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THE FREQUENCY-MAGNITUDE RELATION OF MICROFRACTURING IN ROCK AND ITS RELATION TO EARTHQUAKES

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ABSTRACT

During the deformation of rock in laboratory experiments, small cracking events, i.e., microfractures, occur which radiate elastic waves in a manner similar to earthquakes. These radiations were detected during uniaxial and triaxial compression tests and their frequency-magnitude relation studied. They were found to obey the Gutenberg and Richter relation

$$\log N = a + bM$$

where N is the number of events which occurred of magnitude M, and a and b constants. The dependence of the parameter b on rock type, stress, and confining pressure was studied. It was found to depend primarily on stress, in a characteristic way. The frequency-magnitude relation for events which accompanied frictional sliding and deformation of a ductile rock was found to have a much higher b value than that observed in brittle rock. The Gutenberg and Richter formulation of the frequency-magnitude relation was derived from a statistical model of rock and crustal deformation. This analysis demonstrates the basis of similarity between rock deformation experiments in the laboratory and deformation of the crust.

Introduction

The deformation of brittle rock in laboratory experiments is accompanied by very small scale cracking. Such microscopic cracking, referred to as *microfracturing*, can be studied directly by the detection and analysis of the elastic radiation produced by each event in a manner analogous to that used in studying earthquakes. In contrast to seismic waves, however, microfracturing radiation is typified by very high frequencies. The main part of the power spectrum of such events has been found to lie approximately from 100 khz to 1 Mhz (Scholz, 1968a).

Using an experimental system capable of operating with signals in such a frequency range, we have shown (Scholz, 1968b) that the pattern of microfracturing activity observed in rock deformed at a constant strain rate in compression can be directly related to the inelastic stress-strain behavior typical of rock. This pattern was found to be similar for a variety of rock types over a wide range of confining pressure. The observed microfracturing behavior was found to be adequately described by a statistical model of rock deformation in which rock was treated as an inhomogeneous elastic medium.

In the statistical analysis of a process such as microfracturing, a rather important relation, and one which is fairly simple to study experimentally, is the frequency of occurrence of events as a function of amplitude. This functional dependence, which is known in seismology as the frequency-magnitude or recurrence relation, is a fundamental property of the stochastic process.

Mogi (1962a) and Vinogradov (1959, 1962) studied experimentally the frequency-magnitude relation of microfracturing events in rock. A surprising result was that this relationship was found to be the same as that for earthquakes, i.e., frequency was a power function of maximum trace amplitude. In terms of magnitude, this is given by the Gutenberg and Richter relation

$$\log N = a + bM \tag{1}$$

where N is the frequency of events that occur at a given magnitude M, and a and b are constants.

Mogi's (1962a, 1962b, 1962c, 1963a, 1963b, 1963c) observations were part of a study of the general microfracturing behavior of rock subjected to bending stresses. He demonstrated that in many ways the statistical behavior of microfracturing activity observed in laboratory experiments is similar to that which has been observed for earthquakes. In addition to his observation of the similarity of the frequency-magnitude relations, he showed that the buildup of activity preceding fracture in laboratory tests is similar to earthquake foreshock sequences. In later work, he reproduced foreshock and aftershock sequences in models. By treating microfracturing as a stochastic process, he was able to describe some of the properties of the phenomenon and to relate these qualitatively to the degree of heterogeneity of the model material.

A serious drawback to Mogi's experimental observations and those of other previous microfracturing investigators was that they were limited to the audio frequency range. Because the frequency content of microfracturing events is primarily much higher than this, his conclusions are open to question. In the related investigation (Scholz, 1968b) a system was used which has a bandwidth which extends well into the microfracturing frequency range. It was found that the increased frequency response of this system resulted in a sensitivity of detection which was higher by several orders of magnitude than that of the previous workers. Therefore, it was decided to extend Mogi's study of the frequency-magnitude relation using this system. In addition, it was of interest to see if this relation could also be observed under the more geologically realistic conditions of compressive stress and high confining pressure.

Although the mechanism of shallow earthquakes is apparently some sort of fracture process, Mogi's contention that microfracturing observed in the laboratory is simply a scale model of seismicity is not immediately obvious. If the basis of the similarity can be clearly understood, however, then we shall have in experimental studies of rock deformation a tool for understanding the physical processes responsible for earthquake phenomena. In this study, by observing the dependence of the parameters of the frequency-magnitude relation on several physical variables and by extending the basic model of microfracturing, as outlined by Scholz (1968b), to explain these observations, we shall attempt to clarify some of the problems raised by Mogi's work. What, for example, is the physical significance of the observed frequency-magnitude relation, and what determines the values of the parameters a and b in equation (1)?

