12. Write and prove in tensor notation:

(a) Chapter 6, Problem 3.13.

(b) Chapter 6, Problem 3.14.

(c) Lagrange's identity: $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$.

(d) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{ABD})\mathbf{C} - (\mathbf{ABC})\mathbf{D}$, where the symbol $(\mathbf{XYZ})$ means the triple scalar product of the three vectors.

13. Write in tensor notation and prove the following vector operator identities in the table at the end of Chapter 6: parts (b), (d), (f), (g), (h), (k).

14. Show that the diagonal elements of an antisymmetric tensor are zero and that (5.15) is a correct display of the components of an antisymmetric 2^{nd}-rank tensor in 3 dimensions.

15. Write a 4-by-4 antisymmetric matrix to show that there are 6 different components, not the 4 components of a vector in 4 dimensions.

16. Verify that (5.16) gives (5.17). Also verify that (5.18) gives (5.17).

17. Write out the components of $T_{jk} = A_jB_k - A_kB_j$ to show that T_{jk} is a 2^{nd}-rank antisymmetric tensor with elements which are the components of $\mathbf{A}\times\mathbf{B}$.

6. PSEUDOVECTORS AND PSEUDOTENSORS

So far we have considered only rotations of rectangular coordinate systems in our definitions of tensors. Recall that an orthogonal transformation includes both rotations and reflections (Chapter 3, Sections 7 and 11). Now we want to consider how the entities we have called tensors behave under reflections. Remember that the determinant of an orthogonal matrix is $+1$ for a rotation (sometimes called a "proper" rotation) and the determinant is -1 if a reflection is involved (sometimes called an "improper" rotation).

When $\det \mathrm{A} = -1$, at least one eigenvalue of matrix A is -1 (see Chapter 3, Section 11). The -1 eigenvalue corresponds to the reversal of one principal axis, that is, a reflection through the plane perpendicular to the axis [for example a reflection through the (x, y) plane which reverses the z axis]. The other two eigenvalues correspond to a rotation [see Chapter 3, equation (7.19)]; this includes the case of a 180° rotation which is equivalent to reversal of the other two axes (see Problems 1 and 2). So in thinking about reflections, we can think of reversing all three axes (called an inversion) or reversing just one, since a rotation doesn't affect the sign of $\det \mathrm{A}$. It is important to realize that reversing either one or all three axes changes the coordinate system from a right-handed to a left-handed coordinate system.

▶ **Example 1.** Let's look at a simple example of something we usually think of as a vector (namely a cross product) which doesn't obey the vector transformation laws under reflections. Let $\mathbf{U}$ and $\mathbf{V}$ be displacement vectors. Recall (Section 2) that, by definition, a vector transforms the way displacement vectors do. Also remember that we are considering passive transformations: vectors remain fixed in space while the axes are changed (rotated or reflected). Now if the z axis is reversed [reflected through the (x, y) plane], then the z components of the displacement vectors $\mathbf{U}$ and $\mathbf{V}$ change signs; this is then a requirement for all vectors. But the z component of $\mathbf{U}\times\mathbf{V}$ (which is $U_xV_y - U_yV_x$) does not change sign (Problems 3 and 4). Thus $\mathbf{U}\times\mathbf{V}$ is not a vector under reflections. We call $\mathbf{U}\times\mathbf{V}$ a *pseudovector*. We will discover other pseudovectors as we continue.

Levi-Civita symbols We want to use (5.6) when the matrix A is the matrix of an orthogonal transformation. Remember (Chapter 3, Section 7) that if A is orthogonal, $\det \mathrm{A} = \pm 1$ so $(\det \mathrm{A})^2 = 1$. Multiply (5.6) by $\det \mathrm{A}$ to get the equation $\epsilon_{\alpha\beta\gamma} = (\det \mathrm{A}) a_{\alpha i} a_{\beta j} a_{\gamma k} \epsilon_{ijk}$. Then the transformation which gives $\epsilon' = \epsilon$ (see isotropic tensors in Section 5) is

$$\epsilon'_{\alpha\beta\gamma} = (\det \mathrm{A}) a_{\alpha i} a_{\beta j} a_{\gamma k} \epsilon_{ijk} = \epsilon_{\alpha\beta\gamma}. \tag{6.1}$$

Now this is not the right transformation equation for a 3^{rd}-rank tensor—the factor $\det \mathrm{A}$ would not be there for, say, the direct product of three displacement vectors. Of course, we got away with calling ϵ_{ijk} a 3^{rd}-rank tensor in Section 5 because we were discussing just rotations and $\det \mathrm{A} = 1$ if A is a rotation matrix. But now we are dealing with general orthogonal transformations, and when $\det \mathrm{A} = -1$ (reflection) there is an extra factor -1 in the transformation equation. We call ϵ_{ijk} a 3^{rd}-rank *pseudotensor*. A *pseudovector* or *pseudotensor* obeys the tensor transformation equations under rotations (that is, $\det \mathrm{A} = 1$), but if the transformation includes a reflection (that is, $\det \mathrm{A} = -1$), then the transformation equation contains an extra factor of -1. If we have a direct product of two pseudotensors (or such a product contracted), this will be a tensor because the product of the two $\det \mathrm{A}$ factors is $(\det \mathrm{A})^2 = 1$. (Problem 5).

Polar and Axial Vectors If a vector (under rotations) also satisfies the vector transformation equations (that is, behaves like a displacement vector) under reflections, it is called a *polar vector* (or true vector or just a vector). If there is a change in sign when $\det \mathrm{A} = -1$, it is called an *axial vector* (or pseudovector). In Example 1, $\mathbf{U}$ and $\mathbf{V}$ were polar vectors and $\mathbf{U} \times \mathbf{V}$ was an axial vector.

In order to understand pseudotensors we need to discuss left-handed coordinate systems. These are relatively unfamiliar in elementary work and for good reason. When we define a cross product or specify a vector to represent a rotation, the right hand rule is a part of our definition. It would be confusing to deal with this in a left-handed system so you are always warned to use right-handed systems. But we are now considering the general case of orthogonal transformations which includes reflections and so produces left-handed reference systems which we must learn to cope with.

