3

Mechanics of a Single Particle and of Systems of Particles

Mechanics or the science of motion of material bodies has been a matter of great interest for ages. This science of motion of material bodies involves two aspects—kinematics and dynamics. In kinematics, we describe the motion of the material bodies in terms of quantities such as displacement, velocity, etc. It is a geometrical description of the motion. In dynamics, we investigate the causes of motion and the properties of the moving systems by studying the equation of motion containing force. The name 'classical mechanics' is used in this century to distinguish it from a modern branch of physics called 'quantum mechanics'.

In this chapter, we begin our discussion with the dynamics of a particle. By a particle we mean a point body having some mass or a body of some mass but having negligible physical dimensions in comparison with the dimensions of other bodies involved in the systems under consideration. The concept of a point particle is a mathematical idealisation of an object whose physical dimensions are negligible in a particular description of its motion under consideration. Thus, in the study of the motion of a planet around the sun, the planet can be treated as a particle. However, in some cases, an atom which can go too close to another charged body, such as a molecule, may not be treated as a particle.

The study of particle dynamics can be best developed on the basis of Newton's laws of motion. These laws are ingenious generalizations based on observations and hence are said to be 'empirical laws'. These laws can be looked upon as postulates of classical mechanics.

3.1 NEWTON'S LAWS OF MOTION

Newton's laws of motion are stated in the following form:

(i) "*Every body continues to be in its state of rest or of uniform motion* *in a straight line unless it is compelled to change that state by external forces acting on it.*"

(ii) "*The time rate of change of momentum of a particle is proportional to the external force and is in the direction of the force.*"

(iii) "*To every action there is always an equal and opposite reaction*" or "*the mutual actions of any two bodies are always equal and oppositely directed along the same straight line.*"

We shall try to understand the meaning of the laws by using our experimental knowledge as a guide. The first law contains the words '*body*' and '*straight line*'. We shall take the body to be equivalent to a particle and shall accept the usual geometrical concept of a straight line. The first law introduces two important concepts, namely (a) state of rest or of uniform motion, and (b) force. The concept of state of rest or of uniform motion are kinematical concepts. A body may appear to be at rest to one observer, but it may be in motion for some other observer. Since Newton's laws are based on physical observations, the concept of a 'frame of reference' from which the observations are made is automatically associated with the laws.

The property of a body to remain in the state of rest or of uniform motion when no external force acts on it is known as inertia and a frame of reference in which such a state is observed is known as the inertial frame of reference. We shall consider later the frames of reference in greater details.

The other concept introduced is of a force. Newton's first law of motion tells us about the motion of a body when no force acts on it. This law does not tell us what the force does; but it simply tells us what happens when it is absent. One can interpret the first law as the definition of '*zero force*'.

The meaning of the force in terms of the changes it produces in the momentum

$$\mathbf{p} = m\mathbf{v} \tag{3.1}$$

is given by Newton's second law which can be expressed as

$$\mathbf{F} \propto \frac{d\mathbf{p}}{dt}$$

or

$$\mathbf{F} = k\frac{d\mathbf{p}}{dt} = k\frac{d}{dt}(m\mathbf{v}) = km\frac{d\mathbf{v}}{dt}$$

where k is the constant of proportionality. This constant can be chosen to be equal to unity by defining the unit of the force as that force which while acting on a body of unit mass produces a unit acceleration. We have introduced a quantity mass and assumed it to be a constant, which may not always be true. Thus, the expression of Newton's second law becomes

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} \tag{3.2}$$

From the second law, it follows that when external force $\mathbf{F} = 0$, momentum $m\mathbf{v}$ is a constant, i.e., the body will continue to be in the state of uniform motion. We can consider the state of rest as a special case of state of uniform motion when $\mathbf{v} = 0$. Thus, the first law is a special case of the second law.

One is surprised at the lack of content in the first law and wonders why Sir Isaac Newton has given it the status of a law. The superfluous nature of the first law has been pointed out by Sir Arthur Eddington by saying that 'every particle continues in its state of rest or of uniform motion except in so far as it does not'. Such a comment, however, would be unfair to Newton who might have some definite idea in the statement of the first law.

We are familiar with a simple experiment in which a ball is allowed to roll along a smooth horizontal surface. By making the surface smoother and smoother, we believe that the ball will continue to move even upto an infinitely long distance with constant momentum if the friction is completely eliminated. If, however, one looks at the motion of heavenly bodies which are moving through space without any resistance to their motion, one finds that they move along curved paths. In fact, in our simple experiment the horizontal surface we imagine is also a part of the spherical surface of the earth. Thus, a large scale uniform motion along a straight line (here we neglect cases like dropping stone, rain drop, etc.) is a rarer phenomenon than motion along a curved path. This is why Greek philosophers considered motion along a circle as 'perfect motion'. That motion of a particle under zero force will be along a straight line is no doubt a great generalization from simple observations of small-scale motion. It was, possibly, the necessity to emphasize this concept, which is now of historical importance, that might have led Newton to give the statement the status of a law.

In the discussion on force we were considering the external force acting on a body. If more forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . are acting on the particle, then the total effect in motion produced by these forces can be looked upon as produced by a single force $\mathbf{F}$ which is the vector sum of all these forces. This principle is known as the 'principle of superposition' and is one of the fundamental principles in physical theories.

The third law applies only to the two isolated particles exerting forces on each other when forces due to all other particles are completely absent. The forces of action and reaction or of mutual interaction between the two particles do so along the line joining the two particles. According to the third law, the force acting on particle 1 due to particle 2, viz. $\mathbf{F}_1$ is equal and opposite to the force acting on particle 2 due to particle 1, viz. $\mathbf{F}_2$. Thus, $\mathbf{F}_1 = -\mathbf{F}_2$.

Equation (3.2), viz. $\mathbf{F} = m\mathbf{a}$ can be used to measure the mass of a body. Thus, let the same force $\mathbf{F}$ act on two bodies having masses m_1 and m_2 so as to produce accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively. Thus, we can write

$$m_1\mathbf{a}_1 = m_2\mathbf{a}_2$$

or

$$m_1 a_1 = m_2 a_2 \quad \text{numerically.}$$

Hence,

$$\frac{m_1}{m_2} = \frac{a_2}{a_1} \tag{3.3}$$

If, now, m_2 is known or taken as unit mass, m_1 can be calculated by measuring $\frac{a_2}{a_1}$. The mass of a body measured in this manner is called the inertial mass.

Another way of measuring the mass of a body is by weighing it, i.e., by comparing the gravitational force acting on it with that on a standard mass. In this procedure, we make use of the fact that in the gravitational field the weight of a body is exactly equal to the gravitational force acting on it. In that case, equation $\mathbf{F} = m\mathbf{a}$ becomes the weight, $\mathbf{W} = m\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity. The mass of a body measured in this manner is called the gravitational mass.

In recent experiments performed to find whether these two masses are identical or not it has been established that inertial and gravitational masses are equal within an error of a few parts in 10^{10}. The assertion of the exact equality of the inertial and the gravitational mass is called the *weak principle of equivalence.*

Frames of Reference

The concept of absolute rest or of uniform motion introduced in Newton's first law involves the concept of a frame of reference with respect to which the state of rest or of uniform motion is observed. Consider two frames of reference S and S' moving with relative velocity $\mathbf{v}$, say along x- (or x'-) axis for simplicity (Fig. 3.1). The observers in the two frames will give two different coordinates to the same particle at point

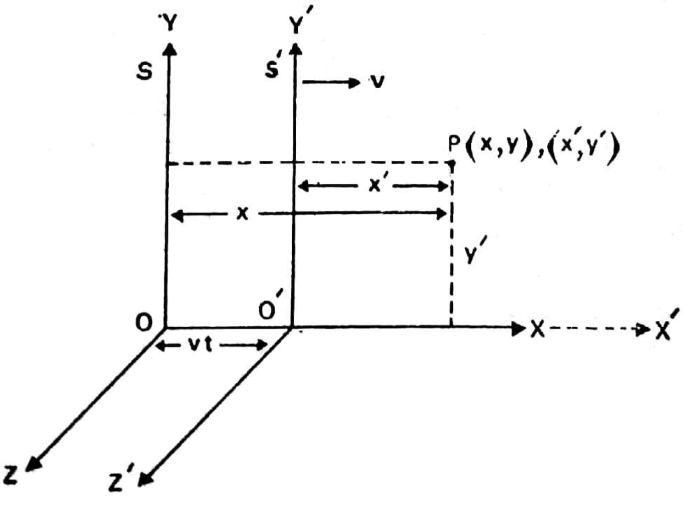


Fig. 3.1 Frames of reference S and S' in uniform relative motion along the x-axis

P which is observed by both. The coordinates are related to each other by transformation equations known as the *Galilean transformations*. Thus,

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
\text{and} \qquad t' &= t
\end{aligned}
\tag{3.4}
$$

Here, we are assuming that initially (i.e. at $t = 0$), the two coordinate systems were coincident and the motion of S' is along the direction of the x-axis only with uniform velocity v. It is clear that Newton's second law of motion will have the same form in these two frames since $d^2x'/dt'^2 = d^2x/dt^2$. Hence, the second law of motion is said to be invariant with respect to the Galilean transformation. Such non-accelerated frames of reference—non-accelerated with respect to one another and in which law of inertia holds—are called *inertial frames of reference*. If the transformation equations contain an acceleration term, i.e., if frame S' is accelerated with respect to S, then the form of the law of motion would be different in the two frames and the nature of the phenomenon observed by the two observers would be different.

Is it possible to have an inertial frame of reference ? A frame of reference fixed to the earth is obviously not an inertial frame since the earth is rotating about its own axis. We can then fix the frame at the centre of the earth. But, the earth is revolving around the sun. One may then fix the frame at the centre of the sun. But, the sun is also not at rest since it is accelerated in the galaxy. In this way, the search for a *point at rest* can be extended. Before the advent of Einstein's Theory of Relativity, it was considered possible to find out a star in the universe which might be at *absolute rest*. A frame of reference fixed on such a *fixed star* may be considered an absolute frame where we can observe *absolute rest* and any motion measured with respect to this frame can be considered an *absolute motion*.

Newton's laws, however, do not refer to *absolute rest* or *absolute uniform motion* and hence do not imply the necessity of a '*fixed star*' frame. The laws demand that if no force acts, the body should move with constant velocity in a reference frame called an inertial frame. Amongst the two frames moving relatively to each other with constant velocity, no particular frame is more important and any one can be used to solve dynamical problems. It is very difficult to have a perfectly inertial frame since bodies like the earth, the moon, the sun, etc. are moving along curved paths and hence are accelerated. We have, therefore, to select a frame of reference for a dynamical problem which is very close to an inertial frame, i.e., acceleration of the frame should be negligibly small as compared to the acceleration involved in the motion under study.

In considering an inertial frame, we implicitly require an Euclidian space, i.e., the usual three-dimensional space, and the assumption that this space is homogeneous and isotropic. By this we mean that the phenomena observed at various points will be identical and will not depend on the orientation of the coordinate axes. We also use the concept of homogeneity or uniformity of time, i.e., a particle moving with a constant velocity or a particle on which no external force is acting will traverse equal intervals of space in equal intervals of time. In the Galilean transformations, time is independent of the frame of reference chosen, i.e. $t' = t$ whatever may be the motion of the frame of reference. This, as will be seen in Chapter 14, is a consequence of the fact that a signal, say a flash of light sent out to observe the position of a body, is assumed to have infinitely large velocity. This situation changes in the Special Theory of Relativity in which time is different in different frames of reference moving relatively to each other, since the velocity of light is assumed to be finite and constant, a fact well known experimentally.

3.2 MECHANICS OF A PARTICLE

The essential problem in mechanics, developed with Newton's laws of motion as the basis, is to solve the differential equation given by Newton's second law of motion viz.

$$\mathbf{F} = m\mathbf{a}$$

or

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} \tag{3.5}$$

under given initial values of position and velocity of the particle on which force $\mathbf{F} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$ is acting. The solution of equation (3.5) is of the form

$$\mathbf{r} = \mathbf{r}(t) \tag{3.6}$$

Equation (3.6) describes a certain path and gives the dependence of the position vector $\mathbf{r}$ on time t. It is not always possible to find explicit solutions of equations (3.5) and we shall only restrict ourselves to simpler problems wherein exact or at least approximate solution can be obtained.

If no external force is acting on the particle, then

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = 0 \tag{3.7a}$$

or

$$\mathbf{p} = m\mathbf{v} = \text{const} \tag{3.7b}$$

Thus, if there is no force acting on the particle, the momentum of the particle is constant. If the mass of the particle is assumed to remain constant, it will continue to move along a straight line. This corresponds to a statement of Newton's first law of motion.

Angular momentum $\mathbf{L}$ of the particle about point O is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{3.8}$$

where $\mathbf{r}$ is the position vector and $\mathbf{p}$ is the linear momentum of the particle at the given instant.

The time rate of change of angular momentum **L** is defined as torque **N**. Thus,

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r}\times\mathbf{p}) = \dot{\mathbf{r}}\times\mathbf{p} + \mathbf{r}\times\dot{\mathbf{p}}$$

But, $$\dot{\mathbf{r}}\times\mathbf{p} = \mathbf{v}\times m\mathbf{v} = 0$$

and $$\mathbf{r}\times\dot{\mathbf{p}} = \mathbf{r}\times\frac{d\mathbf{p}}{dt} = \mathbf{r}\times\mathbf{F}$$

Hence, $$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \mathbf{r}\times\mathbf{F} \tag{3.9}$$

If torque **N** acting on the particle is zero, we have $\frac{d\mathbf{L}}{dt} = 0$ or $\mathbf{L} =$ a constant. Thus, if no torque is acting on a particle, its angular momentum is constant. Planets moving around the sun is the finest example of this conservation law.

