The permeability of porous materials

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The permeability of a porous material to water is a function of the geometry of the boundary between the solid component and the pore space. Expressions of the Kozeny type purporting to represent this function are based upon the particle size or specific surface of the solids, and whilst, for engineering practice, they have given satisfaction for saturated sands, they may fail badly in other cases. By developing a Kozeny type of expression for the particular structure of a bundle of capillary tubes of assorted radii, we demonstrate the cause of the failure.

Such failure may be avoided by relating permeability to pore-size distribution, which is the factor of prime concern and which may be measured directly by even simpler means than are used to determine particle-size distribution. The pore-size distribution is arrived at by an interpretation of the moisture characteristic of the material, i.e. of the curve of moisture content plotted against pressure deficiency. A simple statistical theory, based upon the calculation of the probability of occurrence of sequences of pairs of pores of all the possible sizes, and of the contribution to the permeability made by each such pair, leads to an expression of the permeability as the sum of a series of terms. By stopping the summation at a selected upper limit of pore size one may calculate the permeability at any chosen moisture content and plot it as a function of that content. An example is presented, using a coarse graded sand specified by its moisture characteristic.

To check these calculations, experimental determinations of the permeabilities of unsaturated materials are presented, using two different grades of sand and a sample of slate dust, the results being compared with computed values. The agreement seems good, and is certainly better than that provided by the Kozeny formula as developed, with difficulty, for the purpose.

The limitations and possible improvements of our concept are very briefly discussed, and finally it is shown how a combined use of the moisture characteristic and the permeability (which is itself derivable from the moisture characteristic) leads to an expression for the coefficient of diffusion of water in the material as a function of moisture content. From this it should be possible, in principle, to calculate in suitable cases the course of water movement down a gradient of moisture content. Such a calculation awaits a satisfactory solution of the problem of non-linear diffusion.

The transmission of fluids by porous bodies has widespread relevance to engineering, geological and agricultural problems. The laws of flow and solutions of particular problems have received considerable attention. It is customary to treat problems as essays in the solution of Laplace's equation, it being assumed, sometimes too readily, that the material of the body obeys Darcy's law, which may be written

$$v = -K \operatorname{grad} \phi, \tag{1}$$

where v is the flow velocity commonly expressed as ml./sec./unit area normal to v, ϕ is the hydraulic potential, and the constant K, characteristic of the material, is called the permeability of the material to the fluid. This law seems to be valid for Reynolds numbers of less than unity (Fancher, Lewis & Barnes 1933).

The one physical property of the material which enters into a flow problem is the permeability, and this must be known if a complete solution is to be obtained. In principle a measurement of permeability is a simple matter of measuring the rate of

flow of liquid in a column of the material between planes of measured separation and hydraulic potential. In fact, there is considerable divergence of opinion as to the significance of such measurements. Some engineers report them without comment as a matter of routine, whilst other observers have dilated upon the often uncontrollable variations of permeability measured in this way during the course of any single experiment, and have provided a fairly considerable literature in attempting to account for these apparent variations and to indicate an inter-pretable technique. Permeability has been held to decrease with time owing to the percolating liquid (water) releasing dissolved air into the pores; to swelling of colloidal material; to mechanical blocking by movement of the finest particles of non-cemented materials; to the growth of organisms in the pore space; and to the chemical effect of the flowing liquid upon the material, as, for example, by exchange of bases between the colloidal fraction of the material and saline percolating water, resulting in modification of the colloidal properties of that fraction. Permeability has been held to increase with time due to solution in the percolating water of initially entrapped air. Some of these findings are summarized by Christiansen (1944). In our own experience with low hydraulic heads, the pore space is much modified in the neighbourhood of retaining grids which may be used to support sands in glass tubes, giving rise to anomalously high potential gradients over a portion of the column, whilst manometers measuring the pressure components of such potentials have proved insensitive and capricious. It is therefore of interest to inquire whether the permeability may be inferred from a knowledge of the pore-space geometry by which it is clearly uniquely determined, and which is itself not subject to such experimental uncertainties of determination. There are those who take the argument in reverse and regard a measurement of permeability as the simplest method of inferring the specific surface of a porous material, but we shall show that the argument is not sound.

In the past it has been more common to describe a granular material in terms of particle-size distribution (mechanical composition) than to regard it as a porous material with a given pore-size distribution. As a consequence, current formulas relating permeability to constitution do so in terms of particle size. The best known is perhaps that of Kozeny (1927), but somewhat similar expressions have been derived by Fair & Hatch (1933), Terzaghi (1925) and Zunker (1933); all derive in some degree from Slichter (1899). Kozeny's formula may be given in the form

$$K = Cd^2p^3/(1-p)^2, \tag{2}$$

where p is the porosity (ratio of pore volume to total apparent volume), C is an empirical constant, and d is a measure of the particle size, which is clearly ill-defined in a sand with a wide range of particle sizes. Fair & Hatch substitute for d the factor (V/A), where A is the total surface of a representative sample of particles the sum of whose volumes is V, that is, A/V is the specific surface. Fair & Hatch's form is therefore

$$K = Dp^3(V/A)^2/(1-p)^2. \tag{3}$$

Zunker & Terzaghi present formulas in which the functions of p are somewhat different from (2) and (3).

