Assignment 16

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Abstract—**This document explains the conditions for two vector spaces to be isomorphic.**

Download all python codes from

```
https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment16
    /code
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and latex-tikz codes from

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https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment16
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1 Problem

Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field $\mathbf{F}$. Prove that $\mathbf{V}$ and $\mathbf{W}$ are isomorphic if and only if $\dim \mathbf{V} = \dim \mathbf{W}$

2 Explanation

If $\mathbf{V}$ and $\mathbf{W}$ are vector spaces over the field $\mathbf{F}$, any one to one linear transformation T of $\mathbf{V}$ onto $\mathbf{W}$ is called an isomorphism of $\mathbf{V}$ onto $\mathbf{W}$.

Let T : $\mathbf{V} \to \mathbf{W}$ be an isomorphism and let $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be a basis for $\mathbf{V}$. Since it is isomorphic we know that T is one to one and onto. We need to show that $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$ is a basis for $\mathbf{W}$ so that $\dim \mathbf{V} = \mathrm{n} = \dim \mathbf{W}$.

1. To see that if set of vectors in $\mathbf{W}$ are linearly independent we use the property that T is one to one. Let us consider a linear combination.

$$\alpha_1 T(\mathbf{v_1}) + \alpha_2 T(\mathbf{v_2}) + \cdots + \alpha_n T(\mathbf{v_n}) = \mathbf{0} \quad (2.0.1)$$

$$T(\alpha_1 \mathbf{v_1}) + T(\alpha_2 \mathbf{v_2}) + \cdots + T(\alpha_n \mathbf{v_n}) = \mathbf{0} \quad (2.0.2)$$

$$\implies \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n} = \mathbf{0} \quad (2.0.3)$$

$$\implies \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \quad (2.0.4)$$

From equation (2.0.1) and (2.0.4), the set of vectors $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$ are linearly independent.

2. To check if the set of vectors span $\mathbf{W}$. Since T is onto, for any $\mathbf{y} \in \mathbf{W}$ there exists an $\mathbf{x} \in \mathbf{V}$ such that $T(\mathbf{x}) = \mathbf{y}$.

$$\mathbf{x} = \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n} \quad (2.0.5)$$

$$T(\mathbf{x}) = T(\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n}) = \mathbf{y} \quad (2.0.6)$$

$$\alpha_1 T(\mathbf{v_1}) + \alpha_2 T(\mathbf{v_2}) + \cdots + \alpha_n T(\mathbf{v_n}) = \mathbf{y} \quad (2.0.7)$$

From equation (2.0.7), any vector in $\mathbf{W}$ can be represented as linear combination of $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$.

Therefore from equation (2.0.4) and (2.0.7), $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$ are linearly independent and span $\mathbf{W}$.

$$\implies \dim \mathbf{W} = \mathrm{n} = \dim \mathbf{V}$$

3 Solution

$\mathbf{V}$ and $\mathbf{W}$ are isomorphic if and only if $\dim \mathbf{V} = \dim \mathbf{W}$