Assignment 19

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***Abstract*—This document explains the representation of transformations by matrix.**

Download all python codes from

```
https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment19
    /code
```

and latex-tikz codes from

```
https://github.com/harshachinta/EE5609−Matrix−
    Theory/tree/master/Assignments/Assignment19
```

1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix of rank m with $n > m$. If for some non-zero real number α, we have $\mathbf{x^T A A^T x} = \alpha \mathbf{x^T x}$, for all $x \in \mathbf{R^m}$, then $\mathbf{A^T A}$ has,

1. exactly two distinct eigenvalues.
2. 0 as an eigenvalue with multiplicity $n - m$.
3. α as a non-zero eigenvalue.
4. exactly two non-zero distinct eigenvalues.

2 Explanation

Refer Table 0.

3 Solution

Refer Table 1.

Given	Derivation
Given	$\mathbf{A}$ is a $m \times n$ matrix of rank m with $n > m$. A non-zero real number α. To find eigenvalues of $\mathbf{A^TA}$.
Eigenvalues of $\mathbf{AA^T}$	$\mathbf{AA^T}$ is a $m \times m$ matrix. Given, $$\mathbf{x^TAA^Tx} = \alpha\mathbf{x^Tx} \quad (2.0.1)$$ $$\mathbf{x^T}(\mathbf{AA^T} - \alpha\mathbf{I})\mathbf{x} = 0 \quad (2.0.2)$$ $$\implies \mathbf{AA^T} - \alpha\mathbf{I} = 0 \quad (2.0.3)$$ $$\implies \mathbf{AA^T} = \alpha\mathbf{I} \quad (2.0.4)$$ where $\mathbf{I}$ is a $m \times m$ identity matrix, $$\mathbf{AA^T} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & \alpha \end{pmatrix} \quad (2.0.5)$$ $\mathbf{AA^T}$ has an eigenvalue α with multiplicity m.
Eigenvalues of $\mathbf{A^TA}$	$\mathbf{A^TA}$ is a $n \times n$ matrix. Given $n > m$, We know that, $\mathbf{A^TA}$ and $\mathbf{AA^T}$ have same number of non-zero eigenvalues and if one of them has more number of eigenvalues than the other then these eigenvalues are zero. 1. From above, as α is non-zero, $\mathbf{A^TA}$ has α as its eigenvalue with multiplicity m 2. $\mathbf{A^TA}$ has 0 as its eigenvalue with multiplicity $n - m$ 3. Therefore, the two distinct eigenvalues of $\mathbf{A^TA}$ are α and 0.

TABLE 0: Explanation

$\mathbf{A^TA}$ has exactly two distinct eigenvalues.	True statement
$\mathbf{A^TA}$ has 0 as an eigenvalue with multiplicity $n - m$	True statement
$\mathbf{A^TA}$ has α as a non-zero eigenvalue	True statement
$\mathbf{A^TA}$ has exactly two non-zero distinct eigenvalues.	False statement

TABLE 1: Solution