Matrices

A Matrix is a two dimensional arrangement of numbers in row and column enclosed by a pair of square brackets or can say matrices are nothing but the rectangular arrangement of numbers, expression, symbols which are arranged in column and rows. Matrices find many applications in scientific field and apply to practical real life problem. Matrices can be solved physical related application and in the study of electrical circuits, quantum mechanics and optics, with the help of matrices, calculation of battery power outputs, resistor conversion of electrical energy into another useful energy, these matrices play a role in calculation, with the help of matrices problem related to Kirchhoff law of voltage and current can be easily solved.

Definition

A matrix is an arrangement of numbers in the forms of rows and columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

Types of Matrices

1. **Row Matrix** :- Row matrix (or row vector) is a matrix with one row.
   $$\mathbf{r} = \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{pmatrix}$$

2. **Column Matrix** :- Column vector is a matrix with only one column.
   $$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

3. **Square Matrix** :-When the row and column dimensions of a matrix are equal (m = n) then the matrix is called a square matrix. In other words A matrix is said to be square, if the number of rows and number of columns are equal.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

4. **Scalar Matrix** :- A square matrix in which all non diagonal elements are zero is called scalar matrix.

5. **Diagonal Matrix** :- A Scalar matrix in which all diagonal elements are same is called diagonal matrix.

6. **Unit Matrix** :- A Diagonal matrix in which all diagonal elements are one (1) is called unit matrix.
7. **Identity Matrix** :- A square matrix in which elements of main diagonal are 1 and other elements are zero is called an identity matrix.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. **Transpose of a Matrix :-**The transpose of the (*m* x *n*) matrix ***A*** is the (*n* x *m*) matrix formed by interchanging the rows and columns such that row *i* becomes column *i* of the transposed matrix.

$$\text{If A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ then } \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

9. **Upper Triangular Matrix :-**A matrix in which elements below main diagonal are zero is called Upper Triangular Matrix.

$$U = \begin{bmatrix} u_{11} & 0 & 0 & 0 \\ u_{21} & \mathrm{u}_{22} & 0 & 0 \\ \mathrm{u}_{31} \mathrm{u}_{32} & \mathrm{u}_{33} & 0 & \end{bmatrix}$$

10.Lower Triangular Matrix :-A matrix in which elements below main diagonal are zero is called Lower Triangular Matrix.

$$L = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ 0 & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

11.Singular Matrix :- A square matrix is said to be singular if $|A| = 0$

12.Non –Singular Matrix :- A square matrix is said to be non-singular if $|A| \neq 0$

13. Symmetric Matrix :- A square Matrix A is said to be symmetric matrix if $A=A^T$

14.Skew Symmetric Matrix :- A square matrix A is said to be skew symmetric if $A= -A^T$

Algebra of Matrices

- **Matrix Equality**

  Two (m x n) matrices A and B are equal if and only if each of their elements are equal. i.e. A = B if and only if $a_{ij} = b_{ij}$ for i = 1,...,m; j = 1,...,n.

- **Matrix Addition**
  If A and B are two matrices then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{2n} + b_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

- **Scalar Multiplication**
  Multiplication of a matrix A by a scalar defined as

$$\alpha\mathbf{A} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & & \alpha a_{2n} \\ \vdots & & \ddots & \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{pmatrix}$$

- **Matrix Multiplication**
  The product of two matrices A and B is defined only if the number of columns of A is equal to the number of rows of B. If A is (m x p) and B is (p x n), the product is an (m x n) matrix C. i.e. $\mathbf{C}_{m\times n} = \mathbf{A}_{m\times p}\,\mathbf{B}_{p\times n}$

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & & a_{1p} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{1n} \\ \vdots & & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11}+\cdots+a_{1p}b_{p1} & a_{11}b_{12}+\cdots+a_{1p}b_{p2} & \cdots & a_{11}b_{1n}+\cdots+a_{1p}b_{pn} \\ a_{21}b_{11}+\cdots+a_{2p}b_{p1} & a_{21}b_{12}+\cdots+a_{2p}b_{p2} & & a_{21}b_{1n}+\cdots+a_{2p}b_{pn} \\ \vdots & & \ddots & \vdots \\ a_{m1}b_{11}+\cdots+a_{mp}b_{p1} & a_{m1}b_{12}+\cdots+a_{mp}b_{p2} & \dots & a_{m1}b_{1n}+\cdots+a_{mp}b_{pn} \end{pmatrix}$$

