Crank Nicholson Method

The parabolic equation is

$$\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t} \quad -(1)$$

Using finite difference method

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \quad (2)$$

$$\because \frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \quad (3)$$

$\therefore$ find $\frac{\partial^2 u}{\partial x^2}$ at $i, j+1$

Replace (2) by j to $j+1$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} \quad (4)$$

Take average of (2) and (4)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] \quad (5)$$

put (3) and (5) in (1)

$$\frac{1}{2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] = \frac{a\left[u_{i,j+1} - u_{i,j}\right]}{k}$$

or

$$\left[U_{i,j+1} - U_{i,j}\right] = \frac{K}{2h^2 a}\Big[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}$$

$$+ U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}\Big]$$

Let $\frac{K}{h^2 a} = \lambda$

$$\therefore \left[U_{i,j+1} - U_{i,j}\right] = \frac{\lambda}{2}\Big[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}$$

$$+ U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}\Big]$$

or

$$\left[U_{i,j+1} - U_{i,j}\right] = \Big[\frac{\lambda}{2}U_{i+1,j} - \lambda U_{i,j} + \frac{\lambda}{2}U_{i-1,j}$$

$$+ \frac{\lambda}{2}U_{i+1,j+1} - \lambda U_{i,j+1} + \frac{\lambda}{2}U_{i-1,j+1}\Big]$$

Bringing all the terms containing $j+1$ to LHS and other to the R.H.S

$$U_{i,j+1} - \frac{\lambda}{2}U_{i+1,j} + \lambda U_{i,j+1} - \frac{\lambda}{2}U_{i-1,j+1}$$

$$= \frac{\lambda}{2}U_{i+1,j} - \lambda U_{i,j} + \frac{\lambda}{2}U_{i-1,j} + U_{i,j}$$

$$= \frac{\lambda}{2}U_{i+1,j} + (1-\lambda)U_{i,j} + \frac{\lambda}{2}U_{i-1,j}$$

Take $\lambda = 1$

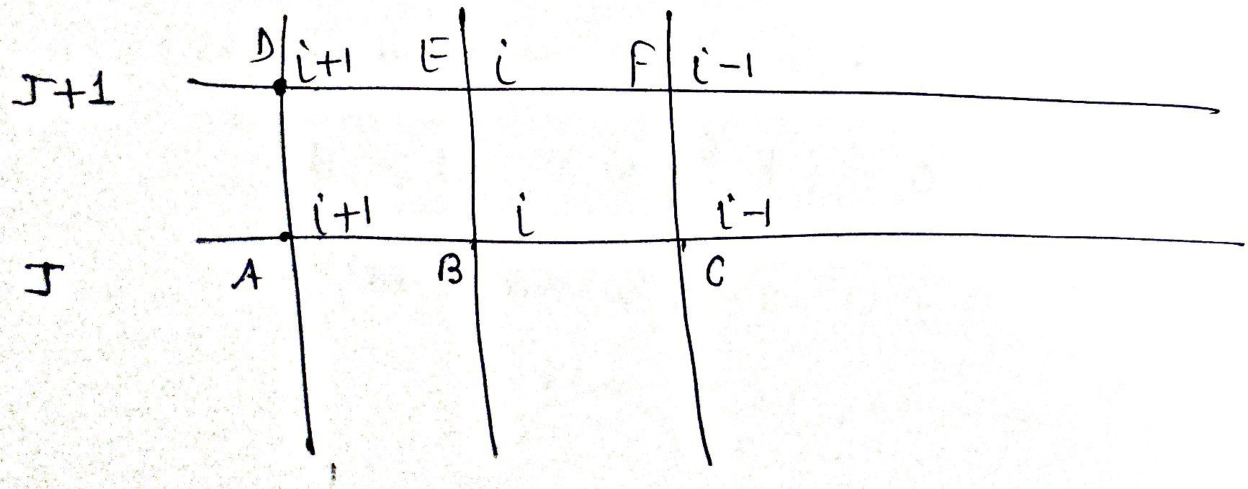
$$u_{i,j+1} - \frac{1}{2} u_{i+1,j+1} + u_{i,j+1} - \frac{1}{2} u_{i-1,j+1}$$
$$= \frac{1}{2} u_{i+1,j} + \frac{1}{2} u_{i-1,j}$$

or

$$2 u_{i,j+1} - u_{i+1,j+1} + 2u_{i,j+1} - u_{i-1,j+1}$$
$$= u_{i+1,j} + u_{i-1,j}$$

OR

$$u_{i+1,j+1} - 4 u_{i,j+1} + u_{i-1,j+1} = - u_{i+1,j} - u_{i-1,j}$$



$$D - 4E + F = - A - C$$

This is crank Nickolson formula.

