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ELASTIC VELOCITY ANISOTROPY IN THE PRESENCE OF AN ANISOTROPIC INITIAL STRESS

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ABSTRACT

The effect of a homogeneous anisotropic initial stress on the propagation of infinitesimal amplitude elastic body waves in a perfectly elastic, homogeneous medium is investigated. If the medium is inherently isotropic in the reference configuration and if the magnitude τ_0 of the deviatoric part of the initial static stress is small compared to the rigidity μ of the medium, then the apparent body-wave velocities of *P* waves are unaffected by the initial stress to first order in τ_0/μ. The apparent body-wave velocities of *S* waves are rendered anisotropic to first order, and this effect is described explicitly. It is concluded that the direct effect of an anisotropic initial stress cannot contribute appreciably to the observed velocity anisotropy of horizontally propagating *P* waves in the oceanic upper mantle. Those observations require an inherent elastic anisotropy of the oceanic upper-mantle material.

INTRODUCTION

The propagation of infinitesimal amplitude elastic body waves in general anisotropic perfectly elastic materials has been extensively investigated. Comprehensive reviews of this subject have been published by Kraut (1963) and by Musgrave (1970); the most important new contribution is that of Backus (1970). Virtually all of this work has dealt explicitly only with homogeneous, perfectly elastic materials which in the initial or reference configuration are at rest under the influence of a purely isotropic initial stress (a hydrostatic pressure). Any such material may be completely described from the point of view of the linearized theory of elasticity by specifying its density ρ and its 21 independent elastic coefficients C_{ijkl}, the Cartesian components of the fourth-order isentropic elastic tensor $\mathbf{C}$. An isotropic elastic material is one for which the elastic tensor $\mathbf{C}$ is an isotropic tensor.

Biot (1940 and 1965) has shown that an otherwise isotropic elastic material may possess an apparent or induced anisotropy if the initial static stress in the reference configuration is not purely isotropic. In this paper, the work of Biot (1940 and 1965) is extended, and the effect of an anisotropic initial static stress on the propagation of infinitesimal amplitude elastic body waves is explicitly described.

The principal seismological motivation for this work is the fact that several investigators (Raitt *et al.*, 1969; Morris *et al.*, 1969; Keen and Barrett, 1971) have recently utilized careful seismic refraction studies to measure anisotropies as large as 8 per cent in the velocities of horizontally propagating *P* waves in the oceanic upper mantle. These measurements are usually interpreted as an inherent material anisotropy of the crystalline aggregate comprising the oceanic upper mantle. This material anisotropy is customarily ascribed, following the original suggestion of Hess (1964), to a preferential alignment of olivine crystals; various mechanisms have been proposed whereby the process of sea-floor spreading could give rise to such an alignment (Francis, 1969; Ave'Lallemant and Carter, 1970). Both Backus (1965) and Morris *et al.* (1969) have, however, pointed out that an alternative or partial explanation could perhaps be advanced in terms of an anisotropic static stress pattern in an otherwise isotropic oceanic upper mantle. It is shown in this paper that the velocity anisotropy of elastic body waves in a medium with an initial anisotropic stress may be divided naturally into two parts. The first part is attributable to the direct effect of the initial stress and can be described explicitly only in terms of that stress. The second part, herein called the inherent anisotropy of the medium (in the reference configuration), may depend indirectly and in some unknown way on the present value of or the history of past changes in the initial static stress, but this effect is observationally indistinguishable from a material anisotropy. It will be shown that the *direct effect* of any initial anisotropic stress cannot explain the magnitude of the observed anisotropies by several orders of magnitude, and that the oceanic upper mantle must, therefore, possess an inherent elastic anisotropy in its present state. The question of the extent to which this inherent anisotropy is a material anisotropy (i.e., caused by a preferential alignment of certain crystals) or is caused by some indirect effect of the present state or past history of stress in the material cannot be determined by seismological methods.

Mathematical Formulation

The theory of Biot (1940 and 1965) dealing with the infinitesimal perfectly elastic deformation of a medium subject to an anisotropic initial static stress has been reviewed and extended by Dahlen (1972), and the notation of that earlier paper will be used here. In that paper, the linearized equations of motion governing infinitesimal elastic-gravitational disturbances away from equilibrium of an arbitrary, uniformly rotating, self-gravitating, perfectly elastic Earth model with an arbitrary initial static stress field are derived. In this paper, those equations of motion will be applied to the much simpler case of the propagation of infinitesimal amplitude elastic body waves in a homogeneous, perfectly elastic medium. The effects of inhomogeneity, boundaries, the Earth's rotation, and self-gravitation will be neglected, and attention will be focused on the direct effect of an initial static stress on the propagation of body waves.

A brief word is in order concerning the nature of the initial static stress in the theory of Biot (1940 and 1965) and Dahlen (1972). A key aspect of the theory is that no assumptions whatsoever are made about the initial stress, except that it satisfy the condition of mechanical equilibrium in the initial configuration. In particular, the initial stress is *not* assumed to be related to some reversible finite strain away from a natural unstressed state; the only reference configuration ever referred to is the initial configuration which is in equilibrium under the influence of the initial static stress. One considers only infinitesimal elastic displacements away from this equilibrium state and the consequent incremental elastic stresses superimposed upon the static initial stress. This point of view is clearly an appropriate one for considering small elastic gravitational displacements of the Earth, because the present state of stress in the Earth was certainly not achieved by any reversible process. The theory of Biot (1940 and 1965) and Dahlen (1972) is in this sense in contrast to the theories of elastic wave propagation superimposed on a state of finite elastic strain which have been developed by Truesdell (1961), Thurston (1965), and others.

