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Determination of gravitational acceleration using a rubber ball

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A rubber ball, with a 4.9-cm diameter and 64-g mass, is dropped from a certain height h to bounce onto the floor, and the time between the first bounce and the nth bounce is measured. In the first version this time is taken by a digital stopwatch with one centisecond resolution, and in the second version, by an integrated-circuit timer with 1-ms resolution, triggered by a microphone. Considering the formula $h = (1/2)gt^2$ and the coefficient of restitution of energy in the collision of the ball against the floor, the acceleration of gravity g may be determined to 2%–3% in the first version and to better than 1% in the second one. The experiment opens many problems, i.e., about the time of contact, the internal vibrational energy of the ball, and the dependence of the elastic properties of the rubber on the temperature.

I. INTRODUCTION

In the scientific literature one may find a number of methods for measuring the acceleration due to gravity g, some of which are very interesting also for pedagogical purposes.[1–11] Our method has this feature, and considers a well-known phenomenon, the bouncing of an elastic ball onto a massive target, a phenomenon which was studied in a number of works, but from points of view different than this.[12–17]

Taking the relation

$$s = \tfrac{1}{2}gt^2 \tag{1}$$

into account, and measuring the time a rubber ball (for example a super ball) takes for n bounces onto a smooth floor, it will be possible to find a good value of g. We know that if we take the time by a stopwatch for a single free fall of a body, this time is very short indeed, unless the height is very large, so the percent uncertainty over this time, and thus over g, will be intolerable. Instead, considering the time for n bounces, it will be almost as long as one wants, hence affected by a much smaller uncertainty. Obviously, at each bounce the height reached by the ball will be lower and lower, but it is possible to consider this by defining a coefficient of restitution of energy in the following way:

$$e = mgh_1/mgh_0 = h_1/h_0 . \tag{2}$$

The theoretical considerations on this experiment are simpler if air drag can be ignored, and this is almost true for a rubber ball only if it is dropped from a very limited height; besides, the coefficient of restitution should depend neither on the height of fall nor on the area of the floor onto which the ball will bounce from time to time. At last, the time of contact between ball and floor should be very short, compared with the free-fall time.

Under the above assumptions, we can use a simplified theory. Suppose one measures the time t_1 elapsed between the first and the second bounce, as indicated by the diagram of Fig. 1: this time is twice the time the ball takes to travel the vertical distance h_1, hence if we are interested in the value of g, we can write

$$g = 2h_1/(t_1/2)^2 = 8h_1/t_1^2$$

or else, using the coefficient e,

$$g = 8h_0e/t_1^2 . \tag{3}$$

But the time t_1 is short too, so using a manual stopwatch one will obtain an intolerable uncertainty. Hence, it will be more useful to consider the time between the first and a subsequent bounce, in general between the first and the nth bounce. This time t_n^* will be the sum of the partial ones:

$$t_n^* = t_1 + t_2 + \cdots t_n ,$$

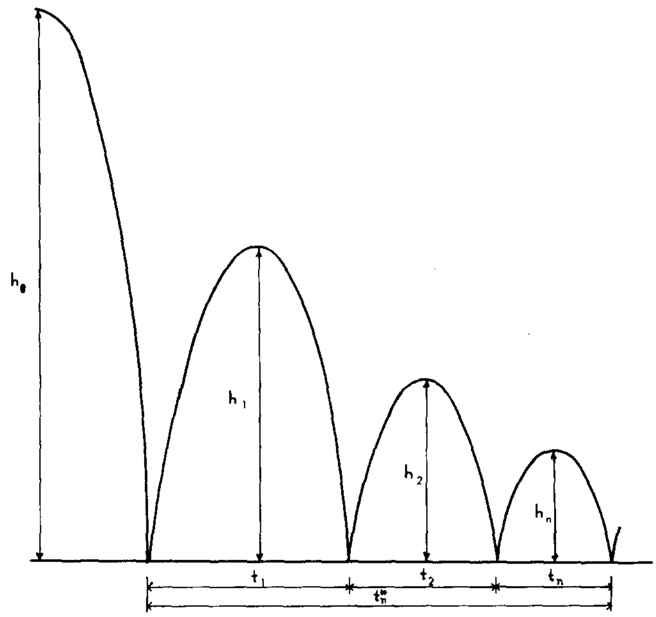

Fig. 1. Schematic representation of the bounces of a ball. In our experiment indeed the ball does not have any horizontal component of velocity.

