

Assignment 15

Sri Harsha CH

Abstract—This document explains the conditions to check for a vector space.

Download all python codes from

<https://github.com/harshachinta/EE5609-Matrix-Theory/tree/master/Assignments/Assignment15/code>

and latex-tikz codes from

<https://github.com/harshachinta/EE5609-Matrix-Theory/tree/master/Assignments/Assignment15>

1 PROBLEM

Let \mathbf{V} be an n -dimensional vector space over the field \mathbf{F} and let \mathbf{T} be a linear transformation from \mathbf{V} into \mathbf{V} such that the range and null space of \mathbf{T} are identical. Prove that n is even. (Can you give an example of such a linear transformation \mathbf{T})?

2 EXPLANATION

Let \mathbf{V} and \mathbf{W} be vector spaces over the field \mathbf{F} and let \mathbf{T} be a linear transformation from \mathbf{V} into \mathbf{W} . Then,

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \dim \mathbf{V} \quad (2.0.1)$$

It is given that range and null space of \mathbf{T} are same, let us assume it to be m . Substituting in equation (2.0.1)

$$m + m = n \quad (2.0.2)$$

$$\implies n = 2m \quad (2.0.3)$$

From equation (2.0.3), we can say that n is even.

Example: Let us consider a vector space \mathbf{V} , such that $\mathbf{V} \in \mathbb{R}^2$ and let us consider a linear transformation $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ defined by $\mathbf{T}(x, y) = (y, 0)$ and is given by matrix \mathbf{M}

$$\mathbf{T}(x, y) = (y, 0) \quad (2.0.4)$$

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.5)$$

Let us consider basis of \mathbb{R}^2 $\{(1, 0), (0, 1)\}$ and apply linear transformation on it.

$$\mathbf{T}(1, 0) = (0, 0) \quad (2.0.6)$$

$$\mathbf{T}(0, 1) = (1, 0) \quad (2.0.7)$$

From (2.0.5),

The range of matrix can be found from row reduced echelon form. But as matrix \mathbf{M} is in RREF form, the basis for range is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The null space of matrix is,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.8)$$

$$\implies x_1 = t \quad x_2 = 0 \quad (2.0.9)$$

$$\implies \mathbf{X} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.10)$$

The basis for null space is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\text{rank}(\mathbf{T}) = 1 \quad \text{nullity}(\mathbf{T}) = 1 \quad (2.0.11)$$

$$\dim(\mathbf{V}) = 2 \quad (2.0.12)$$

Thus the range and null space are equal, and n is even.

3 SOLUTION

The range and null space of \mathbf{T} are equal, and n is even.