Consider then a homogeneous, nongravitating, perfectly elastic medium of density ρ at rest under the influence of a homogeneous but perhaps anisotropic stress tensor $\mathbf{T}_0$. It is convenient to separate this pre-existing static stress field $\mathbf{T}_0$ into an initial hydrostatic pressure p_0 and an initial static stress deviator $\boldsymbol{\tau}_0$ by the relation

$$\mathbf{T}_0 = -p_0\mathbf{I}+\boldsymbol{\tau}_0$$

$$p_0 = -\tfrac{1}{3}tr\,\mathbf{T}_0.$$

Now if this material is subjected to an infinitesimal displacement $\mathbf{s}(\mathbf{r}, t)$, it reacts with an infinitesimal increment $\mathbf{T}(\mathbf{r}, t)$ in the Lagrangian description of its stress. If the initial static stress is purely hydrostatic, $\boldsymbol{\tau}_0 = \mathbf{0}$, then the incremental Lagrangian stress is customarily assumed in the linearized theory to be linearly related to the infinitesimal displacement gradient $\nabla\mathbf{s}(\mathbf{r}, t)$ by means of a fourth-order elastic tensor $\mathbf{C}$. In terms of components with respect to an arbitrary Cartesian axis system $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$,

$$T_{ij} = C_{ijkl}[\tfrac{1}{2}(\partial_k s_l + \partial_l s_k)]. \tag{1}$$

The 81 coefficients C_{ijkl} are the Cartesian components with respect to $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ of the elastic tensor $\mathbf{C}$; if one assumes the existence of a thermodynamic elastic internal energy, then only 21 of the 81 components are independent because, in that case,

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{2}$$

If, however, the initial static stress includes a nonzero deviatoric part $\boldsymbol{\tau}_0$, then the linear stress-strain constitutive relation (1) must be modified. The most general linear relationship consistent with the existence of a thermodynamic elastic internal energy must in that case be of the form (Dahlen, 1972)

$$T_{ij} = \Gamma_{ijkl}\, \partial_k s_l \tag{3}$$

where

$$\Gamma_{ijkl} = C_{ijkl} + \tfrac{1}{2}(\tau^0_{kl}\delta_{ij} - \tau^0_{ij}\delta_{kl} + \tau^0_{ik}\delta_{jl} - \tau^0_{jl}\delta_{ik} + \tau^0_{jk}\delta_{il} - \tau^0_{il}\delta_{jk}) \tag{4}$$

with C_{ijkl} satisfying the elastic tensor symmetry relations (2). The coefficients Γ_{ijkl} and C_{ijkl} in (3) and (4) are the components with respect to $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$ of fourth-order tensors $\boldsymbol{\Gamma}$ and $\mathbf{C}$. The tensor $\boldsymbol{\Gamma}$ does not possess the symmetry (2) of an ordinary fourth-order elastic tensor, although that part of it denoted by $\mathbf{C}$ does. It is clear from (4) that there will, in general, be 26 independent coefficients Γ_{ijkl} (corresponding to 21 independent coefficients C_{ijkl} and five independent components of the traceless initial stress deviator $\boldsymbol{\tau}_0$).

Equation (4) represents the natural division of the elastic anisotropy of this initially stressed medium into a part owing to the direct effect of the initial deviatoric stress $\boldsymbol{\tau}_0$ and another part (represented by the tensor $\mathbf{C}$) which will be called the inherent anisotropy in the initial configuration. In general, all that one can say about the tensor $\mathbf{C}$ is that it has the symmetry (2) of an elastic tensor. A material will be called inherently isotropic (in the reference configuration) if the tensor $\mathbf{C}$ is a fourth-order isotropic tensor; in that case, the coefficients C_{ijkl} may be written in terms of two parameters, the rigidity μ and the isentropic bulk modulus κ

$$C_{ijkl} = (\kappa - \tfrac{2}{3}\mu)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{5}$$

If the inherent anisotropy is small, then C_{ijkl} may conveniently be written as

$$C_{ijkl} = (\kappa - \tfrac{2}{3}\mu)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma_{ijkl} \tag{6}$$

where $\gamma_{ijkl} \ll \mu$. The velocity anisotropy of elastic body waves propagating in an elastic medium free of a deviatoric initial stress, $\boldsymbol{\tau}_0 = \mathbf{0}$, but with a small inherent anisotropy [and hence described by equations (1) and (6)] has been investigated by Backus (1965 and 1970). This paper will consider, for the most part, the direct effect of a deviatoric initial stress $\boldsymbol{\tau}_0$ on the propagation of elastic body waves which in the initial reference configuration is inherently isotropic.