that is,

$$t_n^* = t_1 + 2\sqrt{2h_2/g} + \cdots 2\sqrt{2h_n/g}$$
$$= t_1 + 2(t_1/2)(\sqrt{e}) + \cdots 2(t_1/2)(\sqrt{e})^{n-1}$$
$$= t_1[1 + \sqrt{e} + \cdots (\sqrt{e})^{n-1}]\,. \quad (4)$$

We note that the terms inside square brackets give a series of ratio $\sqrt{e}$, and in a pedagogical context it is a very fortunate circumstance when one meets with so elegant a link between physics and mathematics. Thanks to a well-known property of the polynomials, it is also

$$t_n^* = t_1\{[(\sqrt{e})^n - 1]/(\sqrt{e} - 1)\}$$

from which

$$t_1 = t_n^*\{(\sqrt{e} - 1)/[(\sqrt{e})^n - 1]\}\,.$$

Replacing into Eq. (3), we obtain at last

$$g = \frac{8h_0 e}{t_n^{*2}}\left(\frac{(\sqrt{e})^n - 1}{\sqrt{e} - 1}\right)^2. \quad (5)$$

II. EXPERIMENT

It is essential to release the ball always in the same manner, for the repeatibility of the experiment. The floor beneath must be smooth and homogeneous: we found a floor made out of large marble tiles suitable. Mainly for repeatibility reasons we chose to use a rubber ball, with a 4.9-cm diameter and 64-g mass, bought in a toy shop.

Preliminarily one has to determine the coefficient of restitution of energy, calculating the ratio between the height reached after the first bounce and the initial height. For this part of the experiment, we tried many methods, in order to minimize the uncertainty about the height reached after the first bounce: multiflash photos, videotape recorder, etc., but at last we decided to use two meter sticks put vertically, one in front of the trajectory of the falling ball, the other behind, each of them provided with a horizontal pointer. By trials, one can move the pointers, to find at which height the upper border of the ball does not appear over them. It is necessary to use two pointers instead of one alone, for parallax reasons. We used a digital stopwatch, with 1-centisecond resolution, to measure the times, and took series of at least ten trials for each number of bounces, obtaining the data of Table I. In the third row the values of gravitational acceleration, obtained by Eq. (5), are reported. As regards the scattering of the times, it is due to the fluctuations of the reaction time of the operator together with the fluctuations of the coefficient e, which may change a little from one place of bounce to another. This time also may change for the inhomogeneities of the ball and for the loss of translational energy that turns into rotational energy if the ball begins to spin. However, we discarded the data obtained under the worst conditions, i.e., when the ball moved a long way on the floor in the subsequent bounces.

The uncertainty of g depends, as shown by Eq. (5), upon the uncertainty of e together with that of t_n^*. Now, as regards the uncertainty of t_n^* it was already mentioned above. As for the uncertainty of e, it was computed to be about $\pm 0.2\%$, thanks to the two-pointers method described above. The resulting uncertainty over g can be calculated simply by a numerical method, instead of by an analytical one, thanks to the programmable pocket calculators or personal computers. In fact, from Eq. (5) one can realize that the maximum of g is reached when t_n^* assumes its minimum value and e its maximum.

All the above considerations refer to the random errors, but now it is worthwhile to consider also the possible systematic errors.

III. SOURCES OF SYSTEMATIC ERROR AND NEW EXPERIMENT

In manual timing it is important to become familiar with the kind of phenomenon which is being studied, doing a long series of preliminary trials in order to minimize random together with systematic errors. In a second instance we exploited an electronic timer with 1-ms resolution, triggered by a microphone, in order to take control of the data

Table I. Data obtained by a stopwatch with 1-centisecond resolution. In the third row the values of g are reported, assuming e = constant, while in the last row the values of g are corrected by the air drag, and the dependence of e from height. Ambient temperature: 24 °C.

