3

The Q Equation

"Don't let me catch anyone talking about the Universe in my department."

—*(Lord) Ernest Rutherford*

3.1 INTRODUCTION

A considerable part of our present knowledge of the nuclear structure comes from experiments in which a chosen nucleus is bombarded with different projectiles, such as protons, neutrons, deuterons or α-particles. When these particles come close enough to interact with the target nuclei, either elastic or inelastic scattering may take place or one or more particles which are altogether different may be knocked out of the nucleus, or the incident particle may perhaps be captured and a gamma ray emitted. When the mass number and/or atomic number of the target nuclei changes after the bombardment, we say that a *nuclear reaction* has taken place.

Typically an equation representing a nuclear reaction may be written as

$$x + X = Y + y \qquad \ldots(3.1)$$

In words this would read like: when an incident projectile x hits the target nucleus X, a nuclear reaction takes place and as a result there is a new nucleus Y and an outgoing particle y. The above reaction can also be written in short form as $X\,(x, y)\,Y$.

In 1919, Lord Rutherford found that when nitrogen is bombarded with α-particles from Polonium, protons are produced and these protons are capable of piercing 28 cm. of air. This was the first nuclear reaction to be triggered in the laboratory and to Rutherford goes the credit of breaking through the "inaccessible armour" of normal non-radioactive nuclei and producing a *transmutation*. The nuclear reaction can be put as

$$^{14}_{7}\mathrm{N}(\alpha, p)\,^{17}_{8}\mathrm{O} \qquad \ldots(3.2)$$

In 1930 Cockcroft and Walton used artificially accelerated protons and produced the following nuclear reaction,

$$ {}_{3}^{7}\text{Li} + p \rightarrow {}_{2}^{4}\text{He} + \alpha $$

or

$$ {}_{3}^{7}\text{Li}\,(p, \alpha)\,{}_{2}^{4}\text{He} \qquad \ldots(3.3) $$

Different laboratories in the world have already studied thousands of nuclear reactions and the number is continuously growing. In this chapter we will consider the case in which all of the particles may be treated non-relativistically and the target nucleus is at rest before the collision. Our interest lies in finding out when a particular nuclear reaction becomes energetically possible. This we will know by applying the laws of conservation of momentum and energy.

3.2 TYPES OF NUCLEAR REACTIONS

Nuclear reactions are classified on the basis of the projectile used, the particle detected and the residual nucleus.

(*i*) *Scattering*: In the scattering reaction, the projectile and the detected (outgoing) particle are the same. The scattering is elastic when the residual nucleus is left in the ground state. When the residual nucleus is left in an excited state, the scattering is called inelastic.

(*ii*) *Pickup reactions*: When the projectile gains nucleons from the target, the nuclear reaction is referred to as pickup reaction, e.g.

$$ {}_{8}^{16}\text{O}\left({}_{1}^{2}\text{H}, {}_{1}^{3}\text{H}\right){}_{8}^{15}\text{O} $$

(*iii*) *Stripping reactions*: In this type, the projectile loses nucleons to the target nucleus, e.g.

$$ {}_{8}^{16}\text{O}\left({}_{2}^{4}\text{He}, {}_{1}^{2}\text{H}\right){}_{9}^{18}\text{F} $$

In a pickup or stripping reaction it is assumed that the nucleon involved in the process enters or leaves (the shell of) the target nucleus without disturbing the other nucleons. These reactions are therefore referred to as *direct reactions*. In contrast, we have the following type of compound nuclear reactions.

(*iv*) *Compound nuclear reactions*: Here the projectile and target form a *compound nucleus* which has a typical life span of $\sim 10^{-16}$ sec. When this time is compared with a typical nuclear time, i.e., the time taken by the projectile to traverse the target nucleus ($\sim 10^{-22}$ sec.) as in the case of direct reactions, we can conclude that the decay of a compound nucleus does not depend on the way it was formed.

This situation is often described as: the compound nucleus does not "remember" how it was formed. Usually the same compound nucleus is given rise to by a number of nuclear reactions. This compound nucleus can decay in a number of ways or channels. This is illustrated by taking for example, the compound nucleus $^{64}_{30}\mathrm{Zn}$ formed in an excited state $\left(^{64}_{30}\mathrm{Zn}\right)^*$ by two different methods and then decays as given below:

$$p + {}^{63}_{29}\mathrm{Cu} \rightarrow \left({}^{64}_{30}\mathrm{Zn}\right)^* \rightarrow {}^{63}_{30}\mathrm{Zn} + {}^{1}_{0}n$$

$$\rightarrow {}^{62}_{30}\mathrm{Zn} + {}^{1}_{0}n + {}^{1}_{0}n$$

$$\rightarrow {}^{62}_{29}\mathrm{Cu} + {}^{1}_{0}n + p$$

or

$$\alpha + {}^{60}_{20}\mathrm{Ni} \rightarrow \left({}^{64}_{30}\mathrm{Zn}\right)^* \rightarrow {}^{63}_{30}\mathrm{Zn} + {}^{1}_{0}n$$

$$\rightarrow {}^{62}_{30}\mathrm{Zn} + {}^{1}_{0}n + {}^{1}_{0}n$$

$$\rightarrow {}^{62}_{29}\mathrm{Cu} + {}^{1}_{0}n + p \qquad \ldots(3.4)$$

These reactions were experimentally studied by S.N. Ghoshal. [Phys. Rev. 80939 (1950)]

Figure 3.1 depicts how a compound nucleus in an excited state can have different modes of decay, giving rise to different residual nuclei and detected particles.

