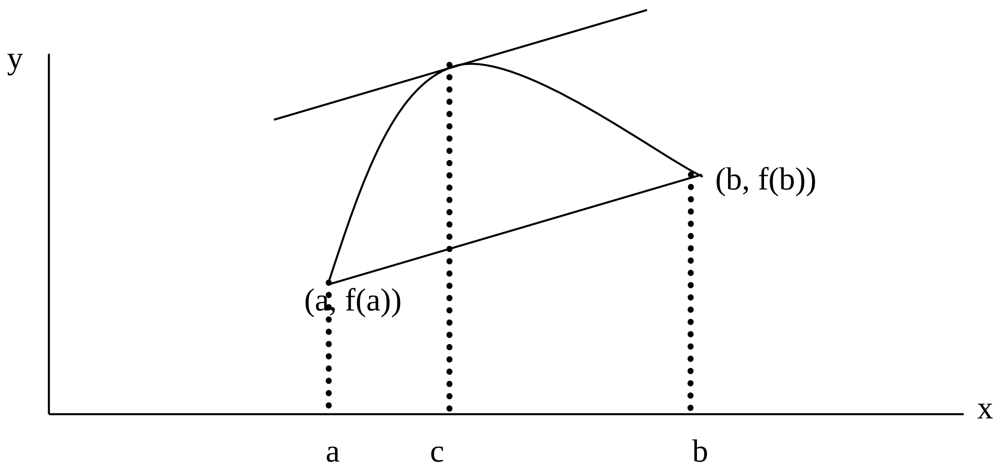
$\therefore$ By Rolle's Theorem, there exists point c in the open interval (a,b) such that $\phi'(c)=0$.

$\phi'(c)=0\Rightarrow f'(c)-k=0\Rightarrow k=f'(c)$..............(3)

From (2) and (3), we conclude that there exists point c in the open interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Thus the Lagrange's Mean Value Theorem (LMVT) is proved.

Geometrical Meaning of Lagrange's Mean Value Theorem: Consider a curve $f(x)$ that satisfies the conditions of the LMVT as shown in figure:



From the figure, we observe that the curve $f(x)$ is continuous in the closed interval $[a,b]$. The curve is smooth i.e. there can be a unique tangent to the curve at any point in the open interval (a,b). Hence by LMVT there exist at least one point c belonging to (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. In other words there exists at least one point at which the tangent drawn to the curve lies parallel to the chord joining the points $[a, f(a)]$ and $[b, f(b)]$.

Other form of LMVT:

Suppose $f(x)$ is continuous in the closed interval $[a, a+h]$ and is differentiable in the open interval $(a, a+h))$ then *there exists* $\theta \in (0,1)$ *such that* $f(a+h) = f(a) + hf'(a+\theta h)$

When $\theta \in (0,1)$ we see that $a+\theta h \in (a, a+h)$ i.e., here $c = a+\theta h$

Using the earlier form of LMVT we may write that $f'(a+\theta h) = \dfrac{f(a+h)-f(a)}{(a+h)-a}$

On simplification this becomes $f(a+h) = f(a) + hf'(a+\theta h)$.

Example 1: Verify LMVT for $f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$.

Solution: (i) $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$ is an algebraic function hence it is continuous in $[0,4]$

(ii) $f'(x) = 3x^2 - 12x + 11$ exists in $(0,4)$ i.e., $f(x)$ is differentiable in $(0,4)$.

Both the conditions of LMVT hold good for $f(x)$ in $[0,4]$.

By LMVT , there exists point c in $(0,4)$ such that $f'(c) = \dfrac{f(4)-f(0)}{4-0}$.

i.e. $3c^2 - 1c + 11 = \dfrac{(3)(2)(1)-(-1)(-2)(-3)}{4-0}$

i.e. $3c^2 - 12c + 11 = 3 \Rightarrow 3c^2 - 12c + 8 = 0 \Rightarrow c = 2 \pm \dfrac{2}{\sqrt{3}} \in (0,4)$

Hence the LMVT is verified.

Example 6: Verify LMVT for $f(x) = \log_e x$ in $[1, e]$.

Solution:

(i) $f(x) = \log_e x$ is an elementary logarithmic function hence continuous in $[1, e]$.

(ii) $f'(x) = \dfrac{1}{x}$ exists in $(1, e)$ i.e. $f(x)$ is differentiable in $(1, e)$.

Both the conditions of LMVT hold good for $f(x)$ in $[1, e]$.

By LMVT , there exists point c in $(1,e)$ such that $f'(c)=\dfrac{f(e)-f(1)}{e-1}$.

$$\Rightarrow \frac{1}{c}=\frac{\log e-\log 1}{e-1}$$

$$\Rightarrow \frac{1}{c}=\frac{1}{e-1}\Rightarrow c=e-1\in(1,e)$$

Hence the LMVT is verified.

Example 3: Verify LMVT for $f(x)=\sin^{-1}x$ in $[0,1]$.

Solution: We find $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$ exists in $(0,1)$, and hence $f(x)$ is continuous in $[0,1]$.

Both the conditions of LMVT hold good for $f(x)$ in $[0,1]$.

By LMVT , there exists point c in $(0,1)$ such that $f'(c)=\dfrac{f(1)-f(0)}{1-0}$.

$$\Rightarrow \frac{1}{\sqrt{1-c^2}}=\frac{\sin^{-1}1-\sin^{-1}0}{1}$$

$$\Rightarrow \frac{1}{\sqrt{1-c^2}}=\frac{\pi}{2}\Rightarrow \sqrt{1-c^2}=\frac{2}{\pi}\Rightarrow c^2=\frac{\pi^2-4}{\pi^2}$$

$$\therefore c=\frac{\sqrt{\pi^2-4}}{\pi}=0.7712\in(0,1)$$

Hence the LMVT is verified.

Example 4: Apply LMVT to show that

$$\frac{b-a}{\sqrt{1-a^2}}<\sin^{-1}b-\sin^{-1}a<\frac{b-a}{\sqrt{1-b^2}},\, where\; 0<a<b$$

Solution: Consider $f(x)=\sin^{-1}x$ in $[a,b]$

We see that $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$ exists in (a,b)

Hence $f(x)$ is differentiable in (a,b) and also it is continuous in $[a,b]$.

Therefore, by LMVT $\exists c\in(a,b): f'(c)=\dfrac{f(b)-f(a)}{b-a}\Rightarrow \dfrac{1}{\sqrt{1-c^2}}=\dfrac{\sin^{-1}b-\sin^{-1}a}{b-a}\rightarrow(1)$

Since $a<c<b\Rightarrow a^2<c^2<b^2\Rightarrow -a^2>-c^2>-b^2\Rightarrow 1-a^2>1-c^2>1-b^2$

$$\therefore \frac{1}{1-a^2} < \frac{1}{1-c^2} < \frac{1}{1-b^2} \text{ or } \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}} \to (2)$$

From (1) and (2), we have

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}} \Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$$

Example 5: Using Lagrange's Mean Value Theorem prove that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$. Hence show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$.

Solution: Let f(x) = $\tan^{-1}(x)$ in the interval (a, b)

$\therefore$ By Lagrange's Mean value Theorem there exist at least one point 'c' such that $a < c < b$

$$\& \quad f'(c) = \frac{f(b) - f(a)}{b-a} \text{ ....(1)}$$

As f(x) = $\tan^{-1}(x) \Rightarrow f'(x) = \frac{1}{1+x^2}$

Equation (1) $\Rightarrow \frac{1}{1+c^2} = \frac{f(b)-f(a)}{b-a} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a}$ ....(2)

But $a < c < b \Rightarrow a^2 < c^2 < b^2 \Rightarrow 1+a^2 < 1+c^2 < 1+b^2 \Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$.............(3)

From Equation (2) replace $\frac{1}{1+c^2}$ by $\frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a}$ in Equation (3)

Therefore
$$\frac{1}{1+b^2} < \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} < \frac{1}{1+a^2}$$
$$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2} \text{..........(4)}$$

For a =1 and b= $\frac{4}{3}$ equation (4) implies

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}(\frac{4}{3}) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1} \Rightarrow \frac{1/3}{25/9} < \tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{1/3}{2}$$

$$\Rightarrow \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}(\frac{4}{3}) < \frac{1}{6} + \frac{\pi}{4}$$

Example 6: If $x > 0$, Apply LMVT to show that

(*i*) $\frac{x}{1+x} < \log_e(1+x) < x$

(ii) $0 < \frac{1}{\log_e(1+x)} - \frac{1}{x} < 1$

Solution: (i) Consider $f(x) = \log_e(1+x)$ in $[0, x]$. We see that $f(x)$ is an elementary logarithmic function hence continuous in $[0, x]$. Also $f'(x) = \frac{1}{1+x}$ exists in $(0, x)$.