- **Matrix Inverse**
  If A is an (n x n) square matrix and there is a matrix B with the property that $\mathbf{A}B = \mathbf{I}$. Then B is defined to be the inverse of A and is denoted $A^{-1.}$
  Inverse of a matrix by adjoint method

$$A^{-1} = \frac{1}{|A|}\,adjA$$

  adjA= transpose of Cofactor matrix

  Note:- Matrix A should be a non singular matrix. i.e. $|A| \neq 0$

- **Elementary Transformation**
  The following three types of transformations, performed on any non zero matrix A, are called elementary transformations.
  1. The interchange of i^{th} and j^{th} row denoted by R_{ij} (same for column i.e. C_{ij})

2. The multiplication of each element of i^{th} row by non zero scalar K is denoted by Kr_i

3. Multiplication of each element of j^{th} row by scalar k and adding to the corresponding element of i^{th} row is denoted by (R_i+kR_j).

- **Minor**
  The minor of an element of matrix A is a determinant obtained by omitting the row and the column in which the element is present.

Rank of a Matrix

The matrix A is said to be rank of r if

1) At least one minor of the order r which is not equal to zero.

2) Every minor of the order r+1 is equal to zero .

The rank of matrix A is the maximum order of its non vanishing minor. it is denoted as $\rho(A) = r$

Rank of Matrix by Echelon form

Echelon form is an upper or lower triangular matrix.

Procedure

Step-1: Convert given matrix A into an upper triangular matrix using only elementary row transformations.

Step-2 : Rank of A = No. of non zero rows

Example:- Find rank of matrix A by reducing the matrix into Echelon form

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \end{bmatrix}$$

Solution :- Given

$$A=\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \end{bmatrix}$$

$R_2 - 4R_1$, $R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)=2$

Rank of Matrix by Normal Form

By performing elementary row and column transformations, any non zero matrix A can be reduced to one of the following four forms, called the normal form

$$(1)\ \begin{bmatrix} I_r \end{bmatrix} \quad (2)\ \begin{bmatrix} \mathrm{I_r} & 0 \end{bmatrix} \quad (3)\ \begin{bmatrix} \mathrm{I_r} \\ 0 \end{bmatrix} \quad (4)\ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Where I_r is identity matrix and 0 denotes zero row or column.

Then $\rho(A)$ = order of identity matrix = r

Example :- Find rank of matrix A by reducing it into normal form where

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

Solution :- Given

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

By R_2-2R_1, R_3-3R_1 $\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & -5 & 0 \end{bmatrix}$

By C_2-C_1, C_3-C_1 $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 2 \\ 0 & -5 & 0 \end{bmatrix}$

By $\frac{-1}{5}(C_2)$, $\frac{1}{2}(C_3)$ $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

By R_3-R_2 $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

By C_3-C_2 $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

By $-1(R_3)$ $\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= [I_3]$

Therefore $\rho(A)=3$

System of Linear Equations

Introduction

Consider a system of m linear equations in n unknowns of the form

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
\vdots \quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m
\end{array}
$$

The matrix form of the system

$$
\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{21} \cdots a_{2n} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ a_{m1} & a_{m1} \cdots a_{mn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

i.e. AX=B

where A is called the coefficient matrix, X is called the variable matrix and B is called the constant matrix.

Non- Homogeneous System of Equations

The system AX=B is called non-homogeneous system if matrix B is not a null matrix or zero matrix.

Augmented Matrix(A,B)

If AX=B is a system of m equations in n unknowns then the matrix (A,B) is called augmented matrix and is written as

$$
(\mathrm{A}, \mathrm{B}) = \begin{bmatrix} a_{11} \; a_{12} \cdots \; a_{1n} & \vdots\, \mathrm{b}_1 \\ a_{21} \; a_{22} \cdots a_{2n} & \vdots\, \mathrm{b}_2 \\ \vdots \quad \vdots \quad \cdots \; \vdots & \vdots \\ a_{m1} \; a_{m1} \cdots a_{mn} & \vdots\, \mathrm{b}_\mathrm{m} \end{bmatrix}
$$

Consistency of the System

The system of equations is said to be consistent if it has solution.