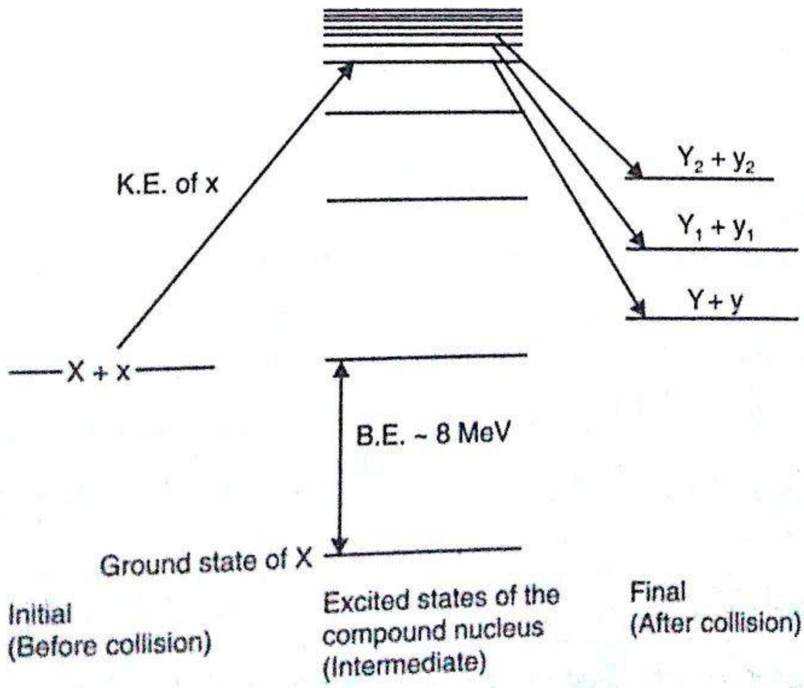

Fig. 3.1. Different modes (channels) of decay of an excited compound nucleus.

The compound nucleus model has been successfully applied for targets of $A > 10$ and projectile energies up to about 15 MeV. For projectile energies between 15 MeV and 50 MeV the *direct reaction model* and the *optical model* are found to be successful. In this chapter, our main emphasis is on understanding energetics of nuclear reactions.

3.3 THE BALANCE OF MASS AND ENERGY IN NUCLEAR REACTIONS

Consider the nuclear reaction,

$$x + X \rightarrow Y + y$$

Since the total mass and energy is conserved, we have

$$(E_x + m_x c^2) + M_X c^2 = (E_Y + M_Y c^2) + (E_y + m_y c^2) \qquad \ldots(3.5)$$

Where,

E_x = kinetic energy of the projectile

$m_x c^2$ = rest energy of the projectile

and similarly E_Y and $M_Y c^2$, E_y and $m_Y c^2$ and $M_X c^2$.

The target nucleus X is assumed to be at rest.

The Q value is expressed as,

$$Q = E_Y + E_y - E_x \qquad \ldots(3.6)$$

i.e. it is the change in the total kinetic energy.

This change in the total kinetic energy in a nuclear reaction is clearly the *nuclear disintegration energy.*

From Eq. 3.5, Eq. 3.6 becomes

$$Q = E_Y + E_y - E_x$$

$$= [(M_X + m_x) - (M_Y + m_y)]c^2 \qquad \ldots(3.7)$$

Thus we see that Q is also the change in the total rest mass. We have an exoergic nuclear reaction if Q is positive. The nuclear reaction is an endoergic reaction if the Q value is negative.

From Eq. 3.7 it is clear that to determine the Q value or nuclear disintegration energy for a nuclear reaction, we must know the kinetic energies of the particles.

E_Y, the kinetic energy of the residual nucleus, is usually small and hard to measure.

In the following section, we will see how E_Y can be eliminated and how the Q equation can be set up.

3.4 THE Q EQUATION

The analytical relationship between the kinetic energy of the projectile and outgoing particle and the nuclear disintegration energy Q is called as the Q equation.

To understand the dynamics of two-body nuclear reactions in laboratory coordinate system, refer to Fig. 3.2.

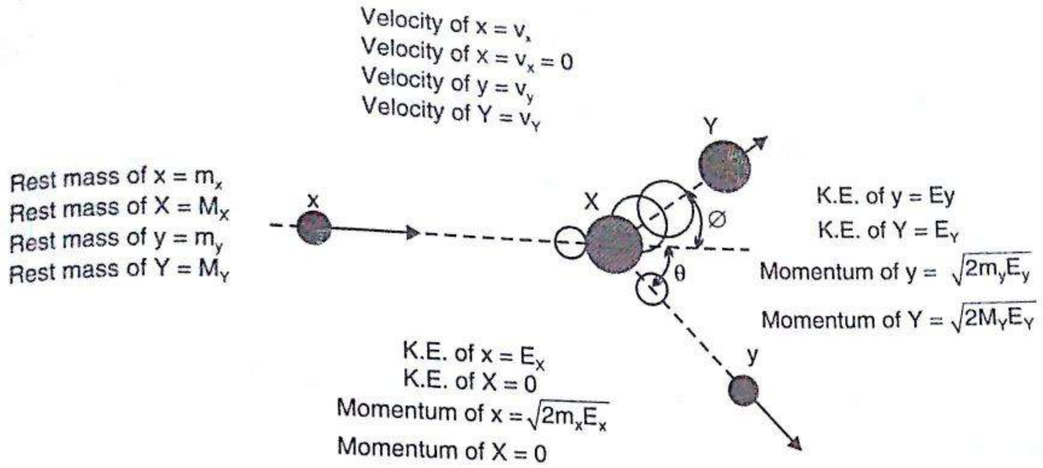

Fig. 3.2. Two-body nuclear reaction in lab. system.