$\therefore$ by LMVT $\exists \theta \in (0,1) : f(x) = f(0) + xf'(0+\theta x)$

$i.e., \log(1+x) = \log 1 + \frac{x}{1+\theta x} \Rightarrow \log(1+x) = \frac{x}{1+\theta x}$...........(1)

Since $x > 0$ and $0 < \theta < 1$, we have $0 < \theta x < x \Rightarrow 1 < 1+\theta x < 1+x$

$\Rightarrow 1 > \frac{1}{1+\theta x} > \frac{1}{1+x} \Rightarrow \frac{1}{1+x} < \frac{1}{1+\theta x} < 1 \Rightarrow \frac{x}{1+x} < \frac{x}{1+\theta x} < x$...........(2)

From (1) and (2) we get $\frac{x}{1+x} < \log_e(1+x) < x$

(ii) Also from (1) we have, $\frac{1}{\log(1+x)} = \frac{1+\theta x}{x} \Rightarrow \frac{1}{\log(1+x)} - \frac{1}{x} = \theta$

Since $0 < \theta < 1$, we can write $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$.

Exercise 2

Q.1) Test whether following functions satisfies Lagrange's Mean Value Theorem.If so find appropriate value of c.

a) f(x)= x(x -1)(x-2) where $0 \le x \le \frac{1}{2}$.

b) f(x) $= 2x^3 - 7x + 10$ where $2 \le x \le 5$.

c) $f(x) = |x|$ in the interval $[-1,1]$.

d) $f(x) = x^{\frac{3}{4}}$ in the interval $[0,16]$.

e) $f(x) = \frac{x^2 - x - 6}{x-1}$ in the interval $[-2,3]$.

f) $f(x) = \log x$ in the interval [1,e].

g) $f(x) = \sqrt{9 - x^2}$ in the interval $[-3, 3]$.

h) $f(x) = tan^{-1} x$ in the interval $[0,1]$.

Q.2) Using Lagrange's Mean Value Theorem prove the following:

a) $\dfrac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \dfrac{b-a}{\sqrt{1-b^2}}$ where $a < b < 1$. Prove that $\dfrac{\pi}{6} - \dfrac{1}{2\sqrt{3}} < \sin^{-1}\dfrac{1}{4} < \dfrac{\pi}{6} - \dfrac{1}{\sqrt{15}}$.

b) $1 - \dfrac{a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b}{a} - 1$ where $0 < a < b$. Hence deduce that $\dfrac{1}{6} < \log\left(\dfrac{6}{5}\right) < \dfrac{1}{5}$.

c) $\dfrac{1}{8} < \sqrt{51} - \sqrt{49} < \dfrac{1}{7}$.

(Hint: Let $f(x) = \sqrt{x}$ in the interval [51 ,49].)

Q.3) Using Lagrange's Mean Value Theorem find approximate value of $\sqrt{88}$.

(Hint: Let $f(x) = \sqrt{x}$ in the interval [81 ,88].)

III) Cauchy's Mean Value Theorem (CMVT):

Statement: Suppose two functions $f(x)$ and $g(x)$ satisfies the following conditions:

(i) $f(x)$ and $g(x)$ are continuous in a closed interval $[a,b]$

(ii) $f(x)$ and $g(x)$ are differentiable in the open interval (a,b)

(iii) $g'(x) \neq 0$ *for all x.*

Then there exists at least one point c in the open interval (a,b) such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Proof: Consider the function $\phi(x)$ defined by $\phi(x) = f(x) - k\, g(x)$ where k is a constant to be found such that $\varphi(a) = \phi(b)$.

Since $f(x)$ and $g(x)$ are continuous in the closed interval $[a,b]$, $\phi(x)$ which is a sum of continuous functions is also continuous in the closed interval $[a,b]$

$\phi'(x) = f'(x) - k\, g'(x)$........(1) exists in the interval (a,b) as $f(x)$ and $g(x)$ are differentiable in (a,b).

We have $\phi(a) = f(a) - kg(a)$ and $\phi(b) = f(b) - k\, g(b)$

$$\therefore \phi(a) = \phi(b) \Rightarrow f(a) - kg(a) = f(b) - k\, g(b) \Rightarrow k = \frac{f(b)-f(a)}{g(b)-g(a)} \text{..........(2)}$$

This means that when k is chosen as in (2) we will have $\phi(a) = \phi(b)$

Hence the conditions of Rolle's Theorem hold good for $\phi(x)$ in $[a,b]$.
$\therefore$ By Rolle's Theorem , there exists point c in the open interval (a,b) such that $\phi'(c)=0$.

$$\phi'(c)=0 \Rightarrow f'(c)-k\,g'(c)=0 \Rightarrow k=\frac{f'(c)}{g'(c)}\quad\text{..............(3)}$$

From (2) and (3), we conclude that there exists point c in the open interval (a,b) such that $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$

Thus the Cauchy's Mean Value Theorem (CMVT) is proved.

Example 1: Verify the CMVT for $f(x)=x^2$ and $g(x)=x^4$ in $[a,b]$

Solution: $f(x)=x^2$ and $g(x)=x^4$ are algebraic polynomials hence continuous in $[a,b]$.

$f'(x)=2x$ and $g'(x)=4x^3$ exist in (a,b).

also we see that $g'(x)\neq 0$ *for all* $x\in(a,b)$ since $0<a<b$

i.e., the conditions of CMVT hold good for $f(x)$ and $g(x)$ in $[a,b]$.
Hence $\exists$ $c\in(a,b)$ such that $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$.

i.e. $$\frac{2c}{4c^3}=\frac{b^2-a^2}{b^4-a^4}\Rightarrow\frac{1}{2c^2}=\frac{1}{b^2+a^2}\Rightarrow c=\sqrt{\frac{b^2+a^2}{2}}\in(a,b)$$

Hence the CMVT is verified.

Example 2: Verify the CMVT for $f(x)=\log x$ and $g(x)=\frac{1}{x}$ in $[1,e]$

Solution: $f(x)=\log x$ and $g(x)=\frac{1}{x}$ are elementary logarithmic and rational algebraic functions that are continuous in $[1,e]$.

$f'(x)=\frac{1}{x}$ and $g'(x)=\frac{-1}{x^2}$ exist in $(1,e)$

also we see that $g'(x)\neq 0$ *for all* $x\in(1,e)$

i.e. the conditions of CMVT hold good for $f(x)$ and $g(x)$ in $(1,e)$.

Hence $\exists c\in(1,e)$ such that $\frac{f'(c)}{g'(c)}=\frac{f(e)-f(1)}{g(e)-g(1)}$

i.e. $\dfrac{1/c}{-1/c^2}=\dfrac{\log e-\log 1}{\frac{1}{e}-1}\Rightarrow -c=\dfrac{1}{\frac{1}{e}-1}\Rightarrow c=\dfrac{e}{e-1}\in(1,e)$

Hence the CMVT is verified.

Example 3: Verify the Cauchy's MVT for $f(x)=e^x$ and $g(x)=e^{-x}$ in $[a,b]$.

Solution:

$f(x)=e^x$ and $g(x)=e^{-x}$ are elementary exponential functions that are continuous in $[a,b]$

$f'(x)=e^x$ and $g'(x)=-e^{-x}$ exist in (a,b)

also we see that $g'(x)\neq 0$ *for all* $x\in(a,b)$, since $0<a<b$.

i.e. the conditions of CMVT hold good for $f(x)$ and $g(x)$ in (a,b).

Hence $\exists\, c\in(a,b)$ such that $\dfrac{f'(c)}{g'(c)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

i.e. $\dfrac{e^c}{-e^{-c}}=\dfrac{e^b-e^a}{e^{-b}-e^{-a}}\Rightarrow -e^{2c}=\dfrac{e^b-e^a}{\left[\frac{1}{e^b}-\frac{1}{e^a}\right]}\Rightarrow -e^{2c}=\dfrac{e^b-e^a}{\left[\frac{e^a-e^b}{e^{a+b}}\right]}$

$$e^{2c}=e^{a+b}\Rightarrow c=\frac{a+b}{2}\in(a,b)$$

Hence the CMVT is verified.

Example 4: Verify the Cauchy's MVT for $f(x)=\sin x$ and $g(x)=\cos x$ in $\left[0,\frac{\pi}{2}\right]$

Solution: $f(x)=\sin x$ and $g(x)=\cos x$ are elementary trignometric functions that are continuous in $\left[0,\frac{\pi}{2}\right]$.