The equations of motion governing the infinitesimal elastic vibration of this homogeneous elastic material are

$$\rho\partial_t{}^2\mathbf{s} = \nabla \cdot \mathbf{T} \tag{7}$$

or rewritten in terms of its Cartesian components

$$\partial_t^{\,2} s_i = \rho^{-1}\Gamma_{ijkl}\, \partial_j \partial_k s_l. \tag{8}$$

A plane elastic body-wave solution of (8) has the form

$$\mathbf{s}(\mathbf{r}, t) = \mathbf{s} \exp i[\mathbf{k}\cdot\mathbf{r} - \omega t] \tag{9}$$

where $\mathbf{s}$ is a constant vector. The angular frequency of the wave is ω, and the wave vector is $\mathbf{k}$ (assumed to be real). If (9) is a solution of (8), then

$$\omega^2 s_i = \rho^{-1}\Gamma_{ijkl}\, k_j k_k s_l. \tag{10}$$

Define $k = (\mathbf{k}\cdot\mathbf{k})^{\frac{1}{2}}$; let $\hat{\mathbf{v}} = \mathbf{k}/k$ be the unit vector in the direction of $\mathbf{k}$, and let $c = \omega/k$ be the phase velocity of the body wave (9). Then equation (10) constitutes an eigenvalue problem for the phase velocity $c(\hat{\mathbf{v}})$. If one defines a second-order tensor $\mathbf{B}(\hat{\mathbf{v}})$ by

$$B_{il}(\hat{\mathbf{v}}) = \rho^{-1}\Gamma_{ijkl}\, v_j v_k \tag{11}$$

then equation (10) is

$$B_{il}(\hat{\mathbf{v}})\, s_l(\hat{\mathbf{v}}) = c^2(\hat{\mathbf{v}})\, s_i(\hat{\mathbf{v}}) \tag{12}$$

or in invariant notation

$$\mathbf{B}(\hat{\mathbf{v}})\cdot\mathbf{s}(\hat{\mathbf{v}}) = c^2(\hat{\mathbf{v}})\,\mathbf{s}(\hat{\mathbf{v}}). \tag{13}$$

The three eigenvalues $c^2(\hat{\mathbf{v}})$ of the tensor $\mathbf{B}(\hat{\mathbf{v}})$ are the squared phase velocities of the three body waves whose wave vector has the direction $\hat{\mathbf{v}}$. The three corresponding eigenvectors $\mathbf{s}(\hat{\mathbf{v}})$ give the polarization of these three body waves.

The tensor $\mathbf{B}(\hat{\mathbf{v}})$ is conveniently separated into a part $\mathbf{B}_0(\hat{\mathbf{v}})$ which describes the inherent anisotropy (if there is any) of the elastic material and another part $\mathbf{b}(\hat{\mathbf{v}})$ which describes the direct effect of the deviatoric initial stress $\boldsymbol{\tau}_0$ and which depends linearly on $\boldsymbol{\tau}_0$. Let $\mathbf{B}_0(\hat{\mathbf{v}})$ be defined by

$$B_{il}^0(\hat{\mathbf{v}}) = \rho^{-1}\, C_{ijkl}\, v_j v_k. \tag{14}$$

Then from (4)

$$\mathbf{B}(\hat{\mathbf{v}}) = \mathbf{B}_0(\hat{\mathbf{v}}) + \mathbf{b}(\hat{\mathbf{v}}) \tag{15}$$

where

$$\mathbf{b}(\hat{\mathbf{v}}) = \frac{1}{2\rho}\,[(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{v}})\,\mathbf{I} - \boldsymbol{\tau}_0]. \tag{16}$$

The tensor $\mathbf{B}_0(\hat{\mathbf{v}})$ is symmetric by virtue of the symmetry (2) of the elastic tensor $\mathbf{C}$, and the tensor $\mathbf{b}(\hat{\mathbf{v}})$ is symmetric by inspection. Hence $\mathbf{B}(\hat{\mathbf{v}})$ is a real symmetric tensor, and all the eigenvalues $c^2(\hat{\mathbf{v}})$ are real, and we may assume that the eigenvectors are real and orthonormal.

Equation (13) is the extension to this case of an anisotropic initial static stress $\mathbf{T}_0$ of the well-known Christoffel equation (Musgrave, 1970); the direct effect of $\mathbf{T}_0$ on the propagation of plane elastic body waves in a homogeneous medium may be completely characterized in terms of the additional tensor $\mathbf{b}(\hat{\mathbf{v}})$. Note that the tensor $\mathbf{b}(\hat{\mathbf{v}})$ depends only on the deviatoric part $\boldsymbol{\tau}_0$ of the initial static stress, as expected.

It is only because the coefficients Γ_{ijkl} do not satisfy the symmetry relations (2) characteristic of an elastic tensor that the well-known classical properties of elastic body waves in anisotropic media (Kraut, 1963; Musgrave, 1970; Backus, 1970) are not immediately applicable. Actually, although Backus (1970) dealt primarily with elastic tensors satisfying (1), he did extend his geometrical representation in terms of Maxwell multipoles to arbitrary fourth-order tensors. Certain of his results may thus be extended immediately to the study of anisotropically stressed media. This paper leans heavily on the work of Backus (1970), but his main result, the Maxwell multipole geometrical picture of a fourth-order tensor, will not be needed here.

We shall, at this point, assume that the elastic medium under consideration is inherently isotropic in the initial configuration so that the coefficients C_{ijkl} are given by (5). In that case,

$$\mathbf{B}_0(\hat{\mathbf{v}}) = (\alpha^2-\beta^2)\hat{\mathbf{v}}\hat{\mathbf{v}}+\beta^2\mathbf{I} \tag{17}$$

where α and β are the P and S body-wave phase velocities of the isotropic elastic medium, $\alpha^2 = (\kappa+4/3\mu)/\rho$ and $\beta^2 = \mu/\rho$. If $\mathbf{b}(\hat{\mathbf{v}}) = \mathbf{0}$ and $\mathbf{B}_0(\hat{\mathbf{v}})$ is given by (17), the solutions of (13) are well known. There is a nondegenerate eigenvalue $c_P{}^2(\hat{\mathbf{v}}) = \alpha^2$ with unit eigenvector $\hat{\mathbf{v}}$ (corresponding to a P wave), and there is a doubly degenerate eigenvalue $c_S{}^2(\hat{\mathbf{v}}) = \beta^2$ whose two-dimensional eigenspace consists of all vectors s that are perpendicular to $\hat{\mathbf{v}}$ (corresponding to S waves).

