1) Big omega notation: Prove that $g(n) = n^3 + n^2 + 4n$ is $\Omega(n^3)$

Sol:

$g(n) \geq cn^3$

$g(n) = n^3 + 2n^2 + 4n$

for finding constants c and n_0

$n^3 + 2n^2 + 4n \geq c \cdot n^3$

Divide both sides with n^3

$1 + \frac{2n^2}{n^3} + \frac{4n}{n^3} \geq c$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq c$

Here $\frac{2}{n}$ and $\frac{4}{n^2}$ approaches 0

$1 + \frac{2}{n} + \frac{4}{n^2} \approx 1$

Example $c = \frac{1}{2}$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2}$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq 1$ $\quad (1 \geq \frac{1}{2}, n \geq 1)$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2}$ $\quad (n \geq 1, n_0 \geq 1)$

Thus $g(n) = n^3 + 2n^2 + 4n$ is indeed $\Omega(n^3)$

2) Big Theta Notation: Determine whether $h(n) = 4n^2+3n$ is $\theta(n^2)$ or not.

$c_1 n^2 \le h(n) \le c_2 n^2$

In upper bound $h(n)$ is $O(n^2)$

In lower bound $h(n)$ is $\Omega(n^2)$

upper Bound $(O(n^2))$

$h(n) = 4n^2+3n$

$h(n) \le c_2 n^2$

$4n^2+3n \le c_2 n^2 \Rightarrow 4n^2+3n \le cn^2$

Let's $c_2 \ge 5$

Divide both sides by n^2

$4 + \frac{3}{n} \le 5$

$h(n) = 4n^2+3n$ is $O(n^2)$ $(c_2 \ge 5, n_0 \ge 1)$

Lower bound:-

$h(n) = 4n^2+3n$

$h(n) \ge c_1 n^2$

$4n^2+3n \ge c_1 n^2$

Let's $c_1 \ge 4 \Rightarrow 4n^2 + 3n \ge 4n^2$

Divide both sides by n^2

$4 + \frac{3}{n} \ge 4$

$h(n) = 4n^2+3n$ $(c_1 \ge 4, n_0 \ge 1)$

$h(n) = 4n^2+3n$ is $\theta(n^2)$

3) Let $f(n) = n^3 - 2n^2 + n$ and $g(n) - n^2$ show whether is $f(n) = \Omega(g(n))$ is true or false and Justify your answer.

$f(n) \geq c \cdot g(n)$

Substituting $f(n)$ and $g(n)$ into this inequality we get.

$n^3 - 2n^2 + n \geq c \cdot (-n^2)$

find c and n_0 holds $n \geq n_0$

$n^3 - 2n^2 + n \geq -cn^2$

$n^3 - 2n^2 + n + cn^2 \geq 0$

$n^3 + (c-2)n^2 + n \geq 0$

$n^3 + (c-2)n^2 + n \geq 0 \quad (n^3 \geq 0)$

$n^3 + (1-2)n^2 = n^3 - n^2 + n \geq 0 \quad (c = 2)$

$f(n) = n^3 - 2n^2 + n$ is $\Omega(g(n)) = \Omega(-n^2$

Therefore the statement $f(n) = \Omega(g(n))$ is True

4) Determine whether $h(n) = n \log n + n$ is $\theta(n \log n)$ Prove a rigorous proof for your conclusion.

$c_1 n \log \leq h(n) \leq c_2 n \log n$

upper Bound:

$h(n) \leq c_2 n \log n$

$h(n) = n \log n + n$

$n \log n + n \leq c_2 n \log n$

Divide both sides by $n \log n$

$1 + \frac{n}{n \log n} \leq 2$

$1 + \frac{1}{\log n} \leq c_2 \quad$ (simplify)

$1 + \frac{1}{\log n} \leq 2 \quad (c_2 \geq 2)$

Then $h(n)$ is $O(n \log n)$ $(c_2 \geq 2, n_0 \geq 2$

Lower Bound!

$h(n) \geq c_1 n \log n$

$h(n) = n \log n + n$

$n \log n + n \geq c_1 n \log n$

Divide both sides by $n \log n$

$1 + \frac{n}{n \log n} \geq c_1 n \log n$

$1 + \frac{n}{\log n} \geq 1$ (simplify)

$1 + \frac{1}{\log n} \geq 1$ $(c_1 = 1)$

$\frac{1}{\log n} \geq 0$ for all $n > 1$

$h(n)$ is $\Omega(n \log n)$ $(c_1 = 1, n_0 = 1)$

$h(n) = n \log n + n$ is $\theta(n \log n)$

5) a) Solve the following recurrence relations and find the order of growth of solutions

$T(n) = 4T(\frac{n}{2}) + n^2, T(1) = 1$

$T(n) = 4T(n/2) + n^2, T(1) = 1$

$T(n) = aT(n/b) + f(n)$

$a = 4, b = 2$ $f(n) = n^2$

Applying master theorem.

$T(n) = aT(n/b) + f(n)$

$f(n) = O(n^{\log_b a - \epsilon})$ $\left(\begin{matrix} \epsilon > 0 \\ T(n) = \theta(n^{\log_b a}) \end{matrix} \right)$

$f(n) = \theta(n^{\log_b a})$, then $T(n) = \theta(n^{\log_b a} \log n)$

$f(n) = \Omega(n^{\log_b a + \epsilon})$, then $T(n) = \theta(n^{\log_b a} \log n + f(n))$

calculating $\log_b a$!

$\log_b a = \log_2 4 = 2$

$f(n) = n^2 = \theta(n^2)$ {comparing $f(n)$ with $n^{\log_b a}$)

$f(n) = \theta(n^2) = \theta(n^{\log_b a})$, (case 2)

$T(n) = 4T(n/2) + n^2$

$T(n) = \theta(n^{\log_b a} \log n) = \theta(n^2 \log n)$

order of growth

$T(n) = 4T(\frac{n}{2}) + n^2$ with $T(1) = 1$

is $\theta(n^2 \log n$.