Work Done The work done by the total external force in moving a particle from position 1 to position 2 is given by

$$W_{12} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} \tag{3.10}$$

If the mass of the particle is constant, we can write

$$\begin{aligned} W_{12} &= \int_1^2 m\frac{d\mathbf{v}}{dt}\cdot d\mathbf{r} \\ &= m\int_1^2 \frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= m\int_1^2 \left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\right)dt \\ &= \tfrac{1}{2}m\int_1^2 d(v^2) \\ &= \tfrac{1}{2}m(v_2^2 - v_1^2) \\ &= T_2 - T_1 \end{aligned} \tag{3.11}$$

where T_1 and T_2 are the kinetic energies of the particle in positions 1 and 2 respectively.

If $T_1 > T_2$, $W_{12} < 0$, i.e., work is done by the particle against the force and its kinetic energy has decreased.

If $T_2 > T_1$, $W_{12} > 0$, i.e., work is done by the force on the particle and the kinetic energy of the particle has increased.

In any case, the work done depends upon the difference in kinetic energies of the particle in the two positions. The work done against dissipative forces like the frictional force is always negative.

If the force-field is such that the work done along a closed path is zero, then the force is said to be *conservative*.

Thus, if $\oint \mathbf{F}\cdot d\mathbf{r} = 0$, then the force **F** is a conservative force.

We now convert this line integral into a surface integral by using

Stokes's theorem. Then, we get

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_\sigma \nabla \times \mathbf{F} \cdot d\boldsymbol{\sigma} \tag{3.12}$$

Thus, for a conservative force-field

$$\int_\sigma \nabla \times \mathbf{F} \cdot d\boldsymbol{\sigma} = 0 \tag{3.13}$$

Since $d\boldsymbol{\sigma}$, the surface element is arbitrary, we must have

$$\nabla \times \mathbf{F} = 0 \tag{3.14}$$

Equation (3.14) is the necessary and sufficient condition for force $\mathbf{F}$ to be conservative. We know that the curl of the gradient of a scalar point function is zero. Hence, we can write

$$\mathbf{F} = -\text{grad } V \tag{3.15}$$

Scalar point function V introduced in equation (3.15) is called the potential energy of the particle at that point. The negative sign on the right-hand side indicates that $\mathbf{F}$ is in the direction of decreasing V. We can write the expression for W_{12} as

$$\begin{aligned} W_{12} &= \int_1^2 \mathbf{F} \cdot d\mathbf{r} = -\int_1^2 \nabla V \cdot d\mathbf{r} \\ &= -\int_1^2 \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) \\ &= -\int_1^2 dV \\ &= V_1 - V_2 \end{aligned} \tag{3.16}$$

Thus, the work done by the force in displacing the particle from position 1 to position 2 is equal to the difference between the potential energies of the particle in those two positions.

Equating equations (3.11) and (3.15), we get

$$T_1 + V_1 = T_2 + V_2 = \text{const} = E \tag{3.17}$$

Thus, the sum of potential and kinetic energies of a particle at every point in a conservative force-field is constant. Gravitational and electrostatic fields are the common examples of conservative fields.

Potential energy V introduced through equation (3.15) makes it clear that it is not unique, and an addition of any constant-energy C to it does not change the equation $\mathbf{F} = -\text{grad } V$. Thus,

$$\mathbf{F} = -\text{grad } V = -\text{grad } (V + C) \tag{3.18}$$

since C is constant. Hence, the absolute value of the potential energy has no meaning. We can determine only the differences in potential energies as in equation (3.16).

It should be remembered that the introduction of potential energy is a convenient device to describe a force-field by a scalar function. Equation of motion (3.2) which describes the physical situation contains $\mathbf{F}$ and not V. The equation of motion can also be written through V as will be done later in Lagrange's and Hamilton's equations of motion.

The determination of the kinetic energy is also relative. As mentioned earlier, we use a certain inertial frame of reference for measuring the velocity and hence the kinetic energy. Consider two inertial frames of reference S and S' moving with relative constant velocity $\dot{\mathbf{R}}$ with respect to each other (Fig. 3.2).

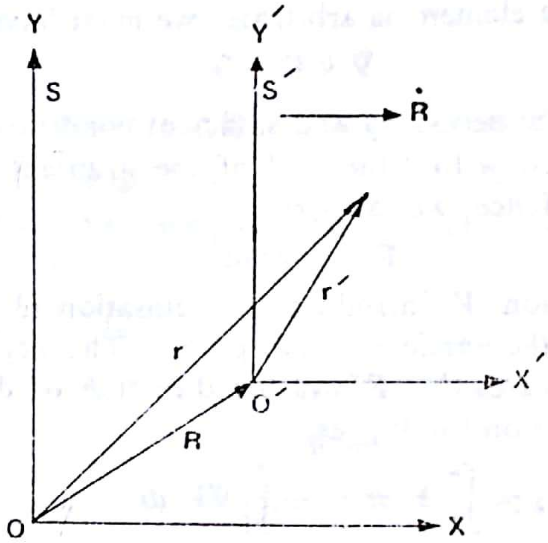


Fig. 3.2 Frames of reference in uniform relative motion (general case)

Let $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors of particle P with respect to S and S' respectively. Then, $\dot{\mathbf{r}}$ and $\dot{\mathbf{r}}'$ are the corresponding velocities of the particle. These values will be related to each other by the equations

$$\mathbf{r} = \mathbf{R} + \mathbf{r}' \tag{3.19}$$

and

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\mathbf{r}}' \tag{3.20}$$

Substitution of $\mathbf{r}$ or $\dot{\mathbf{r}}$ from equations (3.19) and (3.20) shows that the equations of motion, viz. (3.5) or (3.9), have the same form.

As we are using an inertial frame of reference for measuring velocity, the kinetic energy is relative. The absolute kinetic energy could have been measured if we could find a frame of reference which is absolutely at rest. It is, however, impossible to find such a frame of reference in accordance with the Special Theory of Relativity. Hence, the concept of absolute kinetic energy is meaningless. The total energy of a particle (or of a system of particles), viz. $E = T + V$, is not known in the absolute sense. Such a knowledge is unnecessary too and only the differences in the energies are of physical significance.

Consider the time derivative of total energy E,

i.e.
$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt}$$

But,
$$\mathbf{F}\cdot d\mathbf{r} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,dt = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right)dt = dT$$

or
$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \frac{dT}{dt}$$

Further, $$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz + \frac{\partial V}{\partial t}dt$$

$$= (\nabla V \cdot d\mathbf{r}) + \frac{\partial V}{\partial t}dt$$

Hence, $$\frac{dV}{dt} = \left(\nabla V \cdot \frac{d\mathbf{r}}{dt}\right) + \frac{\partial V}{\partial t} dt$$

Substituting these values, we get

$$\frac{dE}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} + \left(\nabla V \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial V}{\partial t}\right)$$

or $$\frac{dE}{dt} = [\mathbf{F} + \nabla V] \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial V}{\partial t} \tag{3.21}$$

The first term on the right-hand side of equation (3.21) will vanish if $\mathbf{F} = -\nabla V$, and

$$\frac{dE}{dt} = \frac{\partial V}{\partial t} \tag{3.22}$$

If, furthermore, the potential energy is not an explicit function of time, $\frac{\partial V}{\partial t} = 0$, then

$$\frac{dE}{dt} = 0 \quad \text{or} \quad E = \text{const}$$

Thus, the total energy of the particle moving in a conservative force-field remains constant, if the potential is not an explicit function of time.

3.3 EQUATION OF MOTION OF A PARTICLE

In general, the force acting on a particle may depend on its position, velocity and time and hence the equation of motion of the particle is

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \tag{3.23}$$

This is a second order differential equation in space coordinates and after integrating it twice, we will get the trajectory of the particle. There will be two constants of integration introduced and these will be determined by knowing the initial conditions of the particle. Integrating equation (3.23) with respect to time, we get

$$m\int_{t_0}^{t} \ddot{\mathbf{r}}\, dt' = \int_{t_0}^{t} \mathbf{F}\, dt'$$

or $$m(\dot{\mathbf{r}} - \mathbf{v}_0) = \int_{t_0}^{t} \mathbf{F}\, dt' \tag{3.24}$$

where we have taken initially, i.e. at $t = t_0$, velocity $\dot{\mathbf{r}}$ to be equal to $\mathbf{v}_0$. The quantity on the right-hand side of equation (3.24) is called the impulse of the force and the integral represents the impulse imparted to the particle during time interval $(t - t_0)$ and is equal to the change in the momentum of the particle. Integrating equation (3.24) once again with respect to time, we get

$$\mathbf{r}-\mathbf{r}_0=\mathbf{v}_0(t-t_0)+\frac{1}{m}\int_{t_0}^{t}dt'\int_{t_0}^{t'}\mathbf{F}\,dt'' \qquad (3.25)$$

where $\mathbf{r}$ and $\mathbf{r}_0$ are the position vectors of the particle at instants $(t=t)$ and $(t=t_0)$ respectively. If force $\mathbf{F}$ is known and integration of equation (3.25) is carried out, we get the explicit form of equation of the trajectory of the particle. In simple forms of $\mathbf{F}$, this integration is possible. In complicated cases, however, we have to resort to numerical integration. While solving problems in mechanics, it is essential to decide first the body whose motion is to be studied and then to consider all the forces applied to that body alone.

We shall now consider some simple forms of forces that we come across in nature. These are,

(a) $\mathbf{F}=m\mathbf{a}$, a constant. For example $\mathbf{F}=m\mathbf{g}$, when $\mathbf{g}$, the acceleration due to earth's gravity can be considered constant, i.e., near the surface of the earth.

(b) $\mathbf{F}=\mathbf{F}(t)$. For example, alternating force of the type $\mathbf{F}=\mathbf{F}_0\sin\omega t$, which is experienced by charged particles in alternating fields.

(c) $\mathbf{F}=\mathbf{F}(\mathbf{r})$. For example, $\mathbf{F}=k\mathbf{r}/r^3$, the force given by inverse square law experienced in the gravitational or electrostatic field, or $\mathbf{F}=-k\mathbf{r}$ which is experienced by a particle performing linear simple harmonic motion.

(d) $\mathbf{F}=\mathbf{F}(\dot{\mathbf{r}})$. For example, $\mathbf{F}=k\mathbf{v}$ or $\mathbf{F}=k\mathbf{v}^n$ which represents the frictional force experienced by a particle moving in a viscous medium.

One or more of the above types of forces might be acting on a particle. We shall consider motion under inverse square law and oscillatory motion in later chapters. In this chapter, we shall consider some simple cases of the remaining types of forces.

(a) Motion Under Constant Force

When constant force $\mathbf{F}=m\mathbf{a}$ is acting on a particle, equation (3.24) becomes

$$\dot{\mathbf{r}}=\mathbf{v}_0+\mathbf{a}(t-t_0) \qquad (3.26)$$

Integration of this equation gives, or from equation (3.25), we get

$$\mathbf{r}-\mathbf{r}_0=\mathbf{v}_0(t-t_0)+\tfrac{1}{2}\mathbf{a}(t-t_0)^2 \qquad (3.27)$$

This is the vector equation corresponding to one-dimensional form $s=ut+\frac{1}{2}at^2$ which is quite familiar to the reader.

Atwood's Machine Consider a system of two masses m_1 and m_2 $(m_2>m_1)$ tied by a light inextensible string of length l. The masses are hanging over a pulley as shown in Fig. 3.3a.

Since the masses can move only in the vertical direction, this is a one-dimensional problem. Let the position of, say m_2 be given by x. This also fixes the position of m_1. As $m_2 > m_1$, velocity $\frac{dx}{dt}$ of m_2 will be downwards whereas that of m_1 will be upwards. We neglect the friction between the string and the pulley.

The force acting on m_1 moving upwards is

$$F_1 = \tau - m_1 g \tag{3.28}$$

and that on m_2 moving downwards is

$$F_2 = -\tau + m_2 g \tag{3.29}$$

where τ is the tension in the string. According to Newton's second law

$$m_1 \ddot{x} = \tau - m_1 g \tag{3.30}$$

and

$$m_2 \ddot{x} = -\tau + m_2 g \tag{3.31}$$

Addition of equations (3.30) and (3.31) gives acceleration

$$\ddot{x} = \frac{m_2 - m_1}{m_1 + m_2} g \tag{3.32}$$

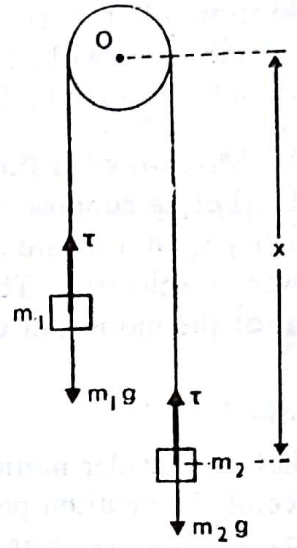


Fig. 3.3a Atwood's machine

Solving equations (3.30) and (3.31) for τ, we get

$$\tau = \frac{2m_1 m_2}{m_1 + m_2} g \tag{3.33}$$

It should be noted that if $m_1 = m_2$, acceleration $\ddot{x} = 0$ and the two masses are stationary. Similarly, if one of the masses is very large, i.e. $m_2 \gg m_1$, then $\ddot{x} \simeq g$ and $\tau \simeq 2m_1 g$ which is negligible as compared to $m_2 g$, and hence mass m_2 undergoes free fall.