Objection may be raised against all of these formulas on various grounds. It is customary for advocates of any one to claim agreement with observed values of permeability, and sometimes superior agreement as compared with others. The fact seems to be that it has been impossible to secure a sufficiently wide range of variation of p and of pore-size distribution to test the formulas severely; and such tests as have been reported indicate a wide range of error (Muskat 1937). That the method of determining the specific surface of powders by measurement of permeability has achieved considerable popularity (Carman 1938, 1939; Lea & Nurse 1939; Rigden 1943, 1947) may reflect only the essential similarity of most industrial powders, and the margin of error which is permissible in the industrial problems involved. As to objections on grounds of principle, it seems to be generally admitted that these formulas all fail utterly to describe structured bodies such as, for example, the 'stiff-fissured' clays. The structural fissures contribute negligibly both to porosity p and to specific surface, and yet they dominate the permeability K. Again, neither p nor (V/A) are directed quantities, and consequently formulas such as (2) and (3) cannot indicate anisotropic permeability, which nevertheless seems to be the rule rather than the exception in nature.

The root of the error in Kozeny-type formulas is readily demonstrated by developing such a formula, by the method of Fair & Hatch, for a hypothetical structure, namely, a bundle of capillary tubes. Consider a tube of radius r conducting a liquid at a rate of δv ml./sec. as a consequence of a potential gradient $d\phi/dl$. By Poisseuille's equation we have

$$\delta v = -(\pi r^4/8\eta)\,(d\phi/dl),$$

where η is the viscosity of the liquid. If there are n of these tubes in an otherwise solid cross-section of unit area, the total flow is

$$v = -(n\pi r^2/8\eta)\,(2\pi r^2/2\pi r)^2\,(d\phi/dl), \tag{4}$$

which may be written

$$v = -(p/2\eta)\{(V/A)\,p/(1-p)\}^2\,d\phi/dl.$$

Comparison with equation (1) yields

$$K = (1/2\eta)\,p^3(V/A)^2/(1-p)^2, \tag{5}$$

which is Fair & Hatch's form of Kozeny's equation; this equation is therefore demonstrably valid for the specified structure.

If now we consider capillaries of varying sizes, providing an overall porosity p, we may take a group of capillaries, each of radius r, associate with it a proportion of the intervening solid to provide the group with porosity p, thus arriving at a total cross-section a_r of conductor permeated with capillaries of radius r. The permeability contribution K_r of this group enters into the total permeability according to the expression

$$K = \Sigma a_r K_r/\Sigma a_r, \tag{6}$$

in which K_r is given by (5) with a value $(V/A)_r$ proper to the group of radius r. Hence we have

$$K = (1/2\eta)\{p^3/(1-p)^2\}\{\Sigma a_r(V/A)_r^2/\Sigma a_r\}. \tag{7}$$

The final bracket may be written $\overline{(V/A)^2}$ and represents the mean value of $(V/A)^2_r$ of all the groups; it is clearly not identical with the overall value of $(V/A)^2$ for all channels together, since the former is dominated by the larger and the latter by the smaller channels. Equation (7) is certainly not Kozeny's equation. However, if we consider two similar structures, i.e. structures λ and μ, such that for every group of channels of radius r in λ there is a corresponding group of channels of radius mr in μ, each group contributing the same fraction to the total respective cross-section, then it is easy to show that

$$\overline{(V/A)^2_\lambda}/\overline{(V/A)^2_\mu} = (V/A)^2_\lambda/(V/A)^2_\mu = 1/m^2. \tag{8}$$

Since, by hypothesis, p is the same for both structures, (7) gives

$$K_\lambda/K_\mu = (V/A)^2_\lambda/(V/A)^2_\mu, \tag{9}$$

and it is only in this restricted sense that Kozeny's equation may be said to be valid for bundles of capillary tubes. We may note that it is necessary to this validity that the porosities of the two structures should be identical, since otherwise the structures cannot be similar. It may be objected that what we have proved for bundles of capillaries may have no relevance to such structures as sand beds, and that must be granted. We would only point out that it has been an accepted procedure to derive Kozeny's equation by a bold development of Poisseuille's (Fair & Hatch 1933; Rigden 1947), without demonstrable justification; it seems to us a far more plausible supposition that the *limitations* we have demonstrated for capillary tubes apply equally to sand beds.

We have sought to relate permeability more rationally with pore-size distribution, which, in recent years, has become almost as common a determination as mechanical composition (Donat 1937; Childs 1940, 1942; Feng & Browning 1946). At the same time we have developed a more stringent test of all such relationships by devising a method of measuring the permeability of a sand column over a wide and controlled range of effective porosity and over a variety of types of pore-size distribution. The latter has been accomplished by measuring the permeability over the complete range of moisture content. A pore which is filled with air is not effective in conducting water; the effective porosity is limited to that which is water-filled. Also, since larger pores lose their water before smaller ones, reduction of moisture content results also in a steady change of pore-size distribution, but this distribution is not, of course, variable at will. In this paper, therefore, we present a comparison of the computed variations of permeability of a number of different packed sands and dusts with moisture content with the variation observed in experiments.

The pore space of a material such as sand is continuous, but presents a set of 'caverns' of various sizes each of which is connected to several others by narrower channels. The 'caverns' or pores have a certain size distribution which may approximate to the normal or may be of Poisson type. They are also randomly distributed in space. The theoretical problem which presents itself is the computation of the permeability of such a random array, and it appears to have received but scant attention from mathematical statisticians. We have therefore perforce proceeded by a bold application of simple concepts. In so far as the results have been satisfactory,

it may perhaps be held that our concepts have merely led to a method of computing permeability, the real justification of which is empirical.