Let's consider the physics and geometry of this by comparing linear velocity and angular velocity, both vectors under rotations. Is there a difference when we consider reflections and so have a left-handed coordinate system? The linear velocity vector indicates a path along which something moves; it has a direct physical meaning, and under passive transformations, it stays fixed in space. In the case of angular velocity, the physical motion is taking place in the plane perpendicular to the angular velocity vector, say a wheel rotating, or a mass or charge moving in a circle. The angular velocity "vector" is something *we choose via the right hand rule* to represent the motion. We might guess (correctly) that linear velocity is a vector (polar vector) and angular velocity is a pseudovector (axial vector). Remember that in Example 1 we found that the cross product (defined using the right hand rule) is a pseudovector. As we continue, watch for this; when the right hand rule is used in the *definition* of a vector, you suspect that it is a pseudovector.

Cross Product In Example 1, we found that the cross product of two displacement vectors does not satisfy the vector transformation equations under reflections. Now we want to write a formula to show exactly how a cross product transforms under a general orthogonal transformation. By (5.11), we write

$$(\mathbf{U}\times\mathbf{V})_i = \epsilon_{ijk}U_jV_k. \tag{6.2}$$

Then using (6.1), (6.2), and the vector transformation equations for the displacement vectors **U** and **V**, we find

$$\begin{aligned}(\mathbf{U}'\times\mathbf{V}')_\alpha &= \epsilon'_{\alpha\beta\gamma}U'_\beta V'_\gamma \\ &= (\det \mathrm{A})a_{\alpha i}a_{\beta j}a_{\gamma k}\epsilon_{ijk}a_{\beta m}U_m a_{\gamma p}V_p \\ &= (\det \mathrm{A})a_{\alpha i}\delta_{jm}\delta_{kp}\epsilon_{ijk}U_mV_p \\ &= (\det \mathrm{A})a_{\alpha i}(\epsilon_{ijk}U_jV_k) = (\det \mathrm{A})a_{\alpha i}(\mathbf{U}\times\mathbf{V})_i.\end{aligned} \tag{6.3}$$

If $\det \mathrm{A} = 1$ (no reflection, just a rotation), then (6.3) is the transformation equation for a vector. If $\det \mathrm{A} = -1$ (reflection) then the transformation has an extra -1 factor. Thus the vector product of two polar vectors is a pseudovector, as we have seen before and as we guessed from the fact that the right hand rule is used in defining cross product.

▶ **Example 2.** Find the triple scalar product of 3 polar vectors.

Here we have one $\det \mathrm{A}$ factor (from the cross product), so the triple scalar product of 3 polar vectors is a pseudoscalar (Problem 7).

▶ **Example 3.** What is the tensor character of $\mathbf{W}\times\mathbf{S}$ if **W** is a polar vector and **S** is a pseudovector?

In the transformation equation for $\mathbf{W}\times\mathbf{S}$, there is one factor of $\det \mathrm{A}$ for **S**, and another $\det \mathrm{A}$ for the cross product as in (6.3). The two minus signs cancel, so $\mathbf{W}\times\mathbf{S}$ is a polar vector (Problem 8).

▶ **Example 4.** Show that acceleration **a** and force **F** are polar vectors.

By definition, the displacement **r** is a polar vector (we define vectors as quantities which transform the way displacements do). Then the velocity $\mathbf{v} = d\mathbf{r}/dt$ and the acceleration $\mathbf{a} = d^2\mathbf{r}/dt^2$ are vectors (since time t is a scalar) and $\mathbf{F} = m\mathbf{a}$ is a vector since m is a scalar.

▶ **Example 5.** Find the tensor character of each symbol in $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$.

By Example 4, **v** is a vector so $\boldsymbol{\omega}\times\mathbf{r}$ must be a vector (both sides of a tensor equation must have the same tensor character). Then $\boldsymbol{\omega}$ must be a pseudovector so that there are two $\det \mathrm{A}$ factors, one from the cross product and one from $\boldsymbol{\omega}$. Recall that we predicted this because the right hand rule is used in defining angular velocity.

PROBLEMS, SECTION 6

1. Show that in 2 dimensions (say the x, y plane), an inversion through the origin (that is, $x' = -x, y' = -y$) is equivalent to a $180°$ rotation of the (x, y) plane about the z axis. *Hint:* Compare Chapter 3, equation (7.13) with the negative unit matrix.

2. In Chapter 3, we said that any 3-by-3 orthogonal matrix with determinant $= -1$ can be written in the form (7.19). Use this and Problem 1 to show that in 3 dimensions, an inversion (that is a reflection through the origin so that all three axes are reversed) is equivalent to a reflection through a plane combined with a rotation about the line perpendicular to the plane [say a reflection through the (x, y) plane—that is, a reversal of the z axis—and a rotation of the (x, y) plane about the z axis]. *Hint:* Consider the matrix B in Chapter 3, (7.19).

3. For Example 1, write out the components of $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{U} \times \mathbf{V}$ in the original right-handed coordinate system S and in the left-handed coordinate system S' with the z axis reflected. Show that each component of $\mathbf{U} \times \mathbf{V}$ in S' has the "wrong" sign to obey the vector transformation laws.

4. Do Example 1 and Problem 3 if the transformation to a left-handed system is an inversion (see Problem 2).

5. Write the tensor transformation equations for $\epsilon_{ijk}\epsilon_{mnp}$ to show that this is a (rank 6) tensor (*not* a pseudotensor). *Hint:* Write (6.1) for each ϵ and multiply them, being careful not to re-use a pair of summation indices.

6. Write the transformation equations to show that $\nabla \times \mathbf{V}$ is a pseudovector if $\mathbf{V}$ is a vector. *Hint:* See equations (5.13), (6.2) and (6.3).

7. Write the transformation equations for the triple scalar product $\mathbf{W}\cdot(\mathbf{U}\times\mathbf{V})$ remembering that now $\det \mathrm{A} = -1$ if the transformation involves a reflection. Thus show that the triple scalar product of three polar vectors is a pseudoscalar as claimed in Example 2. *Hint:* Use the result in (6.3).

8. Write the transformation equations for $\mathbf{W} \times \mathbf{S}$ to verify the results of Example 3.

In the physics formulas of Problems 9 to 14, identify each symbol as a vector (polar vector) or a pseudovector (axial vector). Use results from the text and the fact that both sides of an equation must have the same tensor character. The definition of the symbols used is: $\mathbf{r}$ = displacement, t = time, m = mass, q = electric charge, $\mathbf{v}$ = velocity, $\mathbf{F}$ = force, $\boldsymbol{\omega}$ = angular velocity, $\boldsymbol{\tau}$ = torque, $\mathbf{L}$ = angular momentum, T = kinetic energy, $\mathbf{E}$ = electric field, $\mathbf{B}$ = magnetic field. Assume that t, m, and q are scalars. Note that we are working in 3 dimensional physical space and assuming classical (that is nonrelativistic) physics.