EXPERIMENTAL TECHNIQUE

The experimental method used in this study is similar to that reported by Scholz (1968b). The rocks tested have also been described there. Briefly, the procedure consisted of stressing a rock sample to fracture as in a conventional uniaxial or triaxial compression test. Microfracturing activity occurring within the sample was detected with an attached barium titanate piezoelectric transducer. The signal was amplified, shaped, and admitted to a 100 channel pulse height analyzer. The amplification system had a flat frequency response from 100 hz to 1 Mhz. The major change in technique in the present experiments was that the analyzer was operated in pulse height analyzer mode. In this mode, each channel corresponds to a successively higher increment in amplitude, so that as the rock is stressed, each event occurring within a predetermined period of time (live time) is counted in the channel corresponding to its maximum trace amplitude. The signals are typically of the form of decaying sinusoids with periods of a few microseconds (Scholz, 1968a). The analyzer measures the amplitude of each event 1 μsec after the initial rise. Consequently, in order to insure that the amplitudes measured were nearly the maximum for the events, and to prohibit an event from triggering the analyzer more than once, the signals had to be shaped. This was done with a circuit (Scholz, 1967) which has as its output the envelope of the negative half of the wave train. This was found to give very satisfactory results.

Each channel was adjusted so that it covered a width of 1 mv in amplitude. The upper and lower limits of analysis were set at 100 mv and 1 mv, respectively, referred to input at the analyzer, so that two orders of magnitude in amplitude were analyzed. The number of events that occurred at each amplitude level in the live time was printed out digitally and the frequency-magnitude relation determined simply by plotting this data in logarithmic coordinates.

Previous workers determined the frequency-magnitude relation of microfracturing during fracture experiments by analyzing all the events which were detected over the entire duration of the experiment. Due to the greatly increased sensitivity of the present system, the frequency-magnitude relation could be determined much more accurately, and the change in this behavior during the course of an experiment could be followed in considerable detail. The frequency-magnitude relation was determined as a function of stress, rock type, and confining pressure in uniaxial and triaxial compression tests.

EXPERIMENTAL OBSERVATIONS

Typical results are given in Figure 1, which shows frequency versus maximum trace amplitude in logarithmic coordinates at two different stress levels in a uniaxial compression test on San Marcos gabbro. The data represented by each curve was collected as the rock was stressed over an interval of several hundred bars. The stresses indicated are the simple midpoints of the intervals. Notable is the strong linearity of the curves; as with earthquakes, the data very closely fits a power function. The most convenient way of analyzing this data is to refer it to the Ishimoto-Iida statistical relation

$$n(a)\,da = ka^{-m}\,da \tag{2}$$

where $n(a)$, the amplitude frequency, is the rate that events of amplitude a to $a + da$ occur, a is the maximum trace amplitude, and k and m constants. The most important parameter, m, is simply the slope of the line fitting a plot such as shown in Figure 1. However, the above relation is equivalent to the Gutenberg and Richter instrumental magnitude form, and Suzuki (1959) has shown that the constant b

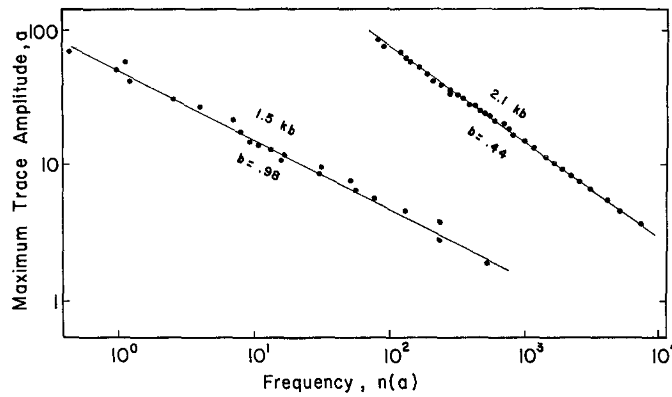

FIG. 1. Amplitude frequency versus maximum trace amplitude at two stress levels in the uniaxial compression of San Marcos gabbro. Note the different b values.

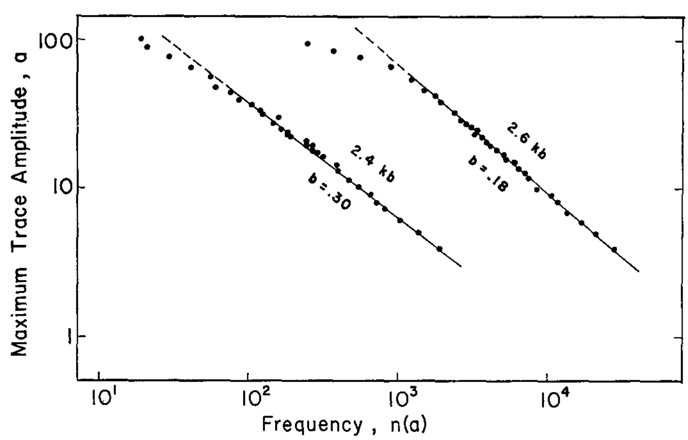

FIG. 2. Frequency versus maximum trace amplitude at two stress levels in the uniaxial compression of Westerly granite. Notice the deviation from linearity at high amplitude.

in the latter form is related to m by

$$b = m - 1. \tag{3}$$

We, therefore, shall use the more familiar parameter b in the discussion of our results. Notice in Figure 1 that b is not constant throughout the experiment but decreases markedly with an increase in stress.

