Nonlinear systems :

(Maps, Flows, Periodicity & chaos)

Flows $\equiv$ Continuous systems. $\Rightarrow$ Differential equations

Maps $\equiv$ Discrete systems $\Rightarrow$ Difference eqns.

Various systems possible.

- Consider a simple model for population growth.

For a realistic touch, make the following assumptions. :

1. Popln. after the end of next generation — proportional to popln. at the end of current gen. — when the population is very small. ie. rate of growth of population is proportional to its size.

2. If population — very large — $\Rightarrow$ all resources will be consumed quickly, & the entire population will die out in the next gen. $\Rightarrow$ extinction!

3. So assume that in a discrete model there is a <u>maximum</u> <u>population</u> level <u>M</u>, which, when reached, results in the extinction of the popln. in next gen.

M $\equiv$ "annihilation parameter"

If popln. ever reaches M $\Rightarrow$ Doom for the species!

Now write down formally what these assumptions correspond to:

$X_n \equiv$ population at the end of generation 'n'.

$X_0 =$ initial population.

$$X_{n+1} = r X_n \left(1 - \frac{X_n}{M}\right)$$

$\therefore$ If $\cancel{X_n}$ is small $\Rightarrow \left(1 - \frac{X_n}{M}\right) \approx 1$

$\Rightarrow$ the difference eqn. becomes:

$X_{n+1} \approx r X_n$ (exponential growth model)

But if $X_n \geq M \Rightarrow X_{n+1} \leq 0$

"-ve population" — interpret it as meaning the species is extinct.

Assume $X_n \equiv$ % or fraction of the max popln. alive at gen. n.

ie. assume $M = 1$ & $0 < X_n \leq 1$.

$\therefore$ $X_n \leq 0$ $\Rightarrow$ extinction

and $X_n = 1$ $\Rightarrow$ max. popln. reached.

$\therefore$ our difference eqn. is

$$\boxed{X_{n+1} = r X_n (1 - X_n)}$$

"Discrete logistic eqn."

or "Logistic Difference eqn."

r — depends on the species or system being studied. For different r, different behaviours.

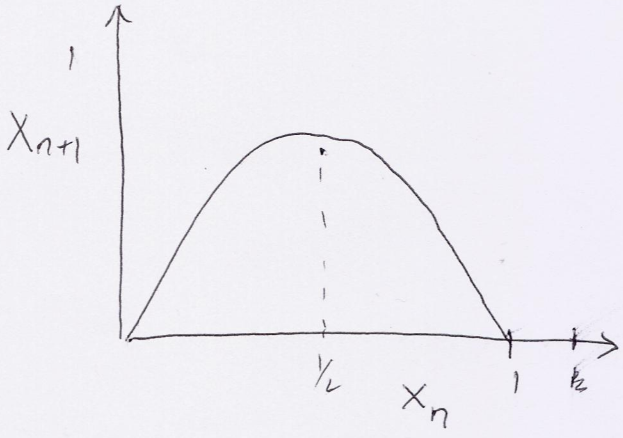
$r \geq 0$ — intrinsic growth rate.

Robert May

"Simple mathematical models with very complicated dynamics."

Nature 261, 459 (1976).

$$\boxed{x_{n+1} = r\, x_n\, (1 - x_n)} \quad \text{①}$$



Graph is a parabola with maxm. val. of $\frac{r}{4}$ at $x = \frac{1}{2}$.

So restrict the <u>control parameter</u> r to the range $0 \leq r \leq 4$ so that the eqn. ① maps the interval $0 \leq x \leq 1$ onto itself. (For other values of x & r, behaviour is less interesting).

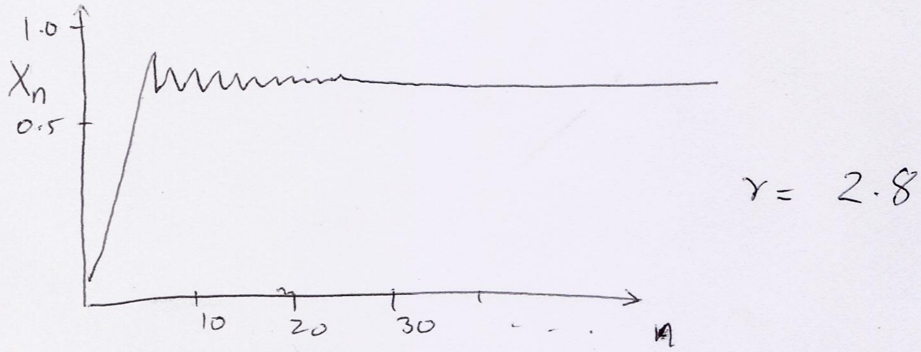
<u>Period - doubling.</u>

Choose some val. of r — fix it.
Choose some initial popln. x_0 & then use ① to generate subsequent x_n. <u>WHAT</u> happens?