n	1	2	3	4	5	6	7
t_n^*(s)	0.87 ± 0.03	1.73 ± 0.03	2.57 ± 0.03	3.39 ± 0.03	4.18 ± 0.03	4.95 ± 0.03	5.67 ± 0.04
g(m/s^2)	9.92 ± 0.74	9.73 ± 0.37	9.62 ± 0.26	9.53 ± 0.21	9.50 ± 0.19	9.46 ± 0.17	9.52 ± 0.20
g(m/s^2) correct	10.0 ± 0.8	9.8 ± 0.4	9.7 ± 0.3	9.6 ± 0.2	9.6 ± 0.2	9.6 ± 0.2	9.7 ± 0.3

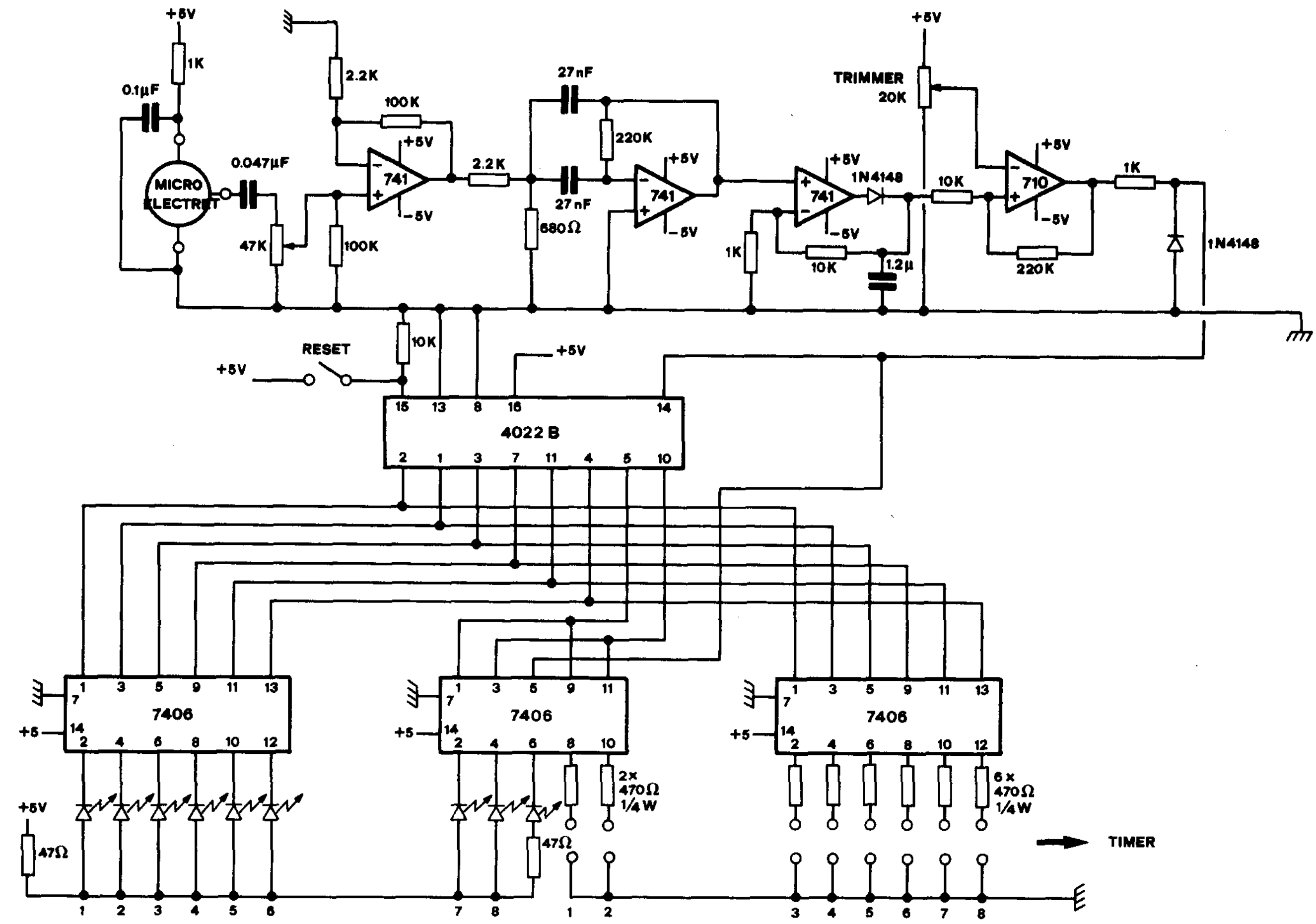

Fig. 2. Diagram of the circuit triggering the electric clock. The frequency of sound controlling the logic gates of the clock is about 600 Hz, in this arrangement.

measured by the manual stopclock. In Fig. 2 the diagram of the electric circuit coupled with the timer is shown.