Conservation of mass-energy gives (see Section 3.3)

$$Q = E_y + E_Y - E_x$$

As remarked before, E_Y is small and hard to measure and is therefore eliminated. Conservation of linear momentum along the direction of projectile x gives,

$$\sqrt{2m_xE_x} = \sqrt{2m_yE_y}\cos\theta + \sqrt{2M_YE_Y}\cos\phi \quad \ldots(3.8)$$

Angles θ and φ are coplanar as linear momentum perpendicular to the θ plane is equal to zero. Therefore, conservation of linear momentum normal to projectile direction (in the θ, φ plane) gives,

$$0 = \sqrt{2M_YE_Y}\sin\phi - \sqrt{2m_yE_y}\sin\theta \quad \ldots(3.9)$$

From these equations E_Y and φ are eliminated. Squaring and adding Eqs. (3.8) and (3.9),

$$M_YE_Y = m_xE_x + m_yE_y - 2\sqrt{m_xm_yE_xE_y}\cos\theta$$

But $$E_Y = Q - E_y + E_x$$

$$\therefore \quad Q = E_y - E_x + \frac{m_x}{M_y}E_x + \frac{m_y}{M_Y}E_y - \frac{2\sqrt{m_xm_yE_xE_y}}{M_Y}\cos\theta$$

$$\boxed{Q = E_y\left(1 + \frac{m_y}{M_Y}\right) - E_x\left(1 - \frac{m_x}{M_Y}\right) - \frac{2\sqrt{m_xm_yE_xE_y}}{M_Y}\cos\theta} \quad \ldots(3.10)$$

This is the standard form of the Q equation.

It is interesting to note that one can also obtain Eq. 3.10 by solving the momentum triangle and substituting for E_Y. This is indicated below:

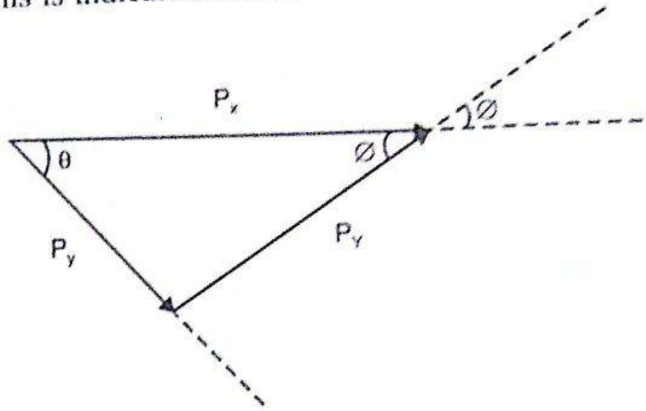

Fig. 3.3. The momentum triangle, using the notation of Fig. 3.2.

We see that

$$P_Y^2 = p_x^2 + p_y^2 - 2p_x p_y \cos\theta \qquad \text{...(3.11)}$$

$$\therefore \quad 2M_Y E_Y = 2m_x E_x + 2m_y E_y - 2.2\sqrt{m_x E_x m_y E_y}\cos\theta$$

giving,

$$E_Y = \frac{m_x}{M_Y}E_x + \frac{m_y}{M_Y}E_y - \frac{2\sqrt{m_x m_y E_x E_y}}{M_Y}\cos\theta \qquad \text{...(3.12)}$$

From Eq. 3.6,

$$E_Y = Q - E_y - E_x$$

Substituting this in Eq. 3.12 we get,

$$Q = E_y\left(1 + \frac{m_y}{M_Y}\right) - E_x\left(1 - \frac{m_x}{M_Y}\right) - \frac{2\sqrt{m_x m_y E_x E_y}}{M_Y}\cos\theta$$

which is the Q equation (3.10).

The kinetic energies E_x, E_y and θ are all measured in the laboratory coordinate system. Since the Q equation is based on mass-energy conservation in a nuclear reaction, it holds for all types of reactions. The exact masses can be replaced by the corresponding mass numbers, in many applications without significant error. For very accurate calculations, the neutral atomic masses are used. Let us see how we can use the isotopic masses (neutral atomic masses) to obtain Q value in alpha and beta decay reactions. For convenience, let the element symbol represent the isotopic mass. We recall that $Q = (m_x + M_X) - (m_y + M_Y)$ where the masses are the *nuclear masses,* and Q is in mass units.

(*i*) Consider the reaction

$$^{238}_{92}\text{U} \rightarrow {}^{234}_{90}\text{Th} + {}^{4}_{2}\text{He}$$

$$Q = (U - 92e) - [(Th - 90e) + (He - 2e)]$$

where e stands for electronic mass.

i.e.,
$$Q = (U - 92e) - (Th + He - 92e)$$
$$= [U - (Th + He)]$$

$\therefore$ Isotopic masses can be used to evaluate Q.

(ii) Consider the reaction,

$$^{14}_{6}C \rightarrow {}^{14}_{7}N + {}^{0}_{-1}e + {}^{0}_{0}\overline{\nu}$$

$$Q = (C - 6e) - [(N - 7e) + e]$$
$$= (C - 6e) - (N - 6e)$$
$$= C - N$$

$\therefore$ Isotopic masses can be used to evaluate Q.

(iii) Consider the reaction,

$$^{63}_{30}Zn \rightarrow {}^{63}_{29}Cu + {}^{0}_{+1}e + {}^{0}_{0}\nu$$

$$Q = (Zn - 30e) - [(Cu - 29e) + e]$$
$$= (Zn - 30e) - (Cu - 30e + 2e)$$
$$= Zn - (Cu + 2e)$$

Thus we see that the isotopic masses can be used to compute Q value in a positron decay reaction, provided two electron masses are included with that of the product particle.