9. $\mathbf{E} = \dfrac{\mathbf{F}}{q}$

10. $\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$

11. $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

12. $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

13. $\dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$

14. $T = \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})$

15. In equation (5.12), find whether $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is a vector or a pseudovector assuming

(a) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are all vectors;

(b) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are all pseudovectors;

(c) $\mathbf{A}$ is a vector and $\mathbf{B}$ and $\mathbf{C}$ are pseudovectors.

Hint: Count up the number of det A factors from pseudovectors and cross products.

16. In equation (5.14), is $\nabla \times (\nabla \times \mathbf{V})$ a vector or a pseudovector?

17. In equation (5.16), show that if T_{jk} is a tensor (that is, not a pseudotensor), then V_i is a pseudovector (axial vector). Also show that if T_{jk} is a pseudotensor, then V_i is a vector (true or polar vector). You know that if V_i is a cross product of polar vectors, then it is a pseudovector. Is its dual T_{jk} a tensor or a pseudotensor?

7. MORE ABOUT APPLICATIONS

Stress Tensor We started our discussion of tensors with a description of the stress tensor (you may want to review this in Section 1). Now let's show that the nine quantities P_{ij} displayed in the matrix (1.1) really are the components of a 2^{nd}-rank tensor. For simplicity in notation (and to use summation convention), we make the replacements indicated in (2.10); we also replace $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ by $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. Our problem is to write the components $P'_{\alpha\beta}$ relative to a rotated coordinate system in terms of the components P_{ij} to show that $P'_{\alpha\beta} = a_{\alpha i}a_{\beta j}P_{ij}$ as in (2.14) or (3.9).

Figure 7.1 shows the unprimed axes and one of the rotated axes. (With $\alpha = 1, 2, 3$, the x'_α axis represents any one of the rotated axes.) We draw a slanted plane, as shown, perpendicular to the x'_α axis, and consider the forces on the small volume element dV bounded by the unprimed coordinate planes and the slanted plane. Recall (Section 1) that pressure is force per unit area, so the force acting across a face is the pressure times the area of the face. Let the area of the slanted face (call it face α) be dS. Then the area of the face perpendicular to the x_i axis (call it face i) is $a_{\alpha i}\,dS$ where $a_{\alpha i}$ [see (2.10)] is the cosine of the angle between the x'_α and x_i axes (Problem 1).

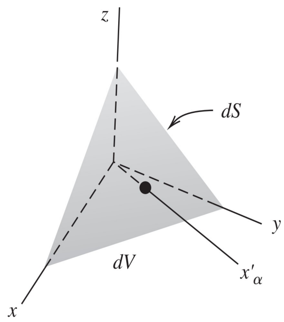

Figure 7.1

(7.1) $$\text{Area of face } i \text{ is equal to } a_{\alpha i}\,dS.$$

The pressure across face i is $P_{ij}\mathbf{e}_j$ (note the sum on j and see Problem 2). Multiplying this by (7.1) (force = pressure times area of face) and summing on i, we find that the total force acting on the material in the volume element dV, across the three faces in the unprimed coordinate planes is

(7.2) $$(P_{ij}\mathbf{e}_j)a_{\alpha i}\,dS.$$

For equilibrium, the sum of these three forces must be equal to the force acting across face α on the neighboring material. This force is

(7.3) $$P'_{\alpha\beta}\mathbf{e}'_\beta\,dS$$

Setting (7.2) and (7.3) equal, taking the dot product of both sides with $\mathbf{e}'_\beta$, and canceling dS, we have (Problem 3)

(7.4) $$P'_{\alpha\beta} = a_{\alpha i}a_{\beta j}P_{ij}$$

Thus we see that the stress P_{ij} is, as claimed, a 2^{nd}-rank tensor.

▶ **Example 1.** Suppose the following matrix is a display of the elements of a stress tensor.

$$\mathrm{P} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

We note that P is symmetric (this is true of stress tensors) so we can diagonalize P by an orthogonal transformation. In Chapter 3, Section 12, Example 2, we found that the eigenvalues of this matrix are 1, −4, 3. Thus a rotation of axes (matrix C in the Chapter 3 example) produces a stress tensor P′ with stress components only along the principal axes. The positive eigenvalues are tensions and the negative are compressions. Relative to the principal axes there are no shear forces.

Strain and Stress; Hooke's Law The strain tensor specifies the deformation of a solid body under stress. For a simple case such as a wire supporting a weight, strain (change in length per unit length) and stress (force per unit cross sectional area) are proportional (Hooke's Law). But for a 3 dimensional problem, stress is a 2^{nd}-rank tensor P_{ij} (as we have seen above), and strain is also a 2^{nd}-rank tensor S_{ij}. If the components of $\mathbf{P}$ are linear combinations of the components of $\mathbf{S}$, then we can write

$$P_{ij} = C_{ijkm} S_{km} \tag{7.5}$$

By the quotient rule, C_{ijkm} is a 4^{th}-rank tensor (Problem 5). The components of C_{ijkm} depend on the kind of material under stress and are called the elastic constants of the material (see Problem 6).

Inertia Tensor Revisited In Section 4 we considered the inertia tensor using vector notation. Now let's look at it using the tensor form for vector identities that we discussed in Section 5.

▶ **Example 2.** In (4.2) we had $\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. Using (5.12) with $\mathbf{A} = \mathbf{C} = \mathbf{r}$ and $\mathbf{B} = \boldsymbol{\omega}$, we find

$$L_n = m[\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_n = m(\delta_{nj}\delta_{ik} - \delta_{nk}\delta_{ij})x_i\omega_j x_k. \tag{7.6}$$

Now sum over i and k to get $\delta_{nj}\delta_{ik}x_i x_k = \delta_{nj}x_k x_k = \delta_{nj}r^2$ and $\delta_{nk}\delta_{ij}x_i x_k = x_j x_n$. Thus we have [compare (4.2)]

$$L_n = m(\delta_{nj}r^2 - x_n x_j)\omega_j. \tag{7.7}$$

The coefficient of ω_j is then the component I_{nj} of the inertia tensor.

$$I_{nj} = m(\delta_{nj}\, r^2 - x_n x_j). \tag{7.8}$$

We can easily verify that these components are the same as we found in Section 4. For example [compare (4.4)]:

$$I_{11} = m(r^2 - x_1^2), \quad I_{12} = -mx_1x_2, \quad I_{13} = -mx_1x_3, \tag{7.9}$$

and similarly for the other components (Problem 7).