(b) Motion Under a Force which Depends on Time Only

Consider time dependent force

$$F = F_0 \sin \omega t \tag{3.34}$$

acting on a particle. For simplicity consider only one-dimensional motion along the x-axis. Then, the equation of motion is

$$\ddot{x} = \frac{F_0}{m} \sin \omega t \tag{3.35}$$

On integration of equation (3.35) for a particle which has at the time $t = 0$, $x = x_0$ and $v = v_0$, we get

$$\dot{x} = v_0 + \frac{F_0}{m\omega}(1 - \cos \omega t)$$

$$= v_0 + \frac{F_0}{m\omega} - \frac{F_0}{m\omega} \cos \omega t \tag{3.36}$$

Integrating equation (3.36) once again, we get

$$x = x_0 + \left(v_0 + \frac{F_0}{m\omega}\right) t - \frac{F_0}{m\omega^2} \sin \omega t \tag{3.37}$$

This problem is of interest in connection with scattering of electromagnetic radiation by free electrons such as those appearing in the ionosphere.

(c) Motion Under a Force Dependent on Distance Only

Motion of a particle moving under the action of gravitational or Coulomb force will be considered in detail in Chapter 5, and that under harmonically varying force in Chapter 6.

(d) Motion of a Particle Subjected to a Resistive Force

1. Let us suppose that a body is projected horizontally with initial velocity $\mathbf{v}_0$ in a medium which offers resistance proportional to the first power of velocity. The equation of motion for the horizontal component of the motion in this case is

$$F = m\frac{dv}{dt} = -kmv \tag{3.38}$$

Let us consider motion along the x-axis only and let k be the resistive force of the medium per unit velocity per unit mass.

From equation (3.38), we get

$$\frac{dv}{v} = -k\,dt \tag{3.39}$$

Integrating equation (3.39), we get

$$\ln v = -kt + C_1 \tag{3.40}$$

If initially $t = 0$, $v = v_0$, then

$$\ln v_0 = C_1$$

Then, equation (3.40) becomes

$$\ln\frac{v}{v_0} = -kt$$

or

$$v = v_0 e^{-kt} \tag{3.41}$$

Integrating equation (3.41) again we will get the trajectory of the body. Thus, equation (3.41) is

$$\frac{dx}{dt} = v_0 e^{-kt}$$

Hence,

$$\int dx = v_0 \int e^{-kt}\,dt$$

or

$$x = -\frac{v_0}{k}e^{-kt} + C_2 \tag{3.42}$$

If we choose initial conditions as $x = 0$ at $t = 0$, then

$$C_2 = \frac{v_0}{k}$$

and equation (3.42) becomes

$$x = \frac{v_0}{k}(1 - e^{-kt}) \tag{3.43}$$

and gives the position of the body at any time t. As time elapses, the exponential factor decreases, i.e. $e^{-kt} \to 0$ as $t \to \infty$, and the body comes to a halt at distance $\frac{v_0}{k}$.

2. Consider the motion of a particle falling under the action of gravity near the surface of the earth. Let us assume that the frictional force of air is proportional to the velocity of the particle. Since the body will be accelerated vertically downwards, we shall take the x-axis along the path of the particle and treat the motion as one-dimensional.

Let the resistive force be $f = mkv$, where m is the mass of the falling body. The equation of motion is

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mg - mkv \tag{3.44a}$$

or

$$\frac{dv}{dt} = g - kv \tag{3.44b}$$

or

$$\frac{dv}{g - kv} = dt \tag{3.45}$$

Let the initial velocity with which the body is dropped be zero, i.e., at $t = 0$, $v = \frac{dx}{dt} = 0$. Integration of equation (3.45) gives

$$-\frac{1}{k}\ln(g - kv) = t + C \tag{3.46}$$

The constant of integration C is obtained from the initial conditions, viz. at $t = 0$, $v = 0$. Thus,

$$-\frac{1}{k}\ln g = C$$

Putting this constant in equation (3.46), we get

$$-\frac{1}{k}[\ln(g - kv) - \ln g] = -\frac{1}{k}\ln\frac{g - kv}{g} = t$$

Hence,

$$g - kv = ge^{-kt}$$

or

$$v = \frac{dx}{dt} = \frac{g}{k}(1 - e^{-kt}) \tag{3.47}$$

This equation gives the velocity of the body in a viscous medium when frictional force is mkv. If the path is long, after a sufficiently long time (as $t \to \infty$, $e^{-kt} \to 0$), the velocity of the body becomes constant and is g/k. This is called the 'terminal velocity'. From equation (3.44a) it is clear that, when the gravitational force is balanced by force of resistance of the medium, the body ceases to have acceleration and $\frac{dv}{dt} = 0$. After a long time the body has the terminal velocity given by

$$g - kv_t = 0$$

or

$$v_t = \frac{g}{k} \tag{3.48}$$

Integrating equation (3.47), we get

$$x = \frac{g}{k}\left(t + \frac{e^{-kt}}{k}\right) + C'$$

To evaluate constant C', let the body be released at point $x = 0$ at time $t = 0$. Then,

$$C' = -\frac{g}{k^2}$$

Hence, the position of the body is given by

$$x = \frac{gt}{k} - \frac{g}{k^2}(1 - e^{-kt}) \tag{3.49}$$

3. If the resistance of the medium is proportional to the square of the velocity, then the equation of motion of a vertically falling body is

$$m\frac{dv}{dt} = mg - mk^2v^2 \tag{3.50}$$

where for convenience we have put resisting force mk^2v^2. Then

$$\frac{dv}{g - k^2v^2} = dt \tag{3.51}$$

This equation can be written as

$$\frac{1}{2k\sqrt{g}}\left(\frac{1}{\sqrt{g} + kv} + \frac{1}{\sqrt{g} - kv}\right)d(kv) = dt \tag{3.52}$$

Integration gives

$$\frac{1}{2k\sqrt{g}}\ln\frac{\sqrt{g} + kv}{\sqrt{g} - kv} = t + C_1 \tag{3.53}$$

With initial conditions $t = 0$, $v = 0$, we get

$$C_1 = 0$$

Hence,
$$kv = \sqrt{g}\frac{e^{2kt\sqrt{g}} - 1}{e^{2kt\sqrt{g}} + 1} = \sqrt{g} - \frac{2\sqrt{g}}{1 + e^{2kt\sqrt{g}}} \tag{3.54}$$

Second integration gives

$$kx = \sqrt{g}t - 2\sqrt{g}t + \frac{1}{k}\ln(1 + e^{2kt\sqrt{g}}) + C_2 \tag{3.55}$$

Let $x = 0$ at $t = 0$, then

$$C_2 = -\frac{\ln 2}{k}$$

Thus,
$$x = \frac{1}{k^2}\ln\left(\frac{1 + e^{2kt\sqrt{g}}}{2}\right) - \frac{\sqrt{g}t}{k} \tag{3.56}$$

which gives the position of the body.

4. *Motion of a projectile—no resistance*: Let a body be projected at angle α with the horizontal with velocity v_0. The motion will remain in the vertical plane of velocity vector v_0. Let us take the x-axis along the horizontal and the y-axis upward along the vertical direction in the plane of motion. We write the initial conditions as

$$\left.\begin{aligned} x(t=0) &= 0, \quad y(t=0) = 0 \\ \dot{x}(t=0) &= v_0 \cos\alpha = U \\ \dot{y}(t=0) &= v_0 \sin\alpha = V \end{aligned}\right\} \tag{3.57}$$

The equations of motion of the projectile are

$$m\ddot{x} = 0, \; m\ddot{y} = -mg \tag{3.58}$$

Integration of equations (3.58) gives

$$\dot{x} = c_1, \; \dot{y} = -gt + c_2$$

With the initial conditions given in equations (3.57), we get $c_1 = U$ and $c_2 = V$. This gives

$$\dot{x} = U \text{ and } \dot{y} = -gt + V \tag{3.59}$$

Integrating again, we get

$$x = Ut, \quad y = Vt - \tfrac{1}{2}gt^2$$

These equations together give the equation of the trajectory of the projectile. It is usually expressed in terms of α, the angle of projection. Thus, eliminating t and remembering $\frac{V}{U} = \tan\alpha$, we get

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2}(1 + \tan^2\alpha) \tag{3.60}$$

Thus, the trajectory of projectile is parabola. The range R of the projectile is obtained from equation (3.60) by putting $x = R$ and $y = 0$. Thus,

$$R\tan\alpha = \frac{gR^2}{2v_0^2}(1 + \tan^2\alpha)$$

or

$$R = \frac{v_0^2 \sin 2\alpha}{g} \tag{3.61}$$

The time required to cover the distance equal to range is

$$T = \frac{2V}{g} \tag{3.62}$$

5. *Motion of a projectile in a resisting medium*: Consider the motion of a projectile in the atmosphere in which the retarding force is offered by the air resistance. Let us assume that the retarding force is proportional to the instantaneous velocity. In this case, the equations of motion of the projectile are, in the component form

$$m\ddot{x} = -km\dot{x} \tag{3.63a}$$

and

$$m\ddot{y} = -mg - km\dot{y} \tag{3.63b}$$

where the symbols have the same meanings as those in the last article. Equations (3.63a) and (3.63b) show that the present problem is equivalent to combinations of the problems discussed in 1 and 2 of this article with different initial conditions, i.e., those of 4.

The solution of equation (3.63a) is

$$x = \frac{U}{k}(1 - e^{-kt}) \tag{3.64a}$$

Similarly, the solution of equation (3.63b) can be written down b evaluating the constants of integration under the new initial condition: The solution is

$$y = -\frac{gt}{k} + \frac{kV + g}{k^2}(1 - e^{-kt}) \tag{3.64b}$$

To find range R', we find time T required by the projectile for the enti trajectory. For this we note that $y = 0$ at the end of the trajectory. Hence, from equation (3.64b), we get

$$T = \frac{kV + g}{gk}(1 - e^{-kT}) \tag{3.65a}$$

This gives time T in terms of e^{-kT} which can be expanded in a series an can be simplified as follows:

$$T = \frac{kV + g}{gk}[1 - (1 - kT + \tfrac{1}{2}k^2T^2 - \tfrac{1}{6}k^3T^3 \ldots)]$$

$$= \frac{kV + g}{gk}[kT - \tfrac{1}{2}k^2T^2 + \tfrac{1}{6}k^3T^3 \ldots]$$

Simplifying this equation, we get

$$1 = \frac{kV + g}{g}\left[1 - \tfrac{1}{2}kT + \tfrac{1}{6}k^2T^2 \ldots\right]$$

$$= \frac{kV + g}{g} - \left(\frac{kV + g}{g}\right)\frac{kT}{2} + \left(\frac{kV + g}{g}\right)\frac{k^2T^2}{6} \ldots$$

Hence,

$$T\left(1 + \frac{kV}{g}\right) = \frac{2}{k}\left[\left(\frac{kV + g}{g} - 1\right) + \left(\frac{kV + g}{g}\right)\frac{k^2T^2}{6} \ldots\right]$$

$$= \frac{2V}{g} + \left(1 + \frac{kV}{g}\right)\frac{kT^2}{3} \ldots \tag{3.65b}$$

This equation in the form of a series gives the value of the total time of flight T and cannot be solved exactly. We have, therefore, to obtain it approximate value by making successive approximations under th assumption that the terms containing higher orders of kT contribute les and less. For this assumption to be true, obviously, $kT < 1$.

Equation (3.65b) shows that if $k \to 0$, the time of flight of the pro jectile is

$$T_0 = \frac{2V}{g} \tag{3.65c}$$

Now, suppose that k is very small (but not equal to zero), then th time of flight will approach value T_0. We shall substitute this value in the second-order term and neglect the contributions from the rest. Thus we get

$$T\left(1 + \frac{kV}{g}\right) = \frac{2V}{g} + \left(1 + \frac{kV}{g}\right)\frac{k}{3}\left(\frac{2V}{g}\right)^2$$

or

$$T \simeq \frac{2V}{g}\left[\left(1 + \frac{kV}{g}\right)^{-1} + \frac{2kV}{3g}\right]$$

$$\simeq \frac{2V}{g}\left[\left(1 - \frac{kV}{g} + \frac{k^2V^2}{g^2} \ldots\right) + \frac{2kV}{3g}\right]$$

Neglecting the terms in k^2 and of higher orders in k and simplifying the equation, we get

$$T \simeq \frac{2V}{g}\left(1 - \frac{kV}{3g}\right) \tag{3.65d}$$

Let us now simplify equation (3.64a) as

$$x = \frac{U}{k}[1 - e^{-kt}]$$

$$= \frac{U}{k}[kt - \tfrac{1}{2}k^2t^2 + \tfrac{1}{6}k^3T^3 \ldots]$$

If we now substitute the value of T as given by equation (3.65d), we get $x_{(t=T)} = R'$, the range of the projectile in the presence of the air resistance. Thus, restricting to the first two terms, we have

$$R' = U[T - \tfrac{1}{2}kT^2]$$

$$\simeq \frac{2UV}{g}\left[1 - \frac{4kV}{3g}\right]$$

But,
$$\frac{2UV}{g} = \frac{2v_0 \cos\alpha \, v_0 \sin\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g} = R$$

Hence,
$$R' = R\left[1 - \frac{4kV}{3g}\right] \tag{3.66a}$$

Therefore, the change in the range of the projectile correct up to first order of k is given by

$$\Delta R = R' - R = -\frac{4kVR}{3g}$$

or
$$\Delta R = -\frac{4kv_0^3 \sin\alpha \sin 2\alpha}{3g} \tag{3.66b}$$

After finding the time of flight and the range of the projectile, let us study the nature of the trajectory. For any position of the projectile such as P (Fig. 3.3b) moving with velocity v and making angle φ with the

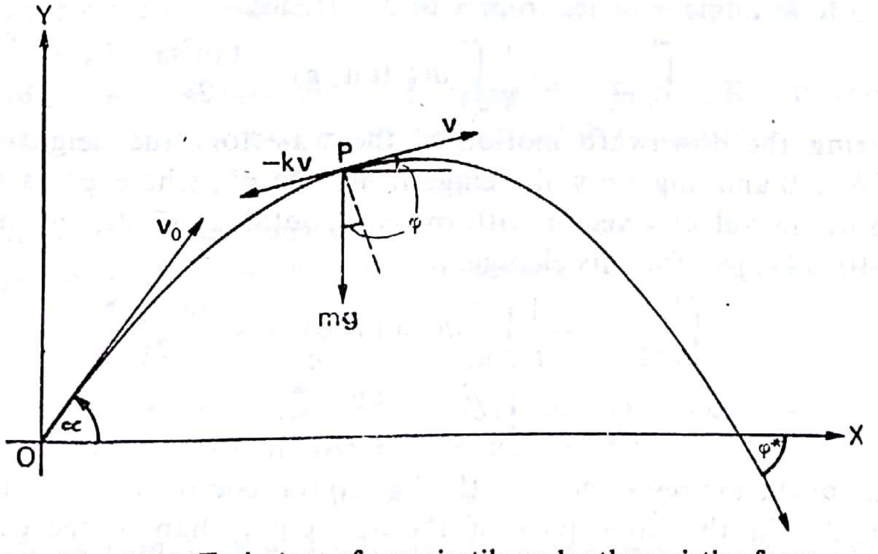


Fig. 3.3b Trajectory of a projectile under the resistive force

horizontal, we can write down the equation of motion by considering the motion along a small arc-length of the trajectory at P. Let the radius of curvature at P be ρ. The equations of motion written in a component form along tangential and radial directions are

$$m\dot{v} = -kv - mg \sin\varphi \tag{3.67}$$

and

$$\frac{mv^2}{\rho} = mg \cos\varphi \tag{3.68}$$

The right-hand sides of equations (3.67) and (3.68) are the components of forces along tangential and radial directions. The radius of curvature is given by

$$\frac{1}{\rho} = -\frac{d\varphi}{ds} = -\frac{d\varphi}{dt}\frac{dt}{ds} = -\frac{1}{v}\frac{d\varphi}{dt}$$

where $ds = \sqrt{dx^2 + dy^2} = v\,dt$.