$f'(x)=\cos x$ and $g'(x)=-\sin x$ exist in $\left(0,\frac{\pi}{2}\right)$

also we see that $g'(x)\neq 0$ *for all* $x\in\left(0,\frac{\pi}{2}\right)$.

i.e., the conditions of CMVT hold good for $f(x)$ and $g(x)$ in $\left(0,\frac{\pi}{2}\right)$.

Hence $\exists\, c \in \left(0, \frac{\pi}{2}\right)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)}$.

$$\Rightarrow \frac{\cos c}{-\sin c} = \frac{\sin\left(\frac{\pi}{2}\right) - \sin(0)}{\cos\left(\frac{\pi}{2}\right) - \cos(0)} \Rightarrow \cot c = 1 \Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence the CMVT is verified.

Exercise 3

Q.1) Verify Cauchy's Mean Value Theorem for the following:

a) $f(x) = \sin x$ and $g(x) = \cos x$ in interval $\left[0, \frac{\pi}{2}\right]$.

b) $f(x) = x^3$ and $g(x) = \tan^{-1} x$ in interval $[0,1]$.

c) $f(x) = x^2 + 2$ and $g(x) = x^3 - 1$ in interval $[1,2]$.

d) $f(x) = x^2$ and $g(x) = x$ in interval $[a,b]$. Show that c is arithmetic mean of a and b.

e) $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x}$ in interval $[a,b]$. Show that c is harmonic mean of a and b.

f) $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ in interval $[a,b]$. Show that c is geometric mean of a and b.

Q.2) Using Cauchy's Mean Value Theorem prove that if $x \neq 0$ then $1 - \dfrac{x^2}{2} < \cos x$.

(Hint: Let $f(x) = 1 - \cos x$ and $g(x) = \dfrac{x^2}{2}$ in interval [0, x])

Expansion of Functions

TAYLOR'S THEOREM

If f(x+h) is a given function of h which can be expanded into a convergent series of positive ascending integral powers of h then,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$$

This can also be expressed in ascending powers of x

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots$$

Putting x = a and h = x – a in the above expansion, we get Taylor's series in powers of (x – a) as

$$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots}$$

Ex 1: Express $f(x) = 2x^3 + 3x^2 - 8x + 7$ in terms of (x − 2).

Solution : Given $f(x) = 2x^3 + 3x^2 - 8x + 7$

By Taylor's Series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

Put a = 2

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) \text{ ………………. (1)}$$

$f(x) = 2x^3 + 3x^2 - 8x + 7$; f(2)= 19

$f'(x) = 6x^2 + 6x - 8$; $f'(2) = 28$

$f''(x) = 12x + 6$; $f''(2) = 30$

$f'''(x) = 12$; $f'''(2) = 12$

Putting the values in (1), we get

$$f(x) = 19 + 28(x-2) + 30\frac{(x-2)^2}{2!} + 12\frac{(x-2)^3}{3!}$$

$$f(x) = 19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3.$$

Ex 2 : Express $f(x) = x^3 + 7x^2 + x - 6$ in ascending powers of $(x-3)$.

Solution : Given $f(x) = x^3 + 7x^2 + x - 6$

By Taylor's Series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

Put a = 3

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) \text{ ……………….(1)}$$

$f(x) = x^3 + 7x^2 + x - 6$; $f(3) = 87$

$f'(x) = 3x^2 + 14x + 1$; $f'(3) = 70$

$f''(x) = 6x + 14$; $f''(3) = 32$

$f'''(x) = 6$; $f'''(3) = 6$

Putting the values in (1), we get

$$f(x) = 87 + 70(x-3) + 32\frac{(x-3)^2}{2!} + 6\frac{(x-3)^3}{3!}$$
$$f(x) = 87 + 70(x-3) + 16(x-3)^2 + (x-3)^3.$$

Ex 3 : Express $f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$ *in powers of* $(x+2)$.
Solution : Given $f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$
By Taylor's Series,
$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$
Put $a = -2$
$$f(x) = f(-2) + (x+2)f'(-2) + \tfrac{(x+2)^2}{2!}f''(-2) + \tfrac{(x+2)^3}{3!}f'''(-2) \text{ ....(1)}$$
$f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$; $f(-2) = 7$
$f'(x) = 69 + 84x + 33x^2 + 4x^3$; $f'(-2) = 1$
$f''(x) = 84 + 66x + 12x^2$; $f''(-2) = 0$
$f'''(x) = 66 + 24x$; $f'''(-2) = 18$
$f^{(4)}(x) = 24$; $f'''(-2) = 24$
Putting the values in (1), we get
$$f(x) = 7 + (x+2)(1) + \frac{(x+2)^2}{2!}(0) + \frac{(x+2)^3}{3!}(18) + \frac{(x+2)^4}{4!}(24)$$
$$= 7 + (x+2) + 3(x+2)^3 + (x+2)^4.$$

Ex 4 : Prove that $\frac{1}{1-x} = \frac{1}{3} + \frac{x+2}{3^2} + \frac{(x+2)^2}{3^3} + \frac{(x+2)^3}{3^4} + \cdots$
Solution: Let f (x) = $\frac{1}{1-x}$
By Taylor's Series,
$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$
Put $a = -2$
$$f(x) = f(-2) + (x+2)f'(-2) + \tfrac{(x+2)^2}{2!}f''(-2) + \tfrac{(x+2)^3}{3!}f'''(-2) \text{ ... .(1)}$$
$f(x) = \frac{1}{1-x}$; $f(-2) = \frac{1}{3}$
$f'(x) = \frac{1}{(1-x)^2}$; $f'(-2) = \frac{1}{3^2}$
$f''(x) = \frac{2}{(1-x)^3}$; $f''(-2) = \frac{2!}{3^3}$
$f'''(x) = \frac{2.3}{(1-x)^4}$; $f'''(-2) = \frac{3!}{3^4}$
$f^{(4)}(x) = \frac{2.3.4}{(1-x)^5}$; $f^{(4)}(-2) = \frac{4!}{3^5}$
Putting the values in (1) , we get
$$\frac{1}{1-x} = \frac{1}{3} + \frac{x+2}{3^2} + \frac{(x+2)^2}{3^3} + \frac{(x+2)^3}{3^4} + \cdots$$

Ex 5 : Express $f(x) = \tan^{-1} x$ in powers of (x − 1).

Solution : Given f (x) = $\tan^{-1} x$

By Taylor's Series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

Put $a = 1$

$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1)$ ……………(1)

$f(x) = \tan^{-1} x$; f(1) = $\frac{\pi}{4}$

$f'(x) = \frac{1}{1+x^2}$; $f'(1) = \frac{1}{2}$

$f''(x) = \frac{-2x}{(1+x^2)^2}$; $f''(1) = \frac{-1}{2}$

$f'''(x) = \frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}$; $f'''(1) = \frac{1}{2}$

Putting the values in (1) , we get

$$\tan^{-1} x = \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) \ldots$$

$$= \frac{\pi}{4} + \frac{x-1}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} \ldots$$

Ex 6 : Express $7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$
in powers of x

Solution : Let f(x+2) = $7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$

$\therefore f(x) = 7 + x + 3x^3 + x^4 - x^5$

By Taylor's Theorem,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$$

Put h = 2

$$f(x+2) = (7 + x + 3x^3 + x^4 - x^5) + 2(1 + 9x^2 + 4x^3 - 5x^4)$$
$$+ \frac{2^2}{2!}(18x + 12x^2 - 20x^3) + \frac{2^3}{3!}(18 + 24x - 60x^2)$$
$$+ \frac{2^4}{4!}(24 + 120x) + + \frac{2^5}{5!}(-120)$$

$\therefore f(x+2) = 17 - 11x - 38x^2 - 29x^3 - 9x^4 - x^5$

Ex 7 : Express $5 + 4(x-1)^2 - 3(x-1)^3 + (x-1)^4$ in powers of x.

Solution: Let $f(x-1) = 5 + 4(x-1)^2 - 3(x-1)^3 + (x-1)^4$

$\therefore f(x) = 5 + 4x^2 - 3x^3 + x^4$

By Taylor's Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$$

Put $h = -1$

$$f(x-1) = (5 + 4x^2 - 3x^3 + x^4) + (-1)(8x - 9x^2 + 4x^3)$$
$$+ \frac{(-1)^2}{2!}(8 - 18x + 12x^2) + \frac{(-1)^3}{3!}(-18 + 24x)$$
$$+ \frac{(-1)^4}{4!}(24)$$

$\therefore f(x-1) = 13 - 21x - 19x^2 - 7x^3 + x^4.$

Ex 8 : Find the expansion of $\tan\left(x+\frac{\pi}{4}\right)$ in ascending power's of x up to x^4 and find approximately the value of $\tan(43^0)$.