- For small $r < 1$, $x_n \to 0$ as $n \to \infty$, population goes extinct.

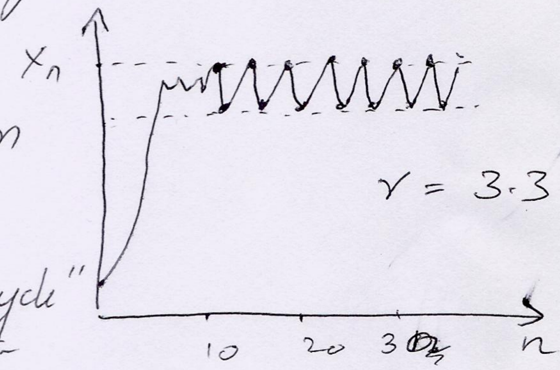
- For $1 < r < 3$, population grows, eventually reaches a non-zero steady state.

→

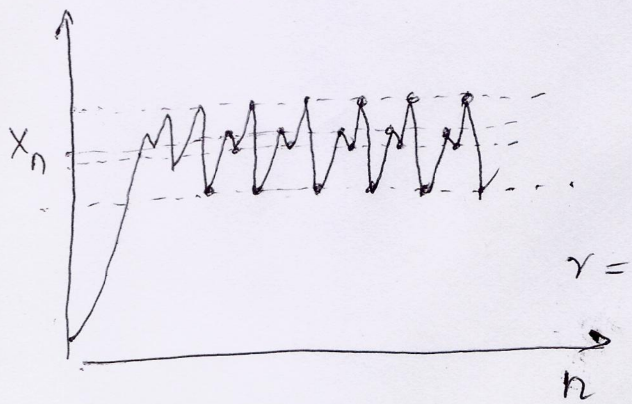


For larger r, eg. $r = 3.3$, population builds up again, but now oscillates about former steady state. — Alternating betwn. a larger popln. in one gen. & smaller popln. in the next.

This type of oscilln where x_n repeats every 2 iterations — "period-2 cycle"



Still larger r — eg. $r = 3.5$ popln. approaches a cycle that repeats every 4 generations — — previous cycle has doubled its period to period-4



Further <u>period doublings</u> to cycles of period 8, 16, 32, ... occur with increasing r.

Let $r_n \equiv$ val. of r where a 2^n cycle first appears. Then:

$r_1 = 3$ period 2 is born

$r_2 = 3.449\ldots$ 4

$r_3 = 3.54409$ 8

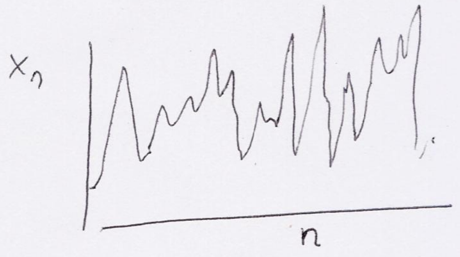
$\vdots$ $\vdots$

$r_\infty = 3.569946$ ∞

<u>Note</u>: Successive <u>bifurcations</u> come closer & closer.

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

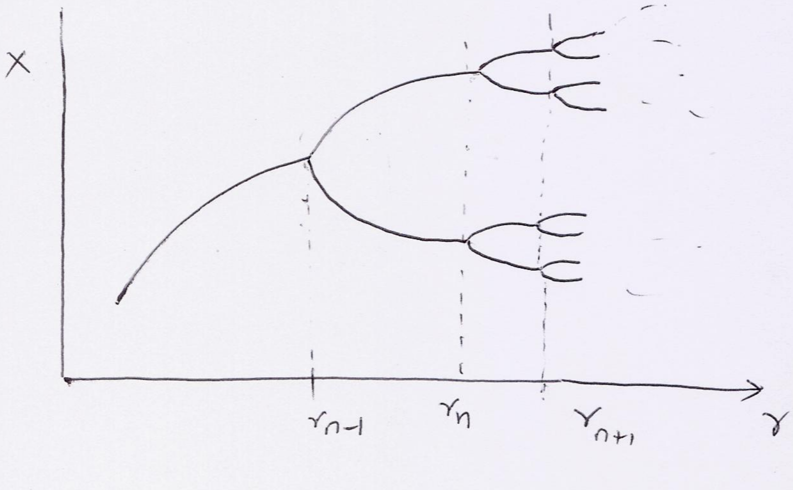
<u>Feigenbaum number</u>.



