Assignment 18

Sri Harsha CH

***Abstract*—This document explains the representation of transformations by matrix.**

Download all python codes from

```
https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment18
    /code
```

and latex-tikz codes from

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https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment18
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1 Problem

Let $\mathbf{F}$ be a subfield of the field of complex numbers and let $\mathbf{V}$ be any vector space over $\mathbf{F}$. Suppose that f and g are linear functionals on $\mathbf{V}$ such that the function h defined by $h(\alpha) = f(\alpha)g(\alpha)$ is also a linear functional on $\mathbf{V}$. Prove that either $f = 0$ or $g = 0$.

2 Explanation

Refer Table 0.

Given	Derivation
f, g, h are linear functionals of $\mathbf{V}$	By contradiction, let us assume $f \neq 0$ and $g \neq 0$. For all $\mathbf{v} \in \mathbf{V}$ $$h(\mathbf{v}) = f(\mathbf{v})g(\mathbf{v}) \quad (2.0.1)$$ $$h(2\mathbf{v}) = f(2\mathbf{v})g(2\mathbf{v}) \quad (2.0.2)$$ $$= 2f(\mathbf{v})2g(\mathbf{v}) \quad (2.0.3)$$ $$= 4f(\mathbf{v})g(\mathbf{v}) \quad (2.0.4)$$ Similarly, $$h(2\mathbf{v}) = 2h(\mathbf{v}) \quad (2.0.5)$$ $$= 2f(\mathbf{v})g(\mathbf{v}) \quad (2.0.6)$$ From equation (2.0.4) and (2.0.6), $$\implies 4f(\mathbf{v})g(\mathbf{v}) = 2f(\mathbf{v})g(\mathbf{v}) \quad (2.0.7)$$ $$\implies f(\mathbf{v}).g(\mathbf{v}) = 0 \quad (2.0.8)$$
Choosing Basis	Let $\mathbf{B}$ be a basis for $\mathbf{V}$. Let, $$\mathbf{B_1} = \{\mathbf{b} \in \mathbf{B} \mid f(\mathbf{b}) = 0\}, \quad (2.0.9)$$ $$\mathbf{B_2} = \{\mathbf{b} \in \mathbf{B} \mid g(\mathbf{b}) = 0\} \quad (2.0.10)$$ Since, $$f(\mathbf{b}).g(\mathbf{b}) = 0 \quad \forall \mathbf{b} \in \mathbf{B} \quad (2.0.11)$$ $$\implies f(\mathbf{b}) = 0 \,\text{or}\, g(\mathbf{b}) = 0 \quad (2.0.12)$$ $$\implies \mathbf{b} \in \mathbf{B_1} \,\text{or}\, \mathbf{b} \in \mathbf{B_2} \quad (2.0.13)$$
Choosing $\mathbf{b_1}$ and $\mathbf{b_2}$ from basis	Let us choose $\mathbf{b_1} \in \mathbf{B_1} - \mathbf{B_2}$ and $\mathbf{b_2} \in \mathbf{B_2} - \mathbf{B_1}$ $\implies f(\mathbf{b_2}) \neq 0$ and $g(\mathbf{b_1}) \neq 0$ $$f(\mathbf{b_1}+\mathbf{b_2}).g(\mathbf{b_1}+\mathbf{b_2}) = (f(\mathbf{b_1}) + f(\mathbf{b_2})).(g(\mathbf{b_1}) + g(\mathbf{b_2})) \quad (2.0.14)$$ $$= f(\mathbf{b_1}).g(\mathbf{b_1}) + f(\mathbf{b_1}).g(\mathbf{b_2}) + f(\mathbf{b_2}).g(\mathbf{b_1}) + f(\mathbf{b_2}).g(\mathbf{b_2}) \quad (2.0.15)$$ $$= 0 + 0 + f(\mathbf{b_2}).g(\mathbf{b_1}) + 0 \quad (2.0.16)$$ $$= f(\mathbf{b_2}).g(\mathbf{b_1}) \neq 0 \quad (2.0.17)$$ Equation (2.0.17) is contradiction to the fact that $f(\mathbf{v}).g(\mathbf{v}) = 0$. $\implies \boxed{f = 0 \,\text{or}\, g = 0}$

TABLE 0: Expanation