Propagation along the Principal Axes

The symmetric stress deviator tensor $\boldsymbol{\tau}_0$ may always be described in terms of its eigenvalues τ_1, τ_2, τ_3 and its associated three mutually perpendicular unit eigenvectors $\hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\eta}}_2, \hat{\boldsymbol{\eta}}_3$ (the principal axes of deviatoric stress).

$$\boldsymbol{\tau}_0 = \tau_1\hat{\boldsymbol{\eta}}_1\hat{\boldsymbol{\eta}}_1+\tau_2\hat{\boldsymbol{\eta}}_2\hat{\boldsymbol{\eta}}_2+\tau_3\hat{\boldsymbol{\eta}}_3\hat{\boldsymbol{\eta}}_3. \tag{18}$$

The case of plane-wave propagation along one of the principal axes of $\boldsymbol{\tau}_0$ is especially simple, and because it will appear later as a separate special case, it will first be treated separately. Without loss of generality, we may consider propagation along $\hat{\boldsymbol{\eta}}_3$, so that $\hat{\mathbf{v}} = \hat{\boldsymbol{\eta}}_3$. Inserting (18) in (16), one obtains for this case

$$\mathbf{B}(\hat{\mathbf{v}}) = \mathbf{B}(\hat{\boldsymbol{\eta}}_3) = \left[\alpha^2-\beta^2-\frac{\tau_3}{2\rho}\right]\hat{\mathbf{v}}\hat{\mathbf{v}}+\left[\beta^2+\frac{\tau_3}{2\rho}\right]\mathbf{I}-\frac{\tau_1}{2\rho}\,\hat{\boldsymbol{\eta}}_1\hat{\boldsymbol{\eta}}_1-\frac{\tau_2}{2\rho}\,\hat{\boldsymbol{\eta}}_3\boldsymbol{\eta}_3. \tag{19}$$

It is readily verified that the unit eigenvectors of $\mathbf{B}(\hat{\mathbf{v}})$ in this case may be chosen to be $\hat{\mathbf{v}}$, $\hat{\boldsymbol{\eta}}_1$, $\hat{\boldsymbol{\eta}}_2$ with associated eigenvalues

$$\begin{aligned} c_P{}^2(\hat{\mathbf{v}}) &= \alpha^2 \\ c_{S1}^2(\hat{\mathbf{v}}) &= \beta^2\left[1+\frac{1}{2\mu}(\tau_3-\tau_1)\right] \\ c_{S2}^2(\hat{\mathbf{v}}) &= \beta^2\left[1+\frac{1}{2\mu}(\tau_3-\tau_2)\right]. \end{aligned} \tag{20}$$

The propagation of P waves along any principal axis of the deviatoric initial stress $\boldsymbol{\tau}_0$ is from (20) completely unaffected by the presence of the stress. The propagation of S waves is, however, altered. Except in the degenerate case $\tau_1 = \tau_2 = -\frac{1}{2}\tau_3$, only two possible polarizations (namely, $\hat{\boldsymbol{\eta}}_1$ and $\hat{\boldsymbol{\eta}}_2$) are allowed to propagate in the $\hat{\boldsymbol{\eta}}_3$ direction. The S-wave polarization is purely transverse (and the P-wave polarization purely longitudinal); it may, in fact, be shown that it is only for this special case of principal axis propagation that this is true. The principal axes $\hat{\boldsymbol{\eta}}_1$, $\hat{\boldsymbol{\eta}}_2$, $\hat{\boldsymbol{\eta}}_3$ of $\boldsymbol{\tau}_0$ could be said to be the pure mode axes of the anisotropically stressed material, in the sense defined, e.g., by Kraut (1963).

The phase velocities of the two possible polarizations of S waves propagating along a principal axis $\hat{\boldsymbol{\eta}}_3$ are given explicitly by (20). Note that under certain conditions, equations

(20) yield an imaginary shear-wave velocity $c_{S1}(\hat{\mathbf{v}})$ or $c_{S2}(\hat{\mathbf{v}})$, corresponding to exponential growth rather than propagation of the S wave. This implies that, if the initial deviatoric stress is large enough compared to the rigidity, then the material is rendered unstable in the sense of infinitesimal oscillations. It might be of interest to examine the consequences of this, but it is most unlikely that such conditions could ever prevail in the Earth. We shall, henceforth, assume that the principal deviatoric stresses τ_1, τ_2, τ_3 are sufficiently less than the rigidity μ so that the tensor $\mathbf{B}(\hat{\mathbf{v}})$ is positive definite, and, thus, so that $c_{S1}^2(\hat{\mathbf{v}})$ and $c_{S2}^2(\hat{\mathbf{v}})$ in (20) are positive. It is then readily shown that both the maximum possible shear-wave velocity and the minimum possible shear-wave velocity in the medium occur for pure mode propagation along the principal axes. The maximum possible shear-wave velocity occurs for propagation $\hat{\mathbf{v}}$ along the axis of greatest deviatoric stress with polarization along the axis of least deviatoric stress, and, conversely, the least possible shear-wave velocity occurs for propagation $\hat{\mathbf{v}}$ along the axis of least deviatoric stress with polarization along the axis of greatest deviatoric stress. The results (20) were first obtained by Biot (1940); this special case of principal axis propagation was the only case considered by him.

