Challenge Problem

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Abstract—**This document explains the conditions when pseudo inverse does not exist.**

Download latex-tikz codes from

https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Challenges/Challenge2

1 Problem

What are the conditions in which pseudo inverse exists?.

2 Explanation

This problem is a part of

https://github.com/Zeeshan−IITH/IITH−EE5609/blob/master/assignment2/assignment2.pdf

where one other method of finding the solution (λ_1 and λ_2) would be by computing the inverse from equation (3.0.8) mentioned in the above document.

The matrix $\mathbf{M}$ was defined as:

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1}^T \\ \mathbf{m_2}^T \end{pmatrix} \tag{2.0.1}$$

where $\mathbf{m_1}$,$\mathbf{m_2}$ are vectors perpendicular to line AB. and,

$$\mathbf{M}\mathbf{M}^T \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M}(\mathbf{x_2} - \mathbf{x_1}) \tag{2.0.2}$$

where λ_1 and λ_2 are points on line.
Here λ_1 and λ_2 can be computed by taking the inverse of $\mathbf{M}\mathbf{M}^T$.

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = (\mathbf{M}\mathbf{M}^T)^{-1}\mathbf{M}(\mathbf{x_2} - \mathbf{x_1}) \tag{2.0.3}$$

The inverse $(\mathbf{M}\mathbf{M}^T)^{-1}$ can be written as:

$$(\mathbf{M}\mathbf{M}^T)^{-1} = (\mathbf{M}^T)^{-1}\mathbf{M}^{-1} \tag{2.0.4}$$

As $\mathbf{M}$ consists of vectors $\mathbf{m_1}$,$\mathbf{m_2}$ which are of order $n \times 1$, the order of matrix $\mathbf{M}$ would be $2 \times n$.
That is matrix $\mathbf{M}$ is a rectangular matrix and computing inverse of a rectangular matrix is not possible.
Hence we use pseudo inverse method to compute inverse.

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1}^T \\ \mathbf{m_2}^T \end{pmatrix} \tag{2.0.5}$$

As, $\mathbf{M}$ is of order $2 \times n$, the rank of this matrix will be $\min(2, n)$. $\implies$ rank will be less than or equal to 2, that is it depends on linear independence of rows of matrix.

Condition1: If $\mathbf{m_1}$ and $\mathbf{m_2}$ are not parallel, then the rows of $\mathbf{M}$ are linearly independent, that is rank of $\mathbf{M}$ is 2, which is full row rank.
Example:

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \mathbf{m_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \tag{2.0.7}$$

As $\mathbf{M}$ is full row rank, right pseudo inverse exists and can be computed as,

$$\mathbf{M}^T(\mathbf{M}\mathbf{M}^T)^{-1} \tag{2.0.8}$$

Condition2: If $\mathbf{m_1}$ and $\mathbf{m_2}$ are parallel, then the rows of $\mathbf{M}$ are linearly dependent, that is rank of $\mathbf{M}$ is 1 or 0, which is rank deficient matrix. **Therefore, pseudo inverse cannot be computed in such situation**.
Example:

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathbf{m_2} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{2.0.9}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \tag{2.0.10}$$

Similarly, if a rectangular matrix has full column rank then left pseudo inverse exists and this is usually for tall matrices.

There is one more way to compute inverse without splitting $(\mathbf{MM}^T)^{-1}$ as in equation (2.0.4). In this case $(\mathbf{MM}^T)$ is a square matrix of order 2×2. As it is a square matrix, if rows or columns are independent then the matrix would not be singular and inverse and pseudo inverse of such a matrix would be same. But if it singular then inverse does not exist.

3 Solution

If a matrix is full column rank then left pseudo inverse exist. If it is a full row rank then right pseudo inverse exist. If a matrix is square and invertible then inverse and pseudo inverse would be same.