$r = 3.9$.

$\Rightarrow$ aperiodic behaviour

Discrete t version of chaos.



SM204P Physics 2 lab.
(Prof. B. Ashok)

Logistic map : numerics

$$x_{n+1} = r\, x_n (1 - x_n)$$

For plotting the orbit diagram (also termed bifurcation diagram), proceed as follows:

- 1st choose a value of r.
- Then, generate an orbit for ~~som~~ – starting from some random initial condition x_0.

[What is an orbit? The sequence $x_0, x_1, x_2, \ldots$ is called the orbit starting from x_0.]

- Iterate for 300 cycles or so – allow the system to settle down, (allow the transient behaviour to decay off).
- Now plot, $x_{301}, x_{302}, \ldots, x_{600}$ for that r.
- Now move to another r value $r \to r + \delta r$, REPEAT the above cycle.

Note that:

at $r = 3.4$. — period 2 cycle.
(—2 branches)

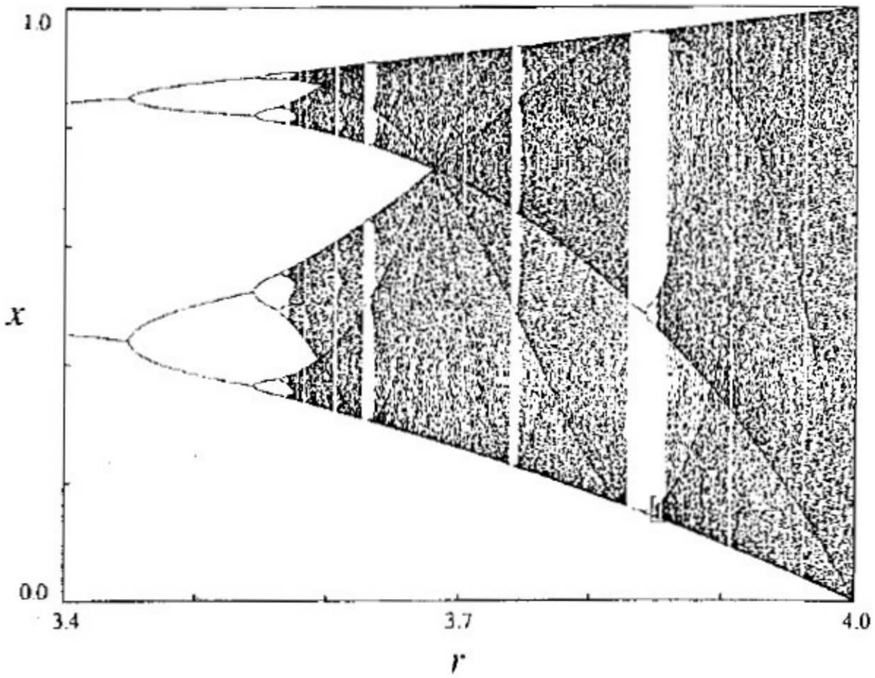
r increases — both branches split simultaneously — period 4 cycle.
(period-doubling bifurcation) —