First-Order Perturbation Theory

Let $\tau_0 = (\tau_1{}^2+\tau_2{}^2+\tau_3{}^2)^{\frac{1}{2}}$ denote the magnitude of the deviatoric initial stress $\boldsymbol{\tau}_0$. It is commonly believed that the magnitude of the deviatoric stresses existing in the Earth is of the order of a few hundred bars. If that is the case, then the approximation that $\tau_0 \ll \mu$ and $\tau_0 \ll \kappa$ is an appropriate one. The condition $\tau_0 \ll \mu$ [for brevity, we will, henceforth, only compare τ_0 to the rigidity μ; for the applications we have in mind $\mu = O(\kappa)$] implies that the tensor $\mathbf{b}(\hat{\mathbf{v}})$ in (15) which describes the effect of $\boldsymbol{\tau}_0$ on the propagation of body waves is much smaller than $\mathbf{B}_0(\hat{\mathbf{v}})$ [to be precise, $\mathrm{tr}(\mathbf{b}\cdot\mathbf{b}) \ll \mathrm{tr}(\mathbf{B}_0\cdot\mathbf{B}_0)$]. Perturbation theory can be used in this case to provide an explicit expression for the velocity anisotropy produced by a small deviatoric initial stress. The theory follows very closely the work of Backus (1965 and 1970), who used perturbation theory to treat the case of a small inherent anisotropy.

We write the eigenvalues of $\mathbf{B}(\hat{\mathbf{v}}) = \mathbf{B}_0(\hat{\mathbf{v}})+\mathbf{b}(\hat{\mathbf{v}})$ in the form $\alpha^2+\delta c_P{}^2(\hat{\mathbf{v}})$, $\beta^2+\delta c_{S1}^2(\hat{\mathbf{v}})$, $\beta^2+\delta c_{S2}^2(\hat{\mathbf{v}})$. For P waves, the perturbation theory is nondegenerate; thus, correct to first order in τ_0/μ,

$$\delta c_P{}^2(\hat{\mathbf{v}}) = \hat{\mathbf{v}}\cdot\mathbf{b}\cdot\hat{\mathbf{v}}. \tag{21}$$

From (16), this is identically zero. Correct to first order in τ_0/μ, the propagation velocities of plane elastic P waves in any direction $\hat{\mathbf{v}}$ are completely unaffected by the presence of an initial static stress. This simple result, the lack of a first-order P-wave anisotropy, is probably the most important conclusion of this work

$$c_P{}^2(\hat{\mathbf{v}}) = \alpha^2+O(\tau_0/\mu)^2. \tag{22}$$

The propagation characteristics of S waves are affected to first order in τ_0/μ. For S waves the perturbation theory is degenerate. Let $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ be any two mutually orthogonal unit vectors in the plane normal to the wave vector $\hat{\mathbf{v}}$. Then, again to first order in τ_0/μ, $\delta c_{S1}^2(\hat{\mathbf{v}})$ and $\delta c_{S2}^2(\hat{\mathbf{v}})$ are the two eigenvalues of the 2×2 symmetric matrix

$$\begin{bmatrix} \hat{\mathbf{s}}_1\cdot\mathbf{b}(\hat{\mathbf{v}})\cdot\hat{\mathbf{s}}_1 & \hat{\mathbf{s}}_1\cdot\mathbf{b}(\hat{\mathbf{v}})\cdot\hat{\mathbf{s}}_2 \\ \hat{\mathbf{s}}_1\cdot\mathbf{b}(\hat{\mathbf{v}})\cdot\hat{\mathbf{s}}_2 & \hat{\mathbf{s}}_2\cdot\mathbf{b}(\hat{\mathbf{v}})\cdot\hat{\mathbf{s}}_2 \end{bmatrix}. \tag{23}$$

These eigenvalues are found most readily by choosing $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ to be such that $\hat{\mathbf{s}}_1\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{s}}_2 = 0$ (i.e., $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ are chosen to be the unit eigenvectors of the tensor in two dimensions obtained by restricting τ_0 to the plane normal to the wave vector $\hat{\mathbf{v}}$). With that choice, the matrix (23) is diagonal with eigenvalues

$$\delta c_{S1}^2(\hat{\mathbf{v}}) = \beta^2 \left[\frac{1}{2\mu} (\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}}) - \frac{1}{2\mu} (\hat{\mathbf{s}}_1 \cdot \tau_0 \cdot \hat{\mathbf{s}}_1) \right]$$
$$\delta c_{S2}^2(\hat{\mathbf{v}}) = \beta^2 \left[\frac{1}{2\mu} (\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}}) - \frac{1}{2\mu} (\hat{\mathbf{s}}_2 \cdot \tau_0 \cdot \hat{\mathbf{s}}_2) \right]. \tag{24}$$