Let us take $f(r)$ as the distribution function which describes the pore space of an uncemented sand, i.e. $f(r)\,\delta r$ is the fraction of the total apparent volume which is occupied by pores of 'radius' range r to $r+\delta r$. A cross-section of this sand will exhibit a fraction of the total area, also given by $f(r)\,\delta r$, devoted to this pore group. Let us now consider a sand column of unit cross-sectional area. A fracture at any chosen plane normal to the length will exhibit two similar faces showing similar pore-size distributions; the continuous column may be regarded as the random juxtaposition of these two faces. If a_ρ is the area devoted to pores of radius ρ to $\rho+\delta r$ and a_σ the area devoted to pores in the range σ to $\sigma+\delta r$, then

$$a_\rho = f(\rho)\,\delta r, \quad a_\sigma = f(\sigma)\,\delta r.$$

Hence, on remaking the continuous column, the area of cross-section devoted to the pore sequence $\rho \to \sigma$ is given by

$$a_{\rho \to \sigma} = f(\rho)\,\delta r f(\sigma)\,\delta r. \tag{10}$$

We now make two assumptions, which introduce errors of opposite sign. The first is that all the effective resistance to flow in the sequence is confined to the smaller of the pores, say σ, owing to the operation of the factor r^4 in Poisseuille's equation. This assumption results in an over-estimation of the contribution of this sequence to the total permeability, but frees us from the necessity of considering the probability of more than one of the smaller pores leading into the same larger pore. The second assumption is that the only contribution to permeability is by a direct sequence of the kind described, i.e. we ignore by-passing sequences of, maybe, several pores. This results in an under-estimation tending to compensate for the previous over-estimation. The distribution of pore sequences given by (10) is repeated wherever the cross-section is taken, and is characteristic of the material. The number of pores of size σ accommodated in the area $a_{\rho \to \sigma}$ is proportional to σ^{-2}, the rate of flow in each is, if we apply Poisseuille's equation, proportional to σ^4 per unit potential gradient, whence the contribution δK which this group of sequences makes to the total permeability K is given by

$$\delta K = \sigma^2 M f(\rho)\,\delta r f(\sigma)\,\delta r, \tag{11}$$

and the total permeability due to all the possible sequences by

$$K = M \sum_{\rho=0}^{\rho=R} \sum_{\sigma=0}^{\sigma=R} \sigma^2 f(\rho)\,\delta r f(\sigma)\,\delta r. \tag{12}$$

In (12) the factor σ^2 must be replaced by ρ^2 in any term of the summation in which $\rho < \sigma$. For any given moisture content the summation is stopped at that pore size, R, appropriate to the largest pore which remains full of water. The constant M can hardly be calculated; it will in fact be determined in this paper by matching theoretical and experimental curves at a single point.

For a non-shrinking material such as sand, the distribution function $f(r)$ may be derived satisfactorily from the moisture characteristic, i.e. from the curve relating moisture content to the hydrostatic pressure at that content (Childs 1940, 1942).

The amount by which this pressure is less than atmospheric is a measure of the curvature of the air-water interface in equilibrium in the pore space, and therefore a measure of the largest cells which have not yet been emptied of water. The volume of water removed by a given increment of suction is thus a measure of the pore space occupied by pores of a known range of sizes.

Where there are very many groups of pores, the calculation indicated by (12) may clearly be a laborious task, and we therefore present a form of tabulation which much reduces the work and which also clearly indicates its course. In the first place, a pore which is emptied at pressure deficiency P will have a 'radius' which we can relate to $2S/P$, since an air-water interface of that radius is involved. We therefore replot the moisture characteristic in the form *moisture content* against $1/P$. There is no need to invoke the surface tension S, since (12) in any case includes the constant M which is obtained by empirical matching, so that for brevity we may refer to $1/P$ as the 'radius'. From the resulting curve we tabulate the pore volume to be assigned to each size group, i.e. $f(r)\,dr$.

Let the pore groups have mean radii, beginning with the smallest, of a, b, c, ... respectively, contributing porosity elements α, β, γ, ... (i.e. α is written for $f(a)\,dr$ and so on). Then (12) becomes

$$K = M(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 + \ldots + 2a^2\alpha\beta + 2a^2\alpha\gamma + \ldots + 2b^2\beta\gamma + \ldots), \qquad (13)$$

the series terminating at the largest water-occupied pore size. It should be noted that the 'r^2' factor, i.e. a^2, b^2, etc., in each term is always the radius of the smaller pore in the sequence. A table is constructed, as shown in table 1. Columns 1 and 2 are abstracted from the transformed moisture characteristic, adopting a suitable pore-size interval dr for the groups. The rest of the table is self-explanatory; it need only be pointed out that column 4 is obtained by summing column 3 down to and including the row being filled in, and column 6 is obtained from column 5 similarly, the result in this case being doubled. Column 8 is obtained by summing column 7. Column 9, multiplied by M, gives the expression (13) appropriate to the limiting pore radius indicated in column 1, i.e. appropriate to a moisture content which may be regarded as the sum of column 2 down to the row indicated but which is naturally most simply read off from the moisture characteristic. As an example, table 2 gives

TABLE 1

1	2	3	4	5
r	$f(r)\,\delta r$	$r^2 f(r)\,\delta r$	$\Sigma r^2 f(r)\,\delta r$	$f(r)\,\delta r \Sigma r^2 f(r)\,\delta r$
a	α	$a^2\alpha$	$a^2\alpha$	$a^2\alpha^2$
b	β	$b^2\beta$	$a^2\alpha + b^2\beta$	$a^2\alpha\beta + b^2\beta^2$
c	γ	$c^2\gamma$	$a^2\alpha + b^2\beta + c^2\gamma$	$a^2\alpha\gamma + b^2\beta\gamma + c^2\gamma^2$
etc.	etc.	etc.	etc.	etc.

