Challenging Problem

Sri Harsha CH

***Abstract*—This document explains the proof of Cayley-Hamilton Theorem.**

Download latex-tikz codes from

https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment3

1 Problem

Prove Cayley-Hamilton Theorem.

2 Explanation

Cayley-Hamilton Theorem: Every Square matrix satisfies its own characteristic equation.
Proof:
Let $\mathbf{A} = \left(a_{ij}\right)$ be any square matrix of order n and $\phi(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$ be its characteristic equation in variable λ and $\mathbf{I}$ is the identity matrix of order n.
The characteristic equation of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{2.0.1}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{nn} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{2.0.2}$$

$$\implies a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n = 0 \tag{2.0.3}$$

According to Cayley-Hamilton theorem, we need to prove that:

$$a_0 + a_1\mathbf{A} + a_2\mathbf{A^2} + \cdots + a_n\mathbf{A^n} = 0 \tag{2.0.4}$$

we know that,

$$\mathbf{A}(adj\mathbf{A}) = \det(\mathbf{A})\mathbf{I} \tag{2.0.5}$$

$$\implies (\mathbf{A} - \lambda\mathbf{I})adj(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})\mathbf{I} \tag{2.0.6}$$

From equation (2.0.6) we already know RHS, but we need to determine LHS and compare the coefficients,
From RHS of (2.0.6),we know that, $adj(\mathbf{A} - \lambda\mathbf{I})$ has a polynomial of degree $n - 1$.

$$adj(\mathbf{A} - \lambda\mathbf{I}) = b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_{n-1}\lambda^{n-1} \tag{2.0.7}$$

Using equation (2.0.7),

$$\begin{aligned}(\mathbf{A} - \lambda\mathbf{I})&adj(\mathbf{A} - \lambda\mathbf{I}) \\ &= (\mathbf{A} - \lambda\mathbf{I})(b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_{n-1}\lambda^{n-1})\end{aligned} \tag{2.0.8}$$

$$\begin{aligned}&= \mathbf{A}b_0 + \mathbf{A}b_1\lambda + \cdots + \mathbf{A}b_{n-1}\lambda^{n-1} \\ &\qquad - b_0\lambda - b_1\lambda^2 - \cdots - b_{n-1}\lambda^n \\ &= \mathbf{A}b_0 + \lambda(\mathbf{A}b_1 - b_0) + \lambda^2(\mathbf{A}b_2 - b_1) + \cdots - b_{n-1}\lambda^n\end{aligned} \tag{2.0.9}$$

Substituting equ (2.0.9) and (2.0.3) in (2.0.6),

$$\begin{aligned}\mathbf{A}b_0 + \lambda(\mathbf{A}b_1 - b_0) + \lambda^2(\mathbf{A}b_2 - b_1) + \cdots - b_{n-1}\lambda^n = \\ a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n\end{aligned} \tag{2.0.10}$$

Comparing coefficients of equal powers of λ in equation (2.0.10),

$$\mathbf{A}b_0 = a_0 \tag{2.0.11}$$

$$\mathbf{A}b_1 - b_0 = a_1 \tag{2.0.12}$$

$$\vdots$$

$$\mathbf{A}b_{n-1} - b_{n-2} = a_{n-1} \tag{2.0.13}$$

$$-b_{n-1} = a_n \tag{2.0.14}$$

Multiplying equ (2.0.11) by $\mathbf{I}$, equ (2.0.12) by $\mathbf{A}$, $\cdots$, (2.0.13) by $\mathbf{A^{n-1}}$ and (2.0.14) by $\mathbf{A^n}$, we get,

$$\mathbf{A}b_0 = a_0\mathbf{I} \tag{2.0.15}$$

$$\mathbf{A^2}b_1 - \mathbf{A}b_0 = \mathbf{A}a_1 \tag{2.0.16}$$

$$\vdots$$

$$\mathbf{A^n}b_{n-1} - \mathbf{A^{n-1}}b_{n-2} = a_{n-1}\mathbf{A^{n-1}} \tag{2.0.17}$$

$$- \mathbf{A^n}b_{n-1} = a_n\mathbf{A^n} \tag{2.0.18}$$

Adding the equations, we get:

$$\implies \boxed{a_0 + a_1\mathbf{A} + a_2\mathbf{A^2} + \cdots + a_n\mathbf{A^n} = 0} \tag{2.0.19}$$

3 Solution

Equation (2.0.19) proves Cayley-Hamilton theorem.