Except for the special case (already treated exactly under "Propagation Along the Principal Axes") of principal axis propagation with $\hat{\mathbf{s}}_1 \cdot \tau_0 \cdot \hat{\mathbf{s}}_1 = \hat{\mathbf{s}}_2 \cdot \tau_0 \cdot \hat{\mathbf{s}}_2 = -\frac{1}{2}(\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}})$, the degeneracy is completely removed to first order in τ_0/μ. Only two possible polarizations of S waves may then propagate in any direction $\hat{\mathbf{v}}$; to zeroeth order, these two possible polarizations are $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ (the two unit eigenvectors of the restriction of τ_0 to the plane normal to $\hat{\mathbf{v}}$). The propagation velocities associated with these two S-wave polarizations $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ are

$$c_{S1}^2(\hat{\mathbf{v}}) = \beta^2 \left[1 + \frac{1}{2\mu} (\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}} - \hat{\mathbf{s}}_1 \cdot \tau_0 \cdot \hat{\mathbf{s}}_1) + O(\tau_0/\mu)^2 \right]$$
$$c_{S2}^2(\hat{\mathbf{v}}) = \beta^2 \left[1 + \frac{1}{2\mu} (\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}} - \hat{\mathbf{s}}_2 \cdot \tau_0 \cdot \hat{\mathbf{s}}_2) + O(\tau_0/\mu)^2 \right]. \tag{25}$$

Except for the special case mentioned, the two S waves propagating in any direction $\hat{\mathbf{v}}$ will have propagation velocities differing by an amount

$$c_{S1}^2(\hat{\mathbf{v}}) - c_{S2}^2(\hat{\mathbf{v}}) = \beta^2 \left[\frac{1}{2\mu} (\hat{\mathbf{s}}_2 \cdot \tau_0 \cdot \hat{\mathbf{s}}_2 - \hat{\mathbf{s}}_1 \cdot \tau_0 \cdot \hat{\mathbf{s}}_1) + O(\tau_0/\mu)^2 \right]. \tag{26}$$

Note, furthermore, that the sum of the two S-wave velocities in any direction $\hat{\mathbf{v}}$ may be expressed very simply in terms of $\hat{\mathbf{v}}$ alone, because

$$c_{S1}^2(\hat{\mathbf{v}}) + c_{S2}^2(\hat{\mathbf{v}}) = \beta^2 \left[2 + \frac{3}{2\mu} (\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}}) + O(\tau_0/\mu)^2 \right]. \tag{27}$$

It is easy to see that the function $\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}}$ appearing in (27) is a homogeneous polynomial in the wave vector $\hat{\mathbf{v}}$ of degree $l = 2$. Inasmuch as τ_0 is a traceless tensor, this homogeneous polynomial is, in fact, a harmonic function (a solution of Laplace's equation in three dimensions) as well. The function $\hat{\mathbf{v}} \cdot \tau_0 \cdot \hat{\mathbf{v}}$ is the unique spherical harmonic (harmonic homogeneous polynomial) of degree $l = 2$ which is generated by the stress deviator tensor τ_0 (Backus, 1970). Equation (27) indicates that, correct to first order in τ_0/μ, the variation of the sum of the two S-wave velocities $\delta c_{S1}^2(\hat{\mathbf{v}}) + \delta c_{S2}^2(\hat{\mathbf{v}})$ may be written as a spherical harmonic of degree $l = 2$ in the wave vector $\hat{\mathbf{v}}$. No such simple spherical harmonic expansion is available for either of the two S-wave velocity anisotropies (24) considered separately.

Second-Order Perturbation Theory

For most geophysical applications, the ratio of the deviatoric initial stress τ_0 to the material rigidity μ is probably sufficiently small that first-order perturbation theory will provide an adequate approximation. It is, however, a simple matter to extend the perturbation theory to second order, and the result is somewhat instructive. In order to determine the second-order corrections to the P- and S-wave velocity anisotropies $c_P{}^2(\hat{\mathbf{v}})$, $c_{S1}^2(\hat{\mathbf{v}})$, and $c_{S2}^2(\hat{\mathbf{v}})$, it is first necessary to solve for the first-order corrections to the associated P- and S-wave polarizations. In the presence of a small, initial deviatoric stress $\boldsymbol{\tau}_0$, the polarization of a plane elastic P wave with wave vector $\hat{\mathbf{v}}$ is not in general (except for principal axis propagation) purely longitudinal, i.e., it is $\hat{\mathbf{v}}+O(\tau_0/\mu)$. Similarly, the polarizations of the two possible plane elastic S waves with wave vector $\hat{\mathbf{v}}$ are not purely transverse; the possible polarizations are $\hat{\mathbf{s}}_1+O(\tau_0/\mu)$ and $\hat{\mathbf{s}}_2+O(\tau_0/\mu)$, where both first-order corrections contain a component parallel to the wave vector $\hat{\mathbf{v}}$. The details of the second-order perturbation calculation are straightforward; only the results will be given here.

