Linear Algebra Notes Gilbert Strang

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April 2024

Chapter-6 Eigenvalues and Eigenvectors

- An eigenvector x lies along the same line as Ax: $Ax = \lambda x$. The eigenvalues is λ
- If $Ax = \lambda x$ then $A^2 x = \lambda^2 x$ and $A^{-1}x = \lambda^{-1}x$ and $(A + cI)x = (\lambda + c)x$:the same x
- If $Ax = \lambda x$ then $(A - \lambda I)$ is singular and $det(A = \lambda I) = 0$. :n eignevalues
- Check $\lambda' s$ by $det(A) = (\lambda_1)(\lambda_2)...(\lambda_n)$ and diagonal sum or trace $a_{11} + a_{22}...a_{nn}$ = sum of $\lambda' s$
- Projections have $\lambda =$ 1 and 0. Reflections have 1 and -1. Rotations have $e^{i\theta}$ and $e^{-i\theta}$: complex!

Diagonalizing matrix

- The columns of $AX = X\Lambda$ are $Ax_k = \lambda_k x_k$. The eigenvalue matrix Λ is diagonal
- n independent eigenvectors in X diagonalize A
  $A = X\Lambda X^{-1}$ *and* $\Lambda = X^{-1}AX$
- The eigenvector matrix X also diagonalizes all powers of A^k:
  $A^k = X\Lambda^k X^{-1}$
- Solve $u_{k+1} = Au_k$ *by* $u_k = A^k u_0 = X\Lambda^k X^{-1} u_0 = c_1(\lambda_1)^k x_1 + ... + c_n(\lambda_n)^k x_n$
- No equal eigenvalues $\Longrightarrow$ X is invertible and diagonalizable
  Equal eigenvalues $\Longrightarrow$ A might have too few independent eignevectors. Then X^{-1} fails
- Every matrix $C = B^{-1}AB$ has the same eigenvalues of A. Then C is similar to A

Symmetric Matrices

- A symmetric matrix S has n real eigenvalues λ_i and n orthonormal eignevectors $q_1 ... q_n$
- Every real symmetric matrix S can be diagonalized: $S = Q \Lambda Q^{-1} = Q \Lambda Q^T$
- The number of positive eigenvalues of S equals the number of positive pivots
- Antisymmetric matrices $A = -A^T$ have imaginary $\lambda' s$ and orthonormal $q' s$

Positive Definite Matrices

- Symmetric S: all eigenvalues $> 0 \iff$ all pivots $> 0 \iff$ all upper left determinants > 0
- The matrix S is then positive definite. Then energy test $x^T Sx > 0$ for all vectors $x \neq 0$
- One more test for positive definitenes is $S = A^T A$ with independent columns of A
- Positive Semi definiteness S allows $\lambda = 0$ pivot=0 and determinant=0 and energy $x^T Sx = 0$
- The equations $x^T Sx = 1$ gives an ellipse in R^n when S is symmetric positive definite.

- The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is* $ad - bc$
- Row exchange reverse signs PA = bc-ad = - det(A)
- Elimination EA $= \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix}$ = product of pivots = det A
- det(AB) = det(A) det(B) and det(A^T) = det(A)
- Minor is the determinant of sub matrix produced by closing ith row and jth column M_{ij}
- Cofactor of $a_{ij} = (-1)^{i+j} M_{ij}$ and det A $= a_{i1}C_{i1} + ... + a_{in}C_{in}$

Inverse

- $A^{-1} = C^T/detA$ $(A^{-1})_{ij} = cofactor\ C_{ji}\ divided\ by\ detA$
- Cramer's rule computes $x = A^{-1}b$ from $x_j =$ det(A with column j changed to b) / det A
- Area of parallelogram $= |ad - bc|$ if four corners are (0,0),(a,b), (c,d) and (a+b,c+d)
- The cross product $w = u * v$ is $det \begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$

- Orthogonal vectors have $v^T w = 0$. Then $||v||^2 + ||w||^2 = ||v+w||^2 = ||v-w||^2$
- Subspaces V and W are orthogonal when $v^T w = 0$ for every v in V and every w in W
- The row space of A is orthogonal to nullspace. The column space is orthogonal to $N(A^T)$
- One pair of dimensions add to r + (n-r) = n. The other pair r + (m-r) = m
- Row space and nullspace are orthogonal complements. Every x in R^n splits into $x_{row} + x_{null}$
- Suppose a space S has dimension d. Then every basis for S consists of d vectors
- If d vectors in S are independent, they span S. If d vectors span S, they are independent

Projections

- The projection of a vector b onto the line through a is the closest point p = $a(a^T b/a^T a)$
- The error e = b - p is perpendicular to a. Right triangle bpe has $||p||^2 + ||e||^2 = ||b||^2$
- The projection of b onto a subspace S is the closest vector p in S; b-p orthogonal to S
- $A^T A$ is invertible (and symmetric) only if A has independent columns $N(A^T A) = N(A)$
- Then the projection of b onto the column space of A is the vector p = $A(A^T A)^{-1} A^T b$
- The projection matrix onto C(A) is P = $A(A^T A)^{-1} A^T$. It has p = Pb and $P^2 = P = P^T$

