Assignment 15

Sri Harsha CH

***Abstract*—This document explains the conditions to check for a vector space.**

Download all python codes from

```
https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment15/code
```

and latex-tikz codes from

```
https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment15
```

1 Problem

Let $\mathbf{V}$ be an n-dimensional vector space over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear transformation from $\mathbf{V}$ into $\mathbf{V}$ such that the range and null space of $\mathbf{T}$ are identical. Prove that n is even. (Can you give an example of such a linear transformation $\mathbf{T}$)?

2 Explanation

Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear transformation from $\mathbf{V}$ into $\mathbf{W}$. Then,

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \dim \mathbf{V} \tag{2.0.1}$$

It is given that range and null space of $\mathbf{T}$ are same, let us assume it to be m. Substituting in equation (2.0.1)

$$m + m = n \tag{2.0.2}$$

$$\implies n = 2m \tag{2.0.3}$$

From equation (2.0.3), we can say that n is even.

Example: Let us consider a vector space $\mathbf{V}$, such that $\mathbf{V} \in \mathbb{R}^2$ and let us consider a linear transformation $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ defined by $\mathbf{T}(x, y) = (y, 0)$ and is given by matrix $\mathbf{M}$

$$\mathbf{T}(x, y) = (y, 0) \tag{2.0.4}$$

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.5}$$

Let us consider basis of $\mathbb{R}^2$ $\{(1, 0), (0, 1)\}$ and apply linear transformation on it.

$$\mathbf{T}(1, 0) = (0, 0) \tag{2.0.6}$$

$$\mathbf{T}(0, 1) = (1, 0) \tag{2.0.7}$$

From equation (2.0.5),

$$\text{rank}(\mathbf{T}) = 1 \quad \text{nullity}(\mathbf{T}) = 1 \tag{2.0.8}$$

$$\dim(\mathbf{V}) = 2 \tag{2.0.9}$$

From equation (2.0.8) and (2.0.9), the range and null space of $\mathbf{T}$ are equal, and n is even.

3 Solution

From equation (2.0.8) and (2.0.9), the range and null space of $\mathbf{T}$ are equal, and n is even.