The presence of an initial static stress deviator $\boldsymbol{\tau}_0$ does induce an anisotropy of second order in τ_0/μ in the propagation of plane elastic P waves. Correct to second order in τ_0/μ, the phase velocities of P waves with wave vector $\hat{\mathbf{v}}$ may be written

$$c_P{}^2(\hat{\mathbf{v}}) = \alpha^2\left[1+\frac{1}{4}\left(\kappa+\frac{4}{3}\mu\right)^{-1}\left(\kappa+\frac{1}{3}\mu\right)^{-1}P(\hat{\mathbf{v}})+O(\tau_0/\mu)^3\right] \tag{28}$$

where the function $P(\hat{\mathbf{v}})$ is given by

$$P(\hat{\mathbf{v}}) = [\hat{\mathbf{v}}\cdot(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_0)\cdot\hat{\mathbf{v}}-(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{v}})^2]. \tag{29}$$

The first term in the P-wave velocity variation $P(\hat{\mathbf{v}})$ is a homogeneous polynomial in $\hat{\mathbf{v}}$ of degree $l = 2$, and the second term is a homogeneous polynomial in $\hat{\mathbf{v}}$ of degree $l = 4$. The method suggested by Backus (1970), following Courant and Hilbert (1953), may be used to decompose $P(\hat{\mathbf{v}})$ into a unique expression containing spherical harmonics in $\hat{\mathbf{v}}$ of degrees $l = 4$, 2 and 0. Let $\mathbf{d}_0$ denote the deviatoric part of the second-order tensor $\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_0$

$$\mathbf{d}_0 = \boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_0-\tfrac{1}{3}\,\mathrm{tr}\,(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_0)\mathbf{I} \tag{30}$$

and define a fourth-order tensor $\mathbf{D}_0$ by the relation

$$\mathbf{D}_0 = \left[-\boldsymbol{\tau}_0\boldsymbol{\tau}_0+\frac{4}{7}\,\mathbf{d}_0\mathbf{I}+\frac{2}{15}\,\mathrm{tr}\,(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_0)\mathbf{II}\right]. \tag{31}$$

Then $P(\hat{\mathbf{v}})$ may be written in the form

$$P(\hat{\mathbf{v}}) = H_4(\hat{\mathbf{v}})+H_2(\hat{\mathbf{v}})+H_0(\hat{\mathbf{v}}) \tag{32}$$

where

$$H_4(\hat{\mathbf{v}}) = D^0_{ijkl}\, v_i\, v_j\, v_k\, v_l$$

$$H_2(\hat{\mathbf{v}}) = \frac{3}{7}\, d^0_{ij}\, v_i\, v_j = \frac{3}{7}\,\hat{\mathbf{v}}\cdot\mathbf{d}_0\cdot\hat{\mathbf{v}}$$

$$H_0(\hat{\mathbf{v}}) = \frac{1}{5}\,\mathrm{tr}\,(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_0). \tag{33}$$

The term $H_0(\hat{\mathbf{v}})$ is a constant (a spherical harmonic in $\hat{\mathbf{v}}$ of degree $l = 0$). The term $H_2(\hat{\mathbf{v}})$ is a spherical harmonic of degree $l = 2$, by virtue of the fact that $\mathbf{d}_0$ is traceless, and it may be readily verified that $H_4(\hat{\mathbf{v}})$ is a spherical harmonic of degree $l = 4$. Equations (28) and (32) indicate that, correct to second order in τ_0/μ, the phase-velocity anisotropy $c_P{}^2(\hat{\mathbf{v}})$ of plane elastic P waves propagating in an initially stressed medium may be expressed as a spherical harmonic expansion in the wave vector $\hat{\mathbf{v}}$ containing only terms of degrees $l = 4$, 2 and 0.

The second-order velocity anisotropy of S waves is also easily obtained, because (except in one special case) the degeneracy is completely removed in the first-order theory. Correct to second order in τ_0/μ, the phase velocities of the two possible S waves with wave vector $\hat{\mathbf{v}}$ are

$$c_{S1}^2(\hat{\mathbf{v}}) = \beta^2[1+\tfrac{1}{2}\mu^{-1}(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{v}}-\hat{\mathbf{s}}_1\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{s}}_1)-\tfrac{1}{4}\mu^{-1}(\kappa+\tfrac{1}{3}\mu)^{-1}(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{s}}_1)^2+O(\tau_0/\mu)^3]$$
$$c_{S2}^2(\hat{\mathbf{v}}) = \beta^2[1+\tfrac{1}{2}\mu^{-1}(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{v}}-\hat{\mathbf{s}}_2\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{s}}_2-\tfrac{1}{4}\mu^{-1}(\kappa+\tfrac{1}{3}\mu)^{-1}(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{s}}_2)^2+O(\tau_0/\mu)^3]. \tag{34}$$

Neither $c_{S1}^2(\hat{\mathbf{v}})$ or $c_{S2}^2(\hat{\mathbf{v}})$ is conveniently written as a simple spherical harmonic expansion containing only a few terms. However, their sum $c_{S1}^2(\hat{\mathbf{v}})+c_{S2}^2(\hat{\mathbf{v}})$ takes the form

$$c_{S1}^2(\hat{\mathbf{v}})+c_{S2}^2(\hat{\mathbf{v}}) = \beta^2[2+\tfrac{3}{2}\mu^{-1}(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{v}})-\tfrac{1}{4}\mu^{-1}(\kappa+\tfrac{1}{3}\mu)^{-1}P(\hat{\mathbf{v}})+O(\tau_0/\mu)^3] \tag{35}$$

and this (an extension of (27) to second order) is a spherical harmonic expansion containing only terms of degrees $l = 4$, 2 and 0.