6	7	8	9
			$(=6-8)$
$2\Sigma\{f(r)\,\delta r \Sigma r^2 f(r)\,\delta r\}$	$r^2\{f(r)\,\delta r\}^2$	$\Sigma r^2\{f(r)\,\delta r\}^2$	K/M
$2a^2\alpha^2$	$a^2\alpha^2$	$a^2\alpha^2$	$a^2\alpha^2$
$2(a^2\alpha^2 + b^2\beta^2 + a^2\alpha\beta)$	$b^2\beta^2$	$a^2\alpha^2 + b^2\beta^2$	$a^2\alpha^2 + b^2\beta^2 + 2a^2\alpha\beta$
$2(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$	$c^2\gamma^2$	$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$	$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$
$+ a^2\alpha\beta + a^2\alpha\gamma + b^2\beta\gamma)$			$+ 2a^2\alpha\beta + 2a^2\alpha\gamma + 2b^2\beta\gamma$
etc.	etc.	etc.	etc.

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TABLE 2. THE MOISTURE CHARACTERISTIC OF 1 TO ½ MM. SAND AND THE DERIVATION OF THE PORE-SIZE DISTRIBUTION

experimental results			data interpolated from the graph of moisture content against reciprocal of tension			
moisture content (ml./100 ml.)	tension (cm. of water)	reciprocal of tension	reciprocal of tension	moisture content (% volume)	volume contribution to pore group $f(r)dr$	mean radius of pore group r
35·67	0·0	∞	0·02625	3·42	0·00	0·0000
35·59	2·4	·4167	0·02875	3·44	0·02	0·0275
35·26	6·5	0·1539	0·03125	3·48	0·04	0·0300
35·10	10·4	0·09615	0·03375	3·53	0·05	0·0325
34·75	14·3	0·06993	0·03625	3·65	0·08	0·0350
31·41	16·3	0·06135	0·03875	4·00	0·35	0·0375
27·21	17·1	0·05848	0·04125	4·61	0·61	0·0400
22·32	17·7	0·05650	0·04375	5·58	0·97	0·0425
19·38	18·05	0·05541	0·04625	7·00	1·42	0·0450
14·15	18·65	0·05362	0·04875	8·72	1·72	0·0475
10·27	19·8	0·0505	0·05125	11·14	2·42	0·0500
6·675	21·8	0·04587	0·05375	15·30	4·16	0·0525
4·385	24·7	0·04049	0·05625	22·00	6·70	0·0550
3·59	28·4	0·03521	0·05875	27·95	5·95	0·0575
3·50	32·5	0·03077	0·06125	31·37	3·42	0·0600
3·43	36·4	0·02747	0·06375	34·35	2·02	0·0625
			0·06625	34·60	0·25	0·0650
			0·06875	34·71	0·11	0·0675
			0·07125	34·80	0·09	0·0700
			0·0775	34·89	0·09	0·0744
			0·0825	34·97	0·08	0·0800
			0·0875	35·02	0·05	0·0850
			0·0925	35·07	0·05	0·0900
			0·1050	35·15	0·08	0·0988
			0·1150	35·20	0·05	0·1100
			0·1250	35·23	0·03	0·1200
			0·1350	35·25	0·02	0·1300
			0·1650	35·30	0·05	0·1500
			0·1950	35·35	0·05	0·1800
			0·2250	35·39	0·04	0·2100
			0·2550	35·43	0·04	0·2400
			0·3250	35·50	0·07	0·2900
			0·3950	35·57	0·07	0·3600
			0·4650	35·61	0·04	0·4300
			0·5350	35·64	0·03	0·5000
			0·6050	35·66	0·02	0·5700
			0·6750	35·67	0·01	0·6400
			∞	35·67	0·00	—

