Assignment 15

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Abstract—**This document explains the conditions to check for a vector space.**

Download all python codes from

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https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment15/code
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and latex-tikz codes from

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https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment15
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1 Problem

Let $\mathbf{V}$ be an n-dimensional vector space over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear transformation from $\mathbf{V}$ into $\mathbf{V}$ such that the range and null space of $\mathbf{T}$ are identical. Prove that n is even. (Can you give an example of such a linear transformation $\mathbf{T}$)?

2 Explanation

Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear transformation from $\mathbf{V}$ into $\mathbf{W}$. Then,

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \dim \mathbf{V} \tag{2.0.1}$$

It is given that range and null space of $\mathbf{T}$ are same, let us assume it to be m. Substituting in equation (2.0.1)

$$m + m = n \tag{2.0.2}$$

$$\implies n = 2m \tag{2.0.3}$$

From equation (2.0.3), we can say that n is even.

Example: Let us consider a vector space $\mathbf{V}$, such that $\mathbf{V} \in \mathbb{R}^2$ and let us consider a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ defined by $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$ and is given by matrix $\mathbf{M}$

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.5}$$

Let us consider basis of $\mathbb{R}^2$ $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and apply linear transformation on it.

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.7}$$

From (2.0.5),
The range of matrix can be found from row reduced echelon form. But as matrix $\mathbf{M}$ is in RREF form, the basis for range is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
The null space of matrix is,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.8}$$

$$\implies x_1 = t \quad x_2 = 0 \tag{2.0.9}$$

$$\implies \mathbf{X} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.10}$$

The basis for null space is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\text{rank}(\mathbf{T}) = 1 \quad \text{nullity}(\mathbf{T}) = 1 \tag{2.0.11}$$

$$\dim(\mathbf{V}) = 2 \tag{2.0.12}$$

Thus the range and null space are equal, and n is even.

3 Solution

The range and null space of $\mathbf{T}$ are equal, and n is even.