Some of the rocks tested, however, do not behave in such a simple way. Figure 2 shows the characteristic behavior of Westerly granite at several stresses. At low and moderate amplitudes, the frequency-magnitude relation is very similar to the more typical case shown in Figure 1, but fewer large amplitude events have occurred than would have been expected from a linear extrapolation of the data. This deviation is very much like that observed for earthquakes of magnitude 8 and above (cf. Gutenberg and Richter, 1949). It will be mentioned below in the discussion of the theoretical results that this deficiency of large events may be a result of the finite dimensions of the sample.

The observation that the parameter b is a function of stress prompted a more detailed examination of this behavior. A group of rocks with widely differing physical properties were fractured in uniaxial compression, and magnitude-frequency curves such as those shown in Figure 1 were determined for a number of stress intervals during loading. A single rock, Westerly granite, was chosen to examine the effect of confining pressure. In these experiments, the frequency-magnitude relation was determined as a function of stress at several confining pressures.

The rocks were stressed at a constant rate of about 10^{-5} sec^{-1}. Events which occurred over successive stress intervals of several hundred bars were sampled and the values of b were computed from curves such as shown in Figure 1. The data obtained for the uniaxial experiments, given in Table 1, is summarized as a plot of b versus normalized stress for the various rocks in Figure 3. The error bar given in the diagram is an average value; the error is larger at lower stresses, where a smaller amount of events were detectable, and smaller at higher stresses.

Perhaps the most striking thing about this data is that the curves for the various rocks are very similar although they are shifted relative to one another.

In order to discuss these results, the general nature of microfracturing activity in brittle rock, which was described by Scholz (1968b), must be reviewed briefly. Activity was divided into three successive stages. At low stress in unconfined experiments, microfracturing activity was attributed to frictional sliding on pre-existing cracks and the crushing of pores. In the second stage, at stresses from about 30 to 50 per cent of the fracture strength, rock was nearly linearly elastic and few events could be detected. In the third state, microfracturing resumes and steadily increases until fracture. In this latter region, the rock becomes dilatant, i.e., if we disregard elastic compression, the rock increases in volume. In this stage, microfracturing radiation was shown to be due to the propagation of new cracks. Therefore, we can consider the behavior shown in Figure 3 in two parts, separated by a central region in which little or no activity was observed. In the region below about 30 per cent of the fracture stress, where the events analyzed are probably due mainly to sliding on pre-existing cracks, the b values are very high, indicating a preponderance of small events. Above about 60 per cent of the strength, where the signals are radiated from propagating cracks, the b values fall between 1 and 0, with a consistent tendency for b to decrease as stress is raised. This indicates that as stress is increased, the events become statistically larger. Note that the b values in this latter region are in the range usually observed for earthquakes. The significance of this will be brought out later when it is suggested that the statistics of the process in this region should be the same as that of earthquakes.

A marble was also studied, with quite different results. This rock, which deforms cataclastically at atmospheric pressure, exhibits a frequency-magnitude relation

TABLE 1

b VALUES FOR VARIOUS ROCKS STRESSED IN UNIAXIAL COMPRESSION

Rock	σ, kb.	σ/C*	b
Westerly granite	.30	.12	1.32
	.40	.25	.68
	1.70	.64	.47
	2.00	.74	.44
	2.20	.82	.41
	2.40	.91	.30
	2.60	.98	.18
San Marcos gabbro	.25	.11	1.70
	.50	.22	1.90
	1.10	.49	1.20
	1.50	.68	.98
	1.90	.84	.72
	2.10	.93	.44
	2.20	.97	.40
Colorado rhyolite tuff	.05	.05	2.58
	.15	.16	1.78
	.20	.21	1.50
	.25	.26	1.28
	.65	.65	.84
	.85	.86	.67
	.90	.92	.60
	.92	.97	.50
Marble	.05	.08	1.48
	.15	.25	1.52
	.25	.41	1.91
	.40	.66	2.50
	.52	.86	1.95
	.57	.95	1.90
Pottsville sandstone	.25	.10	1.55
	.50	.19	2.20
	1.00	.38	.66
	1.60	.60	.64
	1.85	.69	.66
	2.00	.75	.30
	2.20	.83	.38
	2.40	.91	.37
	2.50	.94	.11
Rutland quartzite	.75	.13	.95
	1.25	.23	1.62
	4.00	.73	.60
	4.75	.87	.53
	5.20	.95	.42
	5.25	.96	.41
	5.30	.97	.29

* C refers to stress at fracture.

that is similar in form to that found for the brittle rocks but is characterized by very high b values over the entire range of stress (see Table 1). The similarity of these b values to those found in the low stress region of the brittle rocks re-emphasizes the conclusions (Scholz, 1968b) that cataclastic deformation is indeed due to stable intergranular sliding.