Equation (3.68), after substituting for ρ becomes

$$v\frac{d\varphi}{dt} = -g\cos\varphi \tag{3.69}$$

Now,
$$v_x = \frac{dx}{dt} = v\cos\varphi$$

Hence,
$$dx = v\cos\varphi\,dt = -\frac{v^2\,d\varphi}{g}$$

where we have used equation (3.69).

Similarly,

$$v_y = \frac{dy}{dt} = v\sin\varphi$$

Hence,
$$dy = v\sin\varphi\,dt = v\tan\varphi\cos\varphi\,dt = -\frac{v^2\tan\varphi}{\rho}\,d\varphi$$
$$= -\frac{v_x^2\tan\varphi}{g\cos^2\varphi}\,d\varphi = -\frac{v_x^2}{g}\,d(\tfrac{1}{2}\tan^2\varphi)$$

Let the height of the projectile be h, i.e. $y_{max} = h$. Thus, as y varies from 0 to h, angle φ varies from α to 0. Hence

$$\int_0^h \frac{dy}{v_x^2} = -\frac{1}{g}\int_\alpha^0 d(\tfrac{1}{2}\tan^2\varphi) = \frac{\tan^2\alpha}{2g} \tag{3.70}$$

During the downward motion of the trajectory the height will vary from h to 0 and angle φ will change from 0 to φ^*, where φ^* is the angle made by the velocity vector with the horizontal as it strikes the ground ($y = 0$). Hence, for this change,

$$\int_h^0 \frac{dy}{v_x^2} = -\frac{1}{g}\int_0^{\varphi^*} d(\tfrac{1}{2}\tan^2\varphi) = -\frac{\tan^2\varphi^*}{2g}$$

or

$$\int_0^h \frac{dy}{v_x^2} = \frac{\tan^2\varphi^*}{2g} \tag{3.71}$$

Due to the air resistance, v_x, the horizontal component of the velocity is smaller in the later part of the trajectory than in the earlier one. Hence, the value of the integral in equation (3.70) is less than that in equation (3.71). We have, therefore,

$$\tan \varphi^* > \tan \alpha \quad \text{or} \quad \varphi^* > \alpha$$

Thus, the curvature of the trajectory gets reduced or radius ρ of curvature increases as the projectile progresses and the path is, as shown in Fig. 3.3b, quite different from the parabolic path when the air resistance was neglected.

It is obvious from the equations of motion that the velocity with which the projectile falls on the ground is less than the velocity with which it was projected. This reduction in velocity can be obtained from equation (3.67). Thus, by multiplying equation (3.67) by v, we get

$$v\frac{dv}{dt} = -\frac{k}{m}v^2 - gv\sin\varphi$$

or

$$v\,dv = d(\tfrac{1}{2}v^2) = -\frac{k}{m}v^2\,dt - g\,dy \tag{3.72}$$

since $dy = v \sin \varphi \, dt$. Integrating equation (3.72) from 0 to T, the time of flight during which y changes from 0 through h to 0 again, we get

$$\int_{v_0}^{v^*} d(\tfrac{1}{2}v^2) = -\frac{k}{m}\int_0^T v^2\,dt - g\int_0^T dy$$

where v^* is the velocity of the projectile when it strikes the ground. This gives

$$\frac{v^{*2} - v_0^2}{2} = -\frac{k}{m}\int_0^T v^2\,dt < 0, \text{ since} \int_0^T dy = 0$$

Hence,

$$v^* < v_0$$

3.4 MOTION OF A CHARGED PARTICLE IN ELECTROMAGNETIC FIELD

The force acting on a particle carrying charge e ($e > 0$) in an electromagnetic field is given by Lorentz equation

$$\mathbf{F} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \tag{3.73}$$

where $\mathbf{E}$ is the electric intensity, $\mathbf{B}$ the magnetic induction and $\mathbf{v}$ is the velocity of the particle.

In case of the electrostatic field, i.e., when $\mathbf{B} = 0$; $\nabla \times \mathbf{E} = 0$ and the field is described by the scalar electric potential $\Phi(\mathbf{r})$ defined by

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) \tag{3.74}$$

The force on the charged particle in the magnetostatic field, i.e., when $\mathbf{E} = 0$ depends upon the charge and velocity of the particle and is given by

$$\mathbf{F}_m = e\mathbf{v} \times \mathbf{B} \tag{3.75}$$

Since div $\mathbf{B} = 0$, we have seen earlier that the magnetic induction can be expressed through a vector potential $\mathbf{A}$ defined by

$$\mathbf{B} = \text{curl } \mathbf{A} = \nabla \times \mathbf{A} \tag{3.76}$$

If the particle is moving in a uniform magnetic field, it can easily be verified that the vector potential is given by

$$\mathbf{A} = -\tfrac{1}{2}\mathbf{r} \times \mathbf{B} \tag{3.77}$$

where $\mathbf{r}$ is the position vector of the particle.

The equation of motion of a particle of mass m under the action of Lorentz force is

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \tag{3.78}$$

We assume that the velocity of the particle is small as compared to that of light and hence relativistic variation of mass is neglected.

We shall consider the motion in some special cases.

(a) Motion in a Constant Electric Field

The electrostatic force on a particle of mass m and charge e is $e\mathbf{E}$, and the equation of motion is

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = e\mathbf{E} \tag{3.79}$$

or by introducing the scalar potential, equation (3.79) becomes

$$m\frac{d\mathbf{v}}{dt} = -e\nabla\Phi \tag{3.80}$$

Let us take the dot product of both the sides with velocity $\mathbf{v}$. Then

$$m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = -e\,\nabla\Phi\cdot\mathbf{v}$$

or

$$\frac{d}{dt}(\tfrac{1}{2}mv^2) = -e\left(\frac{\partial\Phi}{\partial x}\frac{dx}{dt} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt} + \frac{\partial\Phi}{\partial z}\frac{dz}{dt}\right)$$

$$= -e\frac{d\Phi}{dt}$$

Thus, we get

$$\frac{d}{dt}(\tfrac{1}{2}mv^2 + e\Phi) = 0$$

or

$$\tfrac{1}{2}mv^2 + e\Phi = \text{const} \tag{3.81}$$

This relation clearly shows conservation of energy which can be stated as "*the sum of kinetic energy $\tfrac{1}{2}mv^2$ and potential energy $e\Phi$ is a constant.*"

If the particle is initially at rest and falls through the potential difference of V volt, its initial potential energy eV is converted into kinetic energy. Hence,

$$\tfrac{1}{2}mv^2 = eV$$

or

$$v = \sqrt{\frac{2eV}{m}}$$

This gives the velocity acquired by the particle in the direction of the field. For an electron having mass $m = 9.11 \times 10^{-31}$ kg and charge $e = 1.60 \times 10^{-19}$ coul. the velocity is

$$v \approx 6 \times 10^5\sqrt{V}\ \text{m/s} = 600\sqrt{V}\ \text{km/s}$$

Thus electron falling through one volt potential difference gains a velocity of about 600 km/s. The kinetic energy acquired by the electron is measured in terms of electron-volt which is defined by

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$$

If we take a constant electric field along the y-axis, i.e., $\mathbf{E} = (0, E, 0)$, then the equations of motion in the cartesian coordinate system are

$$m\ddot{x} = 0 = m\ddot{z}, \quad m\ddot{y} = eE \tag{3.82}$$

If the particle is initially at rest, it will move along the y-direction only. The solutions of equations (3.82) are

$$x = x_0, \quad y = y_0 + \tfrac{1}{2}at^2, \quad z = z_0 \tag{3.83}$$

where (x_0, y_0, z_0) is the original position of the particle, and $a = \dfrac{eE}{m}$ the acceleration due to the electrostatic field along the y-axis.

(b) Motion in a Constant Magnetic Field

Now, the equation of motion of the charged particle is

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{v} \times \mathbf{B} \tag{3.84}$$

Taking the dot product of both sides of equation (3.84) with velocity $\mathbf{v}$, we get

$$m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = e\mathbf{v}\cdot(\mathbf{v} \times \mathbf{B}) \tag{3.85}$$

i.e. $$\frac{d}{dt}(\tfrac{1}{2}mv^2) = 0$$

since the right-hand side is a scalar triple product and $\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = (\mathbf{v}\times\mathbf{v})\cdot\mathbf{B} = 0$. Thus, the kinetic energy of the particle remains unchanged. Hence,

$$\tfrac{1}{2}mv^2 = \text{const} \tag{3.86}$$

Thus, speed v of the particle is unchanged in the magnetic field and field $\mathbf{B}$ changes only the direction of its velocity. The displacement $d\mathbf{r}$ of the particle in the magnetic field is, therefore, always perpendicular to the magnetic force on the particle and hence no work is done during this displacement.

Now consider the case of uniform magnetic field given by $\mathbf{B}$ = constant. Let us split velocity $\mathbf{v}$ into two components $\mathbf{v}_\perp$ which is perpendicular and $\mathbf{v}_\parallel$ which is parallel to the magnetic induction $\mathbf{B}$ (Fig. 3.4a). Thus,

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel \tag{3.87}$$

The parallel and perpendicular components of magnetic force are

$$\mathbf{F}_{m_\parallel} = e\mathbf{v}_\parallel \times \mathbf{B} = 0 \tag{3.88}$$

and $$\mathbf{F}_{m_\perp} = e\mathbf{v}_\perp \times \mathbf{B} \tag{3.89}$$

Thus, from equation (3.88), we have

$$\mathbf{F}_{m_\parallel} = m\frac{d\mathbf{v}_\parallel}{dt} = 0 \quad \text{or} \quad \mathbf{v}_\parallel = \text{const} \tag{3.90}$$

Thus, the velocity of the particle parallel to the field is unaffected. From equation (3.89), we have

$$m\frac{d\mathbf{v}_\perp}{dt} = e\mathbf{v}_\perp \times \mathbf{B} \tag{3.91}$$

But, from equation (3.87), we have

$$v^2 = v_\perp^2 + v_\parallel^2$$

and since $v_\parallel^2$ is constant by equation (3.90) and v^2 is constant by equation (3.81)

$$v_\perp^2 = \text{const} \tag{3.92}$$

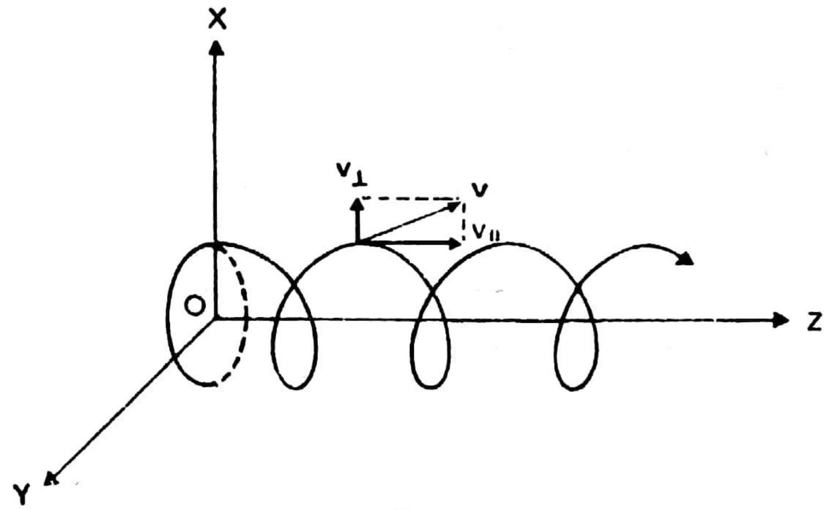


Fig. 3.4a Spiral motion of a charged particle along the constant magnetic induction **B** taken along the z-axis

Thus, the magnitude of perpendicular component is unchanged but its direction changes according to equation (3.91). We are considering motion in a constant magnetic induction and $\mathbf{v}_\perp$ is perpendicular to **B**. Hence $v_\perp B$ is a constant quantity. Thus the particle in a constant magnetic field moves in such a way that the force $ev_\perp B$ and acceleration $\frac{eB}{m}v_\perp$ are always directed perpendicular to velocity. This gives a circular motion in the plane perpendicular to **B** and the force is the centripetal force (Fig. 3.4b). If the circle has radius ρ then equation (3.89) can be written as