Other Applications In your study of electric fields in matter, you will find the equation $\mathbf{P} = \chi\mathbf{E}$; this relates the electric field $\mathbf{E}$ applied to a dielectric and the resulting polarization $\mathbf{P}$ of the dielectric. For some materials it may be true that $\mathbf{P}$ and $\mathbf{E}$ are parallel vectors with $\chi =$ scalar, but for other materials $\mathbf{P}$ and $\mathbf{E}$ are not parallel. Now this should remind you of our work in Section 4 with the equation $\mathbf{L} = I\boldsymbol{\omega}$ when we realized that $\mathbf{L}$ and $\boldsymbol{\omega}$ are not always parallel. Just as we replaced the scalar I by a 2$^{\text{nd}}$-rank tensor, so we replace χ by a 2$^{\text{nd}}$-rank tensor. In the equation $P_i = \chi_{ij}E_j$, the quotient rule (see Section 3) tells us that χ_{ij} is a 2$^{\text{nd}}$-rank tensor. You will find other equations of this sort in various applications.

Tensor Fields Recall from Chapter 6 that a scalar field (temperature, for example) means a single number at each point, that is, a single function $f(x, y, z)$. A vector field (such as the electric field) means a set of three numbers at each point, that is, a set of three functions $V_i(x, y, z)$. Similarly, a 2$^{\text{nd}}$-rank tensor field means a set of 9 numbers at each point, that is, a set of 9 functions $T_{ij}(x, y, z)$. Think of our discussion of stress and strain. At every point in the material under stress, we can think of three vectors giving the force per unit area across the three perpendicular planes through the point, that is, a set of 9 functions. The 4$^{\text{th}}$-rank tensor C_{ijkm} in (7.5) is then a set of $3^4 = 81$ functions, and so on. (Of course, in order to be tensors these sets must transform properly under rotations as discussed in this chapter.)

▶ PROBLEMS, SECTION 7

1. Verify (7.1). *Hints:* In Figure 7.1, consider the projection of the slanted face of area dS onto the three unprimed coordinate planes. In each case, show that the projection angle is equal to an angle between the x'_α axis and one of the unprimed axes. Find the cosine of the angle from the matrix A in (2.10).

2. Write out the sums $P_{ij}\mathbf{e}_j$ for each value of i and compare the discussion of (1.1). *Hint:* For example, if $i = 2$ [or y in (1.1)], then the pressure across the face perpendicular to the x_2 axis is $P_{21}\mathbf{e}_1 + P_{22}\mathbf{e}_{22} + P_{23}\mathbf{e}_3$, or, in the notation of (1.1), $P_{yx}\mathbf{i} + P_{yy}\mathbf{j} + P_{yz}\mathbf{k}$.

3. Carry through the details of getting (7.4) from (7.2) and (7.3). *Hint:* You need the dot product of $\mathbf{e}'_\beta$ and $\mathbf{e}_j$. This is the cosine of an angle between two axes since each $\mathbf{e}$ is a unit vector. Identify the result from matrix A in (2.10).

4. Interpret the elements of the matrices in Chapter 3, Problems 11.18 to 11.21, as components of stress tensors. In each case diagonalize the matrix and so find the principal axes of the stress (along which the stress is pure tension or compression). Describe the stress relative to these axes. (See Example 1.)

5. Show by the quotient rule (Section 3) that C_{ijkm} in (7.5) is a 4$^{\text{th}}$-rank tensor.

6. If P and S are 2$^{\text{nd}}$-rank tensors, show that $9^2 = 81$ coefficients are needed to write each component of P as a linear combination of the components of S. Show that $81 = 3^4$ is the number of components in a 4$^{\text{th}}$-rank tensor. If the components of the 4$^{\text{th}}$-rank tensor are C_{ijkm}, then equation (7.5) gives the components of P in terms of the components of S. If P and S are both symmetric, show that we need only 36 different non-zero components in C_{ijkm}. *Hint:* Consider the number of different components in P and S when they are symmetric. *Comment:* The stress and strain tensors can both be shown to be symmetric. Further symmetry reduces the 36 components of C in (7.5) to 21 or less.

7. In (7.9) we have written the first row of elements in the inertia matrix. Write the formulas for the other 6 elements and compare with Section 4.

8. Do Problem 4.8 in tensor notation and compare the result with your solution of 4.8.

8. CURVILINEAR COORDINATES

Before we discuss non-Cartesian tensors we need to talk about some properties of curvilinear coordinate systems such as spherical or cylindrical coordinates. To make the discussion concrete, we shall illustrate the ideas involved by using two familiar coordinate systems—rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z). The elements of arc length in these two systems are given by

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \qquad \text{(rectangular coordinates)} \\ ds^2 &= dr^2 + r^2\,d\theta^2 + dz^2 \qquad \text{(cylindrical coordinates)} \end{aligned} \tag{8.1}$$

These expressions for ds are called *line elements*; they have much greater significance than just their use in computing arc lengths. First consider how we can find ds^2 for a given coordinate system. In the case of a well-known coordinate system, the answer may be obvious from the geometry. For example in polar coordinates in the plane we have (from Figure 8.1 and the Pythagorean theorem)

$$ds^2 = dr^2 + r^2\,d\theta^2. \tag{8.2}$$

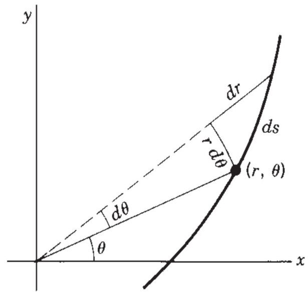

Figure 8.1

For an unfamiliar or complicated change of variables, however, we need a systematic method of finding ds; we illustrate the method by finding the value of ds^2 for cylindrical coordinates as given in (8.1). From the equations