Condition for Consistency of the system :

1. **Consistent System :** If $\rho(A) = \rho(A, B)$, then system is consistent.
   (a)**Unique solution :** If $\rho(A) = \rho(A, B) = \text{No. of unknowns}$
   then system has a unique solution.
   (b)**Infinitely many solutions :** If $\rho(A) = \rho(A, B) < \text{No. of unknowns}$
   then system has Infinitely many solutions.
2. **Inconsistent System :** if $\rho(A) \neq \rho(A, B)$
   then the system is inconsistent and has no solution.

How to solve Non-Homogeneous System of Equation

Step1: Write matrix form of the system. i.e. AX=B.

Step 2 : Write Augmented matrix (A,B).

Step 3 : Find rank of matrix (A,B) by row echelon form.

Step 4 : Check the consistency of the system and then find the solution.

Example : Examine for consistency of the system and if consistent then solve it.

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

Solution : The matrix form of system is

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

i.e. AX=B

Consider augmented matrix

$$(A,B)=\begin{bmatrix} 3 & 3 & 2 & \vdots & 1 \\ 1 & 2 & 0 & \vdots & 4 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix}$$

$$get\ a_{11}=1,\ \mathrm{R}_1 \leftrightarrow \mathrm{R}_2 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 3 & 3 & 2 & \vdots & 1 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix}$$

$$get\ a_{21}, a_{31}, a_{41}=0,$$

$$R_2-3R_1, R_4-2R_1 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 10 & 3 & \vdots & -2 \\ 0 & -7 & -1 & \vdots & -3 \end{bmatrix}$$

$$R_3+3R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & -7 & -1 & \vdots & -3 \end{bmatrix}$$

$$get\ a_{22}=1, R_2 \leftrightarrow R_3 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & -7 & -1 & \vdots & -3 \end{bmatrix}$$

$$R_3+3R_2, R_3+7R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & 0 & 29 & \vdots & -116 \\ 0 & 0 & 62 & \vdots & -248 \end{bmatrix}$$

$$get\ a_{43}=1,\ \frac{1}{29}R_3, \frac{1}{62}R_4 \Rightarrow \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$R_4 - R_3 \Rightarrow \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is in echelon form

$\because \quad \rho(A) = \rho(A,B) = \text{No. of unknowns} = 3$

so system is consistent and has a unique solution

Now by back substitution

$z = -4$ and $y + 9z = -35$

$\Rightarrow \quad y + 9(-4) = -35$

Therefore $y = 1$

Now from $x + 2y = 4$

$x + 2(1) = 4 \Rightarrow x = 2$

so $x = 2, y = 1, z = -4$

which is a unique solution

Example 2: Examine for consistency of the system and if consistent then solve it.

$$2x - 3y + 5z = 1$$
$$3x + y - z = 2$$
$$x + 4y - 6z = 1$$

Solution : Matrix form of the system

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -1 \\ 1 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

i.e. AX=B

Consider the augmented matrix

$$(A, B) = \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{array}\right]$$

$$R_2 - 3R_1, R_3 - 2R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{array}\right]$$

$$R_3 - R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Matrix is in echelon form

here $\rho(A) = \rho(A, B) = 2 < \text{No. of unknowns}$

so system is consistent and has infinitely many solution.

So by back substitution for $-11y + 17z = -1$ and $x + 4y - 6z = 1$

now there are two equations in three unknowns. So we can choose 3-2=1 parameter
Let z = t be the parameter

$$\text{from } R_2: \quad -11y+17t=-1$$
$$11y=1+17t$$
$$y=\frac{1+17t}{11}$$
$$\text{Now from } R_1: \quad x+\left(\frac{1+17t}{11}\right)-6t=1$$
$$11x+4+68t-66t=11$$
$$11x+2t=7$$
$$11x=7\text{-}2t$$
$$x=\frac{7\text{-}2t}{11}$$
$$\text{solution set is} \quad x=\frac{7\text{-}2t}{11},\ y=\frac{1+17t}{11},\ z=t$$

For infinitely many value of t system has infinitely many solutions.