Solution: Let f $\left(x+\frac{\pi}{4}\right) = \tan\left(x+\frac{\pi}{4}\right)$ $\therefore$ f (x) = tan x

By Taylor's Theorem,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots$$

Put $h = \frac{\pi}{4}$

$f\left(x+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right) + \cdots + \frac{x^n}{n!}f^{(n)}\left(\frac{\pi}{4}\right) + \cdots$…………(1)

To find $\tan(43^0)$

$\therefore tan\left(x+\frac{\pi}{4}\right) = tan\left(\frac{\pi}{4}+x\right)$

$\tan(43^0) = tan(45^0 - 2^0) = tan\left(\frac{\pi}{4} - \frac{2\pi}{180}\right) = tan\left(\frac{\pi}{4} - 0.0349\right)$

$= 1 + 2(-0.0349) + 2(-0.0349)^2 + \frac{8}{3}(-0.0349)^3 + \frac{10}{3}(-0.0349)^4$

$\therefore \tan(43^0) = 0.9326\ (approximately)$

Ex 9 : Using Taylor's Theorem evaluate $\sqrt{25.15}$ up to 4 decimals.

Solution: Let f (x) = $\sqrt{x} \Rightarrow f(x+h) = \sqrt{x+h}$

By Taylor's Theorem,

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$

Put $x = 25\ and\ h = 0.15$

$\therefore f(x+h) = \sqrt{x+h} = \sqrt{25+0.15}$

$\sqrt{25.15} = f(25) + (0.15)f'(25) + \frac{(0.15)^2}{2!}f''(25) + \cdots$ ---------------- (1)

$f(x) = \sqrt{x}$; f(25) = 5

$f'(x) = \frac{1}{2\sqrt{x}}$; $f'(25) = \frac{1}{10} = 0.1$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{3/2}}$; $f''(25) = \frac{-1}{500} = -0.002$

Putting the values in (1) and considering the first only three values,

$\sqrt{25.15} = 5 + (0.15)(0.1) + \frac{(0.15)^2}{2!}(-0.002) + \cdots$

$\sqrt{25.15} = 5.0150\ (approximately.)$

Exercise:

1. Express $f(x) = x^4 - 3x^3 + 2x^2 - x + 1$ *in powers of* $(x-1)$.

Ans : $f(x) = -2(x-1) - 2(x-1)^2 + 6(x-1)^3 + 24(x-1)^4$

2. Express $f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$
   In ascending powers of $(x - 1)$.

   Ans : $f(x) = 299 + 286(x-1) + 105(x-1)^2 + 17(x-1)^3 + (x-1)^4$.

3. Express $f(x) = (x+2)^4 + 3(x+2)^3 + (x+2) + 7$ in ascending powers of x . Ans : $f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$.

4. Express $\mathrm{f(x) = sinx}$ about $\mathrm{x} = \frac{\pi}{2}$ using Taylor's theorem .

   Ans : $f(x) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 + \cdots$

5. Express logx in ascending powers of $(x - 1)$.

   Ans : $\mathrm{logx} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$.

6. Exand $\mathrm{f(x) = logcosx}$ about $\mathrm{x} = \frac{\pi}{3}$ using Taylor's theorem .

7. Find approximate value of $\sin(30^0\ 30')$ using Taylor's theorem.

   Ans : $\sin(30^0 30') = 0.50752$

8. Find approximate value of $\cos(48^0)$ using Taylor's theorem.

9. Find approximate value of $\tan^{-1}(1.003)$ using Taylor's theorem.

10. Exand $\mathrm{f(x) = logcos}\left(\mathrm{x} + \frac{\pi}{4}\right)$ in powers of x using Taylor's theorem .

    Hence find the value of $\mathrm{logcos}(48^0)$.

    Ans: $\mathrm{logcos}(48^0) = -0.402$

MACLAURIN'S SERIES:

If $f(x)$ be a given function of x which can be expanded in positive ascending integral powers of x then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

This is known as Maclaurin's Series.

NOTE:

By Taylor's Theorem

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots$$

By putting h = 0 , We get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

which is Maclaurin's Series.

Therefore Maclaurin's Series is special case of Taylor's theorem.

Standard Expansions:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ **Exponential Series.**
2. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots (-1)^n \frac{x^n}{n!} + \cdots$ **Exponential Series.**
3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ **Sine Series.**
4. $\cos x = 1 - \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$ **Cosine Series.**
5. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} - \cdots$ **Tangent Series.**
6. $\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$ **Hyperbolic Sine Series.**
7. $\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$ **Hyperbolic Cosine Series.**
8. $\tanh x = \frac{\sinh x}{\cosh x} = x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots$ **Hyperbolic Tangent Series.**
9. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ **Logarithmic Series.**

   Valid for $-1 < x < 1$
10. $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ **Logarithmic Series.**
11. $(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + ----$ **Binomial Series.**
12. $(1 - x)^m = 1 - mx + \frac{m(m-1)}{2!}x^2 - \frac{m(m-1)(m-2)}{3!}x^3 + ----$
13. $\frac{1}{1+x} = (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 ----$
14. $\frac{1}{1-x} = (1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 ----$

Type 1: Problems on standard expansions:

Ex 1.Expand $\sqrt{1+\sin x}$.

$$Solution: \sqrt{1+\sin x} = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

$$= \left[\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 + \ldots\right] + \left[1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 + \ldots\right]$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \ldots\ldots$$

Ex 2.Expand $\log(1+x+x^2+x^3)$ *upto* x^8.

$$Solution: \log(1+x+x^2+x^3) = \log\left[(1+x)(1+x^2)\right] = \log(1+x) + \log(1+x^2)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \ldots\right] + \left[x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \ldots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + \ldots\ldots$$

Ex 3 : Expand (1+ $x)^{x}$ in ascending powers of x , up to fifth power of x.

Solution: Let $y = (1 + x)^{x}$

$\therefore \log y = x \log(1 + x)$

$$= x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots$$

$$= z \text{ (say)}$$

$$y = e^{z}$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= 1 + \left[x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots\right] + \frac{1}{2!}\left[x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots\right]^2 + \frac{1}{3!}\left[x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots\right]^3 + \cdots$$

$$= 1 + x^2 - \frac{x^3}{2} + \left(\frac{1}{3} + \frac{1}{2}\right)x^4 - \frac{1}{4}x^5 + \cdots$$

$$= 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{1}{4}x^5 + \cdots$$

Ex 4 : Show that $\log\left[\frac{1+e^{2x}}{e^{x}}\right] = log2 + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - - - - - - - -.$

$$\log\left[\frac{1+e^{2x}}{e^{x}}\right] = \log 2\left(\frac{e^{-x}+e^{x}}{2}\right) = \log 2 + \log \cosh x$$

$$= \log 2 + \log\left[1 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + - - - -\right)\right]$$

$$= \log 2 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + - - - -\right)$$

$$- \frac{1}{2}\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + - - - -\right)^2$$

$$+ \frac{1}{3}\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + - - - -\right)^3$$

$$\therefore \log\left[\frac{1+e^{2x}}{e^{x}}\right] = log2 + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - - - - - - - -.$$

Exercise:

1. Expand $\left[\log(1+x)\right]^2$ in ascending powers of x.

$Ans: x^2 - x^3 + \dfrac{11x^4}{12} - \dfrac{10x^5}{12} + ...$

2.Expand $\log\left[\log(1+x)^{1/x}\right]$ in powers of x.

$Ans: -\dfrac{x}{2} + \dfrac{5x^2}{24} - \dfrac{x^3}{8} + ...$

3.Prove that $\log\left[\dfrac{1+e^{2x}}{e^x}\right] = log 2 + \dfrac{x^2}{2} - \dfrac{x^4}{12} + \dfrac{x^6}{45} - .....$

4.Prove that $\log\left(1+\sin x\right) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12} + ...$

5.Prove that $x \cos ecx = 1 + \dfrac{x^2}{6} + \dfrac{7x^4}{360} + ...$

6.Prove that $e^x \cos x = 1 + x - \dfrac{x^3}{3} - \dfrac{x^4}{6} + ...$

7.Prove that $e^{e^x} = e\left(1 + x + x^2 + \dfrac{5x^3}{6} + ...\right).$

8.Expand $e^{\cos x}$ upto x^4.