3.5 SOLUTION OF THE Q EQUATION

We are often interested in E_y, the energy of the detected particle and its variation with E_x, the energy of the bombarding projectile, for a fixed Q. For this purpose, the Q equation (Eq. 3.10) can be regarded as a quadratic in $\sqrt{E_y}$. Then its general solution can be conveniently put in the form,

$$\sqrt{E_y} = v \pm \sqrt{v^2 + w} \qquad ...(3.13)$$

where,

$$v = \frac{\sqrt{m_x m_y E_x}}{m_y + M_Y}\cos\theta \qquad ...(3.14)$$

E_x has its minimum possible value at $\theta = 0$, which is the $(E_x)_{\text{thresh}}$. (see Problem 3.12).

This condition gives:

$$\frac{m_x m_y E_x}{(m_y + M_Y)^2} + \frac{M_Y Q + E_x(M_Y - m_x)}{m_y + M_Y} = 0$$

which gives,

$$(E_x)_{\text{thresh}} = -Q\left(\frac{m_x + M_X}{M_X}\right) \quad \ldots(3.18)$$

In obtaining Eq. 3.18, we have made use of the fact that the value of Q/c^2 is much less than the masses of particles in a nuclear reaction, and so

$$m_y \cong m_x + M_X - M_Y \quad \ldots(3.19)$$

Equation 3.18 can be also obtained in a straightforward way by considering M_c and V_c as mass and velocity of the compound nucleus. Then we have,

$$m_x v_x = M_c V_c$$

or

$$V_c = \frac{m_x}{M_c} v_x$$

Now, the part of kinetic energy of the projectile needed for the excitation of the compound nucleus is,

$$Q = \frac{1}{2} m_x v_x^2 - \frac{1}{2} M_c V_c^2$$

$$= \frac{1}{2} m_x v_x^2 - \frac{1}{2} M_c \left(\frac{m_x}{M_c}\right)^2 v_x^2$$

$$= \frac{1}{2} m_x v_x^2 \left(1 - \frac{m_x}{M_c}\right)$$

But

$$M_c = m_x + M_X$$

$\therefore$

$$-Q = \frac{1}{2} m_x v_x^2 \left(\frac{M_X}{m_x + M_X}\right)$$

The threshold energy is then,

$$(E_x)_{\text{thresh}} = \frac{1}{2} m_x v_x^2 = -Q\left(\frac{m_x + M_X}{M_X}\right)$$

which is Eq. 3.18.

ILLUSTRATIVE EXAMPLES

Example 3.1: Find out the unknown particles in the nuclear reactions given below:

(*i*) ${}^{6}_{3}\text{Li}(d, \alpha)X$

(*ii*) ${}^{122}_{52}\text{Te}(X, d)\,{}^{124}_{53}\text{I}$

Solution: (*i*) In a nuclear reaction, the total charge and the mass number are conserved.

We have,

$${}^{6}_{3}\text{Li} + {}^{2}_{1}d \rightarrow X + {}^{4}_{2}\text{He}$$

∴ X has to be put as ${}^{4}_{2}\text{X}$ i.e., it is an α-particle.

(*ii*) Here,

$${}^{122}_{52}\text{Te} + {}^{4}_{2}\text{X} \rightarrow {}^{2}_{1}d + {}^{124}_{53}\text{I}$$

i.e., X is an α-particle.

Example 3.2: Calculate the Q value of reaction ${}^{14}_{7}\text{N}(\alpha, p)\,{}^{17}_{8}\text{O}$ which occurred in Rutherford's α-range in nitrogen experiment.

Solution: The masses in u of neutral atoms involved in this reaction can be taken from the tables.

It may be kept in mind that,. $1u = 931.5$ MeV

$$Q = [(m_x + M_X) - (m_y + M_Y)]c^2$$

$$= [4.002603u + 14.0031\ u - (1.007825u + 16.9994u)] \times \left(931.5\,\frac{\text{MeV}}{u}\right)$$

$$= -1.418\ \text{MeV}.$$

Example 3.3: A neutron beam is incident on a stationary target of ${}^{19}\text{F}$ atoms. The reaction ${}^{19}\text{F}(n, p)\,{}^{19}\text{O}$ has a Q value of –3.9 MeV. Calculate the lowest *neutron energy* which will make this reaction possible.

Solution: We know that,

$$(E_x)_{\text{thresh}} = -Q\left(\frac{M_X + m_x}{M_X}\right) = (3.9\ \text{MeV})\left(\frac{1+19}{19}\right) = 4.105\ \text{MeV}$$

Example 3.4: Obtain the threshold energy for the reactions.

(*i*) $^{209}_{83}\text{Bi}(p, d)\,^{208}_{83}\text{Bi}$ and

(*ii*) $^{27}_{13}\text{Al}(\alpha, p)\,^{31}_{15}\text{P}$

Solution: For (*i*), the Q value is,

$$Q = [208.980394u + 1.007825u - 207.979731u - 2.014102u] \cdot \left(931.5\frac{\text{MeV}}{u}\right)$$

$$= -5.23 \text{ MeV}$$

Now $$(E_x)_{\text{thresh}} = -Q\left(\frac{M_X + m_x}{M_X}\right)$$

$$= 5.23\left(\frac{209+1}{209}\right)$$

$$= 5.26 \text{ MeV}$$

For (*ii*)

$$Q = [25.986892u + 4.002603u - 30.973765u - 1.007825u] \cdot \left(931.5\frac{\text{MeV}}{u}\right)$$

$$= -2.9 \text{ MeV}$$

$$\therefore \quad (E_x)_{\text{thresh}} = 2.9\left(\frac{27+4}{27}\right) = 3.33 \text{ MeV}$$

Example 3.5: Neutrons from ^{7}Li (p, n) ^{7}Be reaction are observed at a laboratory angle $\theta = 30°$. At what bombarding energy are neutrons of zero energy obtained?