Example 3: Investigate for what values of k the System has infinite no. of solutions.

$$x+y+z=1$$
$$2x+y+4z=k$$
$$4x+y+10z=k^2$$

Solution: Matrix form of the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

i.e. AX=B

Consider the Augmented matrix

$$(A,B)=\left[\begin{array}{ccc|c}1&1&1&1\\2&1&4&k\\4&1&10&k^2\end{array}\right]$$

$$R_2-2R_1, R_3-4R_1 \qquad \Rightarrow\left[\begin{array}{ccc|c}1&1&1&1\\0&-1&2&k-2\\0&-3&6&k^2-4\end{array}\right]$$

$$R_3-3R_2 \qquad \Rightarrow\left[\begin{array}{ccc|c}1&1&1&1\\0&-1&2&k-2\\0&0&0&k^2-4-3k+6\end{array}\right]$$

$$(A,B)=\left[\begin{array}{ccc|c}1&1&1&1\\0&-1&2&k-2\\0&0&0&k^2-3k+2\end{array}\right]$$

so matrix is in echelon form and for system to be consistent and infinitely many solutions

$$k^2-3k+2=0$$
$$(k-2)(k-1)=0$$
$$k=2, k=1$$

For k=1

$$(A,B)=\left[\begin{array}{ccc|c}1&1&1&1\\0&-1&2&-1\\0&0&0&0\end{array}\right]$$

The equations are

by R_1: $\quad x+y+z=1$

and by R_2: $\quad -y+2z=-1$

$\text{let } z = t$

$\Rightarrow y = 2t + 1 \text{ and } x = -3t$

Therefore for k=1

$x = -3t, \ \ y = 2t + 1, \ \ z = t$

Similarly for k = 2

$x = 3t, \ \ y = 2t, \ \ z = t$

Alternate Method
(For Non homogeneous system of equations with 3 equations in 3 unknowns)
Method:

Consider non homogeneous system of equations AX=B

Step1. If $|A| \neq 0$ then the system is consistent and has a unique solution which is given by $X = A^{-1}B, \text{ where } A^{-1} = \frac{1}{|A|} adjA$

'this is known as matrix inversion method.'

Step 2. If $|A| = 0$ then system is consistent or inconsistent and has infinitely many solutions or no solution.

Example: Solve the system of equation

$$x + y + 2z = 8$$
$$-x - 2y + 3z = 1$$
$$3x - 7y + 4z = 10$$

Solution:- Matrix form of the system

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

i.e. AX=B

Since there are 3 equations in 3 unknowns

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{vmatrix} = 52 \neq 0$$

So A^{-1} exists

$$adjA = \begin{bmatrix} 13 & -18 & 7 \\ 13 & -2 & -5 \\ 13 & 10 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adjA}$$

$$A^{-1} = \frac{1}{52}\begin{bmatrix} 13 & -18 & 7 \\ 13 & -2 & -5 \\ 13 & 10 & -1 \end{bmatrix}$$

now $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 13 & -18 & 7 \\ 13 & -2 & -5 \\ 13 & 10 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 156 \\ 52 \\ 104 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Homogeneous System of Equations

The system AX=B is called homogeneous system if matrix B is a null matrix. It is denoted by AX=Z. Here A is coefficient matrix, X is variable matrix and Z is null matrix.

Note:- Homogeneous system is always consistent.

How to solve Homogeneous System of Equation

Step 1. Write given system of equations in matrix form AX=Z.
Step 2. Write augmented matrix (A,Z).
Step 3. Find rank of matrix (A,Z) by Echelon form.
Case 1. If ρ(A)= ρ(A,B) = Number of variables, then the system has trivial solution x = 0, y = 0, z = 0.
Case 2. If ρ(A)= ρ(A,B) < Number of variables, then the system has non-trivial solution
Step 4. Write equations in terms of x,y,z
Step 5. Find the values of x,y,z.