$Ans: e\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{6} + ...\right).$

9.Prove that $\log\left(\dfrac{\tan x}{x}\right) = \dfrac{x^3}{3} + \dfrac{7x^4}{90} + ...$

10.Prove that $(1+x)^{\frac{1}{x}} = e\left[1 - \dfrac{x}{2} + \dfrac{11}{24}x^2 + ...\right]$

11.Prove that $\log\tan\left(x + \dfrac{\pi}{4}\right) = 2x + \dfrac{4}{3}x^3 + \dfrac{4}{3}x^5 + ...$

Hint: $\log\tan\left(x + \dfrac{\pi}{4}\right) = \log\left(\dfrac{1+\tan x}{1-\tan x}\right) = \log(1+\tan x) - \log(1-\tan x)$

Type 2: Problems on Differentiation and Integration:

Step 1: Differentiate given function w.r.to.x.

Step 2: Use standard expansion on R.H.S of $\frac{dy}{dx}$.

Step 3: Then integrate $\frac{dy}{dx}$. After integration R.H.S will contain constant of Integration.

Step 4: Using given function and Step 3, find value of constant of integration.

Ex 1 : Prove that $\log(\sec x) = \frac{x^2}{2} + \frac{1}{3}\frac{x^4}{4} + \frac{2}{15}\frac{x^6}{6} + \cdots$

Let $y = \log(\sec x) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .. (1)$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{\sec x}\sec x \tan x = \tan x.$$

Using the expansion of tanx,

$$\frac{dy}{dx} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .. (2)$$

Integrating (2) w.r.t. x we get,

$$y = \left[\frac{x^2}{2} + \frac{x^4}{3.4} + \frac{2x^6}{15.6} + \cdots\right] + c \dots \dots \dots \dots \dots \dots \dots \dots \dots (3)$$

Put x = 0 in (1) and (3).

$(1) \Longrightarrow y = \log(\sec 0) = \log(1) = 0$

$(3) \Longrightarrow y = 0 + c$

$\therefore\ c = 0$

$$\therefore \log(\sec x) = \frac{x^2}{2} + \frac{1}{3}\frac{x^4}{4} + \frac{2}{15}\frac{x^6}{6} + \cdots$$

Ex 2 : Prove that $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

Let $y = \tan^{-1}x$……………………(1)

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

Expanding $(1+x^2)^{-1}$ using binomial expansion,

$\therefore \frac{dy}{dx} = \ 1 - x^2 + x^4 - x^6 + x^8 - \cdots$ ………………………..(2)

Integrating (2) w.r.t. x we get,

$$y = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right] + c \dots \dots \dots \dots \dots \dots \dots \dots \dots (3)$$

Put x = 0 in (1) and (3).

$(1) \Longrightarrow y = tan^{-1}0 = 0$

$(3) \Longrightarrow y = 0 + c$

$\therefore\ c = 0$

$$\therefore y = \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Exercise:

Prove the following

1. $\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots$
2. $\cos^{-1}x = \frac{\pi}{2} - \left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots\ \right]$

Type 3: Problems on Substitution

Ex 1. Expand $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in ascending powers of x.

Solution: Let $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x=\tan\theta$

$$\therefore y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}\left(\frac{2\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta}\right)=\sin^{-1}(\sin 2\theta)$$

$$\therefore y=2\theta=2\tan^{-1}x=2\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right]$$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right]$$

Ex 2 : Expand $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$ in ascending powers of x.

Solution:

Let $y=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

Put $x=\tan\theta$

$$\therefore y=\cos^{-1}\left(\frac{\tan^2\theta-1}{\tan^2\theta+1}\right)=\cos^{-1}\left[-\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)\right]$$

$$\therefore y=\cos^{-1}[-\cos 2\theta]=\cos^{-1}[\cos(\pi-2\theta)]=\pi-2\theta=\pi-2\tan^{-1}x$$

$$\therefore y=\pi-2\tan^{-1}x=\pi-2\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right]$$

$$\therefore \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=\pi-2\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right]$$

Exercise:

1.Expand $\sin^{-1}(3x-4x^3)$ in ascending power of x. (Hint : Put x = $\sin\theta$)

2.Prove that $\sec^{-1}\left[\frac{1}{1-2x^2}\right]=2\left[n\pi+x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots\right]$. (Hint : Put x = $\sin\theta$)

3.Prove that $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]=\frac{1}{2}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right)$. (Hint : Put x = $\tan\theta$)

4.Prove that $\cos^{-1}\left(\tanh(\log x)\right)=\pi-2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)$.

(Hint : Use formula $\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ and then put x = $\tan\theta$)

5.Prove that $\tan^{-1}\left[\frac{\sqrt{1-x}}{\sqrt{1+x}}\right]=\frac{\pi}{4}-\frac{1}{2}\left(x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots\right)$. (Hint : Put x = $\cos 2\theta$)

Indeterminate Forms:

While evaluating certain limits, we come across expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0 \, and \, 1^\infty$ which do not represent any value. Such expressions are called Indeterminate Forms. We can Evaluate such limits that lead to indeterminate forms by using L'Hospital's Rule (French Mathematician 1661-1704)

L'Hospital's Rule:

If f(x) and g(x) are two functions such that

(i) $\lim_{x\to a} f(x) = 0 \; and \; \lim_{x\to a} g(x) = 0$ (ii) $f'(x) \; and \; g'(x) \; exist \; and \; g'(a) \neq 0$

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

The above rule can be extended, i.e, if

$$f'(a) = 0 \; and \; g'(a) = 0 \; then \; \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = ...$$

Note:

1. We apply L'Hospital's Rule only to Evaluate the limits that in $\frac{0}{0}, \frac{\infty}{\infty}$ forms. Here we differentiate the numerator and denominator separately to write $\frac{f'(x)}{g'(x)}$ and apply the limit to see whether it is a finite value. If it is still in $\frac{0}{0} \; or \; \frac{\infty}{\infty}$ form we continue to differentiate the numerator and denominator and write further $\frac{f''(x)}{g''(x)}$ and apply the limit to see whether it is a finite value. We can continue the above procedure till we get a definite value of the limit.
2. To Evaluate the indeterminate forms of the form $0\times\infty, \infty-\infty,$ we rewrite the functions involved or take L.C.M. to arrange the expression in either $\frac{0}{0} \; or \; \frac{\infty}{\infty}$ and then apply L'Hospital's Rule.
3. To Evaluate the limits of the form $0^0, \infty^0 \; and \; 1^\infty$ i.e. where function to the power of function exists, call such an expression as some constant, then take logarithm on both sides and rewrite the expressions to get $\frac{0}{0} \; or \; \frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.
4. We can use the values of the standard limits like

$$\lim_{x\to 0}\frac{\sin x}{x} = 1; \; \lim_{x\to 0}\frac{\tan x}{x} = 1; \lim_{x\to 0}\frac{x}{\sin x} = 1; \lim_{x\to 0}\frac{x}{\tan x} = 1; \lim_{x\to 0}\cos x = 1; \; etc$$

Limits of the form $\left(\frac{0}{0}\right)$:

Example 1: Evaluate $\lim\limits_{x\to 0}\frac{\sin x - x}{\tan^3 x}$

$$\lim_{x\to 0}\frac{\sin x - x}{\tan^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x - 1}{3\tan^2 x\sec^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x}{6\tan x\sec^4 x+6\tan^3 x\sec^2 x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\cos x}{6\sec^6 x+24\tan^2 x\sec^4 x+18\tan^2 x\sec^4 x+12\tan^4 x\sec^2 x}=-\frac{1}{6}$$

Method 2:

$$\lim_{x\to 0}\frac{\sin x - x}{\tan^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\sin x - x}{x^3}}{\left(\frac{\tan x}{x}\right)^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x - x}{x^3}\left(\frac{0}{0}\right)\quad \because \lim_{x\to 0}\frac{\tan x}{x}=1$$

$$=\lim_{x\to 0}\frac{\cos x - 1}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x}{6x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$$

Example 2: Evaluate $\lim\limits_{x\to 0}\frac{a^x - b^x}{x}$

$$\lim_{x\to 0}\frac{a^x - b^x}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{a^x\log a - b^x\log b}{1}=\log a-\log b=\log\frac{a}{b}$$

Example 3: Evaluate $\lim\limits_{x\to 0}\frac{x\sin x}{\left(e^x-1\right)^2}$

$$\lim_{x\to 0}\frac{x\sin x}{\left(e^x-1\right)^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x + x\cos x}{2\left(e^x-1\right)e^x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x+\cos x - x\sin x}{2\left[e^x.e^x+(e^x-1)e^x\right]}=\frac{1+1-0}{2[1+0]}=\frac{2}{2}=1$$