Q.1

Solve by Crank Nicholson Method

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1 \quad , t > 0$$

$$u(x,0) = 100(x - x^2) , \; u(0,t) = u(1,t) = 0$$

Compute u for one time step with $h = 0.25$

Soln/- parabolic eq is

$$\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t} \qquad \text{Here } a = 1$$

Also from Crank Nicholson Method $\lambda = 1$

$$\because \lambda = \frac{k}{h^2 a} \Rightarrow 1 = \frac{k}{(0.25)^2(1)}$$

$$\therefore k = 0.0625.$$

Use Boundary Conditions.

$$u(0,t) = u(1,t) = 0.$$

Means when $x = 0$, $u = 0$ (first Column)

when $x = 1$, $u = 0$. (last Column)

*. Hence all values in the first Column and last Column are zero.

Also. $u(x,0) = 100(x - x^2)$

Here $h = 0.25$.

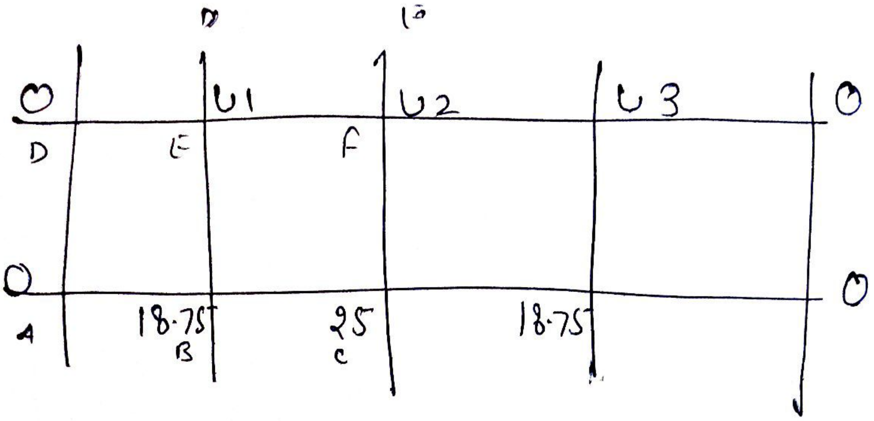
when $x = 0,$ $U = 0$

$x = 1$ $U = 18.75$

$x = 0.5$ $U = 25$

$x = 0.75$ $U = 18.75$

$x = 1$ $U = 0$



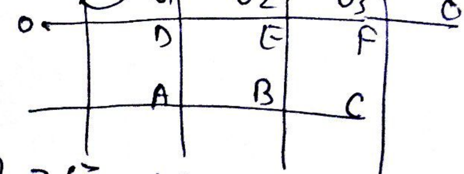
Let U_1, U_2 and U_3 are interior points.

Use Crank Nicholson formula

$$D - 4E + F = -A - C$$

$$0 - 4U_1 + U_2 = -0 - 25$$

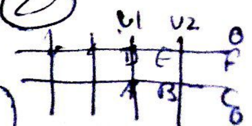
$$-4U_1 + U_2 = -25 \quad \text{— (1)}$$



Also.

$$U_1 - 4U_2 + U_3 = -18.75 - 18.75$$

$$U_1 - 4U_2 + U_3 = -37.5 \quad (2)$$



Also. $U_2 - 4U_3 + 0 = -25 - 0 = -25$ (3)

$\therefore$ we have,

$$\left.\begin{aligned} -4U_1 + U_2 &= -25 \quad (1) \\ U_1 - 4U_2 + U_3 &= -37.5 \quad (2) \\ U_2 - 4U_3 &= -25 \quad (3) \end{aligned}\right\}$$