Under confining pressure, much the same behavior is observed. Microfracturing in triaxial compression of Westerly granite at confining pressures up to 5 kb has the same frequency-magnitude relation as that described in the case of uniaxial

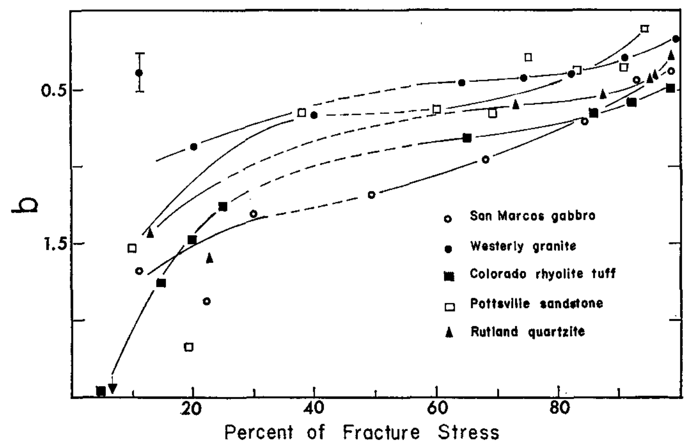

FIG. 3. b as a function of normalized stress for five rocks in uniaxial compression. The dashed part of the curves are in the region where few events were detected.

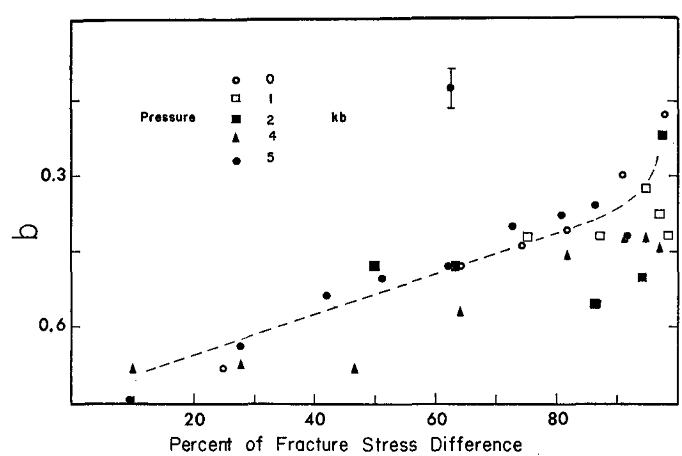

FIG. 4. b versus normalized stress difference for Westerly granite at five different confining pressures.

compression. Figure 4 shows b as a function of stress difference normalized with respect to fracture stress difference at five different confining pressures. Stress difference is defined in the usual way as $(\sigma_1 - \sigma_3)$, where σ_1 and σ_3 are the major and minor compressive stresses respectively. This data is also listed in Table 2.

The stress dependence of b at each pressure has the same general characteristics observed in the uniaxial experiments. Again, b falls between 1 and 0 and shows a steady decrease with stress. The high b values in the low stress region are not observed in the triaxial experiments. This behavior is expected because confining pressure keeps cracks tightly closed and restricts sliding. The most interesting feature of Figure 4 is that within experimental error, pressure seems to have no effect on the stress dependence of b if it is normalized with respect to the fracture stress. For brittle rocks such as Westerly granite, strength increases rapidly with pressure so that at a higher pressure the same value of b will occur at a much higher absolute stress.

TABLE 2
b VALUES FOR MICROFRACTURING IN WESTERLY GRANITE AT VARIOUS CONFINING PRESSURES

Pressure, kb.	$(\sigma_3 - \sigma_1)$, kb.	$(\sigma_3 - \sigma_1)/(\sigma_3 - \sigma_1)_f$*	b
1.0	6.50	.76	.42
	7.50	.88	.42
	8.10	.95	.32
	8.25	.97	.38
	8.40	.99	.43
2.0	5.90	.50	.48
	8.40	.71	.48
	10.20	.87	.56
	11.20	.94	.50
	11.60	.98	.22
4.0	1.58	.10	.69
	4.50	.28	.68
	7.40	.47	.69
	10.10	.64	.58
	12.80	.82	.46
	14.20	.91	.43
	14.90	.95	.43
	15.30	.97	.45
5.0	1.60	.09	.75
	5.00	.28	.64
	7.60	.42	.54
	9.10	.51	.50
	11.10	.62	.48
	13.00	.73	.41
	14.40	.81	.38
	15.40	.86	.36
	16.40	.92	.42

* $(\sigma_3 - \sigma_1)_f$ refers to stress difference at fracture.

As described by Scholz (1968b) the microfracturing activity that accompanied frictional sliding of brittle rock was quite different from that which occurred during deformation of brittle rock but similar to that during cataclastic deformation of marble. Since we have observed above that the frequency-magnitude relation of events detected during cataclastic deformation of marble is quite distinct from that of microfracturing which occurs in the high stress region of brittle rock, it would be interesting to see if the same holds for frictional sliding. We have also noted above that the b values typical of microfracturing which occurs in the low stress region during uniaxial tests are similar to those typical of cataclastic deformation. In the light of our interpretation of the mechanism involved in that region, the b values observed in frictional sliding experiments should be, as in those cases, characteristically high.