Note from (28) and (35) that, correct to second order in τ_0/μ, the sum of all three phase velocities $c_P{}^2(\hat{\mathbf{v}})+c_{S1}^2(\hat{\mathbf{v}})+c_{S2}^2(\hat{\mathbf{v}})$ is, simply

$$c_P{}^2(\hat{\mathbf{v}})+c_{S1}^2(\hat{\mathbf{v}})+c_{S2}^2(\hat{\mathbf{v}}) = \alpha^2+2\beta^2+\frac{3}{2\mu}(\hat{\mathbf{v}}\cdot\boldsymbol{\tau}_0\cdot\hat{\mathbf{v}}). \tag{36}$$

This result is, in fact, exact, because

$$\begin{aligned} c_P{}^2(\hat{\mathbf{v}})+c_{S1}^2(\hat{\mathbf{v}})+c_{S2}^2(\hat{\mathbf{v}}) &= \operatorname{tr}\mathbf{B}(\hat{\mathbf{v}}) \\ &= \operatorname{tr}\mathbf{B}_0(\hat{\mathbf{v}})+\operatorname{tr}\mathbf{b}_0(\hat{\mathbf{v}}). \end{aligned} \tag{37}$$

The variation $c_P{}^2(\hat{\mathbf{v}})+c_{S2}^2(\hat{\mathbf{v}})+c_{S2}^2(\hat{\mathbf{v}})$ of the sum may be written exactly as a single unique spherical harmonic of degree $l = 2$. A corollary of this remark is that measurement of the five independent spherical harmonic coefficients which describe this simple variation allows a direct determination of the initial static stress deviator $\boldsymbol{\tau}_0$.

Group-Velocity Anisotropy

All of the above results are concerned only with the phase-velocity anisotropy $\mathbf{c}(\hat{\mathbf{v}})$ of plane elastic body waves propagating in the initially stressed, homogeneous medium. The travel times of body waves traveling from an impulsive point source toward receivers located in various directions from the source do not give $\mathbf{c}(\hat{\mathbf{v}})$ directly. Such measurements, rather, give for any direction $\hat{\mathbf{G}}$ the magnitude $\boldsymbol{\Gamma}(\hat{\mathbf{G}})$ of the group-velocity $\mathbf{G}$ whose direction is $\hat{\mathbf{G}}$, ($\mathbf{G} = \hat{\mathbf{G}}\Gamma(\hat{\mathbf{G}})$). Backus (1970) has shown that, if the anisotropy is small ($\delta c_P(\hat{\mathbf{v}}) \ll \alpha$, $\delta c_{S1}(\hat{\mathbf{v}}) \ll \beta$, $\delta c_{S2}(\hat{\mathbf{v}}) \ll \beta$), then, correct to first order in the small anisotropy,

$$\Gamma(\hat{\mathbf{v}}) = c(\hat{\mathbf{v}}). \tag{38}$$

This result may be used to show that, correct to second order in τ_0/μ, equation (28) describes the group-velocity anisotropy of P waves as well as the phase-velocity anisotropy. If $\Gamma_P(\hat{\mathbf{v}})$ is the magnitude of the P-wave group velocity whose direction of propagation is $\hat{\mathbf{v}}$, then

$$\Gamma_P{}^2(\hat{\mathbf{v}}) = \alpha^2[1+\tfrac{1}{4}(\kappa+\tfrac{4}{3}\mu)^{-1}(\kappa+\tfrac{1}{3}\mu)^{-1}P(\hat{\mathbf{v}})+O(\tau_0/\mu)^3)]. \tag{39}$$

Similarly, equations (25) provide a description of the S-wave group-velocity anisotropy $\Gamma_{S1}(\hat{\mathbf{v}})$, $\Gamma_{S2}(\hat{\mathbf{v}})$, correct to first order in τ_0/μ. The extension of this result for S waves to second order in τ_0/μ requires an extension (straightforward) of Backus' (1970) argument leading to (38), and will not be given.

Conclusions

An inherently isotropic, perfectly elastic medium may be rendered anisotropic by the presence of a deviatoric initial stress. This paper has used perturbation theory to examine the precise nature of this stress-induced anisotropy for the case where the magnitude τ_0 of the initial static stress deviator is small compared to the material rigidity μ. The most important result for seismology is that, correct to first order in τ_0/μ, both the phase and group velocities of compressional (P) body waves remain independent of the direction of propagation. The velocity anisotropy of P waves is of second order in the ratio τ_0/μ, and is given explicitly as a function of the direction of propagation $\hat{\mathbf{v}}$ by (28) and (39). The magnitude of the deviatoric stresses existing in the Earth is commonly thought to be of the order of a few hundred bars; this would make the ratio τ_0/μ about 10^{-3} to 10^{-2}. If this is the case, the observed anisotropies (as large as 8 per cent) in the velocities of horizontally propagating P waves in the oceanic upper mantle are too large by several orders of magnitude to be explained by this effect. This argues very strongly that the observed velocity anisotropy of the oceanic upper mantle is caused entirely by an inherent anisotropy in its present state of static stress.

The question of the extent to which the presence of an initial deviatoric stress may have contributed to this inherent anisotropy cannot be determined by seismological observations, and would be extremely difficult to treat theoretically. Because the present state of stress within the Earth was certainly not achieved by means of some finite but reversible elastic deformation, the manner in which the present inherent anisotropy of the oceanic upper mantle material might be dependent upon the present state of stress or the past history of stress accumulation is highly uncertain. In view of this uncertainty, the original and simple hypothesis of Hess (1964) that the observed inherent anisotropy is, in fact, a material anisotropy associated with the preferential alignment of olivine crystals remains an attractive one.

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