Assignment 8

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Abstract—**This document explains the concept of affine transformation.**

Download all python codes from

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https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment8/code
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and latex-tikz codes from

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https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment8
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1 Problem

To what point must the origin be moved in order that the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -4 \end{pmatrix} \mathbf{x} = 0 \tag{1.0.1}$$

may become

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.2}$$

and through what angle must the axes be turned in order to obtain

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1 \tag{1.0.3}$$

2 Explanation

The general second order equation can be expressed as follows,

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.1}$$

Comparing (1.0.1) with (**??**),

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$f = 0 \tag{2.0.4}$$

Let the point to which the origin is moved be $\mathbf{c}$
The above equation (**??**) can be modified as

$$(\mathbf{x} + \mathbf{c})^T \mathbf{V}(\mathbf{x} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{x} + \mathbf{c}) = 0 \tag{2.0.5}$$

From equation (2.0.5) consider,

$$\implies (\mathbf{x} + \mathbf{c})^T \mathbf{V}(\mathbf{x} + \mathbf{c}) \tag{2.0.6}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{x}^T\mathbf{V}\mathbf{c} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{2.0.7}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{2.0.8}$$

Substituting (2.0.8) in equation (2.0.5)

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T(\mathbf{x} + \mathbf{c}) = 0 \tag{2.0.9}$$

Comparing equations (2.0.9) and (1.0.2), we can write as,

$$\implies 2\mathbf{c}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} = 0 \tag{2.0.10}$$

$$\implies \mathbf{c}^T\mathbf{V} = -\mathbf{u}^T \tag{2.0.11}$$

$$\implies \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{2.0.12}$$

$$\implies \boxed{\mathbf{c} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}} \tag{2.0.13}$$

From (2.0.13), when the origin is moved to point $\mathbf{c}$, the equation (1.0.1) becomes (1.0.2).

From equations (1.0.1) and (2.0.13), $\mathbf{V}$ doesn't change

$$\det(\mathbf{V}) = -6 \tag{2.0.14}$$

Since $\det(\mathbf{V}) < 0$ the given equation represents the hyperbola

From equation (1.0.2), the equation is of the form,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + f = 0 \tag{2.0.15}$$

The matrix $\mathbf{V}$ can be decomposed as,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.16}$$

where λ_1 and λ_2 are Eigen values of $\mathbf{V}$, and $\mathbf{P}$ contains the Eigen vectors corresponding to the Eigen values λ_1 and λ_2. The affine transformation is given by,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.17}$$

where, $\mathbf{P}$ indicates the rotation of axes and $\mathbf{c}$ indicates the shift of origin.
Eigen values of $\mathbf{V}$ are,

$$\left|\lambda\mathbf{I} - \mathbf{v}\right| = 0 \tag{2.0.18}$$

$$\implies \begin{vmatrix}\lambda - 1 & -2 \\ -2 & \lambda + 2\end{vmatrix} = 0 \tag{2.0.19}$$

$$\implies \lambda^2 + \lambda - 6 = 0 \tag{2.0.20}$$

$$\implies \lambda_1 = -3, \lambda_2 = 2 \tag{2.0.21}$$

$$\mathbf{D} = \begin{pmatrix}-3 & 0 \\ 0 & 2\end{pmatrix} \tag{2.0.22}$$

Eigen vector for λ_1=-3,

$$\lambda_1\mathbf{I} - \mathbf{v} = \begin{pmatrix}-4 & -2 \\ -2 & -1\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{R_1}{2}} \begin{pmatrix}-2 & -1 \\ 0 & 0\end{pmatrix} \tag{2.0.23}$$

$$\implies \mathbf{P_1} = \begin{pmatrix}1 \\ -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}}\end{pmatrix} \tag{2.0.24}$$

Eigen vector for λ_2=2,

$$\lambda_1\mathbf{I} - \mathbf{v} = \begin{pmatrix}1 & -2 \\ -2 & 4\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix}1 & -2 \\ 0 & 0\end{pmatrix} \tag{2.0.25}$$

$$\implies \mathbf{P_2} = \begin{pmatrix}2 \\ 1\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}}\end{pmatrix} \tag{2.0.26}$$

$$\mathbf{P} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{2.0.27}$$

Therefore $\mathbf{V}$ can be written as,

$$\mathbf{V} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}-3 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{2.0.28}$$

Equation (1.0.2) can be written as,

$$\mathbf{x}^T \left[\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}-3 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\right]\mathbf{x} = 1 \tag{2.0.29}$$

$$\left[\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x}\right]^T \begin{pmatrix}-3 & 0 \\ 0 & 2\end{pmatrix}\left[\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x}\right] = 1 \tag{2.0.30}$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{2.0.31}$$

$$\implies \mathbf{x} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{y} \tag{2.0.32}$$

$$\mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^{-1}\mathbf{P}\mathbf{y} \tag{2.0.33}$$

$$\implies \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \tag{2.0.34}$$

$$\text{But, } \mathbf{P}^{-1} = \mathbf{P}^T \tag{2.0.35}$$

$$\implies \mathbf{y} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x} \tag{2.0.36}$$

Using (2.0.36) in (2.0.30), the equation can be rewritten as

$$\mathbf{y}^T \begin{pmatrix}-3 & 0 \\ 0 & 2\end{pmatrix}\mathbf{y} = 1 \tag{2.0.37}$$

Equation (2.0.37) is same as (1.0.3) with p=-3 and q=2.
From equation (2.0.27), the orthogonal matrix represents the rotation matrix in form of,

$$\mathbf{P} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} \tag{2.0.38}$$

Comparing (2.0.27) and (2.0.38),

$$\cos\theta = \frac{1}{\sqrt{5}} \tag{2.0.39}$$

$$\implies \boxed{\theta = 63.43^\circ} \tag{2.0.40}$$

From equation (2.0.38),if the axes is turned by θ then the equation obtained would be (1.0.3).

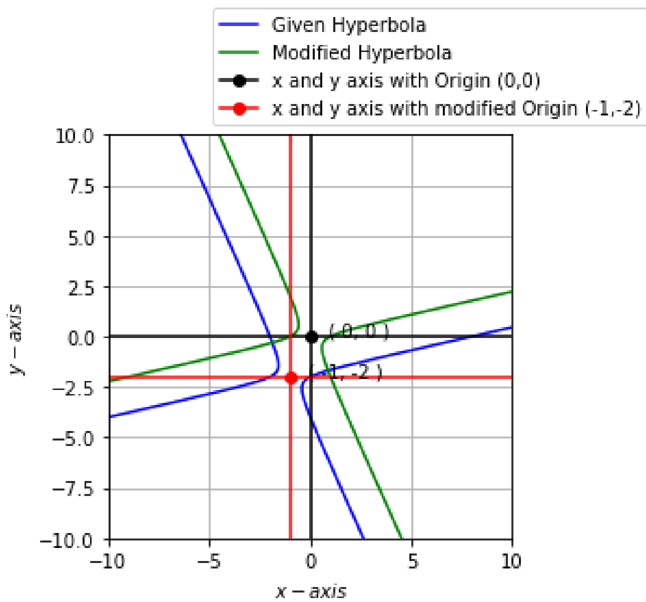

Fig. 1: Hyperbola plot when origin is shifted