TABLE 3. CALCULATION OF PERMEABILITY—MOISTURE CONTENT RELATIONSHIP FROM PORE-SIZE DISTRIBUTION OF 1 TO ½ MM. SAND

1	2	3	4	5	6	7	8	9	10 K(c.g.s.)
0·0000	0·00	—	—	—	—	—	—	—	—
0·0275	0·02	0·000,0153	0·000,0153	0·000,000,305	0·000,000,610	0·000,000,305	0·000,000,305	0·000,000,305	$1·69 \times 10^{-11}$
0·0300	0·04	360	513	2,05	4,71	1,44	1,74	2,97	$1·65 \times 10^{-10}$
0·0325	0·05	528	1041	5,21	15,12	2,64	4,38	10,74	5·97
0·0350	0·08	980	2021	16,17	47,46	7,84	12,22	35,34	$1·96 \times 10^{-9}$
0·0375	0·35	0·000,492	0·000,694	0·000,243	0·000,533	0·000,172	0·000,184	0·000,349	$1·94 \times 10^{-8}$
0·0400	0·61	976	1,670	1,019	2,571	595	779	1,792	9·95
0·0425	0·97	1,752	3,422	3,32	9,21	1,695	2,474	6,74	$3·74 \times 10^{-7}$
0·0450	1·42	2,878	6,300	8,95	27,11	4,09	6,56	20,55	$1·14 \times 10^{-6}$
0·0475	1·72	3,88	10,18	17,52	62,15	6,68	13,24	48,91	2·72
0·0500	2·42	0·006,05	0·016,23	0·039,26	0·140,7	0·014,65	0·027,89	0·112,8	$6·26 \times 10^{-6}$
0·0525	4·16	11,47	27,70	0·115,2	0·371,1	47,70	75,59	0·295,5	$1·64 \times 10^{-5}$
0·0550	6·70	20,28	47,98	0·321,5	1·014	0·136,0	0·211,6	0·802	4·45
0·0575	5·95	19,69	67,67	0·402,8	1·820	0·117,2	0·328,8	1·491	8·28
0·0600	3·42	12,32	79,99	0·273,7	2·367	0·042,2	0·371,0	1·996	$1·11 \times 10^{-4}$
0·0625	2·02	0·007,89	0·087,88	0·177,6	2·722	0·015,9	0·386,9	2·335	$1·296 \times 10^{-4}$
0·0650	0·25	1,06	88,94	22,2	2·767	3	0·387,2	2·380	1·322
0·0675	0·11	51	89,45	9,8	2·786	1	0·387,3	2·399	1·332
0·0700	0·09	44	89,89	8,1	2·803	0	0·387,3	2·416	1·342
0·0744	0·09	50	90,39	8,1	2·819	0	0·387,3	2·432	1·351
0·0800	0·08	0·000,51	0·090,90	0·007,3	2·833	0·000,0	0·387,4	2·446	$1·359 \times 10^{-4}$
0·0850	0·05	36	91,26	4,6	2·843	0	0·387,4	2·456	1·364
0·0900	0·05	41	91,67	4,6	2·852	0	0·387,4	2·465	1·369
0·0988	0·08	78	92,45	7,4	2·866	1	0·387,5	2·478	1·376
0·1100	0·05	61	93,06	4,7	2·875	0	0·387,5	2·487	1·382
0·1200	0·03	0·000,43	0·093,49	0·002,8	2·880	0·000,0	0·387,5	2·492	$1·384 \times 10^{-4}$
0·1300	0·02	34	93,83	1,9	2·883	0	0·387,5	2·495	1·386
0·1500	0·05	1,13	94,96	4,7	2·893	1	0·387,6	2·505	1·391
0·1800	0·05	1,62	96 58	4,8	2·902	1	0·387,7	2·514	1·396
0·2100	0·04	1,76	98,34	3,9	2·911	1	0·387,8	2·523	1·401
0·2400	0·04	0·002,30	0·100,64	0·004,0	2·919	0·000,1	0·387,9	2·531	$1·405 \times 10^{-4}$
0·2900	0·07	5,89	0·106,53	7,5	2·934	4	0·388,3	2·546	1·414
0·3600	0·07	9,08	0·115,61	8,1	2·950	6	0·388,9	2·561	1·423
0·4300	0·04	7,40	0·123,01	4,9	2·960	3	0·389,2	2·571	1·428
0·5000	0·08	7,50	0·130,51	3,9	2·969	2	0·389,4	2·580	1·433
0·5700	0·02	0·006,50	0·137,01	0·002,7	2·974	0·000,1	0·389,5	2·585	$*1·435 \times 10^{-4}$
0·6400	0·01	4,10	0·141,11	1,4	2·977	0	0·389,5	2·587	1·436
∞	0·00	0	0·141,11	0	2·977	0	0·389,5	2·587	1·436

The columns are numbered to agree with those of table 1.   * M chosen to match observed permeability at this point.

the moisture characteristic of the coarsest material shown in figure 1 and its con-version into a pore-size distribution, whilst table 3 shows the calculation of the appropriate permeability against moisture content curve illustrated in figure 2.

THE MEASUREMENT OF PERMEABILITY OF UNSATURATED SAND

The difficulties inherent in the measurement of the permeability of unsaturated porous materials are readily seen. The necessary potential gradient generally in-volves a pressure gradient, and a gradient of pressure deficiencies implies, *ipso facto*, a gradient of moisture content. A flow column can hardly be sampled without spoiling it for subsequent measurements, and so one must resort to indirect measure-ments, such as by measurement of the apparent electric capacitance of a condenser of which the porous material forms the dielectric. The presence of a moisture gradient then complicates matters, and we have ourselves tried intricate systems of guard rings to isolate a restricted region of approximately uniform moisture content. However, the ultimate solution of the problem was very simple, although limited to structureless materials such as sands and dusts. It has been shown (Childs 1945) that when water flows down a sufficiently long soil column to a water table, the moisture content is sensibly uniform over an appreciable length of the column, the zones of variable moisture content being limited at the lower end to the neighbourhood of the water table in a way which depends upon the pore-size distribution and at the upper end to a zone in which is localized any intermittency of water supply. In the zone of uniform moisture content there is no pressure contribution to the potential gradient, which is therefore purely gravitational and exactly known. The moisture content adjusts itself to provide the necessary permeability to conduct the imposed flow with the gravitational gradient of potential. To calculate the permeability we need to measure only the rate of flow and the area of cross-section of the column, whilst the uniform moisture content lends itself to estimation by indirect electrical methods using a simple calibrated cell.