Frictional sliding of brittle rock under high confining pressure is punctuated by jerky stick-slip events (Brace and Byerlee, 1966) which produce large sudden stress drops. Preceding each stick-slip, however, there is usually some stable sliding, during which microfracturing activity is detectable.

Several experiments were done in a manner similar to those of Brace and Byerlee. In these tests, Westerly granite samples with pre-existing ground sliding surfaces were stressed under high confining pressure. In each test, microfracturing was observed prior to several stick-slips (for a more detailed description of the experimental method, see Scholz, (1968b)). As stress was raised during each stick-slip cycle, events were sampled in three regions: the lower, middle, and upper third of the stress cycle prior to stick-slip. A frequency-magnitude relation was determined for the microfracturing events that were detected in each of these regions.

TABLE 3

b VALUES FOR FRICTIONAL SLIDING OF WESTERLY GRANITE

Confining Pressure, kb.	Stick-slip	b		
		1	2	3
2.0	1st	1.60	1.60	1.24
	2nd	2.30	1.17	1.56
1.0	1st	*	0.80	2.70
	2nd	*	1.34	1.30
	3rd	*	1.32	1.15

* Insufficient events were observed.

It was found that these data also fit the Gutenberg and Richter formula. The b values are given in Table 3, where the values in the lower, middle, and upper regions are referred to as 1, 2, and 3, respectively. The results show that b values for frictional sliding are consistently high. This reinforces the earlier view that the microfracturing that occurs during cataclastic deformation and in the low stress region in uniaxial experiments is due to frictional sliding.

The results of this examination show that the frequency-magnitude relation of microfracturing events is a well defined property of brittle rocks with several distinct characteristics. Firstly, although the relation is in general very well approximated by the Gutenberg and Richter equation, the primary statistical parameter b depends strongly on the state of stress, and only to a lesser extent on the physical properties of the rock. The earlier work of Mogi (1962a) and Vinogradov (1959, 1962) was restricted to averaging this behavior over all stresses. Mogi's conclusions that the degree of heterogeneity of the rock is the primary factor determining the value of b is probably a result of this averaging process. He found that more heterogeneous rocks are characterized by much higher b values. The qualitative criterion of heterogeneity which he used, however, was in general related closely to porosity; thus he found that a very porous rock such as pumice has a higher b value than a compact rock, which he considered less heterogeneous, such as granite. As was shown in the related work (Scholz, 1968b) the more porous the rock the larger the proportion of microfracturing events occur in the low stress region relative to the high-stress region. Consequently, if the frequency-magnitude relation is determined by analyzing events over the entire stress range, the average b values obtained for the more porous rocks will be shifted toward the high values characteristic of low stresses.

The second point to be made is that the frequency-magnitude relation sharply distinguishes between the two processes that occur. At low stress, where crack closing and sliding are important, high values of b are observed. Above about 50 per cent of the fracture strength, where new fractures are propagating, b is lower, in the range usually found for earthquakes, and decreases as stress increases. This behavior is similar at all pressures and for all brittle rock types tested.

Theory

Mogi (1962a), as well as Suzuki (1959), have given derivations of the Ishimoto-Iida statistical formulation of the frequency-magnitude relation. In this section we shall extend a basic physical model of microfracturing in a very simple way, and following Mogi's statistical approach, obtain a more complete expression of this relation. The analysis that follows, however, will only deal with the behavior in the region in which new fractures are forming, at high stresses.

Scholz (1968b) introduced a statistical model of rock deformation in which rock was treated as an inhomogeneous elastic medium. This model was found to successfully predict the pattern of microfracturing activity observed in compression tests in the laboratory. If rock under crustal conditions in the laboratory can be considered to be similar to rock on a geological scale, as Mogi suggests, a consideration of the properties of a general inhomogeneous brittle material should reveal the similarities (and dissimilarities) between rock and crustal deformation. The model outlined earlier attempted to describe such a general material, although in that case it was applied to rock behavior observed in the laboratory.

In this study we have shown that the frequency-magnitude relation of microfracturing events in the laboratory is indeed quite similar to that observed for earthquakes. Since in the laboratory we have the advantage of being able to determine the conditions under which our observations are made, we have been able to define the physical parameters which affect this relation. It is still not known, however, if we can directly apply our observations, such as the stress dependence of b, directly to earthquakes. This again depends on how far the similarity can be taken. In order to gain some insight into the basis for the similarity of microfracturing and earthquakes, we shall extend the model to consider the frequency-magnitude relation. To clarify the analysis, however, we shall first briefly review the basic model and discuss its applicability to crustal deformation.