Example : Examine for non trivial solutions the following set of equations and solve them

$$x+y+2z=0$$
$$x+2y+3z=0$$
$$x+3y+4z=0$$
$$x+4y+7z=0$$

Solution : Matrix form of the system

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. AX=Z

Consider the augmented matrix

$$(A,Z)=\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & 0 \\ 3 & 4 & 7 & 0 \end{array}\right]$$

by $R_2-R_1, R_3-R_1, R_4-3R_1$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

now by $R_3 - 2R_1, R_4 - R_1$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Matrix is in echelon form

now $\rho(A) = \rho(A, Z) = 2 <$ no. of unknowns

So the system is consistent and has a non-trivial solution

now from $R_2 \quad y + z = 0$

and from $R_1 \quad x + y + 2z = 0$

there are 2 equations in 3 unknowns so we choose 3-2=1 parameter

let $z = t$

from R_2 : $\quad y + t = 0$

$\Rightarrow \quad y = -t$

from R_1 : $\quad x - t + 2t = 0$

$\Rightarrow \quad x = -t$

Therefore non-trivial solution is $x = -t, y = -t, z = t$

Example: Solve the system of equations

$$x + 2y + 3z = 0$$
$$2x + 3y + z = 0$$
$$4x + 5y + 4z = 0$$
$$x + y - 2z = 0$$

Solution : Matrix form of the system

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. AX=Z

Consider the augmented matrix

$$(A, Z) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 0 \\ 4 & 5 & 4 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

now by $R_2 - 2R_1, R_3 - 4R_1, R_4 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -3 & -8 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

by $R_3 - 3R_2, R_4 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

above matrix is in echelon form and

$\rho(A) = \rho(A, Z) = 3 = \text{number of unknowns}$

So the system is consistent and has a trivial solution.

Solution is $x = 0, y = 0, z = 0$

<u>Alternate Method</u>

For homogeneous system of equations with 3 equations in three unknowns

Method:

Consider the system of equations AX=Z and Find

(a) If $|A|=0$, then system possesses a nontrivial solution. The solution can be obtained by rank method.

(b) If $|A|\neq 0$, then the system possesses trivial solution.

Linear Transformation

A general linear transformation is the mirror image $Q(x_2,y_2)$ of a point $P(x_1,y_1)$ along a straight line ax+by=c. A general linear transformation is represented by

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

In matrix form it can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \cdots \quad \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

i.e. Y=AX

This gives a linear transformation from n variables $x_1, x_2, \ldots. x_n$ to $y_1, y_2, \ldots. y_n$.
The inverse transformation is given by $X=A^{-1}Y$.

Properties of linear Transformation

- Matrix A is called linear operator.
- If $|A|=0$, then the transformation is called singular.
- If $|A|\neq 0$, then the transformation is called non-singular or regular.
- If matrix A is a symmetric matrix then the transformation is called symmetric Transformation.
- If matrix A is a skew symmetric matrix then the transformation is called skew symmetric Transformation.
- If matrix A is orthogonal matrix($AA^T=I$) then the transformation is called orthogonal transformation.

Example: Given the transformation

$$Y = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find the co-ordinates (x_1,x_2,x_3) corresponding to (3,0,8) in Y.

Solution: Consider Y = AX.

Where $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= -5 \neq 0$$

Since A^{-1} exists therefore the inverse transformation is given by $X = A^{-1}Y$

where $A^{-1} = \frac{1}{|A|} adj(A)$

$$A^{-1} = \frac{-1}{5} \begin{bmatrix} 0 & -1 & -1 \\ 5 & 5 & -5 \\ 5 & 2 & -3 \end{bmatrix}$$

$X = A^{-1}Y$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 5 & 5 & -5 \\ 5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

By equality of matrices

$$x_1 = \frac{8}{5}, \; x_2 = 5, \; x_3 = \frac{9}{5}$$

Eigen Values and Eigen Vectors

An eigen vector is a vector that maintains its direction after undergoing a linear transformation. Eigen vectors are vectors that point in directions where there is no rotation . Therefore the eigen vectors of a square matrix are the non-zero vectors that, after being multiplied by the matrix remain parallel to, the original vector.

Eigen values are the change in length of the eigen vector from the original length. Therefore eigen value is the factor by which the eigen vector is scaled when multiplied by a matrix.

