Assignment 19

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***Abstract*—This document explains the representation of transformations by matrix.**

Download all python codes from

https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment19/code

and latex-tikz codes from

https://github.com/harshachinta/EE5609−Matrix−Theory/tree/master/Assignments/Assignment19

1 Problem

Let $\mathbf{A}$ be an $m \times n$ matrix of rank m with $n > m$. If for some non-zero real number α, we have $\mathbf{x^T A A^T x} = \alpha \mathbf{x^T x}$, for all $x \in \mathbf{R^m}$, then $\mathbf{A^T A}$ has,

1. exactly two distinct eigenvalues.
2. 0 as an eigenvalue with multiplicity $n - m$.
3. α as a non-zero eigenvalue.
4. exactly two non-zero distinct eigenvalues.

2 Explanation

Refer Table 0.

3 Solution

Refer Table 1.

Given	Derivation
Given	$\mathbf{A}$ is a $m \times n$ matrix of rank m with $n > m$. A non-zero real number α. To find eigenvalues of $\mathbf{A^T A}$.
Eigenvalues of $\mathbf{AA^T}$	$\mathbf{AA^T}$ is a $m \times m$ matrix and $\mathbf{A^T A}$ is a $n \times n$ matrix. Let, λ be a non-zero eigen value of $\mathbf{A^T A}$. $$\mathbf{A^T A v} = \lambda \mathbf{v} \quad \mathbf{v} \in \mathbf{R^n} \tag{2.0.1}$$ $$\mathbf{AA^T A v} = \lambda \mathbf{A v} \tag{2.0.2}$$ $$\text{Let,} \quad \mathbf{x} = \mathbf{A v} \quad \mathbf{x} \in \mathbf{R^m} \tag{2.0.3}$$ $$\mathbf{AA^T x} = \lambda \mathbf{x} \tag{2.0.4}$$ $$\mathbf{x^T AA^T x} = \lambda \mathbf{x^T x} \tag{2.0.5}$$ $$\text{Given,} \quad \mathbf{x^T AA^T x} = \alpha \mathbf{x^T x} \tag{2.0.6}$$ $$\implies \alpha \mathbf{x^T x} = \lambda \mathbf{x^T x} \tag{2.0.7}$$ From equation (2.0.7), $\lambda = \alpha$ as $\|\mathbf{x}\| \neq 0$ As rank($\mathbf{A^T A}$) = rank($\mathbf{A}$) = m and equation (2.0.7) satisfies the condition in question. Therefore the only non-zero eigen value is α $\mathbf{A^T A}$ has an eigenvalue α with multiplicity m.
Eigenvalues of $\mathbf{A^T A}$	$\mathbf{A^T A}$ is a $n \times n$ matrix. Given $n > m$, We know that, $\mathbf{A^T A}$ and $\mathbf{AA^T}$ have same number of non-zero eigenvalues and if one of them has more number of eigenvalues than the other then these eigenvalues are zero. 1. From above, as α is non-zero, $\mathbf{A^T A}$ has α as its eigenvalue with multiplicity m 2. $\mathbf{A^T A}$ has 0 as its eigenvalue with multiplicity $n - m$ 3. Therefore, the two distinct eigenvalues of $\mathbf{A^T A}$ are α and 0.

TABLE 0: Explanation

$\mathbf{A^T A}$ has exactly two distinct eigenvalues.	True statement
$\mathbf{A^T A}$ has 0 as an eigenvalue with multiplicity $n - m$	True statement
$\mathbf{A^T A}$ has α as a non-zero eigenvalue	True statement
$\mathbf{A^T A}$ has exactly two non-zero distinct eigenvalues.	False statement

TABLE 1: Solution