a given degree fitted together in such a way that the junctions of the successive arcs are as smooth as they could be made without going to a single polynomial over the entire range. Fitting to an empirical data by a spline function offers a numerical method for obtaining a curve similar to the one produced by a French curve. This technique has been used effectively in the areas of computer graphics, flow simulations and for smoothing of satellite data which is received at a tracking station with noise.

Definition 6.1 Suppose, we have $(n + 1)$ data points (x_i, y_i), $i = 0, 1, 2, \ldots, n$; where x_i may not be equally spaced and $x_0 = a$, $x_n = b$ and we wish to determine a cubic spline function $S(x)$, such that it has the following properties:

(i) The cubic spline function has the form

$$S(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

in each interval (x_i, x_{i+1}), $i = 0, 1, 2, \ldots, (n - 1)$.

(ii) $S(x_i) = y_i$, $(i = 0, 1, 2, \ldots, n)$.

(iii) The cubics are so joined that the function $S(x)$ and both its slope $S'(x)$ and curvature $S''(x)$ are continuous in (x_0, x_n). It means that the spline curve $S(x)$ will not have sharp corners and the radius of curvature is defined at each point.

Thus, the cubic spline function will have the form $S(x) = S_i(x)$ in the interval (x_i, x_{i+1}). To get $S(x)$, we have to put together the cubics $S_i(x)$ as shown in Fig. 6.1. A detailed account of the basic properties of the cubic spline can be found in Ahlberg et al. (1967).

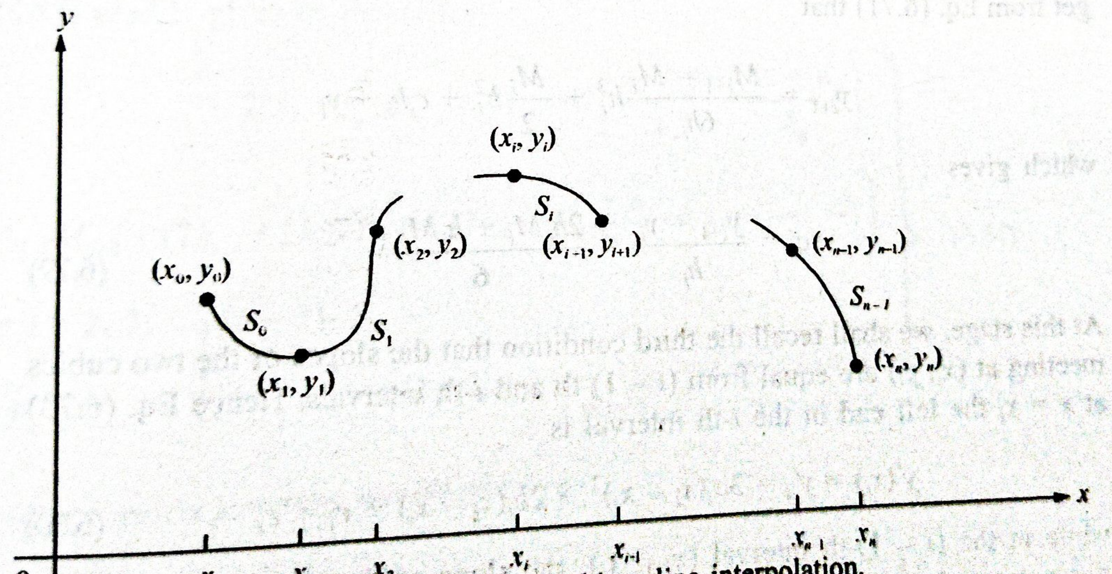

Fig. 6.1 Piecewise cubic spline interpolation.

6.8.1 Construction of Cubic Spline

Spath (1969) suggested a cubic spline $S(x)$ by the piecewise cubic polynomials of the form

$$S(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \tag{6.69}$$

in each interval (x_i, x_{i+1}), $i = 0, 1, 2, \ldots, (n - 1)$.

Since condition (ii) in Definition 6.1 implies that the cubic spline fits exactly at the two end points x_i and x_{i+1} of the i-th interval, we have

$$S(x_i) = y_i = a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i = d_i \tag{6.70}$$

$$S(x_{i+1}) = y_{i+1} = a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + c_i(x_{i+1} - x_i) + d_i \tag{6.71}$$

