

Assignment 16

Sri Harsha CH

Abstract—This document explains the conditions for two vector spaces to be isomorphic.

Download all python codes from

<https://github.com/harshachinta/EE5609–Matrix–Theory/tree/master/Assignments/Assignment16/code>

and latex-tikz codes from

<https://github.com/harshachinta/EE5609–Matrix–Theory/tree/master/Assignments/Assignment16>

1 PROBLEM

Let \mathbf{V} and \mathbf{W} be finite-dimensional vector spaces over the field \mathbf{F} . Prove that \mathbf{V} and \mathbf{W} are isomorphic if and only if $\dim \mathbf{V} = \dim \mathbf{W}$

2 EXPLANATION

Refer Table 0.

3 SOLUTION

Therefore from equation (2.0.4) and (2.0.7) in Table 0, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ are linearly independent and span \mathbf{W} .

$$\implies \dim \mathbf{W} = n = \dim \mathbf{V}$$

\mathbf{V} and \mathbf{W} are isomorphic if and only if $\dim \mathbf{V} = \dim \mathbf{W}$

$T : \mathbf{V} \rightarrow \mathbf{W} \quad \mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \quad \mathbf{W} = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$	
Property Used	Derivation
T is one-one	<p>Linear combination of vectors in \mathbf{W}</p> $\sum_{k=1}^n \alpha_k T(\mathbf{v}_k) = \mathbf{0} \quad (2.0.1)$ $\sum_{k=1}^n T(\alpha_k \mathbf{v}_k) = \mathbf{0} \quad (2.0.2)$ $\implies \sum_{k=1}^n \alpha_k \mathbf{v}_k = \mathbf{0} \quad (2.0.3)$ $\implies \alpha_1 = \alpha_2 = \dots = \alpha_n = 0 \quad (2.0.4)$ <p>From equation (2.0.1) and (2.0.4), the set of vectors $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ are linearly independent</p>
T is onto	<p>For any $\mathbf{y} \in \mathbf{W}$ there exists an $\mathbf{x} \in \mathbf{V}$ such that $T(\mathbf{x}) = \mathbf{y}$.</p> $\mathbf{x} = \sum_{k=1}^n \alpha_k \mathbf{v}_k \quad (2.0.5)$ $T(\mathbf{x}) = T\left(\sum_{k=1}^n \alpha_k \mathbf{v}_k\right) = \mathbf{y} \quad (2.0.6)$ $\sum_{k=1}^n \alpha_k T(\mathbf{v}_k) = \mathbf{y} \quad (2.0.7)$ <p>From equation (2.0.7), any vector in \mathbf{W} can be represented as linear combination of $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$. That is it spans \mathbf{W}.</p>

TABLE 0: Derivation