The flow column used was a glass tube some 3 m. in length, of internal diameter about 2 cm., and containing a removable section approximately 16 cm. in length at about 65 cm. from the upper end. Two electrodes of aluminium foil, of dimensions 3 by 3·4 cm., were sealed to the outside of and half-way along this section, forming a split tube; these electrodes formed the plates of a condenser, the dielectric of which was in part the glass walls of the tube and in part the moist sand or dust. The condenser formed part of the tuned circuit of a triode oscillator tuned to about 1 Mcycle. A small variable condenser was connected in parallel with this cell and with a larger condenser. A second triode oscillator was arranged to beat with the first. Changes of apparent capacitance of the cell resulted in changes of beat frequency, and were measured by bringing the first circuit back to resonance by adjusting the small parallel condenser. Readings on this condenser were interpreted in terms of moisture content of the dielectric of the cell by an initial direct calibration, the cell being removed from the main flow column for determination of moisture content by direct weighing.

The lower end of the tube was immersed below a water surface, and flow to the upper end was controlled by glass syphons from a constant-head reservoir. The tube

was filled in stages with sand or dust by settlement through shallow water; in this way both trapped air and fractionation of the sample were avoided. The rate of flow was then adjusted in steps to secure the required states of moisture content, starting with saturation and proceeding to decreased contents. It was found that, using ordinary tap water, it was impossible to maintain the initial rate of flow in saturated sands. Constancy of flow at the initial maximum rate was possible when boiled distilled water was used, so it was probable that the trouble was due to dissolved air coming out of solution in the sand pores and thereby reducing the moisture content and permeability, a conclusion which was supported by the moisture content 'meter'. It was not practicable to use boiled distilled water for all saturated columns, but tap water was satisfactory once it was known that the true permeability was that at 'zero time'. The trouble did not arise with unsaturated sands, presumably because the continuous air-filled pore space provided a leak path for the escape of momentarily trapped air soon after its emergence from solution.

The whole flow tube and its ancillary equipment was mounted on a beam suspended at its upper end from a ceiling pivot. In this way inclinations from the vertical of up to 60° could be secured, thus providing potential gradients adjustable at will down to $g/2$. By this means one could test for the validity of Darcy's law in the prevailing conditions; if the law is obeyed, then the rate of flow is proportional to the potential gradient, other factors remaining constant.

EXPERIMENTAL RESULTS

The collected results are presented in figures 1 to 3. The moisture characteristics of the materials used, from which the permeabilities were computed, are shown in figure 1. These materials include a sand fraction separated between round-holed sieves of 1 and 0·5 mm. apertures, a second sand fraction separated between the
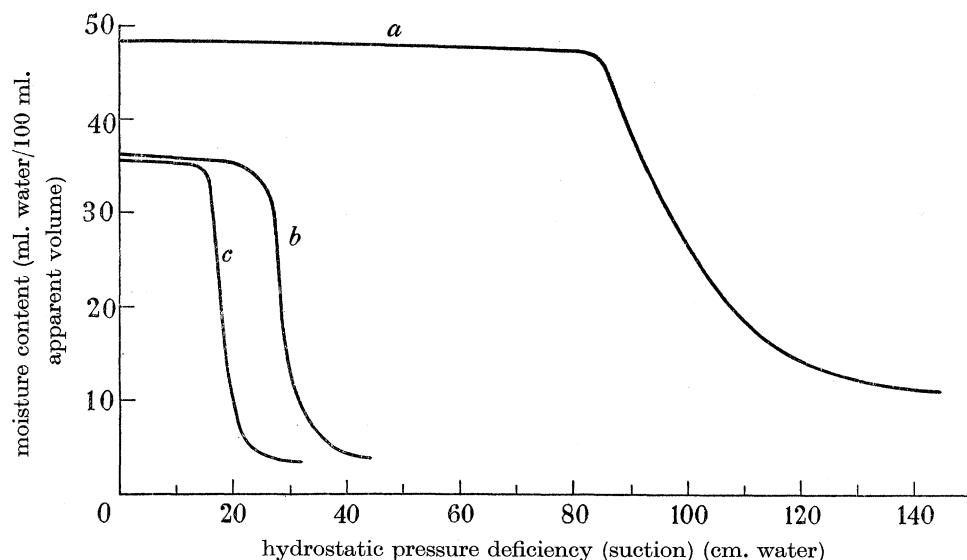

FIGURE 1. Moisture characteristics: a, slate dust; b, $\frac{1}{2}$ to $\frac{1}{4}$ mm. sand fraction; c, 1 to $\frac{1}{2}$ mm. sand fraction.

finer of these two sieves and one with 0·25 mm. square holes (wire-woven), and a slate dust passing a sieve with 0·125 mm. square holes but containing no particles smaller than 0·04 mm., as calculated from the velocity of sedimentation. This information serves only to describe the materials roughly, not to specify them; it is our argument that the specification is provided by the moisture characteristics.

