Assignment 6

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***Abstract*—This document explains the concept of finding the unknown value in an equation such that it is represented as two straight lines .**

Download latex-tikz codes from

https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment6

1 Problem

Find the value of h so that the equation $6x^2+2hxy+12y^2+22x+31y+20=0$ may represent two straight lines.

2 Explanation

Given equation,

$$6x^2+2hxy+12y^2+22x+31y+20=0 \tag{2.0.1}$$

is a second order equation.
The general equation of second degree is given by

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{2.0.2}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0 \tag{2.0.3}$$

$$\mathbf{V}=\mathbf{V}^T=\begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u}=\begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.5}$$

Equation (2.0.3) represents a pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}=0 \tag{2.0.6}$$

Comparing equation (2.0.1) with (2.0.2), we can write in the form of (2.0.4) and (2.0.5) as,

$$\mathbf{V}=\begin{pmatrix} 6 & h \\ h & 12 \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{u}=\begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{2.0.8}$$

$$f=20 \tag{2.0.9}$$

As given in question, the equation represent two straight lines, substitute (2.0.7), (2.0.8), (2.0.9) in (2.0.6) to satisfy the equation.

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix}=0 \tag{2.0.10}$$

Expanding equation (2.0.10) along row 1 gives

$$\implies 6\times(240-\frac{961}{4})-h\times(20h-\frac{341}{2})+$$
$$11\times(\frac{31h}{2}-132)=0$$

$$\implies 20h^2-341h+\frac{2907}{2}=0 \tag{2.0.11}$$

$$\implies \boxed{h=\frac{17}{2}} \tag{2.0.12}$$

$$\implies \boxed{h=\frac{171}{20}} \tag{2.0.13}$$

3 Solution

If $h=\frac{17}{2}$ or $h=\frac{171}{20}$, the equation given will represent two straight lines.

Sub $h=\frac{17}{2}$ in equation (2.0.1) we get,

$$6x^2+17xy+12y^2+22x+31y+20=0 \tag{3.0.1}$$

Equation (3.0.1) can be expressed as,

$$\mathbf{V}=\mathbf{V}^T=\begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{u}=\begin{pmatrix} 11 \\ \frac{31}{2} \end{pmatrix} \tag{3.0.3}$$

$$\mathbf{f}=20 \tag{3.0.4}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x}-c1)(\mathbf{n_2}^T\mathbf{x}-c2)=0 \tag{3.0.5}$$

The slopes of the lines are given by the roots of the polynomial:

$$cm^2 + 2bm + a = 0 \tag{3.0.6}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{3.0.7}$$

$$\tag{3.0.8}$$

Substituting (3.0.1) in the equation (3.0.6),

$$12m^2 + 17m + 6 = 0 \tag{3.0.9}$$

$$m_i = \frac{-\frac{17}{2} \pm \sqrt{\frac{1}{4}}}{12} \tag{3.0.10}$$

$$\implies m_1 = \frac{-2}{3}, m_2 = \frac{-3}{4} \tag{3.0.11}$$

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{3.0.12}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{3.0.13}$$

we know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{3.0.14}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$ From equation (3.0.13)

$$\mathbf{n_1} = \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{3.0.15}$$

$$\implies \begin{pmatrix} -2 & 0 \\ -3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{3.0.16}$$

$\implies$ Equation (3.0.13) satisfies (3.0.14)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{3.0.17}$$

Substituting (3.0.13) in (3.0.17), the augmented matrix is,

$$\begin{pmatrix} -2 & -3 & -22 \\ -3 & -4 & -31 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1-3R_2}{2}]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{pmatrix} \tag{3.0.18}$$

$$\implies c_1 = 4, c_2 = 5 \tag{3.0.19}$$

Substituting (3.0.13) and (3.0.19) in (3.0.5) we get,

$$\implies (-2x - 3y - 4)(3x - 4y - 5) = 0$$

$$\implies \boxed{(2x + 3y + 4)(3x + 4y + 5) = 0} \tag{3.0.20}$$

Equation (3.0.20) represents equations of two straight lines.