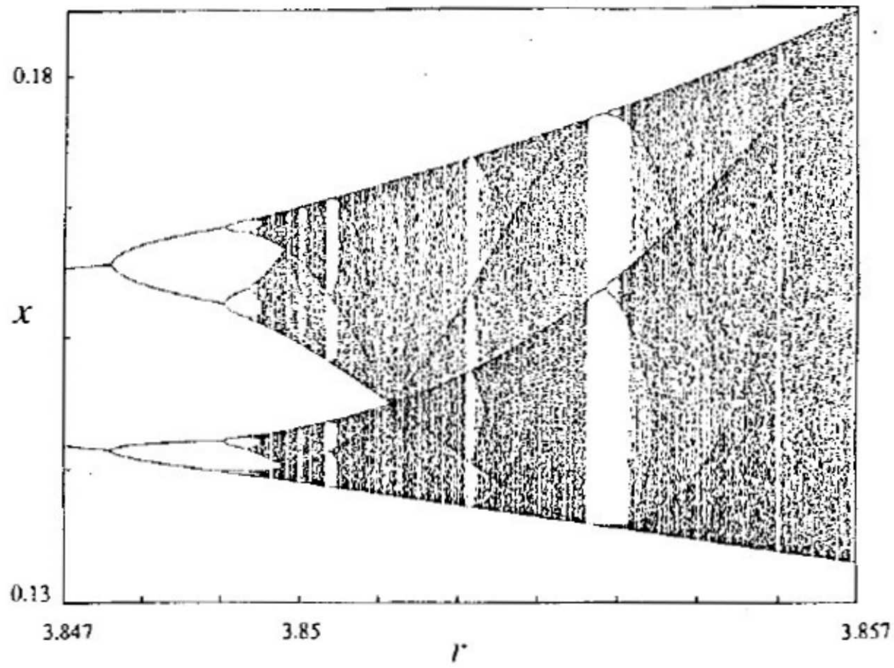
(recall: $r_1 = 3$ period 2 is born
$r_2 = 3.449$ period 4
$\vdots$ $\vdots$
$r_\infty = 3.569946$ ∞)

For $r > r_\infty$ — a mix of order & chaos; periodic windows appearing betw. chaos.

e.g. at $r \approx 3.83$ — a stable period-3 cycle.

See a magnified view of the period-3 window — what do you see ?!

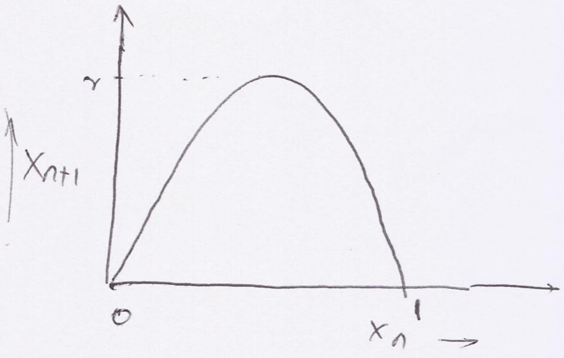

1.0
x
0.0
3.4
3.7
4.0
r


0.18
x
0.13
3.847
3.85
3.857
r

Universality. –

First Consider the sine map:

A unimodal map $\quad x_{n+1} = r \sin \pi x_n$,



for $0 \le r \le 1$, $\quad 0 \le x \le 1$.

The maximum of $r \sin \pi x_n$ is r

(Recall: in the logistic map $rx(1-x)$ has a max. value of $\frac{r}{4}$.)

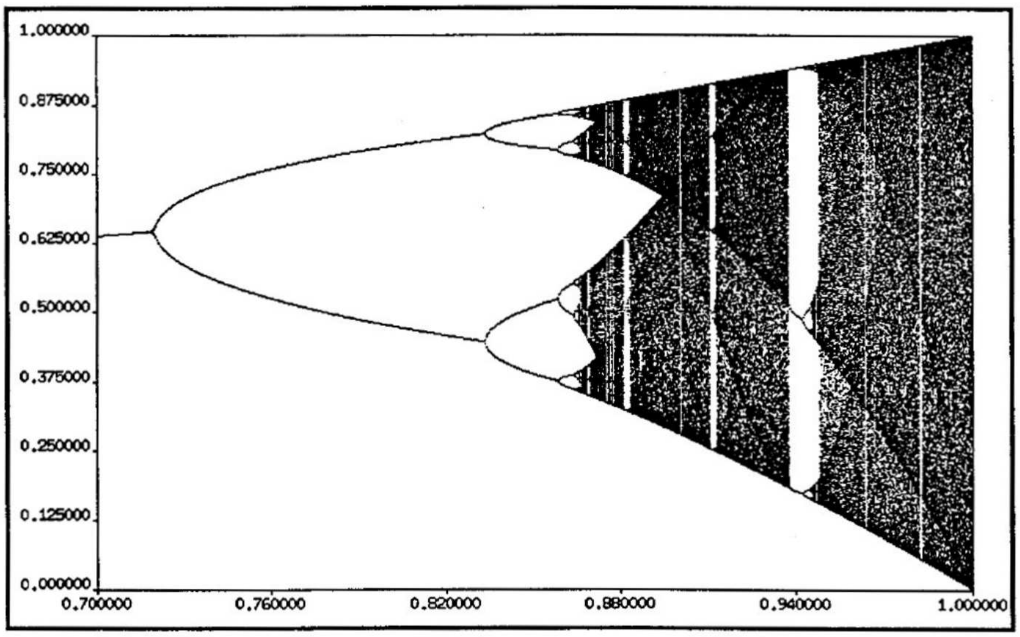
See the orbit diagram for this map, Compare with the logistic map.