Solution: First compute the Q of the reaction

$$Q = [7.016004u + 1.007825u - 7.016929u - 1.008665u] \cdot \left(931.5\frac{\text{MeV}}{u}\right)$$

$$= -1.64 \text{ MeV}$$

Now we have to find the value of E_x for which $E_y = 0$

From Eq. (3.17)

$$0 = v \pm \sqrt{v^2 + w}$$

$$\therefore \quad v^2 = v^2 + w$$

i.e., $$w = 0$$

$$\therefore \quad \frac{M_Y Q + E_x(M_Y - m_x)}{M_Y + m_y} = 0$$

Giving $$E_x = \frac{-M_Y Q}{M_Y - m_x}$$

$$= -(-1.64\text{ MeV})\frac{7}{6}$$

$$= +1.91\text{ MeV}.$$

Example 3.6: ${}^{52}_{25}\text{Mn}$ (6 days) decays to stable ${}^{52}_{24}\text{Cr}$ (51.9571 u) by a positron emission with maximum energy 0.58 MeV and three gamma rays in cascade; viz., 0.73 MeV, 0.94 MeV and 1.46 MeV.

(*i*) Sketch the decay scheme.

(*ii*) Calculate the mass of the neutral atom ${}^{52}\text{Mn}$.

(*iii*) Find the threshold proton bombarding energy for the reaction ${}^{52}\text{Cr}\,(p, n)\,{}^{52}\text{Mn}$.

Solution: (*i*)

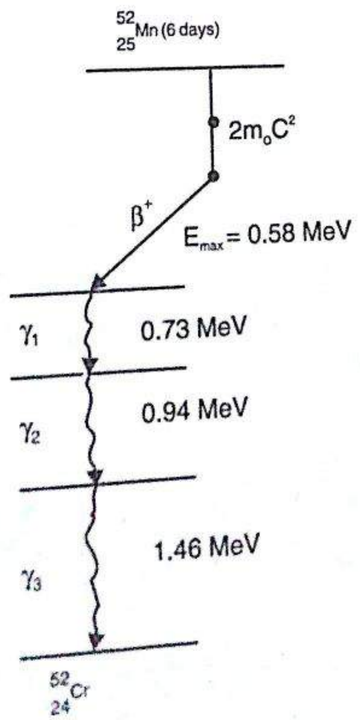

Fig. 3.4. Decay scheme of ${}^{52}_{24}\text{Cr}$.

(ii) $$ {}^{X}_{25}\text{Mn} \rightarrow {}^{51.9571}_{24}\text{Cr} + \beta^{+} + Q $$

$\therefore$ $$ Q = X - [(51.9571 + 2m_0)] \qquad \text{(see Sec. 3.4)} $$

Where m_0 is rest mass of electron.

$\therefore$ $$ Q = (X - 51.9571) \times 931 \frac{\text{MeV}}{u} - 2m_0c^2 $$

But Q = Nuclear disintegration energy

= (0.58 + 0.73 + 0.94 + 1.46) MeV

= 3.71 MeV

Also, $2m_0c^2 = 1.02$ MeV.

$\therefore$ $X = 51.9648u$ i.e., we have ${}^{51.9648}_{25}\text{Mn}$.

(iii) $$ (E_x)_{\text{thresh}} = -Q\left(\frac{m_x + M_X}{M_X}\right) $$

For the reaction,

$$ {}^{52}\text{Cr} + p \rightarrow {}^{52}\text{Mn} + n + Q $$

$$ Q = [(51.9571u - 51.9648u) + (1.007825u - 1.008665u)] \times \left(931.5 \frac{\text{MeV}}{u}\right) $$

$$ = -5.44\text{MeV}. $$

$\therefore$ $$ (E_x)_{\text{thresh}} = 5.44\left(\frac{53}{52}\right)\text{MeV} $$

$$ = 5.54\text{ MeV} $$

Example 3.7: In the case of an elastic collision, use the Q equation to obtain an expression for the kinetic energy of the target particle.

Solution:

$$ Q = E_y\left(1 + \frac{m_y}{M_Y}\right) - E_x\left(1 - \frac{m_x}{M_Y}\right) - \frac{2\sqrt{m_x m_y E_x E_y}}{M_Y}\cos\theta $$

For an elastic collision, $Q = 0$.

$$ \sqrt{E_y} = \frac{\sqrt{m_x m_y E_x}}{m_y + M_Y}\cos\theta \pm \sqrt{\frac{m_x m_y E_x}{(m_y + M_Y)^2}\cos^2\theta + \frac{M_Y Q + E_x(M_Y - m_x)}{(m_y + M_Y)}} $$

(Eqs. 3.13, 3.14, 3.15)

Putting $Q = 0$ and $m_y = M_X$, $M_Y = m_x$,

$$\sqrt{E_X} = \frac{\sqrt{m_x M_X E_x}}{m_x + M_X}\cos\theta \pm \sqrt{\frac{m_x M_X E_x}{(m_x + M_X)^2}\cos^2\theta + 0}$$

$$= \frac{2\sqrt{m_x M_X E_x}}{m_x + M_X}\cos\theta \text{ taking only positive sign.}$$

$$\therefore \quad E_x = 4E_x \frac{m_x M_X}{(m_x + M_X)^2}\cos^2\theta$$

Example 3.8: If $^{235}_{92}U$ captures a thermal neutron and releases 170 MeV and if the resulting fission fragments have mass numbers 138 and 95, what is the kinetic energy of the lighter fragment?