Now, introducing the notation $h_i = x_{i+1} - x_i$, Eq. (6.71) becomes

$$y_{i+1} = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i \tag{6.72}$$

To satisfy the third condition relating to the slope and curvature of the joining cubics, we obtain from Eq. (6.69) that

$$S'(x) = y'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i \tag{6.73}$$

$$S''(x) = y''(x) = 6a_i(x - x_i) + 2b_i \tag{6.74}$$

Using the notation $S_i'' = y''(x_i) = M_i$, we can determine a_i, b_i and c_i in terms of M_i. From Eq. (6.74), we have

$$M_i = 6a_i(x_i - x_i) + 2b_i = 2b_i \tag{6.75}$$

$$M_{i+1} = 6a_i(x_{i+1} - x_i) + 2b_i = 6a_i h_i + 2b_i \tag{6.76}$$

which gives us

$$b_i = \frac{M_i}{2}, \quad a_i = \frac{M_{i+1} - M_i}{6h_i} \tag{6.77, 6.78}$$

$d_i = y_i$

Now, using the values of d_i, b_i and a_i given by Eqs. (6.70), (6,77) and (6.78), we get from Eq. (6.71) that

$$y_{i+1} = \frac{M_{i+1} - M_i}{6h_i} h_i^3 + \frac{M_i}{2} h_i^2 + c_i h_i + y_i$$

which gives

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i M_i + h_i M_{i+1}}{6} \tag{6.79}$$

At this stage, we shall recall the third condition that the slopes of the two cubics meeting at (x_i, y_i) are equal from $(i - 1)$-th and i-th intervals. Hence Eq. (6.73) at $x = x_i$ the left end in the i-th interval is

$$y'(x_i) = y'_i = 3a_i(x_i - x_i)^2 + 2b_i(x_i - x_i) + c_i = c_i \tag{6.80}$$

while in the $(i - 1)$-th interval (x_{i-1}, x_i), the slope at the right end, that is at $x = x_i$ given by

$$\begin{aligned} y'(x_i) = y'_i &= 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1} \\ &= 3a_{i-1}h_{i-1}^2 + 2b_{i-1}h_{i-1} + c_{i-1} \end{aligned} \tag{6.81}$$

Now, equating Eqs. (6.80) and (6.81) and using Eqs. (6.77)–(6.79) for b_{i-1}, a_{i-1} and c_{i-1} respectively, we get

$$\frac{y_{i+1}-y_i}{h_i}-\frac{2h_iM_i+h_iM_{i+1}}{6}=3\,\frac{M_i-M_{i-1}}{6h_{i-1}}\,h_{i-1}^2+M_{i-1}h_{i-1}$$

$$+\frac{y_i-y_{i-1}}{h_{i-1}}-\frac{2h_{i-1}M_{i-1}+h_{i-1}M_i}{6} \tag{6.82}$$

On simplification, we obtain

$$h_{i-1}M_{i-1}+(2h_{i-1}+2h_i)\,M_i+h_iM_{i+1}=6\left(\frac{y_{i+1}-y_i}{h_i}-\frac{y_i-y_{i-1}}{h_{i-1}}\right) \tag{6.83}$$

for $i = 1, 2, \ldots, (n-1)$. From Eq. (6.83), it may be observed that only M_i's are unknowns, while all other terms can be computed from the given data points. In fact, it represents a system of $(n-1)$ linear equations in $(n+1)$ unknowns M_0, $M_1, \ldots, M_n$. Hence, two more additional conditions are required, relating to the end points of the complete spline curve, to generate two more additional equations. Many types of end conditions are specified and discussed in the literature. However, we shall consider only two types of end conditions.

6.8.2 End Conditions

Type I: We specify $S_0 = S_n = 0$, which means $M_0 = 0$, $M_n = 0$. In this case, the end cubics linearly approach to their extremities. This is called *natural spline.* This form of specification of end conditions is very popular. In this case, Eq. (6.83) readily gives us the system

$$\left.\begin{aligned}
2(h_0+h_1)M_1+h_1M_2&=6\left(\frac{y_2-y_1}{h_1}-\frac{y_1-y_0}{h_0}\right)\\
h_{i-1}M_{i-1}+(2h_{i-1}+2h_i)M_i+h_iM_{i+1}&=6\left(\frac{y_{i+1}-y_i}{h_i}-\frac{y_i-y_{i-1}}{h_{i-1}}\right)\\
\text{where } i=2,3,\ldots,(n-2)&\\
h_{n-2}M_{n-2}+2(h_{n-2}+h_{n-1})M_{n-1}&=6\left(\frac{y_n-y_{n-1}}{h_{n-1}}-\frac{y_{n-1}-y_{n-2}}{h_{n-2}}\right)
\end{aligned}\right\} \tag{6.84}$$

In compact matrix notation, we present it as

$$\begin{bmatrix}
2(h_0+h_1) & h_1 & & & \\
h_1 & 2(h_1+h_2) & h_2 & & \\
 & h_2 & 2(h_2+h_3) & h_3 & \\
 & & & \vdots & \\
 & & & h_{n-2} & 2(h_{n-2}+h_{n-1})
\end{bmatrix}
\begin{pmatrix} M_1\\ M_2\\ M_3\\ \vdots\\ M_{n-1}\end{pmatrix}$$

$$= 6 \begin{pmatrix} \frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0} \\ \frac{y_3 - y_2}{h_2} - \frac{y_2 - y_1}{h_1} \\ \frac{y_4 - y_3}{h_3} - \frac{y_3 - y_2}{h_2} \\ \vdots \\ \frac{y_n - y_{n-1}}{h_{n-1}} - \frac{y_{n-1} - y_{n-2}}{h_{n-2}} \end{pmatrix} \quad (6.85)$$

This being an $(n - 1 \times n - 1)$ tridiagonal system can be solved economically using Crout's reduction method as explained in Section 3.4 or Thomas Algorithm as explained in Appendix.