The other principal sources of systematic error can be: (i) the coefficient of restitution of energy may vary with the height of fall; (ii) the air drag may be relevant with respect to the weight of the ball, and (iii) the time of contact with the floor may be longer then one can expect.

Before going ahead, we would like to recall that by repeating the determination of e in different days, the data showed a dependence on the temperature which was not negligible. In other words, we realized that our rubber ball behaved as a thermometer. Indeed, on a day when $t_a = 24\,^\circ\mathrm{C}$, $e = 0.939 \pm 0.001$, while the day after, with $t_a = 23\,^\circ\mathrm{C}$, resulted $e = 0.935 \pm 0.001$. (In both cases the initial height was the same, i.e., 1 m.) But the dependence of e from temperature can be neglected, in this context, provided that one does all the measurements in an ambient with almost constant temperature.

By systematically decreasing the height of release, we determined with extreme care the value of the coefficient e, by means of the two-pointers method mentioned before, and we got the data reported in Fig. 3, which shows a regular increase of e with the decreasing of h.

As regards the time of contact τ in the collision of the ball against the floor,[18,19] it is possible to refer to the following theoretical equation,[20] valid for two colliding balls of the same mass M, radius R, Young's modulus E, and approaching speed v:

$$\tau \cong 2[M^2/E^2Rv]^{1/5}. \tag{6}$$

In our case the situation is different, but the same parameters must play a role in the phenomenon. Now, because the speed is proportional to $\sqrt{h}$ in the free-fall motion, we can rewrite Eq. (6) as

$$\tau \cong K/h^{1/10}, \tag{7}$$

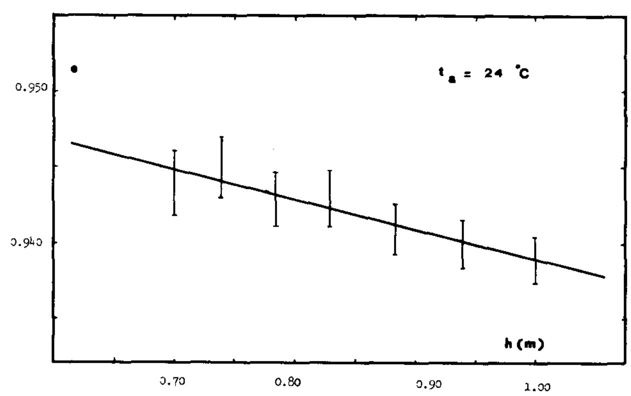

Fig. 3. Coefficient of restitution of energy as a function of release height. The interpolation line was obtained by the least-squares method.

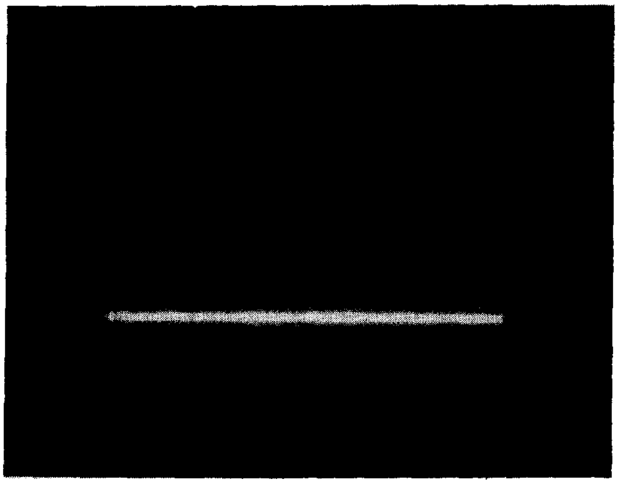

Fig. 4. Time of contact for $h = 0.100$ m. X axis: 1 ms/div; Y axis: 1 V/div.