Definition :
Let A be a square matrix then there exist a scalar λ and a non zero vector X such that $AX=\lambda X$. Then λ is called characteristic value or Eigen value of A and X is called characteristic vector or Eigen vector.

The eigen value λ tells whether the vector X is stretched or shrunk or reversed or left unchanged when it is multiplied by a matrix A.

Applications in Engineering:

- Frequencies are used in electrical systems. When we tune our radio, we change the resonant frequency until it matches the frequency at which the station is broadcasting. Engineers use eigen values when they design the radio.
- Car Designers analyze eigen values in order to damp out the noise so that occupant have a quite ride. Eigen value analysis is also used in the design of car stereo system so that the sounds are directed correctly for the listening pleasure of the passengers and driver.
- Eigen values can be used to test for cracks or deformities in a solid. When a beam is struck, its natural frequency (eigen value) can be heard. If the beam rings then it is not flawed. A dull sound will result from a flawed beam, because the flawed causes the eigen values to change.

Properties:

- The sum of the eigen values of a matrix equals the trace of the matrix. (Trace of matrix=Sum of diagonal elements)
- Product of the eigen values is equal to determinant of matrix.
- The eigen values of an upper (or lower) triangular matrix are the elements on the main diagonal.

- If **λ** is an eigen value of **A** and **A** is invertible, then **1/λ** is an eigen value of matrix $\mathbf{A}^{-1}$.
- The matrix (A - KI) has the eigen values λ – K.
- If $\lambda_1,\lambda_2,----,\lambda_n$ are the eigen values of matrix A, then $k\lambda_1,k\lambda_2,----,k\lambda_n$ are eigen values of matrix kA.
- If $\lambda_1,\lambda_2,----,\lambda_n$ are the eigen values of matrix A, then $\lambda_1^m,\lambda_2^m,----,\lambda_n^m$ are eigen values of matrix A^m.
- The eigen values of a symmetric matrix are real.
- If $\lambda_1,\lambda_2,----,\lambda_n$ are distinct eigen values of matrix A, then corresponding eigen vectors $X_1,X_2,----X_n$ form a linearly independent set .
- A square matrix of order n may have 'n' linearly independent eigen vectors or less than n.
- Eigen vectors of a symmetric matrix corresponding to distinct eigen values are orthogonal. i.e.$X_1^TX_2=0$
- Eigen vector of a square matrix cannot correspond to two distinct eigen values.

Eigen values

Let X be an eigen vector of the matrix **A.** Then there must exist an eigen value **λ** such that $AX = \lambda X$

$$\Rightarrow \quad AX - \lambda X = 0$$

$$\Rightarrow \quad (A-\lambda I)X = 0$$

This is homogeneous system and has solution if, $|A-\lambda I| = 0$

This is called the **characteristic equation** of A. Its roots determine the Eigen values of A.

Note:- 1) For 3×3 matrix $|A-\lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A|$

2) For 2×2 matrix $|A-\lambda I| = \lambda^2 - S_1\lambda + |A|$

Here S_1= Trace of A, S_2= Sum of minors of diagonal elements of A.

Eigen Vectors

Procedure to find Eigen vectors :

- Consider matrix equation $(A-\lambda I)X = 0$
- Substitute value of λ in the equation $(A-\lambda I)X = 0$, Where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- Write equations in terms of x, y, z
- Solve the equations by Cramer's rule.

Examples:

Type 1: When matrix is non-symmetric/symmetric and Eigen values are distinct

Example : Find Eigen values and Eigen vectors for $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution: The characteristic equation is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Here S_1 = Trace of A= sum of diagonal elements= 1+2-1=2

S_2 = Sum of minors of diagonal elements

$$\therefore \ S_2 = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}$$

$$|A| = 1(-2-1) - 1(1-0) - 2(-1-0) = -2$$

Therefore characteristic equation is $\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$

So Eigen values are λ=1,-1,2

Eigen vector for λ=1

Put λ=1 in the equation $(A - \lambda I)X = 0$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equations are

$0x + y - 2z = 0 \qquad \cdots(1)$

$-x + y + z = 0 \qquad \cdots(2)$

$0x + y - 2z = 0 \qquad \cdots(3)$

By using Cramer's Rule for equation(1) and equation(2)

$$\frac{x}{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{-y}{-2} = \frac{z}{1} = t \quad \text{(say)}$$