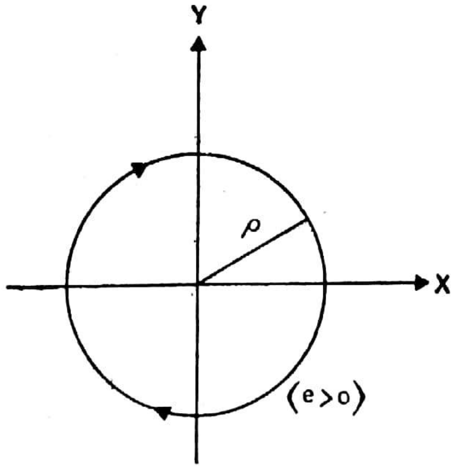


Fig. 3.4b Projection of the spiral motion on the xy-plane

$$\frac{mv_\perp^2}{\rho} = ev_\perp B$$

or

$$v_\perp = \frac{eB}{m}\rho$$

T... circular

motion is given by

$$\omega = \frac{2\pi}{\text{period}} = \frac{2\pi}{(2\pi\rho/v_\perp)} = \frac{v_\perp}{\rho} = \frac{eB}{m}$$

and is known as *cyclotron frequency*. The radius of the circular orbit is given by

$$\rho = \frac{v_\perp}{\omega} = \frac{mv_\perp}{eB}$$

In order to find the orbit of the particle, let magnetic induction **B** be taken along the z-axis so that $\mathbf{B} = \mathbf{k}B$. In the cartesian coordinates, equation of motion (3.84) now becomes

$$\frac{d\mathbf{v}}{dt} = \omega\mathbf{v} \times \mathbf{k} \tag{3.93}$$

where $\omega = \dfrac{eB}{m}$ Or in component form

$$\frac{dv_x}{dt} = \omega v_y, \quad \frac{dv_y}{dt} = -\omega v_x, \quad \frac{dv_z}{dt} = 0 \tag{3.94}$$

The first two equations in (3.94) are coupled equations, i.e., acceleration along the x-direction depends on velocity along the y-direction and vice versa. The third of equations (3.94) expresses the fact that velocity along the z-direction, i.e., along **B** is constant and its solution is

$$z = z_0 + v_{\parallel}t \tag{3.95}$$

where $\dot{z} = v_z = v_{\parallel} =$ constant. Thus, the particle moving from z_0 along the z-axis will be unaffected by the magnetic field and will move with constant velocity.

The two coupled equations can be combined together by multiplying the second equation by $i = \sqrt{-1}$ and then adding them together. Thus,

$$\frac{d}{dt}(v_x + iv_y) = -i\omega(v_x + iv_y) \tag{3.96}$$

The solution of this equation is clearly

$$v_x + iv_y = Ce^{-i\omega t} \tag{3.97}$$

Second integration gives

$$x + iy = C_1e^{-i\omega t} + C_2$$

where C_1 and C_2 are constants of integration to be determined from initial position and velocity. We choose the constants in the form

$$C_1 = Ae^{-i\alpha} \text{ and } C_2 = x_0 + iy_0$$

Then, the solution of the coupled equation is

$$x + iy = Ae^{-i(\omega t+\alpha)} + x_0 + iy_0 \tag{3.98}$$

The x- and y-coordinates of the particle are obviously real and are obtained from equation (3.98). Therefore,

$$\left.\begin{aligned} x &= \text{Re}\,[Ae^{-i(\omega t+\alpha)} + x_0 + iy_0] = A\cos(\omega t + \alpha) + x_0 \\ y &= \text{Im}\,[Ae^{-i(\omega t+\alpha)} + x_0 + iy_0] = -A\sin(\omega t + \alpha) + y_0 \end{aligned}\right\} \tag{3.99}$$

Equations (3.99) together with equation (3.95) give the trajectory of the particle in a constant magnetic field.

From equation (3.99) we have

$$(x - x_0)^2 + (y - y_0)^2 = A^2 \tag{3.100}$$

which is an equation of a circle in the xy-plane with (x_0, y_0) as the origin. If $v_{\parallel} = 0$, then the particle will move in a circular orbit with angular velocity vector

$$\boldsymbol{\omega} = -\frac{e\mathbf{B}}{m} = -\frac{eB}{m}\mathbf{k} = -\omega\mathbf{k} \tag{3.101}$$

The radius $A(=\rho)$ is obtained from equation (3.99) and remembering that $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v_{\perp}^2$. Thus,

$$\dot{x}^2 + \dot{y}^2 = v_{\perp}^2 = A^2\omega^2$$

or

$$A = \frac{v_{\perp}}{\omega} = \rho \tag{3.102}$$

as is already obtained above.

(c) Motion in Crossed Fields

Consider now the motion of a charged particle when constant electric and magnetic fields are simultaneously acting over a region of space. The equation of motion of the particle is

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \tag{3.103}$$

Let the fields be perpendicular to each other, i.e. $\mathbf{E} \perp \mathbf{B}$ (Fig. 3.5). We search for the solution of equation (3.103) in the form

$$\mathbf{v} = \mathbf{v}' + \frac{1}{B^2}\mathbf{E} \times \mathbf{B} = \mathbf{v}' + \mathbf{v}_d \tag{3.104}$$

where we have put $\mathbf{v}_d = \frac{1}{B^2}(\mathbf{E} \times \mathbf{B})$.

Substituting $\mathbf{v}$ from equation (3.104) in equation (3.103) and noting that

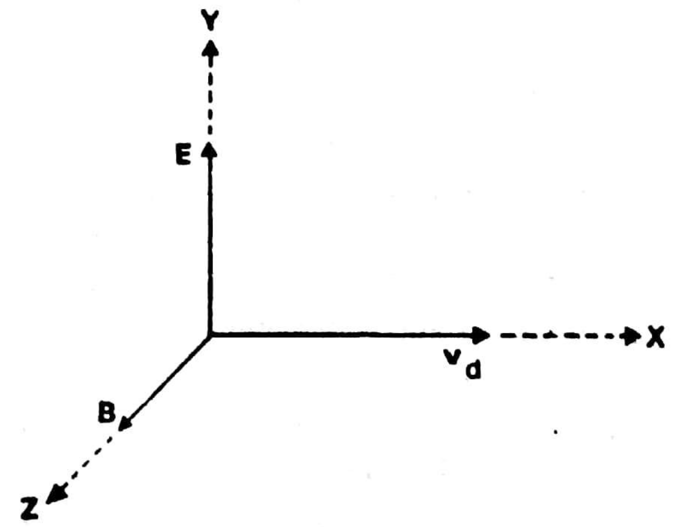


Fig. 3.5 Crossed fields: $\mathbf{E} \perp \mathbf{B}$

E and **B** are constants, we get

$$m\frac{d\mathbf{v}'}{dt} = e\mathbf{E} + e\mathbf{v}' \times \mathbf{B} + \frac{e}{B^2}(\mathbf{E}\times\mathbf{B})\times\mathbf{B} \tag{3.105}$$

since

$$\frac{d}{dt}\frac{\mathbf{E}\times\mathbf{B}}{B^2} = 0.$$

But

$$\frac{e}{B^2}(\mathbf{E}\times\mathbf{B})\times\mathbf{B} = \frac{e}{B^2}(\mathbf{E}\cdot\mathbf{B})\,\mathbf{B} - e\mathbf{E} = -e\mathbf{E}$$

because the **E** and **B** fields are mutually perpendicular. Hence, equation (3.105) becomes

$$m\frac{d\mathbf{v}'}{dt} = e\mathbf{v}' \times \mathbf{B}$$

Thus, we have

$$m\frac{d\mathbf{v}_d}{dt} = 0 \text{ and } m\frac{d\mathbf{v}'}{dt} = e\mathbf{v}' \times \mathbf{B} \tag{3.106}$$

The first of equations (3.106) suggests that $\mathbf{v}_d$ is constant. Velocity $\mathbf{v}_d$ is known as the *drift velocity*. The second equation of (3.106) is similar to equation (3.91). However, it does not describe circular motion since, in this case, the electrostatic field modifies the motion. The magnitude of the drift velocity is

$$v_d = \frac{|\mathbf{E}\times\mathbf{B}|}{B^2} = \frac{E}{B} \tag{3.107}$$

and is fixed by the ratio of electric intensity and magnetic induction.

Let us now obtain the equation for the trajectory of a particle moving in crossed constant fields by solving the equation of motion (3.103) in cartesian coordinates. Consider a general case when **E** and **B** are not perpendicular, but make some angle (Fig. 3.6). We can take **B** along the

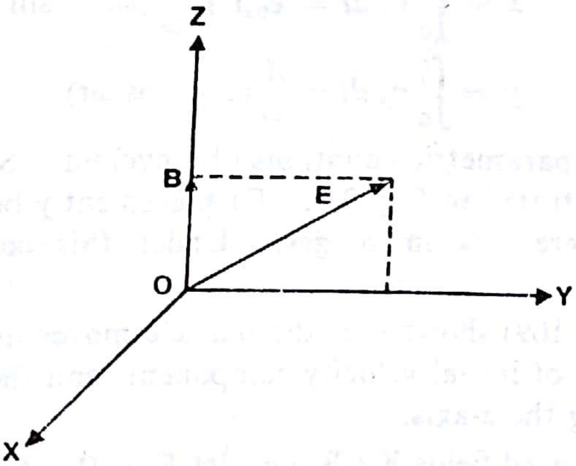


Fig. 3.6 Crossed fields

z-axis and **E** in the yz-plane. Let initially, i.e. at $t = 0$, the particle be at origin and let its initial velocity be $\mathbf{v}_0 = (v_{0x}, v_{0y}, v_{0z})$. In this case, equations (3.103) in the component form are

$$\frac{dv_x}{dt} = \omega v_y \tag{3.108a}$$

$$\frac{dv_y}{dt} = a_y - \omega v_x \tag{3.108b}$$

$$\frac{dv_z}{dt} = a_z \tag{3.108c}$$

where we have put $a_y = \dfrac{eE_y}{m}$, $a_z = \dfrac{eE_z}{m}$ and $\omega = \dfrac{eB}{m}$.

Equation (3.108c) has solutions

$$v_z = v_{0z} + a_z t \text{ and } z = v_{0z}t + \tfrac{1}{2}a_z t^2 \tag{3.109}$$

Coupled equations (3.108a) and (3.108b) can be solved by differentiating any one of them with respect to time and substituting the time derivative of velocity from the other.

$$\frac{d^2v_x}{dt^2} = \omega\frac{dv_y}{dt} = a_y\omega - \omega^2 v_x \tag{3.110}$$

and

$$\frac{d^2v_y}{dt^2} = -\omega\frac{dv_x}{dt} = -\omega^2 v_y \tag{3.111}$$

Equation (3.111) is similar to the equation of a simple harmonic oscillator and has solution

$$v_y = A\sin\omega t \tag{3.112}$$

The other constant, i.e., the phase will be taken as zero.

Substituting solution (3.112) in equation (3.108a) and integrating it with respect to time, we get

$$v_x - v_{0x} = \omega A\int_0^t \sin\omega t\, dt = A(1 - \cos\omega t) \tag{3.113}$$

Further integration of equations (3.113) and (3.112) yields

$$x = \int_0^t v_x\, dt = v_{0x}t + \frac{A}{\omega}(\omega t - \sin\omega t) \tag{3.114}$$

and

$$y = \int_0^t v_y\, dt = \frac{A}{\omega}(1 - \cos\omega t) \tag{3.115}$$

These are the parametric equations of a cycloid. Some forms of trajectories are illustrated in Fig. 3.7. Displacement y becomes zero whenever $\omega t = 2\pi n$, where n is an integer. Under this condition, $v_x = v_{0x}$ and $v_y = 0$.

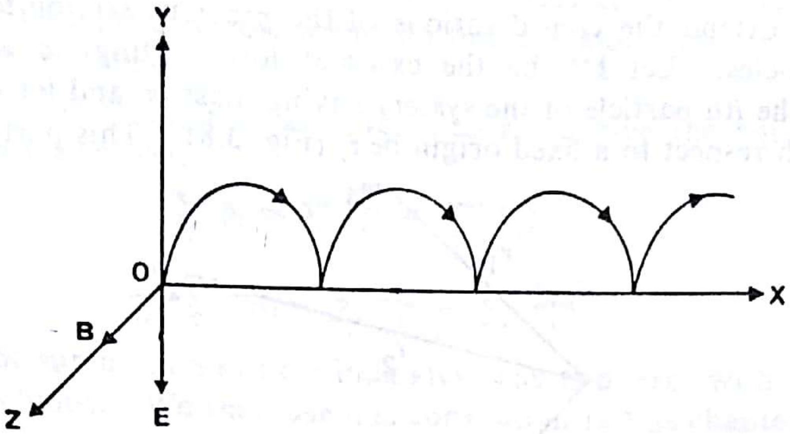
Equation (3.109) shows that the particle moves in the direction of the z-axis because of initial velocity component and the component of electric field along the z-axis.