where K is an appropriate coefficient of proportionality. The measurement of the contact time was done by metallizing—by a conductive paint made out of silver—the parts of the ball and of the floor that come into contact during the collision. We attached to the ball a long and thin metallic wire that does not oppose any resistance to the falling of the body. Then the wire was put in series to a 4.5-V battery and a 1-M Ω resistor, at the end of which the electric signal was taken and sent to the Y axis of an oscilloscope. Taking a photograph of the display for each height of fall, as showed in Fig. 4, the data reported in Table II were obtained. By the least-squares method, we found that these data can be well approximated with the following equation, very near to the theoretical relation (7):

$$\tau = 2.44/h^{0.11} .$$

We note that for the heights which refer to our experiment ($h = 0.939$ m for $n = 1$ and $h = 0.698$ m for $n = 7$), the collision time τ ranges from 2.4 ms to about 2.6 ms, respectively.

Now one can consider the influence of air drag on the motion of the falling ball. This motion is characterized by a Reynolds number enough large so that the air drag D can be expressed by this relation,[5] almost from the beginning of the motion:

$$D = \tfrac{1}{2} C_D \pi R^2 \rho_a v^2 , \tag{8}$$

where C_D is the drag coefficient, which is $\cong 0.5$ for a sphere, ρ_a is the air density, which is about 1.17 kg/m^3 at normal temperature and pressure. Thus, the mean loss of

Table II. Times of ball–floor contact, as a function of release height.

h	(m)	0.100	0.200	0.404	0.590	0.800	1.000
t	(ms)	3.17	2.90	2.72	2.62	2.50	2.41

acceleration is

$$\Delta a = D/m = \tfrac{1}{2} C_D \pi R^2 \rho_a v^2 / m , \tag{9}$$

where m is the mass of the ball. Computing the mean-square velocity $\langle v \rangle^2_{\text{av}}$ for the first bounce, we have

$$\langle v \rangle^2_{\text{av}} = v^2_{\max}/3 = 6.16 \text{ m}^2/\text{s}^2 .$$

Thus, from Eq. (9) we obtain

$$\Delta a = 0.055 \text{ m/s}^2 .$$

In the following bounces the mean-square velocity is less than this value, so we round $\Delta a = 0.06$ m/s^2 for the correction relative to the first two bounces, and $\Delta a = 0.05$ m/s^2 for the following ones. Taking this correction into account, and considering the mean value of e, calculated between the pertinent heights, in the experiment with the stopclock we got the values of g reported in Table I, last row, rounded to the first decimal digit. (We verified that using the mean value of e the degree of approximation is still high.)

In the case of the experiment with the electronic clock, the times obtained are reported in Table III, row 5. Considering only the variability of e with the height and using Eq. (5), we got the values of g reported in row 6, while considering also the corrections due to air drag and time of contact, we got the values reported in row 7. In the same table we reported also the data about the heights that allowed us to calculate the coefficient e.

IV. DISCUSSION

In the first version of this experiment the apparatus needed is extremely simple. In spite of this, good values of g can be obtained, in particular with two and three bounces. (Obviously it is necessary to take a little practice in timing, and to consider at least the corrections due to the dependence of e from height and to air drag.) In this version, the experiment can be done at home or in a laboratory, by a student alone, or in group or as a demonstration, and this is an important pedagogical circumstance.

In the more sophisticated version, the accuracy is higher, but much higher is the cost of the equipment, so that this improvement is not justified in all circumstances. Obviously, with the electronic clock and transducer it is possible to obtain a good value of g also by timing only a bounce. We

Table III. Data obtained for the same series of trials than in Table I, but using an electric clock with 1-ms resolution and an acoustic transducer. Note the difference in uncertainties on the heights h_0 and h_1, because the first ones are obtained by static measurements, while the second ones by dynamical measurements.