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \end{aligned} \tag{8.3}$$

we get

$$\begin{aligned} dx &= \cos\theta\,dr - r\sin\theta\,d\theta, \\ dy &= \sin\theta\,dr + r\cos\theta\,d\theta, \\ dz &= dz. \end{aligned} \tag{8.4}$$

Squaring each equation in (8.4) and adding the results, we find

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\,d\theta^2 + dz^2. \tag{8.5}$$

Notice particularly here that all the cross products ($dr\,d\theta$, etc.) canceled out. This will not always happen, but it often does; when it does we call the coordinate system *orthogonal.* Such coordinate systems have some particularly simple and useful properties. Geometrically, an orthogonal system means that the *coordinate surfaces* are mutually perpendicular. For the cylindrical system (Figure 8.2), the coordinate surfaces are $r =$ const. (set of concentric cylinders), $\theta =$ const. (set of half-planes), and $z =$ const. (set of planes). The three coordinate surfaces through a given point intersect at right angles. The three curves of intersection of the coordinate surfaces in pairs intersect at right

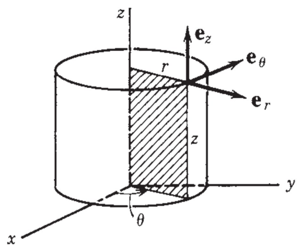

Figure 8.2

angles; these curves are called the *coordinate "lines"* or directions. We draw unit basis vectors tangent to the coordinate directions; for the cylindrical system (Figure 8.2) we might call them $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ ($\mathbf{e}_z$ is identical to $\mathbf{k}$). These unit vectors form an orthogonal triad like $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. We refer to such coordinate systems as *curvilinear coordinate systems* when the coordinate surfaces (or some of them) are not planes and the coordinate lines (or some of them) are curves rather than straight lines. We shall be principally interested in orthogonal curvilinear coordinate systems.

Scale Factors and Basis Vectors In the rectangular system, if x, y, z, are the coordinates of a particle, and x changes by dx with y and z constant, then the distance the particle moves is $ds = dx$. However, in the cylindrical system, if θ changes by $d\theta$ with r and z constant, the distance the particle moves is *not* $d\theta$, but $ds = r\,d\theta$. Factors like the r in $r\,d\theta$ which must multiply the differentials of the coordinates to get distances are known as *scale factors* and are very important as we shall see. A straightforward way to get them is to calculate ds^2 as we did in (8.5); if the transformation is orthogonal, then the scale factors can be read off from ds^2. (Note that the coefficients in ds^2 are the squares of the scale factors.) From (8.5), we see that the scale factors for cylindrical coordinates are 1, r, 1.

It is also useful to consider a vector $d\mathbf{s}$ which (in cylindrical coordinates) has components dr, $r\,d\theta$, dz in the coordinate directions $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$:

$$d\mathbf{s} = \mathbf{e}_r\,dr + \mathbf{e}_\theta\,r\,d\theta + \mathbf{e}_z\,dz. \tag{8.6}$$

Then $ds^2 = d\mathbf{s}\cdot d\mathbf{s}$ which gives (8.1), since the $\mathbf{e}$ vectors are orthonormal.

We can write the unit basis vectors of a curvilinear coordinate system ($\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ in cylindrical coordinates) in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. This is useful when we want to differentiate a vector which is expressed in terms of the curvilinear coordinate basis vectors. The unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are constant in magnitude *and direction*, but $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not fixed in direction, so their derivatives are not zero. We illustrate an algebraic method of finding the relation between two sets of basis vectors by finding them for the cylindrical system. (Compare the geometrical method shown in Chapter 6, Section 4.)

▶ **Example 1.** Using (8.4) and collecting coefficients of dr, $d\theta$, and dz, we find

$$\begin{aligned} d\mathbf{s} &= \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \\ &= \mathbf{i}(\cos\theta\,dr - r\sin\theta\,d\theta) + \mathbf{j}\,(\sin\theta\,dr + r\cos\theta\,d\theta) + \mathbf{k}\,dz \\ &= (\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)\,dr + (-\mathbf{i}r\sin\theta + \mathbf{j}\,r\cos\theta)\,d\theta + \mathbf{k}\,dz. \end{aligned} \tag{8.7}$$

Comparing (8.7) with (8.6), we have

$$\begin{aligned} \mathbf{e}_r &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \\ r\mathbf{e}_\theta &= -\mathbf{i}r\sin\theta + \mathbf{j}\,r\cos\theta \\ \mathbf{e}_z &= \mathbf{k}. \end{aligned} \tag{8.8}$$

Notice that $\mathbf{e}_r$ is already a unit vector since $\sin^2\theta + \cos^2\theta = 1$, but $r\mathbf{e}_\theta$ must be divided by the scale factor r to get the unit vector $\mathbf{e}_\theta$. It is often convenient to use basis vectors which we shall call $\mathbf{a}_r$ and $\mathbf{a}_\theta$ (which are not necessarily of unit length), given by the coefficients of dr and $d\theta$ in (8.7). Then we just have to divide each $\mathbf{a}$ vector by its magnitude to get the corresponding $\mathbf{e}$ vector. Thus from (8.7)

$$\begin{aligned} &\mathbf{a}_r = \mathbf{e}_r \text{ is already a unit vector,} \\ &\mathbf{a}_\theta = -\mathbf{i}r\sin\theta + \mathbf{j}\,r\cos\theta \text{ has magnitude } r, \text{ so} \\ &\mathbf{e}_\theta = \frac{1}{r}\mathbf{a}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta \end{aligned} \tag{8.9}$$

We can use these formulas to find the velocity and acceleration of a particle in cylindrical coordinates, and similar formulas for any coordinate system. The displacement of a particle from the origin at time t is, in cylindrical coordinates (Figure 8.3),

$$\mathbf{s} = r\mathbf{e}_r + z\mathbf{e}_z.$$

Then

$$\frac{d\mathbf{s}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d}{dt}(\mathbf{e}_r) + \frac{dz}{dt}\mathbf{e}_z.$$

By (8.8),

$$\frac{d}{dt}(\mathbf{e}_r) = -\mathbf{i}\sin\theta\frac{d\theta}{dt} + \mathbf{j}\cos\theta\frac{d\theta}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt},$$

so

$$\frac{d\mathbf{s}}{dt} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z. \tag{8.10}$$

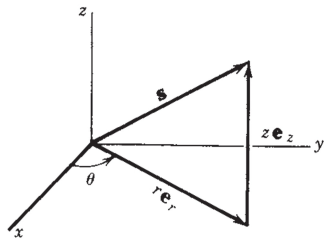

Figure 8.3

By differentiating again with respect to t and using (8.8) to find $(d/dt)(\mathbf{e}_\theta)$, we can find the acceleration $d^2\mathbf{s}/dt^2$ in cylindrical coordinates (Problem 2).