Solution: Since the reaction is due to thermal neutrons we can consider $E_x = 0$, giving from Eq. 3.14, $v = 0$ and so, Eq. 3.13 becomes,

$$E_y = w = \frac{M_Y Q + E_x(M_Y - m_x)}{m_y + M_Y}$$

$$= \frac{M_Y}{m_y + M_Y}Q \qquad \because E_x = 0$$

$\therefore$ E_y = K.E. of the lighter fission fragment

$$= Q\frac{M_Y}{m_y + M_Y} \quad \text{and} \quad Q > 0,$$

otherwise E_y would have been imaginary.

Here $Q = 170$ MeV, since it is the nuclear disintegration energy.

$$M_Y = 138$$

and $$m_y = 95$$

$$\therefore \quad E_y = 170 \times \frac{138}{233}\text{ MeV}$$

$$= 100.69 \text{ MeV}$$

3.6 CENTRE OF MASS FRAME IN NUCLEAR PHYSICS

Often the analysis of collision problems is much simplified by the use of a coordinate system that is moving with the centre of the mass of the colliding bodies, rather than the usual laboratory coordinate system. A nuclear physicist lives in two worlds. One is his laboratory and other is the centre of mass frame in which he studies collisions, interactions. By learning to skip nimbly from one frame to the other, he can get the best of both worlds! Let us have a look at examples of perfectly elastic nuclear collisions to appreciate the view one gets from the centre of mass frame.

(i) Proton-Proton Collisions

Figure 3.5 shows sketch of a typical photoemulsion record of *p-p* collision.

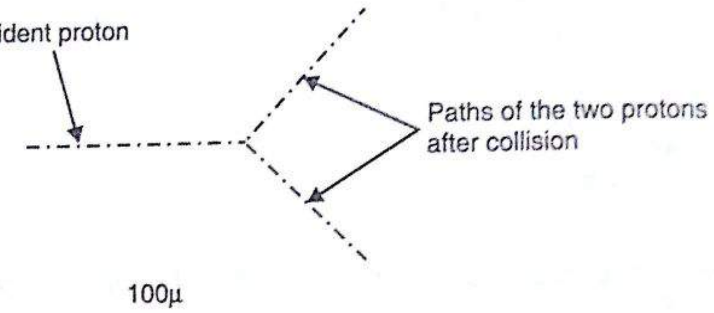

Fig. 3.5. A. *p-p* collision. Sketch of a typical photoemulsion record.

The hit proton is in the photoemulsion and can be considered to be at rest before the collision. The striking feature of Fig. 3.5 is that the paths of the two protons after collision make an angle of 90° with each other. For non-relativistic energies of incident proton, this feature is the same for all *p-p* collisions.

To understand this feature, it is profitable to look at the collision from the centre of mass frame. Let the velocity of the incident proton as observed in the laboratory be $\bar{v}$. Then the centre of mass frame has velocity $\frac{\bar{v}}{2}$. Therefore in this frame the protons approach and recede with equal and opposite velocities as shown in Fig. 3.6.

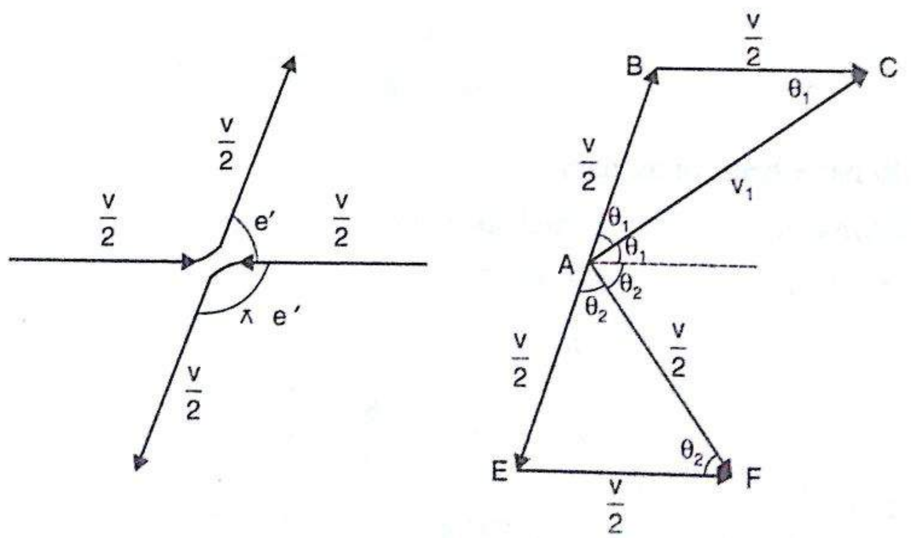

Fig. 3.6. View of *p-p* collision in the centre of mass system is shown on left. How one steps back in the lab frame is depicted on the right.

Suppose that one proton moves in the direction θ' after the collision, then the other would move in the opposite direction at an angle $(\pi - \theta')$. To get back to the laboratory frame, all that we have to do is to add the velocity $\frac{\vec{v}}{2}$ to each proton, in a direction parallel to the original direction of the motion. Now since triangles ABC and AEF are isosceles, the directions θ_1 and θ_2 of the protons as observed in the laboratory are given by,

$$\theta_1 = \frac{\theta'}{2} \quad \text{and} \quad \theta_2 = \frac{(\pi - \theta')}{2}$$

i.e.,

$$\theta_1 + \theta_2 = \frac{\pi}{2} \qquad \ldots(3.20)$$

at once explaining the nature of Fig. 3.5.

Also lab. velocities of two protons after the collision can be easily found out.

$$v_1 = 2\left(\frac{v}{2}\right)\cos\theta_1$$

$$= v\cos\theta_1 \qquad \ldots(3.21)$$

and

$$v_2 = 2\left(\frac{v}{2}\right)\cos\theta_2$$

$$= v\cos\theta_2$$

$$= v\sin\theta_1 \qquad \ldots(3.22)$$

The kinetic energies are,

Initial K.E. $\quad = \frac{1}{2}mv^2$

Final K.E. $\quad = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$

$$= \frac{1}{2}mv^2(\cos^2\theta_1 + \sin^2\theta_1)$$

$$= \frac{1}{2}mv^2 \qquad \ldots(3.23)$$

The reader would appreciate the simplicity and elegance with which the *p-p* scattering dynamics is brought out in the centre of mass frame. Let us consider one more example of nuclear elastic scattering.