Example 4: Evaluate $\lim\limits_{x\to 0}\frac{x\,e^x-\log(1+x)}{x^2}$

$$\lim_{x\to 0}\frac{x\,e^x-\log(1+x)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+x\,e^x-\frac{1}{1+x}}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+e^x+x\,e^x+\frac{1}{(1+x)^2}}{2}=\frac{1+1+0+1}{2}=\frac{3}{2}$$

Example 5: Evaluate $\lim\limits_{x\to 0}\dfrac{\cosh x-\cos x}{x\sin x}$

$$\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sinh x+\sin x}{\sin x+x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cosh x+\cos x}{\cos x+\cos x-x\sin x}=\frac{1+1}{1+1-0}=\frac{2}{2}=1$$

Example 6: Evaluate $\lim\limits_{x\to 0}\dfrac{\cos x-\log(1+x)-1+x}{\sin^2 x}$

$$\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x-\frac{1}{1+x}+1}{\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x+\frac{1}{(1+x)^2}}{2\cos 2x}=\frac{-1+1}{2}=0$$

Example 7: Evaluate $\lim\limits_{x\to 1}\dfrac{x^x-x}{x-1-\log x}$

$$\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)-1}{1-\frac{1}{x}}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)^2+x^{x-1}}{\frac{1}{x^2}}=\frac{1+1}{1}=2$$

$$\left(\begin{array}{l} Since\ y=x^x\Rightarrow \log y=x\log x\Rightarrow \frac{1}{y}y'=1+\log x\Rightarrow y'=y(1+\log x)\\ and\ then\ \frac{d}{dx}(x^x)=x^x(1+\log x)\end{array}\right)$$

Example 8: Evaluate $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\sec^2 x-2\tan x}{1+\cos 4x}$

$$\lim_{x\to \frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}\left(\frac{0}{0}\right)=\lim_{x\to \frac{\pi}{4}}\frac{2\sec^2 x\tan x-2\sec^2 x}{-4\sin 4x}\left(\frac{0}{0}\right)=\lim_{x\to \frac{\pi}{4}}\frac{2\sec^4 x+4\sec^2 x\tan^2 x-4\sec^2 x\tan x}{-16\cos 4x}$$

$$=\frac{2\left(\sqrt{2}\right)^4+4\left(\sqrt{2}\right)^2(1)^2-4\left(\sqrt{2}\right)^2}{16}=\frac{8}{16}=\frac{1}{2}$$

Example 9: Evaluate $\lim\limits_{x\to a}\dfrac{\log(\sin x.\cos ec\ a)}{\log(\cos a.\sec x)}$

$$\lim_{x\to a}\frac{\log(\sin x.\cos ec\ a)}{\log(\cos a.\sec x)}\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{\left[\dfrac{\cos x\cos ec\ a}{\sin x.\cos ec\ a}\right]}{\left[\dfrac{\sec x\tan x.\cos a}{\cos a.\sec x}\right]}=\lim_{x\to a}\frac{\cot x}{\tan x}=\cot^2 a$$

Example 10: Evaluate $\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}$

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x-e^{-x}+2\sin x}{\sin x+x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x}{\cos x+\cos x-x\sin x}=\frac{1+1+2}{1+1-0}=2$$

Example 11: Evaluate $\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$

$$\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x-x\sin x-\frac{1}{1+x}}{2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\sin x-\sin x-x\cos x+\frac{1}{(1+x)^2}}{2}=\frac{-0-0-0+1}{2}=\frac{1}{2}$$

Example 12: Evaluate $\lim_{x\to 0}\frac{\log(1-x^2)}{\log\cos x}$

$$\lim_{x\to 0}\frac{\log(1-x^2)}{\log\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\left[\frac{-2x}{1-x^2}\right]}{\left(\frac{-\sin x}{\cos x}\right)}=\lim_{x\to 0}\frac{2x\cos x}{(1-x^2)\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\cos x-2x\sin x}{(1-x^2)\cos x-2x\sin x}=\frac{2-0}{1-0}=2$$

Example 13: Evaluate $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3\sin^2 x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6\sin x\cos^2 x-3\sin^3 x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6\cos^3 x-12\sin^2 x\cos x-9\sin^2 x\cos x}=\frac{0+2+1}{6-0-0}=\frac{3}{6}=\frac{1}{2}$$

Method 2:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\frac{\tan x-\sin x}{x^3}}{\left(\frac{\sin x}{x}\right)^3}=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\left(\frac{0}{0}\right)\quad\because\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6}=\frac{0+2+1}{6}=\frac{3}{6}=\frac{1}{2}$$

Example 14: Evaluate $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$

$$\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x} = \lim_{x \to 0} \frac{\frac{\tan x - x}{x^3}}{\left(\frac{\tan x}{x}\right)} = \lim_{x \to 0} \frac{\tan x - x}{x^3}\left(\frac{0}{0}\right) \qquad \because \lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \frac{0+2}{6} = \frac{1}{3}$$

Example 15: Evaluate $\lim\limits_{x \to 0} \dfrac{e^{ax} - e^{-ax}}{\log(1+bx)}$

$$\lim_{x \to 0} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{ae^{ax} + ae^{-ax}}{b/(1+bx)} = \frac{a+a}{b} = \frac{2a}{b}$$

Example 16: Evaluate $\lim\limits_{x \to 0} \dfrac{a^x - 1 - x\log a}{x^2}$

$$\lim_{x \to 0} \frac{a^x - 1 - x\log a}{x^2}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{a^x \log a - \log a}{2x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{a^x (\log a)^2}{2} = \frac{1}{2}(\log a)^2$$

Example 17: Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - \log(e+ex)}{x^2}$

$$\lim_{x \to 0} \frac{e^x - \log(e+ex)}{x^2} = \lim_{x \to 0} \frac{e^x - \log e(1+x)}{x^2} = \lim_{x \to 0} \frac{e^x - \log e - \log(1+x)}{x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - \frac{1}{1+x}}{2x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^x + \frac{1}{(1+x)^2}}{2} = \frac{1+1}{2} = 1$$

Exercise 1:

Evaluate the following limits.

(*i*) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

(*ii*) $\lim\limits_{x \to 0} \dfrac{\log(1+x^3)}{\sin^3 x}$

(*iii*) $\lim\limits_{x \to 0} \dfrac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$

(*iv*) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\log \sin x}{(x - \frac{\pi}{2})^2}$

(*v*) $\lim\limits_{x \to 0} \dfrac{\cosh x + \log(1-x) - 1 + x}{x^2}$

(*vi*) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin x \sin^{-1} x}{x^2}$

(*vii*) $\lim\limits_{x \to 0} \dfrac{e^{2x} - (1+x)^2}{x \log(1+x)}$

Limits of the form $\left(\frac{\infty}{\infty}\right)$:

Example 18: Evaluate $\lim\limits_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}$

$$\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{(2\cos 2x/\sin 2x)}{(\cos x/\sin x)}=\lim_{x\to 0}\frac{2\cot 2x}{\cot x}=\lim_{x\to 0}\frac{2\tan x}{\tan 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x}{2\sec^2 2x}=\frac{2}{2}=1$$

Example 19: Evaluate $\lim\limits_{x\to 0}\frac{\log x}{\cos ecx}$

$$\lim_{x\to 0}\frac{\log x}{\cos ecx}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{1\backslash x}{-\cos ec\ x.\cot x}=\lim_{x\to 0}\frac{-\sin^2 x}{x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-2\sin 2x}{\cos x-x\sin x}=\frac{0}{1-0}=0$$

Example 20: Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}$

$$\lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=-\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin x\cos x}{1}=\frac{-0}{1}=0$$

Example 21: Evaluate $\lim\limits_{x\to 1}\frac{\log(1-x)}{\cot\pi x}$

$$\lim_{x\to 1}\frac{\log(1-x)}{\cot\pi x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 1}\frac{-1/(1-x)}{-\pi\cos ec^2\pi x}=\lim_{x\to 1}\frac{\sin^2\pi x}{\pi(1-x)}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{2\pi\sin\pi x\cos\pi x}{-\pi}=\frac{0}{-\pi}=0$$

Example 22: Evaluate $\lim\limits_{x\to 0}\log_{\tan 2x}\tan 3x$

$$\lim_{x\to 0}\log_{\tan 2x}\tan 3x=\lim_{x\to 0}\left(\frac{\log\tan 3x}{\log\tan 2x}\right)\left(\frac{\infty}{\infty}\right)\qquad \because \log_b a=\frac{\log_e a}{\log_e b}$$

$$=\lim_{x\to 0}\left(\frac{3\sec^2 3x/\tan 3x}{2\sec^2 2x/\tan 2x}\right)=\lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right)=\lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{6/\sin 6x}{4/\sin 4x}\right)=\lim_{x\to 0}\left(\frac{6\sin 4x}{4\sin 6x}\right)\left(\frac{0}{0}\right)=\lim_{x\to 0}\left(\frac{24\cos 4x}{24\cos 6x}\right)=\frac{24}{24}=1$$

Example 23: Evaluate $\lim\limits_{x\to a}\dfrac{\log(x-a)}{\log(e^x-e^a)}$

$$\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}\left(\frac{\infty}{\infty}\right)=\lim_{x\to a}\frac{1/(x-a)}{e^x/(e^x-e^a)}=\lim_{x\to a}\frac{(e^x-e^a)}{e^x(x-a)}\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x}=\frac{e^a}{e^a}=1$$

Exercise 2:

Evaluate the following limits.