Consider crossed fields $\mathbf{E} \perp \mathbf{B}$, i.e., let $E_z = 0$ Now, the particle move along the z-axis as given by

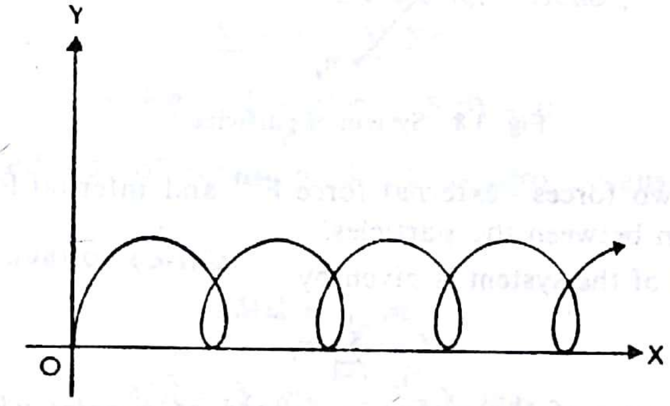
$$z = v_{0z}t \tag{3.116}$$

Let $v_{0x} = 0$, $v_{0y} = 0$. In this case the trajectory of the particle is a cycloid and constant A can be obtained from equations (3.112) and (3.108b) by taking values at $t = 0$. Thus, from

$$\frac{dv_y}{dt} = a_y - \omega v_x = A\omega\cos\omega t$$



(a)



(b)

Fig. 3.7b Different paths of a charged particle in crossed fields

at $t = 0$, we get

$$A = \frac{a_y}{\omega} - v_{0x} = \frac{E_y}{B} - v_{0x} \tag{3.117}$$

Since $v_{0x} = 0$ and $E_y = E$, we have

$$A = \frac{E}{B}$$

The case of crossed electric and magnetic fields is of great practical application. If initially, the particle is moving along the x-direction with velocity v_{0x}, so that we have at $t = 0$, $x = y = z = 0$, $v_{0y} = 0 = v_{0z}$, and $v_{0x} \neq 0$, the solution (3.112) in this case is

$$v_y = \left(\frac{E_y}{B} - v_{0x}\right) \sin \omega t \tag{3.118}$$

If $v_{0x} = \frac{E_y}{B}$, the particle will not be deflected along the y-axis. Thus, the particles with velocity (E_y/B) will go undeviated and are said to be filtered out. The perpendicular combination of electric and magnetic fields thus works as a velocity filter for charged particles, and particles of desired velocity can be obtained by choosing suitable values of the electric intensity **E** and magnetic induction **B**.

3.5 MECHANICS OF SYSTEMS OF PARTICLES

We now extend the considerations of the previous section to a system of N particles. Let $\mathbf{F}^{ext}$ be the external force acting on the system. Consider the ith particle of the system having mass m_i and let its position vector with respect to a fixed origin be $\mathbf{r}_i$ (Fig. 3.8). This particle will be

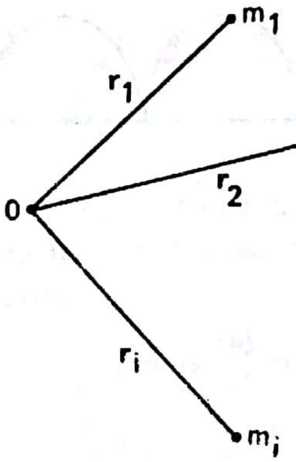


Fig. 3.8 System of particles

acted upon by two forces—external force $\mathbf{F}_i^{ext}$ and internal force $\mathbf{F}_i^{int}$ due to the interaction between the particles.

Total mass M of the system is given by

$$M = \sum_{i=1}^{N} m_i \tag{3.119}$$

The *centre of mass* of the system is defined as *a point whose position vector is given by*

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{1}{M} \sum m_i \mathbf{r}_i \tag{3.120}$$

Let $\mathbf{F}_{ij}^{int}$ be the internal force acting on the ith particle as a result of its interaction with the jth particle. Then, the total internal force acting on the ith particle is

$$\mathbf{F}_i^{int} = \sum_j \mathbf{F}_{ij}^{int}, \; i \neq j$$

Then, the equation of motion of the ith particle can be written as

$$\dot{\mathbf{p}}_i = \mathbf{F}_i^{ext} + \sum_j \mathbf{F}_{ij}^{int} \tag{3.121}$$

The first term on the right-hand side of equation (3.121) represents the external force on the ith particle, while the second term is a vector sum of all the internal forces due to the interaction of the remaining $N-1$ particles with the ith particle. In this summation we shall take $\mathbf{F}_{ii}^{int} = 0$, i.e., the force of self-interaction is zero.

Now, according to Newton's third law of motion

$$\mathbf{F}_{ij}^{int} = -\mathbf{F}_{ji}^{int} \tag{3.122}$$

i.e., the mutual interaction forces between the ith and jth particle are equal and opposite. Illustrations of such force-fields are the gravitational and the electrostatic fields. The electromagnetic forces are to be excluded as they are velocity-dependent forces and do not act along the line joining the ith and jth particles.

Equation (3.121) is summed up over i so as to give the equation of motion of the system viz.

$$\sum_i \dot{\mathbf{p}}_i = \sum_i \mathbf{F}_i^{\text{ext}} + \sum_{ij}{}' \mathbf{F}_{ij}^{\text{int}} \tag{3.123a}$$

or

$$\frac{d^2}{dt^2} \sum_i m_i \mathbf{r}_i = \sum_i \mathbf{F}_i^{\text{ext}} + \sum_{ij}{}' \mathbf{F}_{ij}^{\text{int}} \tag{3.123b}$$

Prime over the summation symbol indicates that the term with $i = j$ in the sum is excluded. We shall use this convention in this chapter.

In the double summation term $\sum_{ij}' \mathbf{F}_{ij}^{\text{int}}$ suffixes i and j are dummy and can be interchanged without changing the sum. Hence,

$$\sum_{ij} \mathbf{F}_{ij}^{\text{int}} = \sum_{ij} \mathbf{F}_{ji}^{\text{int}}$$

Further

$$\sum_{ij} \mathbf{F}_{ij}^{\text{int}} = \tfrac{1}{2} \sum_{ij} (\mathbf{F}_{ij}^{\text{int}} + \mathbf{F}_{ji}^{\text{int}}) = 0 \tag{3.124}$$

The right-hand side of equation (3.124) is zero because of equation (3.122).

Now, from equation (3.120)

$$M\mathbf{R} = \sum_i m_i \mathbf{r}_i$$

or

$$M\ddot{\mathbf{R}} = \sum_i m_i \ddot{\mathbf{r}}_i = \frac{d^2}{dt^2} \sum_i m_i \mathbf{r}_i \tag{3.125}$$

On using equations (3.124) and (3.125) along with equation (3.123b), we get

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}} \tag{3.126}$$

where $\mathbf{F}^{\text{ext}} = \sum_i \mathbf{F}_i^{\text{ext}}$, i.e., the total external force acting on the system of particles.

The total linear momentum of the system is given by

$$\mathbf{P} = \sum m_i \dot{\mathbf{r}}_i = \frac{d}{dt} \sum_i m_i \mathbf{r}_i = \frac{d}{dt}(M\mathbf{R}) = M\dot{\mathbf{R}}$$

Then, equation (3.126) becomes

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}} \tag{3.127}$$

Equations (3.126) and (3.127) show that *in the case of a system of N particles, under the action of external forces, the centre of mass of the system behaves like a particle whose mass is equal to the total mass of the system and is acted upon by total external force* $\mathbf{F}^{\text{ext}}$. The motion of the centre of mass of the system is independent of the internal forces that exist among the particles of the system. *If the total external force acting on the system is zero, then the total linear momentum of the system is conserved.*

Thus, if $\mathbf{F}^{ext} = 0$, $\dot{\mathbf{P}} = 0$ or $\mathbf{P} = \text{const}$ (3.128)

(a) Angular Momentum of the System

The total angular momentum of the system about any point will be equal to the vector sum of the angular momenta of individual particles. Let $\mathbf{l}_i$ represent the angular momentum of the ith particle about some point. Then,

$$\mathbf{l}_i = \mathbf{r}_i \times \mathbf{p}_i \tag{3.129}$$

where $\mathbf{r}_i$ is the position vector of the ith particle from the given point.

Hence the total angular momentum of the system is obtained by taking vector sum of individual momenta.

$$\mathbf{L} = \sum_i \mathbf{l}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i \tag{3.130}$$

Let $\mathbf{N}$ be the total torque acting on the system. Then,

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \frac{d}{dt} \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

$$= \sum_i \dot{\mathbf{r}}_i \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i$$

But, $$\sum_i \dot{\mathbf{r}}_i \times \mathbf{p}_i = \sum_i \dot{\mathbf{r}}_i \times m_i \dot{\mathbf{r}}_i = \sum_i m_i \dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i = 0$$

Further, $$\sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times (\mathbf{F}_i^{ext} + \sum_j{}' \mathbf{F}_{ij}^{int})$$

$$= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext}$$

Since $$\sum_i \sum_j{}' \mathbf{r}_i \times \mathbf{F}_{ij}^{int} = +\sum_j \sum_i{}' \mathbf{r}_j \times \mathbf{F}_{ji}^{int}$$

by the same argument as given above and

$$\sum_{ij}{}' \mathbf{r}_i \times \mathbf{F}_{ij}^{int} = \tfrac{1}{2} \sum_{ij}{}' [\mathbf{r}_i \times \mathbf{F}_{ij}^{int} + \mathbf{r}_j \times \mathbf{F}_{ji}^{int}]$$

$$= \tfrac{1}{2} \sum_{ij}{}' [(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{int}]$$

$$= \tfrac{1}{2} \sum_{ij}{}' \mathbf{r}_{ij} \times \mathbf{F}_{ij}^{int}$$

Now, $\mathbf{F}_{ij}^{int}$ is also proportional to $(\mathbf{r}_i - \mathbf{r}_j) \equiv \mathbf{r}_{ij}$, since we are considering the forces of action and reaction only. Hence, the right-hand side of the above equation will be zero. Then, we have

$$\sum_{ij}{}' \mathbf{r}_i \times \mathbf{F}_{ij}^{int} = 0$$

This is true in the case of internal forces acting along the line joining the two particles but not in the case of forces acting on the moving charged particles.

Thus, total torque $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \sum_i \mathbf{N}_i \tag{3.131}$$

Equation (3.131) shows that the total torque on the system is equal to the vector sum of the torques acting on the individual particles.

If $\mathbf{N} = 0$, $\dfrac{d\mathbf{L}}{dt} = 0$ or $\mathbf{L} = \text{const}$ (3.132)

Thus, if the total external torque acting on the system is zero, then the total angular momentum of the system is conserved.

Now, we prove that *the angular momentum of the system about a fixed point is equal to the sum of the angular momentum of the total mass concentrated at the centre of mass about that point and the angular momentum of the system about its centre of mass.*

Let m_i and $\mathbf{r}_i$ denote the mass and the position vector of the ith particle with reference to point O (Fig. 3.9). Let $\mathbf{r}'_i$ be the position

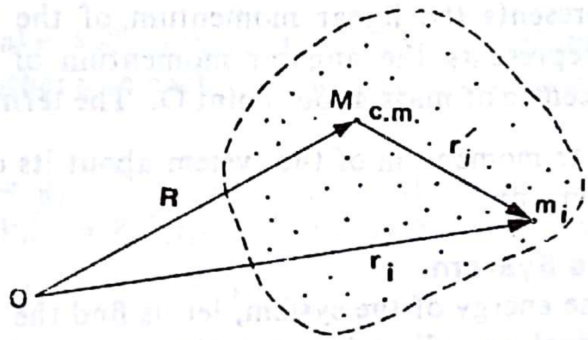


Fig. 3.9 System of particles with total mass M and centre of mass at **R**

vector of the ith particle with reference to the centre of mass of the system. Then

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i \tag{3.133}$$

The corresponding equation relating the velocities can be written as

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}'_i$$

or

$$\mathbf{v}_i = \mathbf{V} + \mathbf{v}'_i \tag{3.134}$$

where $\mathbf{V}$ is the velocity of the centre of mass of the system and $\mathbf{v}'_i$ is the velocity of the ith particle with reference to the centre of mass of the system.

The total angular momentum is, then, given by

$$\begin{aligned}\mathbf{L} &= \sum \mathbf{r}_i \times \mathbf{p}_i \\ &= \sum_i (\mathbf{R} + \mathbf{r}'_i) \times m_i(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_i) \\ &= \sum_i m_i \mathbf{R} \times \dot{\mathbf{R}} + \sum_i \mathbf{R} \times m_i \dot{\mathbf{r}}'_i + \sum_i m_i \mathbf{r}'_i \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{r}'_i \times \dot{\mathbf{r}}'_i\end{aligned} \tag{3.135}$$

The second term on the right-hand side of equation (3.135) can be written as

$$\begin{aligned}\sum_i \mathbf{R} \times m_i \dot{\mathbf{r}}'_i &= \mathbf{R} \times \sum_i m_i \dot{\mathbf{r}}'_i \\ &= \mathbf{R} \times \frac{d}{dt} \sum_i m_i \mathbf{r}'_i\end{aligned}$$

But, $\sum_i m_i \mathbf{r}'_i = 0$, since distances $\mathbf{r}'_i$ are measured with respect to the centre of mass of the system.

The third term on the right-hand side of equation (3.135) also vanishes for the same reason. Then, we are left with

$$\begin{aligned} \mathbf{L} &= \sum_i m_i \mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{r}'_i \times \dot{\mathbf{r}}'_i \\ &= \mathbf{R} \times \dot{\mathbf{R}} \sum_i m_i + \sum_i \mathbf{r}'_i \times m_i \dot{\mathbf{r}}'_i \\ &= \mathbf{R} \times \dot{\mathbf{R}} M + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i \end{aligned}$$

or
$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \mathbf{L}' \tag{3.136}$$

where $\mathbf{P} = M\dot{\mathbf{R}}$ represents the linear momentum of the centre of mass and hence $\mathbf{R} \times \mathbf{P}$ represents the angular momentum of the total mass concentrated at the centre of mass about point O. The term $\mathbf{L}' = \sum_i \mathbf{r}'_i \times \mathbf{p}'_i$ represents the angular momentum of the system about its centre of mass. This proves the statement.

