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Assignment 8

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Abstract—This document explains the concept of affine transformation.

Download all python codes from

https://github.com/harshachinta/EE5609-Matrix-Theory/tree/master/Assignments/Assignment8/ code

and latex-tikz codes from

https://github.com/harshachinta/EE5609-Matrix-Theory/tree/master/Assignments/Assignment8

1 Problem

To what point must the origin be moved in order that the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -4 \end{pmatrix} \mathbf{x} = 0 \tag{1.0.1}$$

may become

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.2}$$

and through what angle must the axes be turned in order to obtain

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1 \tag{1.0.3}$$

2 Explanation

The general second order equation can be expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing (1.0.1) with (2.0.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$f = 0 \tag{2.0.4}$$

Let the point to which the origin is moved be c The above equation (2.0.1) can be modified as

$$(\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{x} + \mathbf{c}) = 0$$
 (2.0.5)

From equation (2.0.5) consider,

$$\implies (\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) \tag{2.0.6}$$

$$\implies \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{x}^T \mathbf{V} \mathbf{c} + \mathbf{c}^T \mathbf{V} \mathbf{c}$$
 (2.0.7)

$$\implies \mathbf{x}^T \mathbf{V} \mathbf{x} + 2 \mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} \tag{2.0.8}$$

Substituting (2.0.8) in equation (2.0.5)

$$\implies \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T (\mathbf{x} + \mathbf{c}) = 0$$
(2.0.9)

Comparing equations (2.0.9) and (1.0.2), we can write as,

$$\implies 2\mathbf{c}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} = 0 \tag{2.0.10}$$

$$\implies \mathbf{c}^T \mathbf{V} = -\mathbf{u}^T \tag{2.0.11}$$

$$\implies \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} \qquad (2.0.12)$$

$$\implies \left| \mathbf{c} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right| \tag{2.0.13}$$

From (2.0.13), when the origin is moved to point \mathbf{c} , the equation (1.0.1) becomes (1.0.2).

From equations (1.0.1) and (2.0.13), **V** doesn't change

$$\det(\mathbf{V}) = -6 \tag{2.0.14}$$

Since det(V) < 0 the given equation represents the hyperbola

From equation (1.0.2), the equation is of the form,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + f = 0 \tag{2.0.15}$$

The matrix V can be decomposed as,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.16}$$

where λ_1 and λ_2 are Eigen values of **V**, and **P** contains the Eigen vectors corresponding to the Eigen values λ_1 and λ_2 . The affine transformation is given by,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.17}$$

where, \mathbf{P} indicates the rotation of axes and \mathbf{c} indicates the shift of origin.

Eigen values of V are,

$$\left| \lambda \mathbf{I} - \mathbf{v} \right| = 0 \tag{2.0.18}$$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda + 2 \end{vmatrix} = 0 \tag{2.0.19}$$

$$\implies \lambda^2 + \lambda - 6 = 0 \tag{2.0.20}$$

$$\implies \lambda_1 = -3, \lambda_2 = 2 \tag{2.0.21}$$

$$\mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \tag{2.0.22}$$

Eigen vector for λ_1 =-3,

$$\lambda_1 \mathbf{I} - \mathbf{v} = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{R_1}{2}} \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix} \quad (2.0.23)$$

$$\implies \mathbf{P_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix} \quad (2.0.24)$$

Eigen vector for $\lambda_2=2$,

$$\lambda_1 \mathbf{I} - \mathbf{v} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \stackrel{R_2 \leftarrow R_2 + 2R_1}{\longleftrightarrow} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \quad (2.0.25)$$

$$\implies \mathbf{P_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (2.0.26)$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$
 (2.0.27)

Therefore V can be written as,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$
(2.0.28)

Equation (1.0.2) can be written as,

$$\mathbf{x}^{T} \begin{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} = 1 \quad (2.0.29)$$

$$\begin{bmatrix} \left(\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\right) \mathbf{x} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \mathbf{x} \end{bmatrix}^{T} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \begin{bmatrix} \left(\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\right) \mathbf{x} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \mathbf{x} \end{bmatrix} = 1 \quad (2.0.30)$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{2.0.31}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{y}$$
 (2.0.32)

$$\mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^{-1}\mathbf{P}\mathbf{y} \tag{2.0.33}$$

$$\implies \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \tag{2.0.34}$$

But,
$$\mathbf{P}^{-1} = \mathbf{P}^T$$
 (2.0.35)

$$\implies \mathbf{y} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x}$$
 (2.0.36)

Using (2.0.36) in (2.0.30), the equation can be rewritten as

$$\mathbf{y}^T \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = 1 \tag{2.0.37}$$

Equation (2.0.37) is same as (1.0.3) with p=-3 and q=2.

From equation (2.0.27), the orthogonal matrix represents the rotation matrix in form of,

$$\mathbf{P} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \tag{2.0.38}$$

Comparing (2.0.27) and (2.0.38),

$$\cos \theta = \frac{1}{\sqrt{5}} \tag{2.0.39}$$

$$\implies \theta = 63.43^{\circ} \tag{2.0.40}$$

From equation (2.0.38),if the axes is turned by θ then the equation obtained would be (1.0.3).

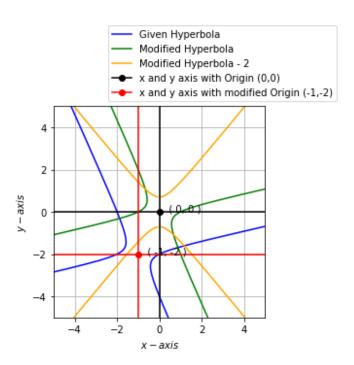


Fig. 1: Hyperbola plot when origin is shifted and rotated