Assignment 20

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Download latex-tikz codes from

https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment20

1 Problem

Let $\mathbf{N_1}$ and $\mathbf{N_2}$ be 6×6 nilpotent matrices over the field $\mathbf{F}$. Suppose that $\mathbf{N_1}$ and $\mathbf{N_2}$ have the same minimal polynomial and the same nullity. Prove that $\mathbf{N_1}$ and $\mathbf{N_2}$ are similar. Show that this is not true for 7×7 nilpotent matrices.

2 Explanation

Statement	**Derivation**
Given	$\mathbf{N_1}$ and $\mathbf{N_2}$ be 6×6 nilpotent matrices. $\mathbf{N_1}$ and $\mathbf{N_2}$ have the same minimal polynomial and the same nullity. To prove $\mathbf{N_1}$ and $\mathbf{N_2}$ are similar.
From given statement	Two matrices are similar if they have the same Jordan Canonical form. 1. As $\mathbf{N_1}$ and $\mathbf{N_2}$ are nilpotent matrices, 0 is the only eigen value. 2. As minimal polynomial is same, $p\mathbf{N_1} = p\mathbf{N_2}$, the two matrices should have the same maximum block size. 3. As they have same nullity, they will have same total number of blocks. If $\mathbf{J_1}$ and $\mathbf{J_2}$ are similar, then $\mathbf{N_1}$ and $\mathbf{N_2}$ are similar. Let us consider all the possibilities for the dimensions of the block matrices for both Jordan forms.
Matrix size - 6 and Jordan size - 6	If Jordan form $\mathbf{J_1}$ consists of one block of dimension 6, then by (3) above $\mathbf{J_2}$ also has one block of dimension 6. $$\begin{aligned} \mathbf{J_1} = \begin{pmatrix}\mathbf{J}_{11}\end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix}\mathbf{J}_{21}\end{pmatrix} \\ \mathbf{J}_{11} : 6 \times 6 \quad \mathbf{J}_{21} : 6 \times 6 \\ \mathbf{J}_{11}, \mathbf{J}_{21} \text{ are similar ,} \\ \mathbf{J}_1, \mathbf{J}_2 \text{ are similar} \\ \implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar} \end{aligned}$$
Matrix size - 6 and Jordan size - 5 + 1	If Jordan form $\mathbf{J_1}$ consists of one block of dimension 5 and other 1, then by (2), $\mathbf{J_2}$ also has same maximum block of dimension 5 and by (3) have other block of size 1.

	$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$$ $$\mathbf{J}_{11} : 5 \times 5 \quad \mathbf{J}_{21} : 5 \times 5$$ $$\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \text{ and } \mathbf{J}_{12}, \mathbf{J}_{22} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$
Matrix size - 6, Jordan size - 4+2, Jordan size - 4+1+1	Although there are two different possibilities for Jordan blocks, From (2) $\mathbf{J}_{11}, \mathbf{J}_{21}$ are of dimension 4, From (3) $\mathbf{J}$ have same number of Jordan blocks Case 1: $$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$$ $$\mathbf{J}_{11} : 4 \times 4 \quad \mathbf{J}_{21} : 4 \times 4$$ $$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \text{ and } \mathbf{J}_{12}, \mathbf{J}_{22} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$ Case 2: $$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{13} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{pmatrix}$$ $$\mathbf{J}_{11} : 4 \times 4 \quad \mathbf{J}_{21} : 4 \times 4$$ $$\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$$ $$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$
Matrix size - 6, Jordan size - 3+3, Jordan size - 3+2+1, Jordan size - 3+1+1+1	There are three different possibilities for Jordan blocks, From (2) $\mathbf{J}_{11}, \mathbf{J}_{21}$ are of dimension 3, From (3) $\mathbf{J}$ have same number of Jordan blocks Case 1:

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$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$$

$$\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$$

$$\mathbf{J}_{12} : 3 \times 3 \quad \mathbf{J}_{22} : 3 \times 3$$

$$\mathbf{J}_{11}, \mathbf{J}_{21} \text{ and } \mathbf{J}_{12}, \mathbf{J}_{22} \text{ are similar },$$

$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$

$$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Case 2:

$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{13} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{pmatrix}$$

$$\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$$

$$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$

$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$

$$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \text{ are similar },$$

$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$

$$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Case 3:

$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} \end{pmatrix}$$

$$\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$$

$$\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$$

$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$

$$\mathbf{J}_{14} : 1 \times 1 \quad \mathbf{J}_{24} : 1 \times 1$$

$$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \text{ are similar },$$

$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$

$$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

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<td>Matrix size - 6, Jordan size - 2+2+2, Jordan size - 2+2+1+1, Jordan size - 2+1+1+1+1</td>
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There are three different possibilities for Jordan blocks,

From (2) $\mathbf{J}_{11}, \mathbf{J}_{21}$ are of dimension 2,
From (3) $\mathbf{J}$ have same number of Jordan blocks
Case 1:

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	$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J_{13}} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J_{23}} \end{pmatrix}$$ $$\mathbf{J}_{11} : 2\times 2 \quad \mathbf{J}_{21} : 2\times 2$$ $$\mathbf{J}_{12} : 2\times 2 \quad \mathbf{J}_{22} : 2\times 2$$ $$\mathbf{J}_{13} : 2\times 2 \quad \mathbf{J}_{23} : 2\times 2$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$ Case 2: $$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} \end{pmatrix}$$ $$\mathbf{J}_{11} : 2\times 2 \quad \mathbf{J}_{21} : 2\times 2$$ $$\mathbf{J}_{12} : 2\times 2 \quad \mathbf{J}_{22} : 2\times 2$$ $$\mathbf{J}_{13} : 1\times 1 \quad \mathbf{J}_{23} : 1\times 1$$ $$\mathbf{J}_{14} : 1\times 1 \quad \mathbf{J}_{24} : 1\times 1$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$ Case 3: $$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{15} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{25} \end{pmatrix}$$ $$\mathbf{J}_{11} : 2\times 2 \quad \mathbf{J}_{21} : 2\times 2$$ $$\mathbf{J}_{12} : 1\times 1 \quad \mathbf{J}_{22} : 1\times 1$$ $$\mathbf{J}_{13} : 1\times 1 \quad \mathbf{J}_{23} : 1\times 1$$ $$\mathbf{J}_{14} : 1\times 1 \quad \mathbf{J}_{24} : 1\times 1$$ $$\mathbf{J}_{15} : 1\times 1 \quad \mathbf{J}_{25} : 1\times 1$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \quad \mathbf{J}_{15}, \mathbf{J}_{25} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$
Matrix size - 6, Jordan size- 1+1+1+1+1+1	$\mathbf{J_1}$ and $\mathbf{J_2}$ will have same number of blocks because of (3).

	$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{J}_{16} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{J}_{26} \end{pmatrix}$ $\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \quad \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \quad \mathbf{J}_{15}, \mathbf{J}_{25}$ and $\mathbf{J}_{16}, \mathbf{J}_{26}$ are similar , $\mathbf{J}_1, \mathbf{J}_2$ are similar $\implies \mathbf{N}_1, \mathbf{N}_2$ are similar
7×7 Nielpotent matrix	Let us take a counter example, Matrix size - 7, Jordan block size - 3+3+1 and 3+2+2 $\mathbf{J_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad (2.0.1)$ From above, $\mathbf{J_1}$ and $\mathbf{J_2}$ are not same. Hence $\mathbf{N_1}$ and $\mathbf{N_2}$ are not similar.

Table1:Solution