Suppose that a rock in the laboratory, or on a much larger scale a portion of the Earth's crust, is subjected to a uniform applied stress, $\bar{\sigma}$, as shown in Figure 5. Due to the presence of elastic and structural inhomogeneities, however, the local stress at a point, σ, will not in general be equal to $\bar{\sigma}$ but will vary in some complex way from this mean value. For such an inhomogeneous medium the local stress σ

cannot, in theory, be predicted without a detailed knowledge of the configuration of inhomogeneities in the body. At the present state of our knowledge, the stress distribution within such a body cannot be treated with conventional elasticity theory. Therefore we take a heuristic approach and, assuming that the inhomogeneities will be small relative to the body itself, consider σ as a random variable. Consequently, we can define the probability that the stress within a region (small enough such that the stress on it can be considered uniform) is some given value σ in terms of probability density function $f(\sigma; \bar{\sigma})$. This notation implies that the probability function of σ will depend on $\bar{\sigma}$. For example, the mean of $f(\sigma; \bar{\sigma})$ can be shown from equilibrium to be $\bar{\sigma}$. Each small region is also characterized by a strength, S, that is, fracture will occur within the region if the local stress exceeds S. The presence of some sort of weakness, e.g., "Griffith" cracks, is included in this definition of S. Variations in S are equivalent to variations in σ.

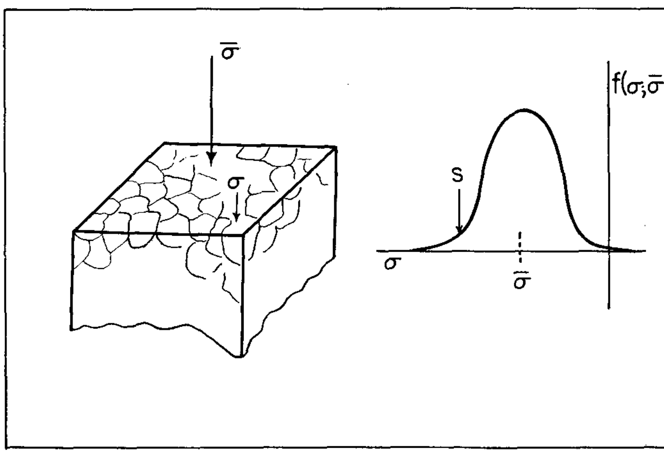

FIG. 5. The diagram to the left indicates the response of an inhomogeneous medium to a uniform applied stress $\bar{\sigma}$. The local stress σ may vary significantly from $\bar{\sigma}$. On the right is shown an arbitrary form of $f(\sigma; \bar{\sigma})$, which describes the probability that the stress at a point is σ. S is the average local strength.

How will this material react when it is stressed? In Figure 5 we show this situation, with an arbitrary density function. As the applied stress is increased, $f(\sigma; \bar{\sigma})$ will gradually change, since it is a function of $\bar{\sigma}$, and move to the left, intercepting S so that the probability of local fracture steadily increases. Fracture will occur in regions where the local stress exceeds S, and we will assume that these fractures will be arrested if they propagate into adjacent regions of lower stress. In order to describe a frequency-magnitude relation for these events, we need to consider how such fracture propagation takes place.

Suppose we have a fracture of area A, such as shown in Figure 6, growing in a heterogeneous stress field. It follows from the properties of our model that a fracture can only propagate through regions where σ exceeds S. In the case of uniform stresses, i.e., the mean stresses are not functions of position, the probability that the local stress exceeds the strength is uniform in space and is given by $F(S; \bar{\sigma})$, the distribution function over $S(F(S; \bar{\sigma})$ is the integral of $f(\sigma; \bar{\sigma})$ to the left of S. See e.g., Parzen, 1962; Volkov, 1962). Therefore the probability that a crack will be arrested somewhere within an arbitrary area is constant and is given by $[1 - F(S; \bar{\sigma})]$. It then follows that $g(A)\, d(a)$, the probability that a fracture will stop as it grows from size A to $A + dA$, is given by

$$g(A)\, dA = \frac{[1 - F(S;\bar{\sigma})]}{A}\, dA. \tag{4}$$

Physically, this says simply that this probability varies directly with the probability that the stress at a point is less than S, and inversely to the area swept out by the fracture. This model takes into account two basic properties of fracture

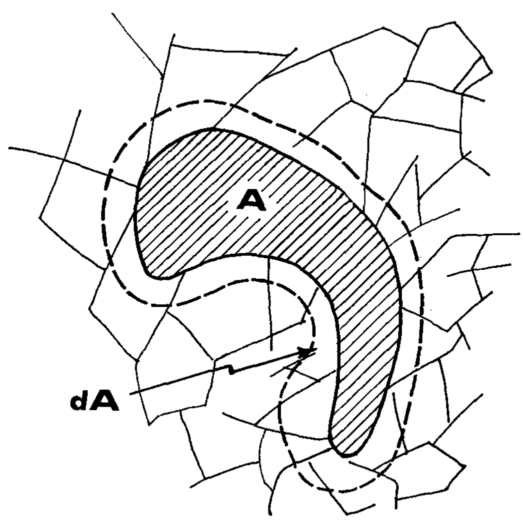

FIG. 6. Top view of a fracture propagation through an inhomogeneous medium. We treat the case of a fracture of area A (crosshatched region) that is arrested as it propagates an additional increment dA (bounded by the dashed line).

propagation. Firstly, since the definition of $F(S; \bar{\sigma})$ requires the presence of a fracture, we imply that a fracture weakens the region it penetrates (i.e., the stress concentration due to the fracture augments the stress in the region into which the fracture is growing). Secondly, (4) shows that the probability of a fracture growing an additional increment increases as the fracture becomes larger since it is sampling a larger number of regions.