(i) $\lim\limits_{x\to 0}\dfrac{\log\tan x}{\log x}$ (ii) $\lim\limits_{x\to 0}\dfrac{\log\sin x}{\cot x}$

(iii) $\lim\limits_{x\to 0}\dfrac{\cot 2x}{\cot 3x}$ (iv) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\sec x}{\tan 3x}$ (v) $\lim\limits_{x\to 0}\log_{\sin x}\sin 2x$

Limits of the form $(0\times\infty)$:

To Evaluate the limits of the form $(0\times\infty)$, we rewrite the given expression to obtain either $\left(\dfrac{0}{0}\right)$ or $\left(\dfrac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 24: Evaluate $\lim\limits_{x\to\infty}(a^{\frac{1}{x}}-1)x$

$$\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x\ \ (0\times\infty\ \ form)=\lim_{x\to\infty}\frac{(a^{\frac{1}{x}}-1)}{\left(\frac{1}{x}\right)}\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{a^{\frac{1}{x}}(\log a)\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$=\lim_{x\to\infty}a^{\frac{1}{x}}(\log a)=a^0\log a=\log a$$

Example 25: Evaluate $\lim\limits_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

$$\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x\ \ (0\times\infty\ \ form)=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)}{\cot x}\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\cos ec^2 x}=\frac{0}{1}=0$$

Example 26: Evaluate $\lim\limits_{x\to 1}\sec\dfrac{\pi}{2x}.\log x$

$$\lim_{x\to 1}\sec\frac{\pi}{2x}.\log x\ \left(\infty\times 0\ \ form\right)=\lim_{x\to 1}\frac{\log x}{\cos\dfrac{\pi}{2x}}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{1/x}{-\dfrac{\pi}{2}\left(\sin\dfrac{\pi}{2x}\right)\left(\dfrac{-1}{x^2}\right)}=\lim_{x\to 1}\frac{2x}{\pi\sin\dfrac{\pi}{2x}}=\frac{2}{\pi}$$

Example 27: Evaluate $\lim\limits_{x\to 0}x\log\tan x$

$$\lim_{x\to 0}x\log\tan x\ \left(0\times\infty\ \ form\right)=\lim_{x\to 0}\frac{\log\tan x}{(1/x)}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{\sec^2 x/\tan x}{\left(\dfrac{-1}{x^2}\right)}=\lim_{x\to 0}\frac{-x^2}{\sin x.\cos x}$$

$$=\lim_{x\to 0}\frac{-2x^2}{\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-4x}{2\cos 2x}=\frac{0}{2}=0$$

Example 28: Evaluate $\lim\limits_{x\to 1}(1-x^2)\tan\dfrac{\pi x}{2}$

$$\lim_{x\to 1}(1-x^2)\tan\frac{\pi x}{2}\ \left(0\times\infty\ \ form\right)=\lim_{x\to 1}\frac{1-x^2}{\cot\dfrac{\pi x}{2}}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{-2x}{-\dfrac{\pi}{2}\cos ec^2\dfrac{\pi x}{2}}=\frac{2}{\left(\dfrac{\pi}{2}\right)}=\frac{4}{\pi}$$

Example 29: Evaluate $\lim\limits_{x\to 0}\tan x.\log x$

$$\lim_{x\to 0}\tan x.\log x\left(0\times\infty\ \ form\right)=\lim_{x\to 0}\frac{\log x}{\cot x}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{1/x}{-\cos ec^2 x}=\lim_{x\to 0}\frac{-\sin^2 x}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin 2x}{1}=\frac{0}{1}=0$$

Limits of the form $(\infty-\infty)$:

To Evaluate the limits of the form $(\infty-\infty)$, we take L.C.M. and rewrite the given expression to obtain either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 30: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x}-\cot x\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]=\lim_{x\to 0}\left[\frac{1}{x}-\frac{\cos x}{\sin x}\right]\left(\infty-\infty\ \textit{form}\right)$$

$$=\lim_{x\to 0}\left[\frac{\sin x-x\cos x}{x\sin x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right]$$

$$=\lim_{x\to 0}\left[\frac{x\sin x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}\right]=\frac{0+0}{1+1-0}=0$$

Example 31: Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\left[\sec x-\tan x\right]$

$$\lim_{x\to\frac{\pi}{2}}\left[\sec x-\tan x\right]=\lim_{x\to\frac{\pi}{2}}\left[\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right]\left(\infty-\infty\ \textit{form}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\cos x}{-\sin x}\right]=\frac{0}{1}=0$$

Example 32: Evaluate $\lim\limits_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]$

$$\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]\left(\infty-\infty\ \textit{form}\right)=\lim_{x\to 1}\left[\frac{(x-1)-x\log x}{(x-1)\log x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\left[\frac{1-1-\log x}{\frac{x-1}{x}+\log x}\right]=\lim_{x\to 1}\left[\frac{-\log x}{1-\frac{1}{x}+\log x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 1}\left[\frac{-1/x}{\frac{1}{x^2}+\frac{1}{x}}\right]=\frac{-1}{1+1}=\frac{-1}{2}$$

Example 33: Evaluate $\lim\limits_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]\left(\infty-\infty\ \textit{form}\right)=\lim_{x\to 0}\left[\frac{(e^x-1)-x}{x(e^x-1)}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\left[\frac{e^x-1}{(e^x-1)+xe^x}\right]\left(\frac{0}{0}\right)=\lim_{x\to0}\left[\frac{e^x}{e^x+e^x+xe^x}\right]=\frac{1}{1+1+0}=\frac{1}{2}$$

Example 34: Evaluate $\lim_{x\to0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]$

$$\lim_{x\to0}\left[\frac{1}{\sin x}-\frac{1}{x}\right](\infty-\infty\ \ form)=\lim_{x\to0}\left[\frac{x-\sin x}{x\sin x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\left[\frac{1-\cos x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)==\lim_{x\to0}\left[\frac{\sin x}{\cos x+\cos x-x\sin x}\right]=\frac{0}{1+1}=0$$

Example 35: Evaluate $\lim_{x\to0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]$

$$\lim_{x\to0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\lim_{x\to0}\left[\frac{x-\log(1+x)}{x^2}\right]\left(\frac{0}{0}\right)=\lim_{x\to0}\left[\frac{1-\frac{1}{1+x}}{2x}\right]\left(\frac{0}{0}\right)=\lim_{x\to0}\left[\frac{\frac{1}{(1+x)^2}}{2}\right]=\frac{1}{2}$$

Example 36: Evaluate $\lim_{x\to0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]$

$$\lim_{x\to0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]=\lim_{x\to0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right](\infty-\infty\ \ form)=\lim_{x\to0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to0}\left[\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{x\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to0}\left[\frac{a.\frac{1}{a}\cos\frac{x}{a}-\cos\frac{x}{a}+\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]$$

$$=\lim_{x\to0}\left[\frac{\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to0}\left[\frac{\frac{1}{a}\sin\frac{x}{a}+\frac{x}{a}.\frac{1}{a}.\cos\frac{x}{a}}{\frac{1}{a}\cos\frac{x}{a}+\frac{1}{a}\cos\frac{x}{a}-\frac{x}{a^2}\sin\frac{x}{a}}\right]=\frac{0+0}{\frac{1}{a}+\frac{1}{a}-0}=0$$

Exercise 3:

Evaluate the following limits.

(i) $\lim_{x\to a}\left(2-\frac{x}{a}\right)\cot(x-a)$ (ii) $\lim_{x\to 0}\left(\operatorname{cos}ecx-\cot x\right)$

(iii) $\lim_{x\to\frac{\pi}{2}}\left[x\tan x-\frac{\pi}{2}\sec x\right]$ (iv) $\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)$

(v) $\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{x\tan x}\right]$ (vi) $\lim_{x\to\frac{\pi}{2}}\left[2x\tan x-\pi\sec x\right]$

Problems on constants:

Example 37: Find the value of 'a' such that $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$ is finite. Also find the value of the limit.