In figure 2 are shown the observed and calculated permeabilities of these three materials, as functions of moisture content. The constant M in equation (12), required in order to calculate the permeability in absolute terms, is obtained for all materials by matching the observed and calculated curves for the coarsest material at the point indicated in figure 2. This set of curves includes also, for comparison, a computed curve for the coarsest material based on Kozeny's equation. This will be discussed further in due course, but it will be obvious that the agreement with observation is not close.
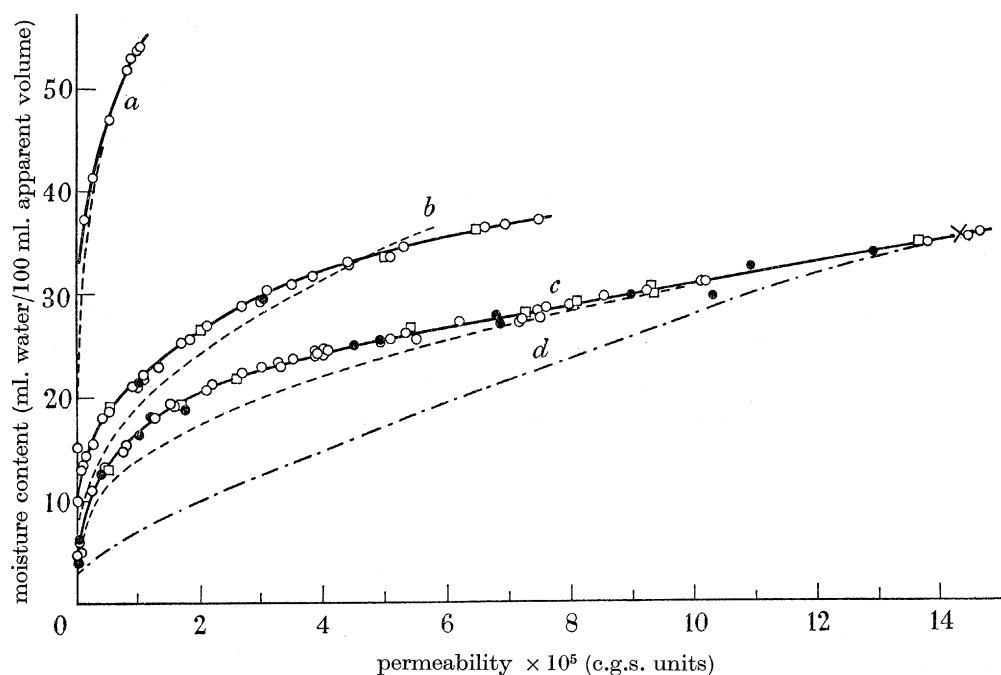

FIGURE 2. Observed and calculated permeabilities: a, slate dust; b, $\frac{1}{2}$ to $\frac{1}{4}$ mm. sand fraction; c, 1 to $\frac{1}{2}$ mm. sand fraction; d, 1 to $\frac{1}{2}$ mm. sand fraction. Experimental curve ———. Computed curve: Childs & George - - - -; after Kozeny — · — · —. Experimental points potential gradient: g ○; $\frac{3}{4}g$ ●; $\frac{1}{2}g$ □. Experimental and computed curves matched at ×.

DISCUSSION

We have referred already to the imperfect nature of the agreement between permeabilities calculated from Kozeny's equation (3) and those observed. The difficulty of making the comparison at all is another ground for doubting the validity of the equation. The term (V/A) in equation (3) was calculated for each stage from the moisture characteristic, it being taken as axiomatic that empty

pores were as ineffective conductors of water as if they had been filled with solids, i.e. V includes solids and empty pores. Referring both V and A to unit apparent volume of sand, we may substitute $(1 - p)$ for V, whence equation (3) becomes

$$K = Dp^3/A^2, \tag{14}$$

where A is now the area of the solid-liquid interface per unit apparent volume and p is the effective (water-filled) porosity. This area A is derived by summing the contributions of each pore-size group and involves a knowledge of the contributions of those groups which lie outside the range of the moisture characteristic, contributions which, owing to small pore size, are far from negligible, even though the contributions to porosity are minute. The trend of the permeability against moisture content curve at the low moisture content end and within the range of the moisture characteristic depends very much upon the way in which one approximates in these circumstances. In deriving the curve shown in figure 2 we have assumed, from the fact that the observed permeability is negligible at a certain lower limit of moisture content, that this limit is the effective datum for the estimate of porosity. It is the necessity for such an assumption which we refer to as indicating the inadequacy of the Kozeny type formula. No such assumption is required for the calculation of permeability from (12), the group contributions becoming sensibly zero within the range of the moisture characteristics.

The limitations of our conception will be obvious. We have confined ourselves to a cross-sectional factor and have ignored the effect of the different lengths of cells of different sizes. A sequence of two small pores, for example, occupies a much smaller proportion of the total volume than does a sequence of two larger pores, and should presumably be weighted less on that account. This may well account for the relative over-estimate of permeability at low moisture contents, where the small pores become significant. It might seem that if one were to consider sequences of very many pores instead of pore pairs, the frequency of occurrence would be concentrated about an average sequence, departures from this being rare, so that weighting of different sequences would not arise. Other difficulties, however, have prevented much progress on these lines; it is clearly very artificial to ignore all but the smallest pore in a sequence of many, and the problem of series-parallel pore connexions arises in an acute form. However, as a matter of empirical trial, it is found that a better agreement between observed and calculated permeability against moisture-content curves is obtained with our method of computation for pore sequences up to four. The error then tends to reverse in sign, the calculated permeabilities at low moisture content being relatively too low.