Due to the conditional nature of $g(A)\, dA$,

$$g(A)\, dA = \frac{-\, dN(A)}{N(A)} \tag{5}$$

where $N(A)$, the accumulated frequency of fractures larger than A, is given by

$$N(A) = \int_A^\infty n(A)\, dA \tag{6}$$

and $n(A)\,dA$ is the frequency of fractures that occur within the range A to $A + dA$. Rearranging the right sides of (4) and (5) and setting them equal, we have

$$[1 - F(S; \bar{\sigma})]\, d\,(\log A) = -d\,[\log N(A)] \tag{7}$$

which upon integration yields

$$A^{-[1-F(S;\,\bar{\sigma})]} = N(A). \tag{8}$$

Differentiating (8) with respect to A, we obtain

$$n(A)\,dA = [1 - F(S; \bar{\sigma})]A^{-[1-F(S;\,\bar{\sigma})]-1}\,dA. \tag{9}$$

So far we have made only the most basic assumptions regarding the mechanism of fracture propagation. On this basis we have deduced, in equation (9), the size distribution of fractures that will occur in an inhomogeneous medium. This result should apply equally to extension or shear fractures, for example, and to both earthquakes and microfracturing. The scale of the body does not enter into the model. In order to calculate from equation (9) the distribution of the maximum trace amplitude, the quantity we actually measure, however, we must introduce a specific fracture mechanism.

With the assumption that the fractures are in the form of narrow, penny-shaped cracks, and that cracks of different size are similar, we can approximate the stress-relieved volume, V, by a sphere enclosing the fracture and

$$A = \alpha V^{2/3} \tag{10a}$$

where α is a constant. Also, we have the relation

$$V = \tau\bar{\sigma}^{-2}E \tag{10b}$$

where E is the strain energy released. The constant τ includes the elastic constants and the relative stress drop coefficient. From the general relation of seismology, we also have

$$E = \gamma a^{\nu} \tag{10c}$$

where a is the maximum trace amplitude of the elastic radiations, γ and ν constants. This relation is known empirically for earthquakes, and can also be demonstrated theoretically (K. Aki, personal communication, 1967).

From (10a), (10b), and (10c), a relation between the fractures size A and the maximum trace amplitude a can be obtained. Substituting this into equation (9), it can be shown that $n(a)\,da$, the frequency of events that occur within the amplitude range a to $a + da$, is given by

$$n(a)\,da = Ka^{-m}\,da \tag{11}$$

where

$$K = (\tfrac{2}{3})[1 - F(S;\bar{\sigma})]\alpha^{1-F(S;\bar{\sigma})}(\gamma\tau)^{-(\frac{2}{3})[1-F(S;\bar{\sigma})]}(\bar{\sigma})^{(4/3)[1-F(S;\bar{\sigma})]-1} \tag{12}$$

and

$$m = (\tfrac{2}{3})\nu[1 - F(S;\bar{\sigma})] + 1. \tag{13}$$

Equation (11) is the Ishimoto-Iida relation, expressed in terms of the physical parameters of our model. The Gutenberg and Richter parameter b is given from equations (3) and (13) by

$$b = (\tfrac{2}{3})\nu[1 - F(S;\bar{\sigma})]. \tag{14}$$

Several definite things can be said about b from this expression. First, since $F(S;\bar{\sigma})$ increases with increasing stress, b must decrease as stress is increased, as our experimental data indicates (cf., Figures 3 and 4). Secondly, because $F(S;\bar{\sigma})$ is a distribution function, b is limited between $2\nu/3$ and 0. For earthquakes, from the well known empirical relation between energy and magnitude (Richter, 1958)

$$\log E = 11.4 + 1.5M$$

we know that $\nu = 1.5$ and accordingly the limits of b for earthquakes are 1.0 and 0. This agrees very well with observation, since we know that all earthquakes except possibly some of volcanic origin are characterized by b values between 0.5 and 1.

For simplicity, the above calculations have been made for the special case of a single uniform component of mean stress. The theory will of course also hold for general uniform stresses, except that a joint distribution function defined in terms of all components of mean stress must be used. For nonuniform stresses, however, the situation is quite different. Qualitatively, since fracture will tend to occur where the stresses are highest, $\bar{\sigma}$, and hence $F(S;\bar{\sigma})$, will always tend to decrease away from the origin of a fracture. Therefore the fractures will always tend to be smaller, and b will be larger, than if the stresses were uniform. This is partially confirmed by comparing the data of Mogi (1962a), 1962b, who studied the magnitude-frequency relation of microfracturing of rock in bending, with the present data for uniform compression. His b values, for similar rocks, are consistently higher.