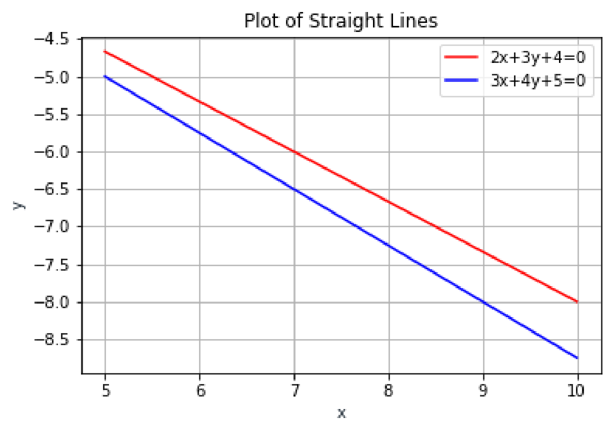

Fig. 1: Plot of Straight lines when $h = \frac{17}{2}$

Similarly, Sub $h = \frac{171}{20}$ in equation (2.0.1) we get,

$$20x^2 + 57xy + 40y^2 + \frac{220}{3}x + \frac{310}{3}y + \frac{200}{3} = 0 \tag{3.0.21}$$

Equation (3.0.21) can be expressed as,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 20 & \frac{57}{2} \\ \frac{57}{2} & 40 \end{pmatrix} \tag{3.0.22}$$

$$\mathbf{u} = \begin{pmatrix} \frac{220}{6} \\ \frac{310}{6} \end{pmatrix} \tag{3.0.23}$$

$$\mathbf{f} = \frac{200}{3} \tag{3.0.24}$$

The pair of straight lines are given by,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = 0 \tag{3.0.25}$$

Substituting (3.0.21) in the equation (3.0.6),

$$40m^2 + 57m + 20 = 0 \tag{3.0.26}$$

$$m_i = \frac{-\frac{57}{2} \pm \sqrt{\frac{49}{4}}}{40} \tag{3.0.27}$$

$$\implies m_1 = \frac{-5}{8}, m_2 = \frac{-4}{5} \tag{3.0.28}$$

$$\mathbf{m_1} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \tag{3.0.29}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{3.0.30}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a toeplitz matrix and multiplying with $\mathbf{n_2}$ From equation (3.0.30)

$$\mathbf{n_1} = \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -4 \\ -5 \end{pmatrix} \tag{3.0.31}$$

$$\implies \begin{pmatrix} -5 & 0 \\ -8 & -5 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 57 \\ 40 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{3.0.32}$$

$\implies$ Equation (3.0.30) satisfies (3.0.14)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{3.0.33}$$

Substituting (3.0.30) in (3.0.33), the augmented matrix is,

$$\begin{pmatrix} -5 & -4 & -\frac{220}{3} \\ -8 & -5 & -\frac{310}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{-R_1 - 4R_2}{5}]{R_2 \leftarrow \frac{5R_2 - 8R_1}{7}} \begin{pmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & 10 \end{pmatrix} \tag{3.0.34}$$

$$\implies c_1 = 10, c_2 = \frac{20}{3} \tag{3.0.35}$$

Substituting (3.0.30) and (3.0.35) in (3.0.25) we get,

$$\implies \boxed{(5x + 8y + 10)(4x + 5y + \frac{20}{3}) = 0} \tag{3.0.36}$$

Equation (3.0.36) represents equations of two straight lines.

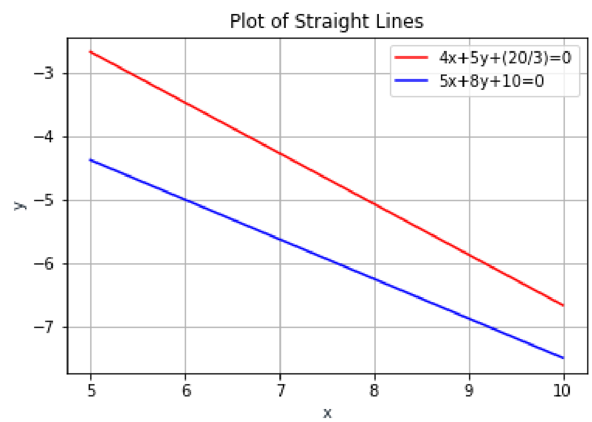

Fig. 2: Plot of Straight lines when $h = \frac{171}{20}$