You'll see:

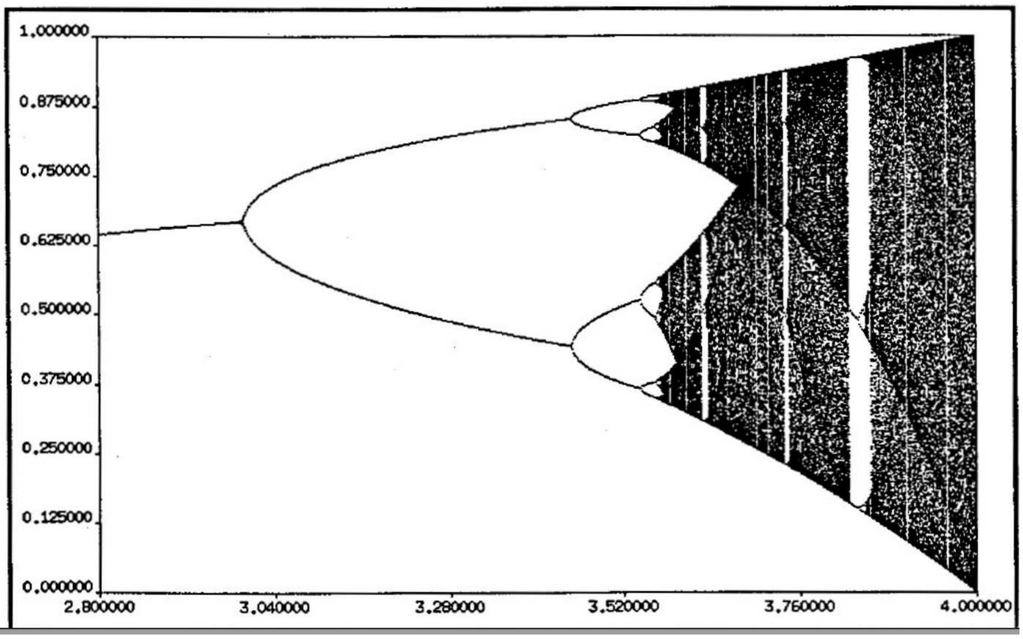
1. Qualitative dynamics — identical.
   - Both maps — undergo period-doubling routes to chaos; followed by periodic windows mixed with chaotic bands.
   - Periodic windows occur in the same order, with the same relative sizes
     - period 3 window is the largest in both & the next largest windows preceding it are period 5, & period 6.

Quantitative differences: eg. Periodic windows — thinner. Period doubling occurs later in logistic map

9b



SINE MAP



LOGISTIC
MAP

Qualitative Universality: the U sequence.

Consider all unimodal maps of the form

$$x_{n+1} = r f(x_n), \quad \text{with } f(x) \text{ satisfying } f(0) = f(1) = 0$$

(for full details, see
N. Metropolis, M.L. Stein & P.R. Stein,
J. Combin. Theor. 15, 25 (1973))

As r is varied, the order in which stable periodic solutions appear is independent of the unimodal map being iterated.

i.e. Periodic attractors always occur in the same sequence, a Universal Sequence, or U-Sequence.

Upto period-6, this sequence is (omitting higher-order periods of 8, 16, etc.)

1, 2, 2×2, 6, 5, 3, 2×3, 5, 6, 4, 6, 5, 6

CHECK!

Period doubling in the real world:

- in hydrodynamics
- in acoustics
- laser feedback
- Electronics, etc.

U-sequence found in expts. on the Belousov-Zhabotinsky reaction, in chemical reactors, etc..