It is also interesting to note that volcanic earthquakes, which often are typified by anomalously high b values of 2 to 3, must have occurred in a nonuniform stress field according to this theory.

Conclusions

Mogi (1962a, 1962b, 1962c, 1963a, 1963b, 1963c) demonstrated the striking similarities between the statistics of microfracturing events observed in the laboratory and the statistics of earthquakes. In the light of this observation, he suggested that rock deformation is a scale model of crustal deformation and interpreted his results accordingly. His conclusion was at once very profound and very puzzling. What possible physical similarities exist between the two processes to account for the strong similarity in behavior?

In this study we have begun to understand the importance of physical parameters such as stress and confining pressure in determining the frequency-magnitude relation in microfracturing. We have yet to decide how far these results can be extended to earthquakes. Can we, for example, use observed b values to determine, say, the stress level during an earthquake sequence?

In order to understand such problems, we have developed a general model which is applicable to rock deformation on both a laboratory and crustal scale. In the present analysis, we have found, from equation (9), that the size distribution of fractures in an inhomogeneous medium is a power function of fracture size, and that the exponent of this relation must vary inversely with stress. At that stage, we had introduced only two basic assumptions: first, stress varies significantly from the mean value within the body, and second, fracture occurs when the local stress exceeds some critical value. These conditions seem reasonable both for rocks in the laboratory and for portions of the Earth's crust. The only fundamental difference between the two is in the scale of the inhomogeneities. Therefore equation (9), which should be applicable to both cases, implies that the stress dependence of b for earthquakes will be similar to that observed in the laboratory.

On the other hand, the values of the parameters a and b in equation (1) under given conditions depend on the fracture mechanism, transmission properties of the medium, and instrumental characteristics. We have assumed in the analysis that these are independent of fracture size (equations 10a, b, and c). Mogi (1962a) suggested that the frequency-magnitude may hold over the entire spectrum of fracture size from earthquakes to microfracturing. Certainly it seems to hold over several orders of magnitude for earthquakes and for microfracturing. Recently, however, Smith *et al* (1967) noted that microearthquakes and earthquakes which occurred in the same region of California are not similar. Fault lengths for the microearthquakes were found to be much larger than would have been predicted by extrapolation from data on larger shocks. Also, stress drops for the microearthquakes were unusually low. Therefore, the parameters in equations (10a), (10b), and (10c) will not be the same for both microearthquakes and earthquakes in that region. In this light, it is not surprising that they found that the b value for microearthquakes was not the same as that for the larger earthquakes. If similarity does not hold over all sizes, then the form of the frequency-magnitude relation given here only approximates the relation over a limited range of interest. In order to predict the behavior over the entire range, we must know how equations (10a), 10b), and (10c) change with source dimensions.

Aside from these problems, Mogi's suggestion of the similarity of microfracturing and earthquakes is largely upheld in the present work. It was found, however, that the state of stress, rather than the heterogeneity of the material, plays the most important role in determining the value of b. The b value for earthquakes has been extensively studied and has been found to be characteristic of the seismicity within a given region. Consequently, regional variations of b may reflect variations of the state of stress. For example, McEvilly and Casaday (1967) noted that after-shock sequences of magnitude 5 earthquakes in different regions of northern California are two distinct types, characterized by widely disparate b values. This may reflect a difference in the mean stresses or in the stress gradients.

In concluding this discussion, a general limitation of the model should be set forth. We have assumed an infinite medium, but in an actual case in which we study a sample of finite dimensions, or in the case of the Earth a finite stressed region, the behavior should be limited at large amplitudes. We might expect that as a fracture approaches these limiting dimensions it will begin to significantly reduce the strain energy stored in the system. If this occurs, large fractures will tend to attain a shorter terminal length than predicted by the model, and consequently the frequency of very large events reduced. This may be the effect that we have observed earlier, illustrated in Figure 2, in which there is a deficiency of large events in a manner analogous to that observed for great earthquakes. Certainly the large earthquakes which have been well studied, such as the Alaskan earthquake of 1964, tend to encompass nearly the entire tectonic region in which they lie.

An alternate explanation of this phenomenon is also possible. Brace and Bombolakis (1963) and Hoek and Bieniawski (1965) have shown in two dimensional photolastic studies that cracks in compression do not propagate in an instable manner indefinitely, but become stable and stop after propagating some fraction of their original length. This self-stablizing property of cracks in compression may produce a limitation at large amplitude. That is, if a fracture is not stopped by inhomogeneities, it may eventually stop of its own accord.

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Note added in proof: R. Ikegami, *Bull. Earth. Res. Inst.* **45**, 1967, pp. 327–338, reports that in Japan secular variations of b occur with a period of from 10 to 20 years with a range in b from 0.77 to 1.25. He finds that periods of great seismic energy release correspond to periods during which b is a minimum. From the present analysis this can be interpreted as secular stress variation resulting in seismic episodes.