Type II: In this case, we specify the slopes at the end of entire spline curve; that is, we are given $y'(x_0) = A$ and $y'(x_n) = B$. This is called *clamped cubic spline*. From Eqs. (6.79) and (6.80), we have

$$y_i' = -\frac{h_i}{3} M_i - \frac{h_i}{6} M_{i+1} + \frac{y_{i+1} - y_i}{h_i} \quad (6.86)$$

using the left-end condition for $i = 0$, we obtain

$$-\frac{h_0}{3} M_0 - \frac{h_0}{6} M_1 + \frac{y_1 - y_0}{h_0} = A$$

That is,

$$2M_0 + M_1 = \frac{6}{h_0}\left(\frac{y_1 - y_0}{h_0} - A\right) \quad (6.87)$$

Similarly from Eq. (6.82), we have

$$y_i' = \frac{M_i h_{i-1}}{3} + \frac{M_{i-1} h_{i-1}}{6} + \frac{y_i - y_{i-1}}{h_{i-1}}$$

Now, using the right-end condition for $i = n$, the above equation becomes

$$B = \frac{M_n h_{n-1}}{3} + \frac{M_{n-1} h_{n-1}}{6} + \frac{y_n - y_{n-1}}{h_{n-1}}$$

Further simplification yields

$$M_{n-1} + 2M_n = \frac{6}{h_{n-1}}\left(B - \frac{y_n - y_{n-1}}{h_{n-1}}\right) \quad (6.8\ldots)$$

For $i = 1$, Eq. (6.83) gives

$$h_0 M_0 + (2h_0 + 2h_1) M_1 + h_1 M_2 = 6\left(\frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}\right) \quad (6.\ldots)$$

Eliminating M_0 from Eqs. (6.87) and (6.89), we get

$$\left(\frac{3}{2}h_0 + 2h_1\right) M_1 + h_1 M_2 = 6\,\frac{y_2 - y_1}{h_1} - 9\,\frac{y_1 - y_0}{h_0} + 3A \qquad (6.90)$$

Also, for $i = n - 1$, Eq. (6.83) gives

$$h_{n-2}M_{n-2} + (2h_{n-2} + 2h_{n-1})M_{n-1} + h_{n-1}M_n = 6\left(\frac{y_n - y_{n-1}}{h_{n-1}} - \frac{y_{n-1} - y_{n-2}}{h_{n-2}}\right) \qquad (6.91)$$

Eliminating M_n from Eqs. (6.88) and (6.91), we get

$$h_{n-2}M_{n-2} + \left(2h_{n-2} + \frac{3}{2}h_{n-1}\right)M_{n-1} = 9\frac{y_n - y_{n-1}}{h_{n-1}} - 6\frac{y_{n-1} - y_{n-2}}{h_{n-2}} - 3B \qquad (6.92)$$

★ n+1

0 — n

0 n — n+1

★

Hence, in type II, we have to solve Eqs. (6.83), (6.90) and (6.92). These equations together constitute an $(n - 1 \times n - 1)$ tridiagonal system in unknowns $M_1, M_2, \ldots, M_{n-1}$, which can be solved using Crout's reduction technique or Thomas Algorithm. In order to see the sequence of steps involved to construct a cubic spline $S(x)$ for a given data set, using cubic spline interpolation, we shall consider below a couple of simple examples.

Example 6.20 Fit a cubic spline curve that passes through (0, 1), (1, 4), (2, 0), (3, –2) with the natural end boundary conditions $S''(0) = S''(3) = 0.0$.

Solution From the given data, we observe that there are three intervals, in each of which we can construct a cubic spline function. These piecewise cubic spline polynomials when put together determine the cubic spline curve $S(x)$ in the entire interval (0, 3).

At the outset, we observe that $h_0 = h_1 = h_2 = 1$. For natural spline, we obtain Eqs. (6.84) as

$$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = 6\begin{pmatrix} -4 & -3 \\ -2 & +4 \end{pmatrix} = \begin{pmatrix} -42 \\ 12 \end{pmatrix}$$

$h_0 = x_1 - x_0$

$h_1 = x_2 - x_1$

$h_2 = x_3 - x_2$

$h_3 = x_4 - x_3$

That is,

$$4M_1 + M_2 = -42$$
$$M_1 + 4\,M_2 = 12$$

Its solution is $M_1 = -12$, $M_2 = 6$. Natural-end conditions imply $M_0 = M_3 = 0.0$.

Let the natural cubic spline is given by

$$S(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

where the coefficients are given by the relations

$$a_i = \frac{M_{i+1} - M_i}{6h_i}, \qquad b_i = \frac{M_i}{2}$$

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i M_i + h_i M_{i+1}}{6}, \qquad d_i = y_i$$