Therefore $x = 3t, y = 2t, z = t$

So $X_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} t$ or $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Similarly for $\lambda = -1$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$ or $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and for $\lambda = 2$, $X_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

- **Type 2**: When matrix is non-symmetric and Eigen values are repeated

Q. Find Eigen values and Eigen vectors for $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution:- The characteristic equation is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A|$

Here $S_1 = 9, S_2 = 15, |A| = 7$

Therefore characteristic equation is

$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$

$\Rightarrow \quad \lambda = 7,1,1$

Therefore Eigen values are $\lambda = 7,1,1$

Eigen vector for $\lambda = 7$

Substitute $\lambda = 7$ in the equation $(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The equations are

$$-5x + y + z = 0 \qquad \cdots(1)$$

$$2x - 4y + 2z = 0 \qquad \cdots(2)$$

$$3x + 3y - 3z = 0 \qquad \cdots(3)$$

By using Cramer's rule for equation (2) and equation (3)

$$\frac{x}{\begin{vmatrix} -4 & 2 \\ 3 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -4 \\ 3 & 3 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x}{6} = \frac{-y}{-12} = \frac{z}{18}$$

$$\Rightarrow \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t \qquad \text{(say)}$$

Therefore Eigen vector for λ=7 is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t$ or $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Now put λ=1 in the equation $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From above we have only one equation $x + y + z = 0$

Note: Here we have only one equation and three unknowns so we choose

[3-1=2] parameters

let $y = t, z = u \Rightarrow x = -t - u$

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t-u \\ t \\ u \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -u \\ 0 \\ u \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Therefore $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Type 3: When matrix is symmetric and Eigen values are distinct

Example: Find Eigen values and vectors for $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Solution:- Since matrix is symmetric therefore Eigen vectors are orthogonal.

The characteristic equation is $|A - \lambda I| = 0$

Therefore $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Here S_1=3, S_2=-9, $|A|$=5

So the characteristic equation is $\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$

$\Rightarrow \quad \lambda = 5, -1, -1$

Therefore Eigen values are $\lambda = 5, -1, -1$

Eigen vector for λ=5

Substitute λ=5 in the equation $(A-\lambda I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The equations are

$$-4x+2y+2z=0 \qquad \cdots(1)$$
$$2x-4y+2z=0 \qquad \cdots(2)$$
$$2x+2y-4z=0 \qquad \cdots(3)$$

By using Cramer's rule for equation (1) and equation (2)

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ -4 & 2 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} -4 & 2 \\ 2 & 2 \end{vmatrix}}=\frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\Rightarrow \qquad \frac{x}{12}=\frac{-y}{-12}=\frac{z}{12}$$

OR

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=t \qquad \text{(say)}$$

Therefore eigen vector is $X_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}t$ or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Eigen vector for λ=-1

Substitute λ= -1 in the equation $(A-\lambda I)X=0$

$$\Rightarrow \qquad \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So we have only one equation $2x+2y+2z=0$ or $x+y+z=0$

substitute $y=t, z=u \Rightarrow x=-t-u$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -t-u \\ t \\ u \end{bmatrix}=\begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}+\begin{bmatrix} -u \\ 0 \\ u \end{bmatrix}=t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+u\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $X_2=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Choose Eigen vector $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ such that $X_1^T X_3 = 0$ and $X_2^T X_3 = 0$

Therefore $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$ and $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$

$\therefore\ l + m + n = 0$ and $-l + m + 0n = 0$

Solving equations by Cramer's rule

$$\frac{l}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}}$$

$$\therefore \quad \frac{l}{-1} = \frac{m}{-1} = \frac{n}{2} = t \quad \text{(say)}$$

So $\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} t$ or $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

Therefore Eigen vector for $\lambda = 5$ is $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

And for $\lambda = -1$, $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

Type 4: Only one Eigen vector for repeated Eigen value

By substituting repeated eigen value in the equation $(A - \lambda I)X = 0$ if we get two distinct equation then there will be only one Eigen vector corresponding to the repeated Eigen value.