We have been concerned until now with permeability, i.e. with the velocity of flow per unit potential gradient. One is very often more conscious of moisture movement as a result of a gradient of moisture content, as, for example, when irrigation water redistributes itself down the soil profile during the first few days after the irrigation. The problem is then one of diffusion with a coefficient of diffusion which varies with both time and depth. We may write Darcy's law for one dimension (vertical flow)

$$dQ/dt = -K\,d\phi/dz, \tag{15}$$

where dQ/dt is the rate of flow (ml./sec.) in the vertical direction z. The hydraulic potential is the sum of gravitational and 'capillary' components, and may be written

$$\phi = P + gsz,$$

whence

$$d\phi/dz = dP/dz + gs. \tag{16}$$

Here P is the pressure component (a suction in unsaturated materials) and s is the density of the water. The moisture profile represents the moisture content, m, as a function of z, whilst, from the moisture characteristic, it is also expressible as a function of P. Hence, combining (14) and (15), we may write

$$dQ/dt = -K(dP/dm \,.\, dm/dz + gs)$$

$$= -\{\kappa \, dm/dz + Kgs\}. \tag{17}$$

The first term, indicating a rate of movement of moisture down a concentration gradient, is clearly a diffusion term with a coefficient of diffusion, κ, which can be calculated, for a given moisture content, from the moisture characteristic, since this latter permits us to calculate both factors in κ, namely the permeability K and the slope of the characteristic dP/dm. In figure 3 are shown the diffusion coefficients of the coarser sand and the slate dust, plotted as functions of moisture content. It will often arise, in heavier soils with low K and high dP/dm (and always with horizontal movement), that the second term of (16) will be negligible compared with the first; in such a case there would appear to be a prospect of forecasting the movement, for example, of irrigation water from measurements of the moisture characteristics of profile samples, always bearing in mind that the problem may also be
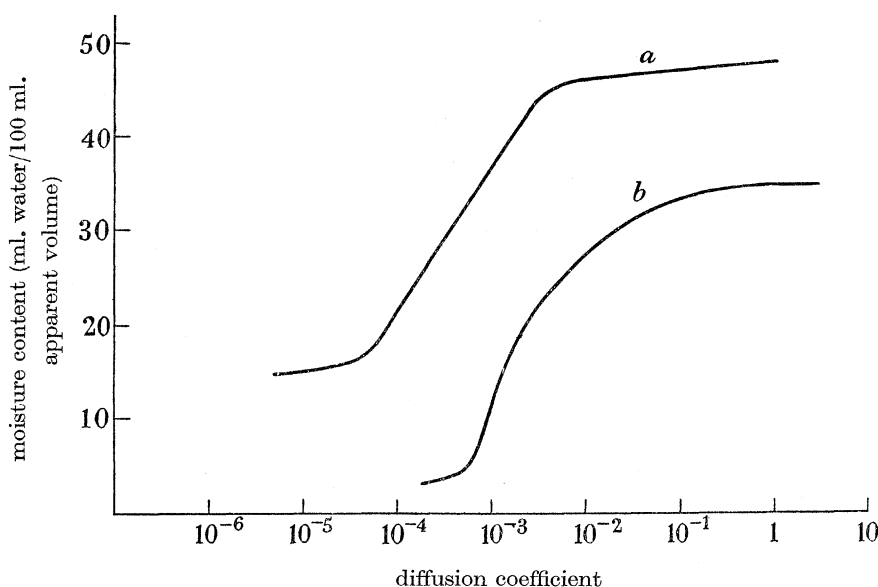

FIGURE 3. Moisture content of a, slate dust, and b, 1 to $\frac{1}{2}$ mm. sand fraction plotted against the diffusion coefficients.

complicated by hysteresis in the characteristic. Such a forecast requires the solution of the non-linear diffusion problem, and attention is now being turned in that direction.

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On the spontaneous magnetic field in a conducting liquid in turbulent motion

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Several recent investigations in geophysics and astrophysics have involved a consideration of the hydrodynamics of a fluid which is a good electrical conductor. In this paper one of the problems which seem likely to arise in such investigations is discussed. The fluid is assumed to be incompressible and in homogeneous turbulent motion, and externally imposed electric and magnetic fields are assumed to be absent. The equations governing the interaction of the electromagnetic field and the turbulent motion are set up with the same assumptions as are used to obtain the Maxwell and current flow equations for a metallic conductor. It is shown that the equation for the magnetic field is identical in form with that for the vorticity in a non-conducting fluid; immediate deductions are that lines of magnetic force move with the fluid when the conductivity is infinite, and that the small-scale components of the turbulence have the more powerful effect on the magnetic field.

The first question considered is the stability of a purely hydrodynamical system to small disturbing magnetic fields, and it is shown that the magnetic energy of the disturbance will increase provided the conductivity is greater than a critical value determined by the viscosity of the fluid. The rate of growth of magnetic energy is approximately exponential, with a doubling time which can be simply related to the properties of the turbulence. General mechanical considerations suggest that a steady state is reached when the magnetic field has as much energy as is contained in the small-scale components of the turbulence. Estimates of this amount of energy and of the region of the spectrum in which it will lie are given in terms of observable properties of the turbulence.