(b) Energy of the System

In order to find the energy of the system, let us find the work done by all the forces—external as well as internal—in moving the system from initial configuration 1 to final configuration 2. The total work done in moving the system is equal to the sum of the work done in moving all the particles from configuration 1 to configuration 2. Thus,

$$\begin{aligned} W_{12} &= \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i \\ &= \sum_i \int_1^2 \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{r}_i + \sum_{ij}{}' \int_1^2 \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_i \end{aligned} \tag{3.137}$$

If the internal and the external forces are conservative, then they can be expressed in terms of corresponding potential energies. Thus, total force $\mathbf{F}_i$ on the ith particle can be written as

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_j \mathbf{F}_{ij}^{\text{int}} = -\nabla_i V_i \tag{3.138a}$$

where the potential energy

$$V_i = V_i^{\text{ext}} + V_i^{\text{int}} \tag{3.138b}$$

is the sum of potential energy functions of the external and internal forces. In equation (3.138a), symbol ∇_i is

$$\nabla_i = \sum_i \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \tag{3.139}$$

and represents the gradient operator performing differentiation with respect to x_i, components of position vector $\mathbf{r}_i$ of the ith particle. The operator can be written separately as

$$\left.\begin{aligned} \mathbf{F}_i^{\text{ext}} &= -\nabla_i V_i^{\text{ext}} \\ \text{and} \qquad \mathbf{F}_{ij}^{\text{int}} &= -\nabla_{ij} V_{ij}^{\text{int}} \end{aligned}\right\} \tag{3.140}$$

Quantity V_{ij}^{int} is the potential energy arising due to internal forces $\mathbf{F}_{ij}^{\text{int}}$ and $\nabla_{ij} = \sum_i \hat{e}_i \dfrac{\partial}{\partial(x_i - x_j)}$. From this, it will be clear that $\nabla_{ij} = -\nabla_{ji}$.

Now the potential energy of the ith particle arising due to internal forces is given by

$$V_i^{\text{int}} = \sum_j V_{ij}^{\text{int}}$$

Hence, the total potential energy due to internal forces is

$$V^{\text{int}} = \sum_i V_i^{\text{int}} = \sum_{\substack{ij \\ i<j}} V_{ij}^{\text{int}} \tag{3.141}$$

We have to take $V_{ii}^{\text{int}} = 0$, to have $\mathbf{F}_{ii}^{\text{int}} = 0$. Condition $i<j$ is necessary because otherwise each term will be taken twice in summing over i and j.

Potential energy V_{ij}^{int} depends upon the relative positions of the two particles, i.e. $V_{ij}^{\text{int}} = V_{ij}^{\text{int}}(\mathbf{r}_{ij})$. Then, by Newton's third law, we have

$$\begin{aligned} \mathbf{F}_{ij}^{\text{int}} &= -\nabla_{ij} V_{ij}^{\text{int}} \\ &= -\mathbf{F}_{ji}^{\text{int}} = +\nabla_{ji} V_{ji}^{\text{int}} = -\nabla_{ij} V_{ji}^{\text{int}} \end{aligned} \tag{3.142}$$

From equation (3.142), we find that $V_{ij}^{\text{int}} = V_{ji}^{\text{int}}$ and hence

$$V^{\text{int}} = \tfrac{1}{2} \sum_{ij} V_{ij}^{\text{int}} \tag{3.143}$$

Factor $\frac{1}{2}$ occurring on the right-hand side of equation (3.143) is due to the fact that each term is being taken twice while summing over i and j. The same has been incorporated in equation (3.141) by writing $i<j$. This condition avoids the duplication of terms.

The work done by the external force is given by

$$\begin{aligned} \sum_i \int_1^2 \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{r}_i &= -\sum_i \int_1^2 \nabla_i V_i^{\text{ext}} \cdot d\mathbf{r}_i \\ &= -\sum_i \int_1^2 dV_i^{\text{ext}} \\ &= -\sum_i V_i^{\text{ext}} \Big|_1^2 \\ &= V_1^{\text{ext}} - V_2^{\text{ext}} \end{aligned} \tag{3.144}$$

where V_1^{ext} and V_2^{ext} represent the potential energies of the system arising due to external forces acting on the system in configuration 1 and 2 respectively.

The work done by the internal forces is given by

$$\sum_{ij}' \int_1^2 \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_i$$

But,
$$\sum_{ij}' \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_i = \sum_{ij}' \mathbf{F}_{ji}^{\text{int}} \cdot d\mathbf{r}_j = -\sum_{ij}' \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_j$$

Hence,
$$\sum_{ij}{}' \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_i = \tfrac{1}{2} \sum_{ij}{}' \mathbf{F}_{ij}^{\text{int}} \cdot (d\mathbf{r}_i - d\mathbf{r}_j)$$
$$= \tfrac{1}{2} \sum_{ij} \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_{ij}$$

where $d\mathbf{r}_{ij} = d\mathbf{r}_i - d\mathbf{r}_j$.

Substituting this value, we get

$$\sum_{ij}{}' \int_1^2 \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_i = \tfrac{1}{2} \sum_{ij}{}' \int_1^2 \mathbf{F}_{ij}^{\text{int}} \cdot d\mathbf{r}_{ij}$$
$$= -\tfrac{1}{2} \sum_{ij}{}' \int_1^2 \nabla_{ij} V_{ij}^{\text{int}} \cdot d\mathbf{r}_{ij}$$
$$= -\tfrac{1}{2} \sum_{ij}{}' \int_1^2 dV_{ij}^{\text{int}}$$
$$= -\tfrac{1}{2} \sum_{ij}{}' V_{ij}^{\text{int}} \Big|_1^2$$
$$= -V^{\text{int}} \Big|_1^2, \text{ by equation (3.143)}$$
$$= V_1^{\text{int}} - V_2^{\text{int}} \tag{3.145}$$

The total potential energy of the system is then given by

$$V = V^{\text{ext}} + V^{\text{int}}$$
$$= \sum_i V_i^{\text{ext}} + \tfrac{1}{2} \sum_{ij}{}' V_{ij}^{\text{int}}$$
$$= \sum_i V_i^{\text{ext}} + \sum_{\substack{ij \\ i<j}} V_{ij}^{\text{int}}$$

In terms of the total potential energy of the system, the work done is.

$$W_{12} = -V \Big|_1^2 = V_1 - V_2 \tag{3.146}$$

Work done W_{12} can be expressed in terms of the difference between the kinetic energy of the system in the initial and final configurations as follows:

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$$
$$= \sum_i \int_1^2 \frac{d}{dt}(m_i \mathbf{v}_i) \cdot \frac{d\mathbf{r}_i}{dt}\, dt = \sum_i \int_1^2 m_i \left(\frac{d\mathbf{v}_i}{dt} \cdot \mathbf{v}_i \right) dt$$
$$= \sum_i \int_1^2 m_i (\mathbf{v}_i \cdot d\mathbf{v}_i)$$
$$= \sum_i \int_1^2 d(\tfrac{1}{2} m_i v_i^2)$$
$$= \sum_i T_i \Big|_1^2 = T \Big|_1^2 = T_2 - T_1 \tag{3.147}$$

Combining equations (3.146) and (3.147), we get

$$T_1 + V_1 = T_2 + V_2$$

or

$$E_1 = E_2 \tag{3.148}$$

Equation (3.148) states that the total energy of the system is conserved.

This is true only when *all* the forces—internal and external—are derivable from the potential energy functions that do not depend explicitly on time. A system in which all the *forces acting on it are derivable from the potential energy functions is called a conservative system.*

In a conservative system, potential energy V_{ij}^{int} depends entirely upon the separation between the ith and jth particle. If this separation is constant, V_{ij} is also constant and can be taken to be zero. A system in which the distance between any two particles remains constant is called a *rigid body*.

(c) Kinetic Energy of the System

We now express the kinetic energy of the system as a sum of (i) the kinetic energy of a particle at the centre of mass and having mass $M = \sum m_i$ and moving with velocity $\mathbf{V} = \dot{\mathbf{R}}$, with respect to origin O, and (ii) the kinetic energy of the system referred to the centre of mass as the origin. The total kinetic energy of the system is

$$T = \sum T_i = \sum_i \tfrac{1}{2} m_i v_i^2 \tag{3.149}$$

But by equation (3.134), we have, $\mathbf{v}_i = \mathbf{V} + \mathbf{v}_i'$.

Hence,

$$\begin{aligned} v_i^2 &= \mathbf{v}_i \cdot \mathbf{v}_i \\ &= (\mathbf{V} + \mathbf{v}_i') \cdot (\mathbf{V} + \mathbf{v}') \\ &= V^2 + v_i'^2 + 2\mathbf{V} \cdot \mathbf{v}_i' \end{aligned}$$

Substituting this value in equation (3.149), we get

$$\begin{aligned} T &= \sum_i \tfrac{1}{2} m_i [V^2 + v'^2 + 2\mathbf{V} \cdot \mathbf{v}_i'] \\ &= \sum_i \tfrac{1}{2} m_i V^2 + \sum_i \tfrac{1}{2} m_i v_i'^2 + \sum_i \tfrac{1}{2} m_i 2\mathbf{V} \cdot \mathbf{v}_i' \end{aligned} \tag{3.150}$$

The third term on the right-hand side of equation (3.150) can be shown to be zero as follows:

$$\begin{aligned} \sum \tfrac{1}{2} m_i 2\mathbf{V} \cdot \mathbf{v}_i' &= \mathbf{V} \cdot \sum_i m_i \mathbf{v}_i' \\ &= \mathbf{V} \cdot \frac{d}{dt} \sum m_i \mathbf{r}_i' \\ &= 0 \end{aligned}$$

Hence,

$$T = \tfrac{1}{2} M V^2 + T_c \tag{3.151}$$

where $T_c = \sum_i \tfrac{1}{2} m_i v_i'^2 =$ kinetic energy of the system about the centre of mass of the system.

This proves the statement.

(d) Laws of Conservation

In the previous articles, we have obtained the laws of conservation of momentum and energy of a particle or of a system of particles as a consequence of Newton's laws of motion. These laws of conservation are helpful in analysing the motion of a particle or of a system, particularly when the nature of the force is not known. The conservation laws relate the momentum or the energy of a system at two different instants and help us to obtain the kinematical relations for the system. These laws are used extensively in studying collisions of particles.

The conservation laws have far wider applicability and are not restricted to Newtonian mechanics alone. The laws of conservation are exact, i.e., they are true in the case of systems with any type of interaction between the particles. In fact, so great is the conviction in the laws of conservation that it led W. Pauli in 1930 to postulate a new particle called *neutrino* to account for the missing energy in the process of beta-decay.

The laws of conservation must obviously have an intimate relationship with the physical nature of space and time. The relation of the laws with symmetry is discussed in the Lagrangian formulation of mechanics.

3.6 MOTION OF A SYSTEM WITH VARIABLE MASS

So far, we have considered the equations of motion and the laws of conservation in such cases when the mass of the system was constant during the motion of the system. We now wish to consider the motion of a system when the mass varies with time. We often come across such systems in nature and also in technology. A drop of water falling through a cloud will gain in mass as it descends. A rocket will lose mass in its flight as a result of the burning of the fuel and the exhaust gas which provides acceleration to the rocket to attain high velocities. We can apply the laws of conservation in the case of such systems to obtain the equations of motion of the system and solve them.

It should be noted that we are not considering the variation of mass of a particle with velocity which is the well-known relativistic effect. The velocities involved in the problems under discussion are very small as compared to the velocity of light. Hence, the discussion that follows is a non-relativistic discussion.

A rocket fired from the earth will always be affected by the gravitational pull of the earth. Other celestial objects are at great distances from the rocket and the effect of such objects on the motion of the rocket can be ignored. To simplify the problem still further, we neglect the effect of rotation and the gravitational pull of the earth also and consider a free flight of the rocket. We assume, therefore, that a rocket fired to move along the x-axis will continue to move along the x-axis itself.

Consider a rocket propelled by burning fuel. To write its equation of motion, we find the change in the momentum of the whole system in time interval Δt. Let M be the mass of the rocket and v its speed at time t. Then, in time interval Δt, the mass of the system is reduced by amount ΔM due to a burning of the fuel and expulsion of an equal amount of mass of gas. As a result of this reduction in mass, the velocity of the system increases by amount Δv. Let u be the velocity of the exhaust gases relative to the rocket (Fig. 3.10). Then, the law of conservation of momentum gives

$$Mv = (M - \Delta M)(v + \Delta v) - \Delta M(u - v) \tag{3.152}$$

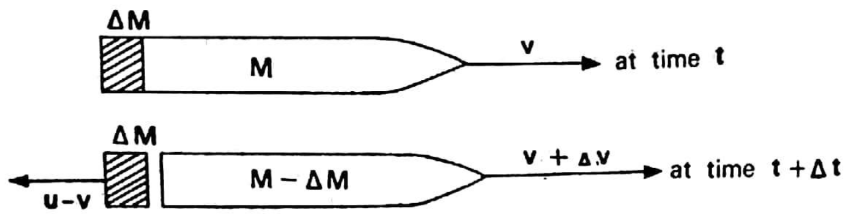


Fig. 3.10 Rocket motion

Simplifying equation (3.152) and retaining only the terms containing first-order infinitesimal quantities, we get

$$M\,\Delta v = u\,\Delta M$$

Dividing throughout by Δt and taking the limit as $\Delta t \to 0$, we get

$$M\dot{v} = -u\frac{dM}{dt} \tag{3.153}$$

where $\dot{v} = \frac{dv}{dt}$. The negative sign is added on the right-hand side to indicate that velocity v increases as mass M decreases.

Integrating equation (3.153) with respect to time, we get

$$\int_{v_0}^{v} dv = -u\int_{M_0}^{M_t}\frac{dM}{M}$$

or

$$v = v_0 - u\ln\frac{M_t}{M_0} \tag{3.154}$$

where v and M_t are the velocity and mass of the system at instant t and v_0 and M_0 those at $t = 0$.