Solving these we get

$$U_2 = 14.2$$

$$U_1 = 9.8$$

$$U_3 = 9.8$$

Q.2 Solve by Crank - Nicolson Method

$$\frac{\partial u}{\partial t} = \frac{1}{16}\frac{\partial^2 u}{\partial x^2} \quad ; 0 < x < 1 \, , \quad t > 0$$

$$u(x,0) = 0 \, , \quad u(0,t) = 0 \, , \quad u(1,t) = 100t$$

Compute for one time step with $h = \frac{1}{4}$

Sol^n/- Here $a = 16$ & $\lambda = 1$

But $\lambda = \frac{k}{h^2 a} = 1 = \frac{k}{\left(\frac{1}{4}\right)^2 \cdot 1}$

$\Rightarrow \quad k = 1$

Using Boundary Conditions

$U(0,t) = 0$

$\Rightarrow$ When $x = 0$, $U = 0$ and $t = ??$

$\Rightarrow$ All values in the first column are zeros.

Also $U(x,0) = 0$

Means $t = 0$, $U = 0$

All values in the first row are zero.

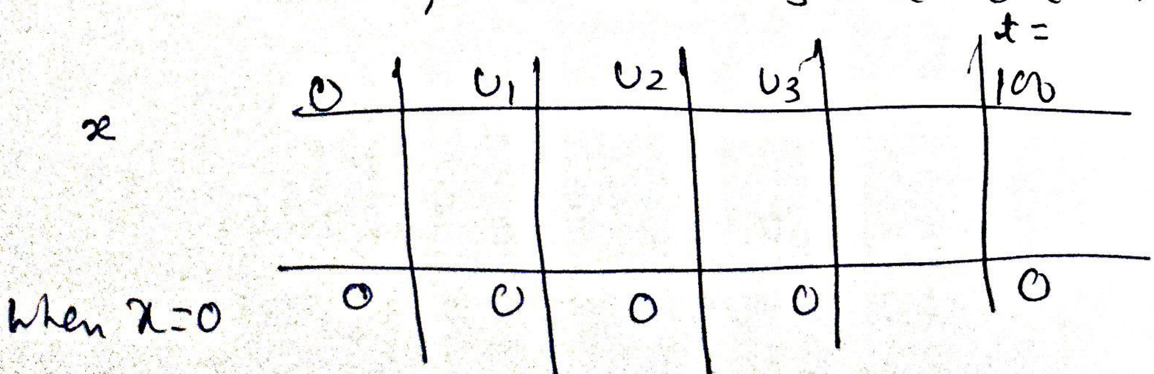
Since $x = 1$, put $t = 0, 1$, in $100\,t$.

When $t = 0$, $U = 100(0) = 0$

$t = 1$, $U = 100(t) = 100$

(Boundary at last column)

Let U_1, U_2 and U_3 be the interior



Now Apply Crank Nickolson Method

$0 - 4U_1 + U_2 = 0 - 0$

$U_1 - 4U_2 + U_3 = -0 - 0$

$U_2 - 4U_3 + 100 = -0 - 0.$

$\therefore$

$$\left.\begin{aligned} -4U_1 + U_2 &= 0 \\ U_1 - 4U_2 + U_3 &= 0 \\ U_2 - 4U_3 &= -100 \end{aligned}\right\}$$

Solving these we get

$U_1 = 1.786, \quad U_2 = 7.143, \quad U_3 = 26.786$

Imp

Practice Ques (of C.N. Method)

(1) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ when $u(x,0) = 0$

$u(0,t) = 0$

$u(1,t) = t$

take $h = 1/2$, $k = 1/8$ using C.N. Method

[Hint $x=0, x=\frac{1}{8}, x=\frac{1}{4}$] Ans $U_1 = 0.021$

$\lambda = 1/2$

(2) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ given $u(x,0) = 0$

$u(0,t) = 0$

$u(1,t) = t$

take $h = 1/4$, $k = 1/8$ using C.N. Method

[Hint: $\lambda = 2$, $t=0, u=0$, $t=1/8, u=1/8$]