Least Squares Approximations

- Solve $A^T A\hat{x} = A^T b$ gives the projection p $= A\hat{x}$ of b onto the column space of A
- When Ax $=$ b has no solution, $\hat{x}$ is the "least squares solution": $||b - A\hat{x}||^2 = minimum$
- Setting partial derivatives of E $= ||Ax - b||^2$ to zero $\frac{\partial E}{\partial x_i} = 0$ also produces $A^T A\hat{x} = A^T b$
- To fit points $(t_1, b_1), ..., (t_m, b_m)$ by a straight line, A has columns (1,..1) and $(t_1, ..., t_m)$
- In that case $A^T A$ is the 2 by 2 matrix $\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}$ and $A^T b$ is the vector $\begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$

Orthonormal Bases and Gram-Scmidt

- The columns $q_1, ..., q_n$ are orthonormal if
  $q_i^T q_j = \begin{cases} 0 & \textit{for } i \neq j \\ 1 & \textit{for } i = j \end{cases}$
- If Q is also square, then $QQ^T = I$ and $Q^T = Q^{-1}$. Q is an "orthonormal matrix"
- The least squares solution to Qx = b is $\hat{x} = Q^T b$. Projection of b: p = $QQ^T b = Pb$
- The Gram-Schmidt process takes independent a_i to orthonormal q_i. Start with $q_1 = a_1/||a_1||$
- $q_i = (a_i - \textit{ projection } p_i)/||a_i - p_i||$; projection $p_i = (a_i^T q_i)q_1 + ... + (a_i^T q_{i-1})q_{i-1}$
- Each a_i will be a combination of q_1 *to* q_i. Then $A = QR$; orthogonal Q and traiangular R

Chapter-3 Vector spaces and subspaces

- The standard n-dimensional space R^n contains all real columns vectors with n components
- If v and w are in a vector space S, every combination $cv + dw$ must be in S
- The "vectors" in S can be matrices or functions of x. The 1-point space Z consists of x=0
- A subspace of R^n is a vector space inside R^n

Ex

The line y=3x inside R^2

- The column space of A contains all combinations of the columns of A: a subspace of R^m
- The column space contains all the vectors Ax. So Ax= b is solvable when b is in C(A)

Nullspace

- The nullspace N(A) in R^n contains all solutions x to Ax=0. This includes x=0
- Elimination (from A to U to R) does not change the nullspace; N(A) = N(U) = N(R)
- The reduced row echelon form R = rref(A) has all pivots=1, with zeros above and below
- If column j of R is free (no pivot), there is a "special solution" to Ax=0 with $x_j = 1$
- Number of pivots = number of nonzero rows in R = rank r. There are n-r free columns
- Every matrix with $m < n$ has nonzero solutions to Ax=0 in its nullspace

Solution to Ax=b

- Complete solution to Ax=b: x = (one particular solution x_p) + (any x_n in the nullspace)
- Ax=b and Rx=d are solvable only when all zero rows of R have zeros in d
- When Rx=d is solvable, only very particular solution to x_p has all free variables equal to zero
- A has full column rank r=n when its nullspace N(A) = zero vector : no free variables
- A has full row rank r= m when its column space C(A) is R^m : Ax=b is always solvable

$r = m = n$	Square, Invertible	1	Perfect
$r = m < n$	Short and Wide	∞	More variable
$r = n < m$	Tall and Thin	0 or 1	More equatior
$r < min(m, n)$	Not full rank	0 or ∞	2 same eq b1=b2 or

Table: Caption

Independence

- Independent columns of A: The only solution to Ax=0 is x=0. The nullspace is Z
- Independent vectors: The only zero combination $c_1 v_1 + ... + c_k v_k$ has all c's =0
- A matrix with $m < n$ has dependent columns; At least n-m free variable /special columns
- The vectors $v_1, ..., v_k$ span the space S if S = all combinations of the $v's$
- The vectors $v_1, .., v_k$ are basis for S if they are independent and they span S
- The dimension of a space S is the number of vectors in every basis for S
- If A is 4 by 4 and invertible, its columns are basis for R^4. The dimension of R^4 is 4

Dimensions of Four Subspaces

- The column space C(A) and the row space $C(A^T)$ both have dimension r (rank of A)
- The nullspace N(A) has dimension of n-r. The left nullspace $N(A^T)$ has dimension m-r, left mean $A^T y = 0$
- Elimination produces bases for the row space and nullspace of A. They are same as for R
- Elimination often changes the column space and left nullspace (but dimensions don't change)
- Rank one matrices A $= uv^T =$ column times row ; C(A) has basis u and $C(A^T)$ has basis v

- The column picture of Ax=b : a combination of n columns of A produces the vector b
- This is a vector equation Ax $= x_1a_1 + ... + x_na_n = b$: the columns of A are $a_1, .., a_n$
- (AB)C = A(BC) , $A^{-1}A = AA^{-1} = I$, A must have n (non zero pivots)
- Ax=0 $\rightarrow$ x = 0 is the only solution then A is invertible, $(AB)^{-1} = B^{-1}A^{-1}$, Gauss Jordan [A I] to [I A^{-1}]
- A = LU factorization (lower triangular)(upper triangular)
- $(AB)^T = B^TA^T$, the dot product $x.y = x^Ty$, the outer product xy^T
- The idea behind A^T is that $Ax.y = (Ax)^Ty = x^TA^Ty = x^T(A^Ty) = x.(A^Ty)$
- A symmetric matrix has $S^T = S$ (and the product A^TA is always symmetric)
- A orthogonal matrix has $Q^T = Q^{-1}$. The columns of Q are orthonormal unit vectors

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Remark

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Important theorem

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Examples

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