Let us suppose that the fuel is burnt at constant rate $\frac{dM}{dt} = b$ and it lasts for time T. If the mass of the vehicle is M_v and that of the fuel at $t = 0$ is M_f, then

$$M_0 = M_v + M_f \tag{3.155}$$

The mass of the vehicle-fuel system at any instant t can be written as

$$M_t = M_v + M_f\left(1 - \frac{t}{T}\right) = M_0 - M_f\frac{t}{T}, \text{ for } 0 \leqslant t \leqslant T$$

and

$$M_t = M_v, \quad \text{for} \quad t \geqslant T$$

Substituting the value of M_t in equation (3.154), we get

$$v = \frac{dx}{dt} = v_0 - u\ln\left(1 - \frac{M_f}{M_0}\frac{t}{T}\right) \tag{3.156}$$

Integrating equation (3.156), with respect to time, we get

$$x = x_0 + v_0 t - u\int_0^t \ln\left(1 - \frac{M_f}{M_0}\frac{t}{T}\right) dt \tag{3.157}$$

The last integral of equation (3.157) can be evaluated by parts and we obtain

$$\int_0^t \ln\left(1-\frac{M_f}{M_0}\frac{t}{T}\right)dt = \left(t-\frac{M_0 T}{M_f}\right)\ln\left(1-\frac{M_f}{M_0}\frac{t}{T}\right)-t$$

Thus, the distance covered by the rocket in time t is given by

$$x = x_0 + v_0 t - u\left[\left(t-\frac{M_0 T}{M_f}\right)\ln\left(1-\frac{M_f}{M_0}\frac{t}{T}\right)-t\right] \tag{3.158}$$

The rocket attains maximum velocity at $t = T$ when all its fuel is burnt out. This maximum velocity calculated from equation (3.156) is given by

$$\begin{aligned} v_{\max} &= v_0 - u\ln\left(1-\frac{M_f}{M_0}\right) \\ &= v_0 + u\ln\frac{M_0}{M_v} \\ &= v_0 + u\ln\left(1+\frac{M_f}{M_v}\right) \end{aligned} \tag{3.159}$$

From equation (3.159), it is clear that the larger the value of ratio $\frac{M_f}{M_v}$, the greater will be the maximum velocity attained by the rocket..

If the rocket is moving vertically upward and if the gravitational pull of the earth on it is assumed to be constant, then the equation of motion of the rocket can be written as

$$M\dot{v} = -u\frac{dM}{dt} - Mg$$

or

$$\dot{v} = -\frac{u}{M}\frac{dM}{dt} - g \tag{3.160}$$

Integrating equation (3.160) with respect to time, we get

$$v = -u\ln\frac{M_t}{M_0} - gt \tag{3.161}$$

Assuming the initial conditions as $x_0 = 0$ and $v_0 = 0$, we can derive an expression for the height attained by a rocket at time t and is given by

$$x = ut - \tfrac{1}{2}gt^2 - \left(t-\frac{M_0 T}{M_f}\right)\ln\left(1-\frac{M_f}{M_0}\frac{t}{T}\right) \tag{3.162}$$

The rockets are normally expected to carry some load called the payload. Payload may be the load of a satellite which is to be placed in an orbit around the earth, or of bombs in the case of missiles. The payload and the body of the rocket have a fixed mass. Ratio $\frac{M_f}{M_v}$ has a practical limit and it cannot be increased indefinitely. A single rocket, i.e., one-stage rocket, therefore, will not attain high velocities that are required. Multistage rockets are used for this purpose.

QUESTIONS

1. What is a particle ? Can atom or earth be treated as a particle ? Explain.
2. In the equation of motion of system $\mathbf{F} = m\mathbf{a}$, what is represented by each side ? To answer this, consider a motion of a body falling towards the earth through the atmosphere.
3. When does Newton's third law break down ?
4. What is the force for which potential function $V(\mathbf{r})$ does not exist ?
5. How does Newton's second law govern the behaviour of (a) the linear momentum, and (b) the angular momentum, of a particle ? When are $\mathbf{p}$ and $\mathbf{L}$ conserved ?
6. In a projectile motion, when air resistance is negligible, is it necessary to consider three-dimensional motion instead of two-dimensional motion ?
7. Can the acceleration of a projectile be represented by a radial and a tangential component at each point of the motion ? If so, is it advantageous to represent it in this manner ?
8. A particle of mass m is moving with velocity $\mathbf{v}$. Under what conditions of m and $\mathbf{v}$ will the following apply ? (a) Classical mechanics, (b) quantum mechanics, and (c) relativistic mechanics.
9. Distinguish between centre of mass and centre of gravity.
10. Explain the idea of Newtonian relativity.
11. The determination of potential energy and kinetic energy is relative. Explain.
12. What is meant by an 'inertial mass' and 'gravitational mass' ? Is there any difference between the two ?
13. Newton's second law of motion is $\mathbf{F} = \frac{d}{dt}(m\mathbf{v})$. Under what condition can we write $\mathbf{F} = m\frac{d\mathbf{v}}{dt}$ and $\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \frac{dm}{dt}\mathbf{v}$?
14. When is a force-field said to be conservative ? Give illustrations.
15. What is meant by terminal velocity ? Give illustrations.
16. You are given a system of N particles on which external force $\mathbf{F}$ is acting. Show that the centre of mass of the system behaves like a particle whose mass is equal to the total mass of the system and is acted upon by total external force $\mathbf{F}$.
17. Multistage rockets are used while launching satellites. Explain why
18. In the case of rocket motion, show that the greater the ratio M_f/M_u the greater is the maximum speed attained by the rocket.
19. Show that the centre of gravity coincides with the centre of mas when a body is in a uniform gravitational field. What will happe when force-field is non-uniform ?

20. Does frictional loss occur in the collision of one molecule with the other ? Explain.
21. Give some examples of motion that are approximately simple harmonic. Why are motions that are exactly simple harmonic, rare ?

PROBLEMS

1. A particle tied to an inextensible light string is rotating in a vertical plane. If its speed at the lowest point of the circle is v_0, find the minimum value of the speed v_m so that the particle will not leave the track. If v_0 is 0.775 m/s, find the point where the particle leaves the track.
2. A ball is tied to a string of length l and suspended from a nail in the wall. The ball is displaced from its equilibrium position and released when the string was horizontal. There is another nail vertically below the suspension and d is the distance between the nails. Show that $d \geqslant 0.6l$ if the ball completes a circle around the lower nail.
3. Consider a simple pendulum having massless inextensible rod of length l and having speed v_0 at displacement θ_0 less than $\frac{\pi}{2}$. Find the maximum values of v_0 at θ_0 (a) for $\theta = \frac{\pi}{2}$ to be reached by the pendulum, and (b) to keep the pendulum going in a vertical circle.
4. If the rod in problem 3 is elastic, show that it will be stretched at the lowest point by amount $\Delta l \simeq 3mg/k$, if $\Delta l \ll l$ and $\theta_0 = \frac{\pi}{2}$, $v_0 = 0$.
5. A particle of mass m falls along a frictionless track, the lower part of the track being circular. The particle starts from rest from point P which is at a vertical height of $5R$, where R is the radius of the circular part of the track. (a) What is the resultant force on it at point Q at a vertical height R and situated on the outer part of the circular part of the track? (b) At what height from the bottom should m be released if its force against the track at the top of the loop is equal to mg? Express the answers in terms of the speed at the bottom.
6. (a) Calculate the work done by force $\mathbf{F} = 4y\mathbf{i} - 2x\mathbf{j} - \mathbf{k}$ along helix $x = 4 \cos \theta$, $y = 4 \sin \theta$ and $x = 2\theta$ from $\theta = 0$ to $\theta = 2\pi$. (b) Calculate the work done by force $\mathbf{F} = 2x\mathbf{i} - 3z^2\mathbf{j} - y^2\mathbf{k}$ along the line $x = 2y = 4z$ from the origin to point (4, 2, 1).
7. A particle having initial velocity $v\mathbf{i}$ passes through a region in which there is electric field $E\mathbf{j}$ and magnetic field $B\mathbf{k}$. If the mass and the charge of the particle are m and e respectively, for what value of velocity $\mathbf{v}$ will the particle move along the straight line?

8. A projectile shot from the ground has range R and the maximum height it reaches is H. Find the magnitude and the direction of its initial velocity (in the plane of its trajectory).
9. A particle has total energy E and the force on it is due to potential field $V(x)$. Show that the time taken by the particle to go from x_1 to x_2 is

$$t_2 - t_1 = \int_{x_1}^{x_2} \left[\frac{2E}{m} - V(x)\right]^{-1/2} dx$$

if the motion is one-dimensional. Show that this is true only if the particle does not reverse its motion.
10. A billiard ball is dropped on a table with velocity v_0 and angular speed ω_0. At what time does slipping cease and rolling begin? Describe the subsequent motion of the ball.
11. A body is sliding down an inclined plane which is moving horizontally with constant velocity. Find the position of the body as a function of time.
12. A rotating sphere contracts slowly, due to internal forces, to $\frac{1}{n}$ of its original radius. What happens to its angular velocity? Show that increase in its energy is equal to work done during contraction.
13. Three particles of masses 2, 3 and 5 units move under the influence of a force-field so that their position vectors relative to a fixed coordinate system are given, respectively, by

$$\mathbf{r}_1 = 2t\mathbf{i} - 3\mathbf{j} - t^2\mathbf{k}$$
$$\mathbf{r}_2 = (t + 1)\mathbf{i} - 3t\mathbf{j} - 4\mathbf{k}$$

and
$$\mathbf{r}_3 = t^2\mathbf{i} - t\mathbf{j} - (2t - 1)\mathbf{k}$$

where t is the time. Find (a) the total angular momentum of the system, and (b) the total external torque applied to the system with respect to the origin.
14. Three particles of mass 2, 1 and 3 units have the following position vectors:

$$\mathbf{r}_1 = 5t\mathbf{i} - 2t^2\mathbf{j} + (3t - 2)\mathbf{k}$$
$$\mathbf{r}_2 = (2t - 3)\mathbf{i} + (12 - 5t^2)\mathbf{j} + (4 + 6t + 3t^2)\mathbf{k}$$

and
$$\mathbf{r}_3 = (2t - 1)\mathbf{i} + (t^2 + 2)\mathbf{j} - t^3\mathbf{k}$$

where t is time. Find (a) the velocity of the centre of mass of the system at $t = 1$, and (b) the total linear momentum of the system at $t = 1$.
15. A system of particles consists of particles of mass $3g$ located at point $P(1, 0, -1)$, $5g$ at point $Q(-2, 1, 3)$ and $2g$ at point $R(3, -1, +1)$. Find the coordinates of the centre of mass of the system.
16. In a 2-stage rocket, the first stage gets detached after its fuel is used up. Each empty rocket (with neither fuel nor pay load) weighs 1/10th of the mass of the fuel it can contain. A pay load of 100 kg

is to be accelerated to a speed of 6000 m/s in a region free of external forces. The speed of exhaust gases is 1500 m/s. Find the choice of the masses of the two stages, including fuel so that the total mass at take-off is minimum. Also show that the required speed cannot be attained with a single stage rocket.

17. A rocket is initially at rest. It is driven by emitting photons. What fraction of the initial rest mass should be converted into energy if it is to reach speed v?

18. A raindrop of initial mass m_0 falls from rest through a cloud whose thickness is a. As the raindrop falls, it gains mass at rate b. The droplets of the cloud are at rest relative to the ground. The motion of the drop is resisted by a force proportional to its velocity. (a) Write down the differential equation of motion of the raindrop. (b) Find the velocity of the drop as it emerges from the cloud, if, during the passage, its mass has been doubled. (c) What will be the limiting velocity of the drop after it leaves the cloud, assuming that the air resistance outside the cloud is the same as that within?

19. Given force $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j}$, find the work done in moving a particle from (0, 0) to (2, 1).

20. Find the nature of the force-conservative or non-conservative if the work done is given by $W = x^2y - xz^3 - z$.

21. The equation of motion of a particle is

$$\frac{d^2\mathbf{r}}{dt^2} = \omega\mathbf{j} \times \frac{d\mathbf{r}}{dt}$$

where $\mathbf{j}$ is a constant unit vector and ω is a constant. Determine the motion.

22. A particle of mass m is moving under central force $f = \frac{k\mathbf{r}}{r^3}$, where $\mathbf{r}$ is the radius vector from the centre and k is a constant. Show that the vectors

$$\text{(i) } m\mathbf{r} \times \frac{d\mathbf{r}}{dt} \quad \text{and} \quad \text{(ii) } m\frac{d\mathbf{r}}{dt} \times \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) + \frac{k\mathbf{r}}{r}$$

are constant and that they are orthogonal.

23. A bead is able to slide along a small wire in the form of a parabola. The parabola is rotating with constant angular velocity about its vertical symmetrical axis. Write down the equation of motion of the bead.

24. A particle of mass m moves under the influence of force $\mathbf{F}$ on the surface of a sphere of radius r. Write down the equation of motion of the particle.

25. A projectile is fired with velocity v_0 from a gun adjusted for a maximum range. It passes through two points P and Q whose heights above the horizontal are h each. Show that the separation of the

points is

$$x = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

26. A particle is projected vertically upward with velocity v_0 in a constant gravitational field. The medium offers a resistive force proportional to the square of the instantaneous velocity of the particle. If v_t denotes the terminal velocity, show that the velocity of the particle when it returns to the point from which it was projected is

$$v_0 v_t / \sqrt{v_0^2 + v^2}$$

27. Suppose that electrons could be added to earth and moon until repulsive force thus produced is just equal to balance the gravitational attraction. What would be the smallest total mass of electrons that would achieve this?