General Curvilinear Coordinates In general, let x_1, x_2, x_3 be the set of variables or coordinates we are considering (for example, in cylindrical coordinates, $x_1 = r$, $x_2 = \theta$, $x_3 = z$). Then the three sets of coordinate surfaces are $x_1 =$ const., $x_2 =$ const., $x_3 =$ const. The three coordinate surfaces through a given point intersect in three coordinate lines.

Example 2. Given x, y, z as functions of x_1, x_2, x_3, we can find $d\mathbf{s}$ and the $\mathbf{a}$ vectors as we did for cylindrical coordinates in (8.7) and (8.9).

$$\begin{aligned} d\mathbf{s} &= \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \\ &= \mathbf{i}\frac{\partial x}{\partial x_n}\,dx_n + \mathbf{j}\frac{\partial y}{\partial x_n}\,dx_n + \mathbf{k}\frac{\partial z}{\partial x_n}\,dx_n \\ &= \mathbf{a}_1\,dx_1 + \mathbf{a}_2\,dx_2 + \mathbf{a}_3\,dx_3 = \mathbf{a}_n\,dx_n, \end{aligned} \tag{8.11}$$

where

$$\mathbf{a}_n = \frac{\partial}{\partial x_n}\mathbf{s} = \mathbf{i}\frac{\partial x}{\partial x_n} + \mathbf{j}\frac{\partial y}{\partial x_n} + \mathbf{k}\frac{\partial z}{\partial x_n}. \tag{8.12}$$

Now defining $g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$, we can write $ds^2 = d\mathbf{s}\cdot d\mathbf{s}$ in matrix form as follows:

$$ds^2 = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}, \tag{8.13}$$

Note that g_{ij} is symmetric since the dot product of two vectors is the same in either order. In simpler form using summation convention (8.13) becomes

$$ds^2 = g_{ij}\,dx_i\,dx_j. \tag{8.14}$$

We will see later (Section 10) that the g_{ij} are the components of a tensor known as the *metric tensor.*

If the coordinate system is orthogonal, that is, if the basis vectors ($\mathbf{e}$ or $\mathbf{a}$) form an orthogonal triad, then $d\mathbf{s}$ and ds^2 can be written in terms of the scale factors as follows:

$$d\mathbf{s} = \mathbf{e}_1 h_1\,dx_1 + \mathbf{e}_2 h_2\,dx_2 + \mathbf{e}_3 h_3\,dx_3, \tag{8.15}$$

$$ds^2 = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}. \tag{8.16}$$

Also note that the volume element in an orthogonal system is $h_1h_2h_3\,dx_1\,dx_2\,dx_3$ (volume of a small rectangular parallelepiped with edges $h_1\,dx_1$, $h_2\,dx_2$, $h_3\,dx_3$). For example, in cylindrical coordinates, the volume element is $dr \cdot r\,d\theta \cdot dz = r\,dr\,d\theta\,dz$.

▶ PROBLEMS, SECTION 8

1. Find ds^2 in spherical coordinates by the method used to obtain (8.5) for cylindrical coordinates. Use your result to find for spherical coordinates, the scale factors, the vector $d\mathbf{s}$, the volume element, the basis vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, $\mathbf{a}_\phi$ and the corresponding unit basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$. Write the g_{ij} matrix.

2. Observe that a simpler way to find the velocity $d\mathbf{s}/dt$ in (8.10) is to divide the vector $d\mathbf{s}$ in (8.6) by dt. Complete the problem to find the acceleration in cylindrical coordinates.

3. Use the results of Problem 1 to find the velocity and acceleration components in spherical coordinates. Find the velocity in two ways: starting with $d\mathbf{s}$ and starting with $\mathbf{s} = r\mathbf{e}_r$.

4. In the text and problems so far, we have found the $\mathbf{e}$ vectors for various coordinate systems in terms of $\mathbf{i}$ and $\mathbf{j}$ (or $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in three dimensions). We can solve these equations to find $\mathbf{i}$ and $\mathbf{j}$ in terms of the $\mathbf{e}$ vectors, and so express a vector given in rectangular form in terms of the basis vectors of another coordinate system. Carry out this process to express in cylindrical coordinates the vector $\mathbf{V} = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$. *Hint:* Use matrices (as in Chapter 3) to solve the set of equations for $\mathbf{i}$ and $\mathbf{j}$.

5. Using the results of Problem 1, express the vector in Problem 4 in spherical coordinates.

As in Problem 1, find ds^2, the scale factors, the vector $d\mathbf{s}$, the volume (or area) element, the $\mathbf{a}$ vectors, and the $\mathbf{e}$ vectors for each of the following coordinate systems.

6. Parabolic cylinder coordinates u, v, z:

$$x = \frac{1}{2}(u^2 - v^2),$$
$$y = uv,$$
$$z = z.$$

7. Elliptic cylinder coordinates u, v, z:

$$x = a\cosh u \cos v,$$
$$y = a\sinh u \sin v,$$
$$z = z.$$

8. Parabolic coordinates u, v, ϕ:

$$x = uv\cos\phi,$$
$$y = uv\sin\phi,$$
$$z = \frac{1}{2}(u^2 - v^2).$$

9. Bipolar coordinates u, v:

$$x = \frac{a\sinh u}{\cosh u + \cos v},$$
$$y = \frac{a\sin v}{\cosh u + \cos v}.$$

10. Sketch or computer plot the coordinate surfaces in Problems 6 to 9.

Using the expression you have found for $d\mathbf{s}$, and for the $\mathbf{e}$ vectors, find the velocity and acceleration components in the coordinate systems indicated.