Solution: Let A$=\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\cos 2x+a\cos x}{3x^2}=\frac{2+a}{0}\neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if $2+a=0$ $i.e.$ $a=-2$

$For\ a=-2,$

$$A=\lim_{x\to 0}\frac{2\cos 2x-2\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-4\sin 2x+2\sin x}{6x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-8\cos 2x+2\cos x}{6}=\frac{-8+2}{6}=-1$$

$\therefore$ *The given* lim *it will have a finite value when* $a=-2$ *and it is* -1.

Example 38: Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}=\frac{1}{3}$.

Solution:

$Let\ A=\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}=\frac{1-a+b}{0}\neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if 1-a+b=0 i.e., a-b = 1.

$For\ a-b=1,$

$$A=\lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2a\sin x+ax\cos x-b\sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{3a\cos x - ax\sin x - b\cos x}{6} = \frac{3a-b}{6} = \text{finite}$$

$$\text{This finite value is given as } \frac{1}{3}.\ i.e., \frac{3a-b}{6} = \frac{1}{3} \Rightarrow 3a-b=2$$

$$\text{Solving the equations } a-b=1 \text{ and } 3a-b=2 \text{ we obtain } a=\frac{1}{2} \text{ and } b=-\frac{1}{2}.$$

Example 39: Find the values of ‘a’ and ‘b’ such that $\lim_{x \to 0} \frac{a\cosh x - b\cos x}{x^2} = 1$**.**

Solution: $\text{Let } A = \lim_{x \to 0} \frac{a\cosh x - b\cos x}{x^2} = \frac{a-b}{0} \neq \text{finite}$

We can continue to apply the L’Hospital’s Rule, if a-b=0, since the denominator=0 .

$\text{For } a-b=0$**,**

$$A = \lim_{x \to 0} \frac{a\cosh x - b\cos x}{x^2}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{a\sinh x + b\sin x}{2x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{a\cosh x + b\cos x}{2} = \frac{a+b}{2}$$

$\text{But this is given as } 1.$

$\therefore a+b=2$

$$\text{Solving the equations } a-b=0 \text{ and } a+b=2 \text{ we obtain } a=1 \text{ and } b=1.$$

Exercise 4:

Q.1) Find the value of a, b and c, if

(i) $\lim_{x \to 0} \frac{x(a+b\cos x) - c\sin x}{x^5} = 1;$ (ii) $\lim_{x \to 0} \frac{a\tanh x + b\sin x + cx}{x^5} = \frac{7}{60};$

(iii) $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2.$

Q.2) Find the value of a, b if

(i) $\lim_{x \to 0} \frac{a\sin^2 x + b\log\cos x}{x^4} = -\frac{1}{2}.$ (ii) $\lim_{x \to 0} \frac{a\sinh x + b\sin x}{x^3} = \frac{5}{3}.$

Limits of the form $0^0, \infty^0$ and 1^∞ :

To Evaluate such limits, where function to the power of function exists, we call such an expression as some constant, then take logarithm on both sides and rewrite the expressions to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply the L’Hospital’s Rule.

Example 40: Evaluate $\lim_{x\to 0} x^x$

$$Let\ A=\lim_{x\to 0} x^x\ (0^0\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log x^x=\lim_{x\to 0} x.\log x\ (0\times\infty\ \ form)=\lim_{x\to 0}\frac{\log x}{1/x}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0}\frac{1/x}{(-1/x^2)}=\lim_{x\to 0}\frac{-x}{1}=\frac{0}{1}=0$$

$$\log_e A=0\Rightarrow A=e^0=1\quad \therefore \lim_{x\to 0} x^x=1$$

Example 41: Evaluate $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

$$Let\ A=\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}\ (1^\infty\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log(\cos x)^{\frac{1}{x^2}}=\lim_{x\to 0}\frac{1}{x^2}\log\cos x\ (\infty\times 0\ \ form)=\lim_{x\to 0}\frac{\log\cos x}{x^2}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\tan x}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sec^2 x}{2}=\frac{-1}{2}$$

$$\log_e A=-\frac{1}{2}\Rightarrow A=e^{\frac{-1}{2}}=\frac{1}{\sqrt{e}}\quad \therefore \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}=\frac{1}{\sqrt{e}}$$

Example 42: Evaluate $\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$

$$Let\ A=\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}\ (\infty^0\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to\frac{\pi}{2}}\log(\tan x)^{\cos x}=\lim_{x\to\frac{\pi}{2}}\cos x\log(\tan x)\ (0\times\infty\ \ form)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\log\tan x}{\sec x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x/\tan x}{\sec x.\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}=\frac{0}{1}=0$$

$$\log_e A=0\Rightarrow A=e^0=1\quad \therefore \lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}=1$$

Example 43: Evaluate $\lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}}$

$$Let\ A=\lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}}\ (1^{\infty}\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log(\frac{\tan x}{x})^{\frac{1}{x^2}}=\lim_{x\to 0}\frac{1}{x^2}\log(\frac{\tan x}{x})\ (\infty\times 0\ form)$$

$$=\lim_{x\to 0}\frac{\log(\frac{\tan x}{x})}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\sin x.\cos x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{\frac{2}{\sin 2x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{2x-\sin 2x}{2x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2-2\cos 2x}{4x\sin 2x+4x^2\cos 2x}\left(\frac{0}{0}\right)==\lim_{x\to 0}\frac{-4\sin 2x}{4\sin 2x+16x\cos 2x-8x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-8\cos 2x}{24\cos 2x-48x\sin 2x-16x^2\cos 2x}=\frac{-8}{24}=\frac{-1}{3}$$

$$\log_e A=-\frac{1}{3}\Rightarrow A=e^{-\frac{1}{3}}\quad \therefore \lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}}=e^{-\frac{1}{3}}$$

Example 44: Evaluate $\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}$

$$Let\ A=\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}\ (1^{\infty}\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to 0}\log(a^x+x)^{\frac{1}{x}}=\lim_{x\to 0}\frac{1}{x}\log(a^x+x)\ (\infty\times 0\ form)$$

$$=\lim_{x\to 0}\frac{\log(a^x+x)}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(a^x\log a+1)/(a^x+x)}{1}$$

$$=\log a+1=\log a+\log e=\log ae$$

$$\therefore \log_e A=\log ea\Rightarrow A=ea\qquad Hence\ \lim_{x\to 0}(a^x+x)^{\frac{1}{x}}=ea.$$

Example 45: Evaluate $\lim\limits_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

$$Let\ A=\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} \quad (1^\infty\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to a}\log\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}=\lim_{x\to a}\tan\frac{\pi x}{2a}.\log\left(2-\frac{x}{a}\right)\ \ (\infty\times 0\ \ form)$$

$$=\lim_{x\to a}\frac{\log(2-\frac{x}{a})}{\cot\frac{\pi x}{2a}}\left(\frac{0}{0}\right)=\lim_{x\to a}\left(\frac{\frac{(-1/a)}{2-\frac{x}{a}}}{-\frac{\pi}{2a}\cos ec^2\frac{\pi x}{2a}}\right)=\lim_{x\to a}\frac{2}{\pi}.\frac{\sin^2\frac{\pi x}{2a}}{2-\frac{x}{a}}=\frac{2}{\pi}$$

$$\therefore \log_e A=\frac{2}{\pi}\Rightarrow A=e^{\frac{2}{\pi}} \qquad Hence\ \lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}=e^{\frac{2}{\pi}}.$$

Exercise 5:

Evaluate the following limits.

$(i)\ \ \lim\limits_{x\to 0}(\cos ax)^{\frac{b}{x^2}}$ $\qquad (ii)\ \ \lim\limits_{x\to 0}\left(\frac{1+\cos x}{2}\right)^{\frac{1}{x^2}}$

$(iii)\ \ \lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $\qquad (iv)\ \ \lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$(v)\ \ \lim\limits_{x\to 0}(\sin x)^{\tan x}$ $\qquad (iv)\ \ \lim\limits_{x\to 0}(1+\sin x)^{\cot x}$

$(vii)\ \ \lim\limits_{x\to 0}(\cos x)^{\cos ec^2 x}$ $\qquad (viii)\ \ \lim\limits_{x\to \frac{\pi}{4}}(\tan x)^{\tan 2x}$

$(ix)\ \ \lim\limits_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^{x}$ $\qquad (x)\ \ \lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$