Challenge Problem

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***Abstract*—This document explains the concept of finding the determinant of a vandermonde matrix.**

Download latex-tikz codes from

https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Challenges/challenge_6

1 Problem

Derive an expression for the determinant of a vandermonde matrix.

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix} \tag{1.0.1}$$

for real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$.

2 Explanation

For simplification let us consider a 2×2 matrix and find the determinant for that.

$$\det \begin{pmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{pmatrix} = \alpha_2 - \alpha_1 = \prod_{1 \le i < j \le 2} (\alpha_j - \alpha_i) \tag{2.0.1}$$

From (2.0.1), let us assume that the result is true for $n \ge 2$ (inductive step), that is

$$\det \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i) \tag{2.0.2}$$

Let us consider a $(n+1) \times (n+1)$ matrix,

$$A = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_{n+1} & \alpha_{n+1}^2 & \cdots & \alpha_{n+1}^{n} \end{pmatrix} \tag{2.0.3}$$

$$\xleftrightarrow[R_{n-1} \leftarrow R_{n-1} - R_1]{R_n \leftarrow R_n - R_1} \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n} \\ 0 & \alpha_2 - \alpha_1 & \alpha_2^2 - \alpha_1^2 & \cdots & \alpha_2^{n} - \alpha_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{n+1} - \alpha_1 & \alpha_{n+1}^2 - \alpha_1^2 & \cdots & \alpha_{n+1}^{n} - \alpha_1^n \end{pmatrix} \tag{2.0.4}$$

$$\xleftrightarrow[C_{n-1} \leftarrow C_{n-1} - \alpha_1 C_{n-2}]{C_n \leftarrow C_n - \alpha_1 C_{n-1}} \tag{2.0.5}$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 - \alpha_1 & (\alpha_2 - \alpha_1)\alpha_2 & \cdots & (\alpha_2 - \alpha_1)\alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha_{n+1} - \alpha_1 & (\alpha_{n+1} - \alpha_1)\alpha_{n+1} & \cdots & (\alpha_{n+1} - \alpha_1)\alpha_{n+1}^{n-1} \end{pmatrix} \tag{2.0.6}$$

$$= \prod_{1 < j \le n+1} (\alpha_j - \alpha_1) \det \begin{pmatrix} 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_{n+1} & \alpha_{n+1}^2 & \cdots & \alpha_{n+1}^{n-1} \end{pmatrix} \tag{2.0.7}$$

Sub equation (2.0.2) in (2.0.7) by the inductive hypothesis,

$$\det A = \prod_{1 < j \le n+1} (\alpha_j - \alpha_1) \prod_{2 \le i < j \le n+1} (\alpha_j - \alpha_i) \tag{2.0.8}$$

$$\implies \det A = \prod_{1 \le i < j \le n+1} (\alpha_j - \alpha_i) \tag{2.0.9}$$

3 Solution

$$\det \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)$$