(ii) Neutron-Nucleus Collisions

Suppose a neutron of mass m makes an elastic collision with a nucleus of mass M. Let the initial velocity of the neutron in the laboratory frame be $\vec{u}$, while the target nucleus is stationary. Figure 3.7 shows this collision in the centre of mass frame.

The centre of mass frame has a velocity $\vec{v}$ relative to the laboratory frame, given by,

$$(m+M)\vec{v} = m\vec{u} \quad \ldots(3.24)$$

i.e.,

$$\vec{v} = \frac{m}{m+M}\vec{u} \quad \ldots(3.25)$$

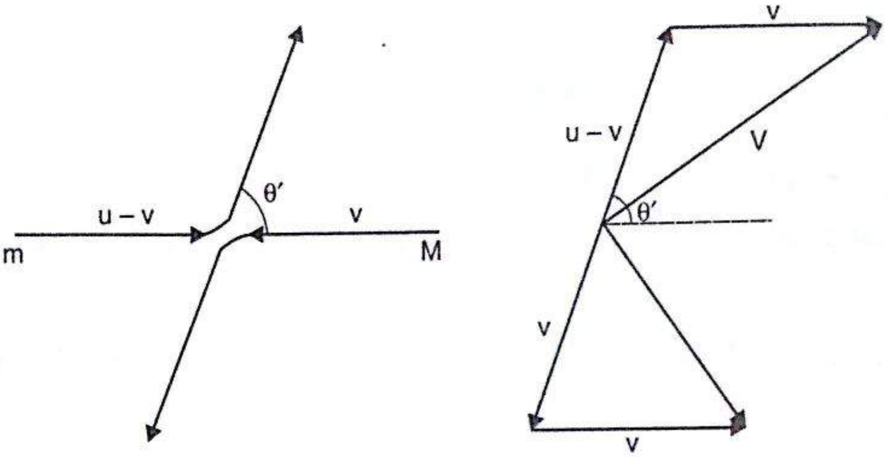

Fig. 3.7. View of neutron—nucleus collision in the centre of mass system is shown on the left. How to get back to the lab. system is shown on the right.

Let the neutron come out in direction θ' after the collision, in the centre of mass frame. The final velocity of the neutron in the laboratory system is $\vec{V}$. as shown. $\vec{V}$, for any given θ' can be at once calculated.

The neutrons suffer greatest energy loss, as seen in the laboratory frame, when it gets scattered backward, corresponding to $\theta' = \pi$. In this case we have,

$$\vec{V}(\pi) = \vec{v} - (\vec{u} - \vec{v})$$

$$= -(\vec{u} - 2\vec{v})$$

$$= -\vec{u}\left(1 - 2\frac{\vec{v}}{\vec{u}}\right)$$

$$= -\vec{u}\left(1 - \frac{2m}{M+m}\right) \text{from Eq. 2.34}$$

$$= -\left(\frac{M-m}{M+m}\right)\vec{u} \quad \ldots(3.26)$$

For $\theta' = 0$, the neutron loses no energy at all. Thus the neutron's kinetic energy after the collision lies between the limits,

$$(\text{K.E.})_{\text{max.}} = \frac{1}{2}mu^2$$

$$(\text{K.E.})_{\text{min.}} = \frac{1}{2}m\left(\frac{M-m}{M+m}\right)u^2 \qquad \text{...(3.27)}$$

We see that, $M \cong m$ makes $(\text{K.E.})_{\text{min.}} \cong 0$.

Fission reaction releases neutrons of energy ~1 MeV. To introduce further fissions, these neutron energies have to be brought down to thermal energies (~10^{-2} eV). Different moderators have to be used for this purpose. From Eq. 3.27 hydrogen would have been the best moderator, but protons capture slow neutrons readily. Therefore, deuterium, carbon etc. are used as moderators.

EXERCISES

1. Complete the following nuclear reactions:

$${}^{6}_{3}\text{Li} + {}^{2}_{1}\text{H} \rightarrow {}^{7}_{4}\text{Be} + X$$

$${}^{23}_{11}\text{Na} + {}^{1}\text{H} \rightarrow {}^{20}_{10}\text{Ne} + X$$

$${}^{35}_{17}\text{Cl} + X \rightarrow {}^{32}_{16}\text{S} + {}^{4}_{2}\text{He}$$

2. A beam of deutrons is incident of ${}^{31}_{15}\text{P}$ target. Write down the nuclear reaction equations, when the emitted particle is (*i*) a proton, (*ii*) a neutron and (*iii*) an alpha particle.
3. Calculate the Q value for the reactions,

   (*i*) ${}^{42}_{20}\text{Ca}\,(p, d)\,{}^{41}_{20}\text{Ca}$

   (*ii*) ${}^{15}_{8}\text{O}(d, p)\,{}^{16}_{8}\text{O}$
4. Show that in a two-body elastic collision, each particle's speed will remain unchanged by the collision when measured in the centre of mass system.
5. Obtain the threshold energy for the reactions,

   (*i*) ${}^{93}_{40}\text{Nb}\,(p, d)\,{}^{92}_{41}\text{Nb}$

   (*ii*) ${}^{19}_{9}\text{F}\,(n, p)\,{}^{19}_{8}\text{O}$
6. The Q value of the reaction,

   ${}^{14}_{7}\text{N}\,(\alpha, p)\,{}^{17}_{8}\text{O}$ is -1.16 MeV.