n	1	2	3	4	5	6	7
h_0(m)	1.000 $\pm$ 0.0005	0.9390 $\pm$ 0.0005	0.8830 $\pm$ 0.0005	0.8310 $\pm$ 0.0005	0.7840 $\pm$ 0.0005	0.7390 $\pm$ 0.0005	0.6980 $\pm$ 0.0005
h_1(m)	0.939 $\pm$ 0.001	0.883 $\pm$ 0.001	0.831 $\pm$ 0.001	0.784 $\pm$ 0.001	0.739 $\pm$ 0.001	0.698 $\pm$ 0.001	0.659 $\pm$ 0.001
$e = h_1/h_0$	0.9390 $\pm$ 0.0016	0.9400 $\pm$ 0.0016	0.9410 $\pm$ 0.0017	0.9430 $\pm$ 0.0018	0.9430 $\pm$ 0.0018	0.9450 $\pm$ 0.0020	0.9440 $\pm$ 0.0021
t^*_n(s)	0.879 $\pm$ 0.001	1.732 $\pm$ 0.001	2.564 $\pm$ 0.002	3.378 $\pm$ 0.002	4.168 $\pm$ 0.003	4.930 $\pm$ 0.006	5.665 $\pm$ 0.005
g(m/s^2)	9.72 $\pm$ 0.04	9.71 $\pm$ 0.04	9.67 $\pm$ 0.05	9.62 $\pm$ 0.06	9.59 $\pm$ 0.07	9.60 $\pm$ 0.09	9.63 $\pm$ 0.09
g(m/s^2) correct	9.83 $\pm$ 0.06	9.82 $\pm$ 0.05	9.78 $\pm$ 0.06	9.73 $\pm$ 0.07	9.70 $\pm$ 0.08	9.71 $\pm$ 0.10	9.73 $\pm$ 0.10

recall that the measured value of g must be compared with the nominal value of gravity, that for our latitude (46 °N) is 9.81 m/s^2.

We must note that in both versions of the experiment the uncertainty of e resulting from the changing area of bounce on the floor was not taken into account. Also not taken into account was the vibrational energy that the rubber ball retains after a bounce,[14] till the next bounce, causing the restitution coefficient e to be different from that measured by us, i.e., in the case where the ball is dropped without vibrational energy.

As regards the number of bounces, one could think these may be increased to infinity, gaining in accuracy on the time t_n^*. But an obstacle arises because the ball goes wandering on the floor,[16] so that the coefficient e may become quite different.

At last, we note that some aspects of this experiment could be studied deeply, such as the dependence of e on the temperature and the energy retained by the rubber ball after every collision.

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Newton's law of motion for variable mass systems applied to capillarity

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The standard Newton's equation of motion for variable masses is set up and solved to describe the time-dependent rise of a liquid in a capillary tube. The findings in the nonviscous case are also supported by quantitative estimates of the potential energy changes suffered by the interfacial films. The theoretical oscillations in the viscous case are confirmed experimentally for tubes having a radius larger than a critical value.

I. INTRODUCTION

During the 60s and 70s several papers have appeared in this Journal dealing with the properties of variable mass systems and the applicability of Newton's laws of motion to the same. The status of this subject until 1969 has been well summarized in the paper by Tiersten[1] and many interesting examples[1–4] such as falling of ropes, descent of drops, motion of rockets, etc. employing the variable mass technique (VMT) have also been given. In the present paper we describe the application of VMT to a highly illuminating and important phenomenon, viz., the rise of a liquid in a capillary tube. To be more specific, we set up the appropriate Newtonian equation of motion, find out the maximum height up to which the liquid rises in the first jump, and finally deduce the kind of motion it executes with time.

For the sake of completeness the analysis is done both for viscous and nonviscous liquids. Some interesting findings of the formulation done in Sec. II below are that the maximum height attained by the column in the first jump is $2h + a$ if $\eta/\sin\alpha = 0$, and the time period of small oscillations (Sec. III) about h is approximately $2\pi\sqrt{[(h+a)/g]}$ if $\eta/\sin\alpha \neq 0$, where h is the standard equilibrium height,[5] α the contact angle, η the coefficient of viscosity and a the immersion depth. An alternative physical explanation for the maximum height attained in the nonviscous case is offered in Sec. IV in terms of the potential energy changes suffered by the solid-liquid, liquid-vapor, and solid-vapor