11. Parabolic cylinder

12. Elliptic cylinder

13. Parabolic

14. Bipolar

15. Let $x = u + v$, $y = v$. Find $d\mathbf{s}$, the $\mathbf{a}$ vectors, and ds^2 for the u, v coordinate system and show that it is not an orthogonal system. *Hint:* Show that the $\mathbf{a}$ vectors are not orthogonal, and that ds^2 contains $du\,dv$ terms. Write the g_{ij} matrix and observe that it is symmetric but not diagonal. Sketch the lines $u =$ const. and $v =$ const. and observe that they are not perpendicular to each other.

9. VECTOR OPERATORS IN ORTHOGONAL CURVILINEAR COORDINATES

We have previously (Chapter 6, Sections 6 and 7) defined the gradient (∇u), the divergence ($\nabla \cdot \mathbf{V}$), the curl ($\nabla \times \mathbf{V}$), and the Laplacian ($\nabla^2 u$) in rectangular coordinates x, y, z. Since in many practical problems it is better to use some other coordinate system (cylindrical or spherical, for example), we need to see how to express the vector operators in terms of general orthogonal coordinates x_1, x_2, x_3. (We consider only orthogonal coordinate systems here; see Section 10 for the more general case.) We shall outline proofs of the formulas; some of the details of the proofs are left to the problems.

Gradient, ∇u. In Chapter 6, Section 6, we showed that the directional derivative du/ds in a given direction is the component of ∇u in that direction.

Example 1. In cylindrical coordinates, if we go in the r direction (θ and z constant), then by (8.5) $ds = dr$. Thus the r component of ∇u is du/ds when $ds = dr$, that is, $\partial u/\partial r$. Similarly, the θ component of ∇u is du/ds when $ds = r\,d\theta$, that is, $(1/r)(\partial u/\partial\theta)$. Thus ∇u in cylindrical coordinates is

$$\nabla u = \mathbf{e}_r\frac{\partial u}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial u}{\partial\theta} + \mathbf{e}_z\frac{\partial u}{\partial z}. \tag{9.1}$$

In general orthogonal coordinates x_1, x_2, x_3, the component of ∇u in the x_1 direction (x_2 and x_3 constant) is du/ds if $ds = h_1\,dx_1$ [from (8.11)]; that is, the component of ∇u in the direction $\mathbf{e}_1$ is $(1/h_1)(\partial u/\partial x_1)$. Similar formulas hold for the other components and we have

$$\begin{aligned}\nabla u &= \mathbf{e}_1\frac{1}{h_1}\frac{\partial u}{\partial x_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial u}{\partial x_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial u}{\partial x_3}\\ &= \sum_{i=1}^{3}\mathbf{e}_i\frac{1}{h_i}\frac{\partial u}{\partial x_i}.\end{aligned} \tag{9.2}$$

Divergence, $\nabla \cdot \mathbf{V}$ Let

$$\mathbf{V} = \mathbf{e}_1 V_1 + \mathbf{e}_2 V_2 + \mathbf{e}_3 V_3 \tag{9.3}$$

be a vector with components V_1, V_2, V_3 in an orthogonal system. We can prove (Problem 1) that

$$\nabla \cdot \left(\frac{\mathbf{e}_3}{h_1 h_2}\right) = 0, \qquad \nabla \cdot \left(\frac{\mathbf{e}_2}{h_1 h_3}\right) = 0, \qquad \nabla \cdot \left(\frac{\mathbf{e}_1}{h_2 h_3}\right) = 0. \tag{9.4}$$

Let us write (9.3) as

$$\mathbf{V} = \frac{\mathbf{e}_1}{h_2 h_3}(h_2 h_3 V_1) + \frac{\mathbf{e}_2}{h_1 h_3}(h_1 h_3 V_2) + \frac{\mathbf{e}_3}{h_1 h_2}(h_1 h_2 V_3). \tag{9.5}$$

We find $\nabla \cdot \mathbf{V}$ by taking the divergence of each term on the right side of (9.5). Using (7.6) of Chapter 6, namely

$$\nabla \cdot (\phi \mathbf{v}) = \mathbf{v} \cdot (\nabla \phi) + \phi \nabla \cdot \mathbf{v}, \tag{9.6}$$

with $\phi = h_2 h_3 V_1$ and $\mathbf{v} = \mathbf{e}_1 / h_2 h_3$, we find that the divergence of the first term on the right side of (9.5) is

$$\nabla \cdot \left(h_2 h_3 V_1 \frac{\mathbf{e}_1}{h_2 h_3}\right) = \frac{e_1}{h_2 h_3} \cdot \nabla (h_2 h_3 V_1) + h_2 h_3 V_1 \nabla \cdot \left(\frac{\mathbf{e}_1}{h_2 h_3}\right). \tag{9.7}$$

By (9.4), the last term in (9.7) is zero. In the first term on the right side of (9.7), the dot product of $\mathbf{e}_1$ with $\nabla(h_2 h_3 V_1)$ is the first component of $\nabla(h_2 h_3 V_1)$. By (9.2), this is

$$\frac{1}{h_1} \frac{\partial}{\partial x_1}(h_2 h_3 V_1).$$

Calculating the divergence of the other terms of (9.5) in a similar way, we get

$$\nabla \cdot \mathbf{V} = \frac{1}{h_2 h_3} \frac{1}{h_1} \frac{\partial}{\partial x_1}(h_2 h_3 V_1) + \frac{1}{h_1 h_3} \frac{1}{h_2} \frac{\partial}{\partial x_2}(h_1 h_3 V_2) + \frac{1}{h_1 h_2} \frac{1}{h_3} \frac{\partial}{\partial x_3}(h_1 h_2 V_3)$$

or

$$\boxed{\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1}(h_2 h_3 V_1) + \frac{\partial}{\partial x_2}(h_1 h_3 V_2) + \frac{\partial}{\partial x_3}(h_1 h_2 V_3)\right].} \tag{9.8}$$

▶ **Example 2.** In cylindrical coordinates, $h_1 = 1$, $h_2 = r$, $h_3 = 1$. By (9.8), the divergence in cylindrical coordinates is

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{1}{r} \left[\frac{\partial}{\partial r}(r V_r) + \frac{\partial}{\partial \theta}(V_\theta) + \frac{\partial}{\partial z}(r V_z)\right] \\ &= \frac{1}{r} \frac{\partial}{\partial r}(r V_r) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}. \end{aligned} \tag{9.9}$$

Laplacian, $\nabla^2 u$. Since $\nabla^2 u = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}u$ we can find $\nabla^2 u$ by combining (9.2) and (9.8) with $\mathbf{V} = \boldsymbol{\nabla}u$. We get