Example: Find Eigen values and Eigen vectors for $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$

Solution: Given matrix is an upper triangular matrix so Eigen values are $\lambda=2,2,2$

Eigen vector for λ=2

Substitute λ=2 in the equation (A-λI)X=0

$$\Rightarrow \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Equations are

$$0x+y+0z=0 \quad \cdots(1)$$
$$0x+0y+z=0 \quad \cdots(2)$$

By using Cramer's rule

$$\frac{x}{\begin{vmatrix}1&0\\0&1\end{vmatrix}}=\frac{-y}{\begin{vmatrix}0&0\\0&1\end{vmatrix}}=\frac{z}{\begin{vmatrix}0&1\\0&0\end{vmatrix}}$$

$$\Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z}{0}=t \quad \text{(say)}$$

$\therefore\ X=\begin{bmatrix}1\\0\\0\end{bmatrix}$ is the only Eigen vector for repeated Eigen value λ=2

Caley Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

Example: Verify Cayley-Hamilton Theorem for matrix $A=\begin{bmatrix}2&1&1\\3&-1&2\\2&1&3\end{bmatrix}$

and use it to find A^{-1}

Solution:- The characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 1 & 1 \\ 3 & -1-\lambda & 2 \\ 2 & 1 & 3-\lambda \end{vmatrix}=0$$

$$|A-\lambda I|=\lambda^3-S_1\lambda^2+S_2\lambda-|A|=0$$

Here $S_1=4,\ S_2=-6, |A|=-10$

Therefore characteristic equation is $\lambda^3-4\lambda^2-6\lambda+10=0$

Using Cayley-Hamilton Theorem we have

$$A^3-4A^2-6A+10I=0$$

Now we have to show that

$$A^3-4A^2-6A+10I=0 \qquad \text{---(1)}$$

Now

$$A^2=\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}=\begin{bmatrix} 9 & 2 & 7 \\ 7 & 6 & 7 \\ 13 & 4 & 13 \end{bmatrix}$$

$$A^3=A^2\times A=\begin{bmatrix} 9 & 2 & 7 \\ 7 & 6 & 7 \\ 13 & 4 & 13 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}=\begin{bmatrix} 38 & 14 & 34 \\ 46 & 8 & 40 \\ 64 & 22 & 60 \end{bmatrix}$$

Now substitute values of A^3, A^2, A and I in L.H.S. of equation (1)

L.H.S. $=A^3-4A^2-6A+10I$

$$\therefore \text{ L.H.S.}=\begin{bmatrix} 38 & 14 & 34 \\ 46 & 8 & 40 \\ 64 & 22 & 60 \end{bmatrix}-4\begin{bmatrix} 9 & 2 & 7 \\ 7 & 6 & 7 \\ 13 & 4 & 13 \end{bmatrix}-6\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}+10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \text{ L.H.S.}=\begin{bmatrix} 38 & 14 & 34 \\ 46 & 8 & 40 \\ 64 & 22 & 60 \end{bmatrix}-\begin{bmatrix} 36 & 8 & 28 \\ 28 & 24 & 28 \\ 52 & 16 & 52 \end{bmatrix}-\begin{bmatrix} 12 & 6 & 6 \\ 18 & -6 & 12 \\ 12 & 6 & 18 \end{bmatrix}+\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$\therefore \quad \text{L.H.S} = \begin{bmatrix} 38-36-12+10 & 14-8-6+0 & 34-28-6+0 \\ 46-24-18+0 & 8-24+6+10 & 40-28-12+0 \\ 64-52-12+0 & 22-16-6+0 & 60-52-18+10 \end{bmatrix}$$

$$\text{L.H.S.} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ L.H.S. = R.H.S

Therefore Cayley-Hamilton theorem is verified.

Now from equation (1)

$A^3 - 4A^2 - 6A + 10I = 0$

Multiply both the sides by A^{-1}

$$\Rightarrow \quad A^2 - 4A - 6I + 10A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{10}\left[-A^2 + 4A + 6I\right]$$

$$\therefore \quad A^{-1} = \frac{1}{10}\begin{bmatrix} 5 & 2 & -3 \\ 5 & -4 & 1 \\ -5 & 0 & 5 \end{bmatrix}$$