   Calculate (*i*) the threshold energy for the reaction, and (*ii*) the difference in mass between target and residual nuclei.

7. Alpha particles having a kinetic energy of 5 MeV in the lab. system, start the reaction,

$$ {}^{9}_{4}\text{Be}\,(\alpha, n)\,{}^{12}_{6}\text{C}. $$

Calculate the maximum energy of the neutrons emitted in the lab. system.

8. Accelerated protons from a Van de Graaff generator are used to bombard nuclei having mass number $A \sim 20$. Assuming elastic scattering find what energies do the protons have for exit angles 45°, 90°, 135° and 179°. The incident proton energies used are, 1.0, 2.0, 4.0, 6.0, 8.0, and 10.0 MeV. (in lab. system).

9. Show that the reduction in energy ΔE of a neutron in an elastic collision with a nucleus of mass number A is given by

$$ \Delta E = E\left(1 - \left(\frac{A-1}{A+1}\right)^2\right) $$

Calculate the fractional loss of energy of a neutron colliding with (*i*) a proton, (*ii*) a deuteron, (*iii*) a carbon nucleus ^{12}C and (*iv*) a beryllium nucleus. ^{7}Be.

10. The Q value of $^{18}\text{O}(p,\alpha)^{15}\text{N}$ is +3.97 MeV and that of $^{15}\text{N}(p,\alpha)^{12}\text{C}$ is +4.96 MeV. Calculate the atomic mass of ^{15}N from these reactions.

11. Calculate the Q value for the reaction

$$ ^{3}\text{H}(d, n)^{4}\text{He}. $$

(*i*) Write an expression for the neutron energy for this reaction in terms of deuteron energy E_x and angle θ.

(*ii*) In what direction will the neutron energy be smallest?

(*iii*) If neutrons of energy between 1 MeV and 6 MeV are available, at what deuteron energy will the neutron energy be the smallest? Explain.

(*iv*) What is the smallest possible energy for neutrons?

(*v*) Why does this *not* occur at the smallest value of deuteron energy?

12. Consider an endoergic nuclear reaction. The smallest value of bombarding energy which induces such a reaction is called threshold energy $(E_x)_{\text{thresh}}$ (see Sec. 3.5). for that reaction. Show that E_x has its minimum possible value at $\theta = 0$, which is $(E_x)_{\text{thresh}}$.

(*Hint:* From Eq. 3.13, write $(E_x)_\theta$ and then show that for $\theta = 0$, E_x has minimum possible value.)

13. A particle of mass M_1, kinetic energy E_1, velocity v_1, momentum p_1 is captured by a nucleus (M_2, E_2, v_2, p_2) at rest. A light particle (M_3, E_3, v_3, p_3) is ejected and the heavy particle (M_4, E_4, v_4, p_4) recoils. When the light particle is emitted at an angle of 90° with the direction of the incident particle, show that the energy E_3 is given by,

$$ E_3 = \frac{E_4}{M_3 + M_4}\left(Q - \frac{M_1 - M_4}{M_4} E_1\right) $$

(*Hint:* Use conservation of energy and momentum.)

14. When ${}^{11}_{5}\text{B}$ is bombarded by 200 keV protons alpha particles are emitted at right angles to the path of the incident beam. The mass of ${}^{8}_{4}\text{Be}$ is 8.00794 u. Calculate the energy of the alpha particles.

(*Hint:* Use the result of problem 13.)

15. The radioactive isotope ${}^{37}_{18}\text{A}$ can be produced by ${}^{37}_{17}\text{Cl}\ (d, 2n)\ {}^{37}_{18}\text{A}$ reaction. ${}^{37}\text{A}$ disintegrates back in to ${}^{37}\text{Cl}$, either by emitting a very soft radiation which may be a very low energy positron or may be X rays or a possible soft γ ray, following orbital electron capture. Show that the threshold deuteron energy must be:
   (*i*) greater than 4.25 MeV if ${}^{37}\text{A}$ emits positrons.
   (*ii*) between 3.17 and 4.25 MeV if ${}^{37}\text{A}$ decays by electron capture only.
   (*iii*) why is the difference (4.25 – 3.17) MeV not equal to $2m_0c^2$?

16. Consider the reaction, ${}^{19}_{9}\text{F}(\alpha, p)\,{}^{22}_{10}\text{Ne}$ for which $Q = +1.58$ MeV. If the observed particle is chosen to be the residual nucleus ${}^{22}_{10}\text{Ne}$, show that in the forward direction (in lab. system) it is possible to have two monoenergetic groups of ${}^{22}\text{Ne}$ nuclei. Obtain the condition that the bombarding energy (of α-particles) must satisfy for this to happen.

17. ${}^{233}\text{Th}$ captures a thermal neutron and releases 190 MeV. If the resulting fission fragments have mass numbers 136 and 93. what is the kinetic energy of the lighter fragment?

18. π^+ mesons at rest decay spontaneously into $\mu^+ + \nu$. The μ^+ are observed to have a kinetic energy of 4.0 MeV. If the rest mass of μ^+ is 212 m_0 and that of the neutrino is zero, what is the rest mass of π^+ in units of electron rest mass m_0? If kinetic energy of μ^+ is taken as 4.1 MeV, what would be m_{π^+} ?

19. Consider head-on and glancing collisions in the centre of mass coordinate system. Show that the ratio of the total kinetic energy T' in the centre of mass system and the total kinetic energy T in the laboratory system is

$$\frac{T'}{T} = \frac{m_2}{m_1 + m_2}$$

Represent a completely inelastic collision in both the centre of mass and laboratory systems. Draw neat diagrams.