Assignment 10

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***Abstract*—This document explains the concept of finding the distance between a given point and a plane using Singular Value Decomposition.**

Download all python codes from

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https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment10
    /code
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and latex-tikz codes from

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https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment10
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1 Problem

Find the distance of the given point $\begin{pmatrix}3\\-2\\1\end{pmatrix}$ from the plane $\begin{pmatrix}2 & -1 & 2\end{pmatrix}\mathbf{x} = 3$.

2 Explanation

Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}2\\-1\\2\end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies 2a - b + 2c = 0 \tag{2.0.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{2.0.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\\frac{1}{2}\end{pmatrix} \tag{2.0.5}$$

Let us solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.6}$$

Substituting (2.0.4) and (2.0.5) in (2.0.6),

$$\begin{pmatrix}1 & 0\\0 & 1\\-1 & \frac{1}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{2.0.7}$$

To solve (2.0.7), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}2 & \frac{-1}{2}\\\frac{-1}{2} & \frac{5}{4}\end{pmatrix} \tag{2.0.9}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & -1\\0 & 1 & \frac{1}{2}\\-1 & \frac{1}{2} & \frac{5}{4}\end{pmatrix} \tag{2.0.10}$$

Substituting (2.0.8) in (2.0.6),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.11}$$

$$\implies \mathbf{x} = \mathbf{VS_{+}U^Tb} \tag{2.0.12}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Let us calculate eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & -1\\0 & 1-\lambda & \frac{1}{2}\\-1 & \frac{1}{2} & \frac{5}{4}-\lambda\end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \tag{2.0.15}$$

From equation (2.0.15) eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{2.0.16}$$

The eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \tag{2.0.17}$$

Normalizing the eigen vectors in equation (2.0.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{2.0.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \tag{2.0.19}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.20}$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.21}$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{2.0.22}$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{2.0.23}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \tag{2.0.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.0.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.27}$$

From (2.0.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \tag{2.0.28}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.29}$$

From (2.0.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{11}{3\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ \frac{10}{3} \end{pmatrix} \tag{2.0.30}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{22}{9\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \tag{2.0.32}$$

Verifying the solution of (2.0.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.33}$$

Evaluating the R.H.S in (2.0.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \tag{2.0.34}$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \tag{2.0.35}$$

Solving the augmented matrix of (2.0.35) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & 2 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \tag{2.0.36}$$

$$\xleftrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{9}{8} & -1 \end{pmatrix} \tag{2.0.37}$$

$$\xleftrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \tag{2.0.38}$$

$$\xleftrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \tag{2.0.39}$$

From equation (2.0.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \tag{2.0.40}$$

3 Solution

Comparing results of $\mathbf{x}$ from (2.0.32) and (2.0.40), we can say that the solution is verified.