(9.10)
$$\nabla^2 u = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right].$$

Example 3. In cylindrical coordinates, the Laplacian is then

$$\begin{aligned}\nabla^2 u &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}\right)\right]\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}.\end{aligned}$$

Curl, $\boldsymbol{\nabla}\times\mathbf{V}$. By methods similar to those used in finding $\boldsymbol{\nabla}\cdot\mathbf{V}$ we can find $\boldsymbol{\nabla}\times\mathbf{V}$ (Problem 2). The result is

(9.11)
$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{V} &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\\ h_1V_1 & h_2V_2 & h_3V_3\end{vmatrix}\\ &= \frac{\mathbf{e}_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3V_3) - \frac{\partial}{\partial x_3}(h_2V_2)\right]\\ &\quad + \frac{\mathbf{e}_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1V_1) - \frac{\partial}{\partial x_1}(h_3V_3)\right]\\ &\quad + \frac{\mathbf{e}_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2V_2) - \frac{\partial}{\partial x_2}(h_1V_1)\right]\end{aligned}$$

Example 4. In cylindrical coordinates, we find

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{V} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z}\\ V_r & rV_\theta & V_z\end{vmatrix}\\ &= \mathbf{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial\theta} - \frac{\partial V_\theta}{\partial z}\right) + \mathbf{e}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \frac{1}{r}\mathbf{e}_z\left(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right).\end{aligned}$$

PROBLEMS, SECTION 9

1. Prove (9.4) in the following way. Using (9.2) with $u = x_1$, show that $\boldsymbol{\nabla}x_1 = \mathbf{e}_1/h_1$. Similarly, show that $\boldsymbol{\nabla}x_2 = \mathbf{e}_2/h_2$ and $\boldsymbol{\nabla}x_3 = \mathbf{e}_3/h_3$. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in that order form a right-handed triad (so that $\mathbf{e}_1\times\mathbf{e}_2 = \mathbf{e}_3$, etc.) and show that $\boldsymbol{\nabla}x_1\times\boldsymbol{\nabla}x_2 = \mathbf{e}_3/(h_1h_2)$. Take the divergence of this equation and, using the vector identities (h) and (b) in the table at the end of Chapter 6, show that $\boldsymbol{\nabla}\cdot(\mathbf{e}_3/h_1h_2) = 0$. The other parts of (9.4) are proved similarly.

2. Derive the expression (9.11) for curl $\mathbf{V}$ in the following way. Show that $\boldsymbol{\nabla} x_1 = \mathbf{e}_1/h_1$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} x_1) = \boldsymbol{\nabla} \times (e_1/h_1) = 0$. Write $\mathbf{V}$ in the form

$$\mathbf{V} = \frac{\mathbf{e}_1}{h_1}(h_1 V_1) + \frac{\mathbf{e}_2}{h_2}(h_2 V_2) + \frac{\mathbf{e}_3}{h_3}(h_3 V_3)$$

and use vector identities from Chapter 6 to complete the derivation.

3. Using cylindrical coordinates write the Lagrange equations for the motion of a particle acted on by a force $\mathbf{F} = -\boldsymbol{\nabla} V$, where V is the potential energy. Divide each Lagrange equation by the corresponding scale factor so that the components of $\mathbf{F}$ (that is, of $-\boldsymbol{\nabla} V$) appear in the equations. Thus write the equations as the component equations of $\mathbf{F} = m\mathbf{a}$, and so find the components of the acceleration $\mathbf{a}$. Compare the results with Problem 8.2.

4. Do Problem 3 in spherical coordinates; compare the results with Problem 8.3.

5. Write out $\boldsymbol{\nabla} U$, $\boldsymbol{\nabla} \cdot \mathbf{V}$, $\nabla^2 U$, and $\boldsymbol{\nabla} \times \mathbf{V}$ in spherical coordinates.

Do Problem 3 for the coordinate systems indicated in Problems 6 to 9. Compare the results with Problems 8.11 to 8.14.

6. Parabolic cylinder

7. Elliptic cylinder

8. Parabolic

9. Bipolar

Do Problem 5 for the coordinate systems indicated in Problems 10 to 13.

10. Parabolic cylinder

11. Elliptic cylinder

12. Parabolic

13. Bipolar

In each of the following coordinate systems, find the scale factors h_u and h_v; the basis vectors $\mathbf{e}_u$ and $\mathbf{e}_v$; the u and v Lagrange equations, and from them the acceleration components (see Problem 3).

14. $\begin{cases} x = u - v, \\ y = 2\sqrt{uv}. \end{cases}$

15. $\begin{cases} x = uv, \\ y = u\sqrt{1 - v^2}. \end{cases}$

Use equations (9.2), (9.8), and (9.11) to evaluate the following expressions

16. In cylindrical coordinates, $\boldsymbol{\nabla} \cdot \mathbf{e}_r$, $\boldsymbol{\nabla} \cdot \mathbf{e}_\theta$, $\boldsymbol{\nabla} \times \mathbf{e}_r$, $\boldsymbol{\nabla} \times \mathbf{e}_\theta$.

17. In spherical coordinates, $\boldsymbol{\nabla} \cdot \mathbf{e}_r$, $\boldsymbol{\nabla} \cdot \mathbf{e}_\theta$, $\boldsymbol{\nabla} \times \mathbf{e}_\theta$, $\boldsymbol{\nabla} \times \mathbf{e}_\phi$.

18. In cylindrical coordinates, $\boldsymbol{\nabla} \times \mathbf{k} \ln r$, $\boldsymbol{\nabla} \ln r$, $\boldsymbol{\nabla} \cdot (r\mathbf{e}_r + z\mathbf{e}_z)$.

19. In spherical coordinates, $\boldsymbol{\nabla} \times (r\mathbf{e}_\theta)$, $\boldsymbol{\nabla}(r \cos\theta)$, $\boldsymbol{\nabla} \cdot \mathbf{r}$.

20. In cylindrical coordinates, $\nabla^2 r$, $\nabla^2(1/r)$, $\nabla^2 \ln r$.

21. In spherical coordinates, $\nabla^2 r$, $\nabla^2(r^2)$, $\nabla^2(1/r^2)$, $\nabla^2 e^{ikr\cos\theta}$.