4. ... .2653 $u_3 = 9.5238$

$u_2 = 0.0045$ $u_6 = 27.4376$

$u_5 = 0.0168$

HYPERBOLIC EQUATIONS
(One dimensional wave equations)

The hyperbolic equation is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{... (1)}$$

By definition of finite difference approximations we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Substituting for $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$ in (1)

$$a^2 \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right] = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Interchanging L.H.S into R.H.S.

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = a^2 \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right]$$

Cross multiplying

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{k^2 a^2}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$= \left(\frac{k}{h}\right)^2 a^2 [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

Let $\frac{\mathbf{k}}{\mathbf{h}} = \lambda$

$$\therefore \quad u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \lambda^2 a^2 [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

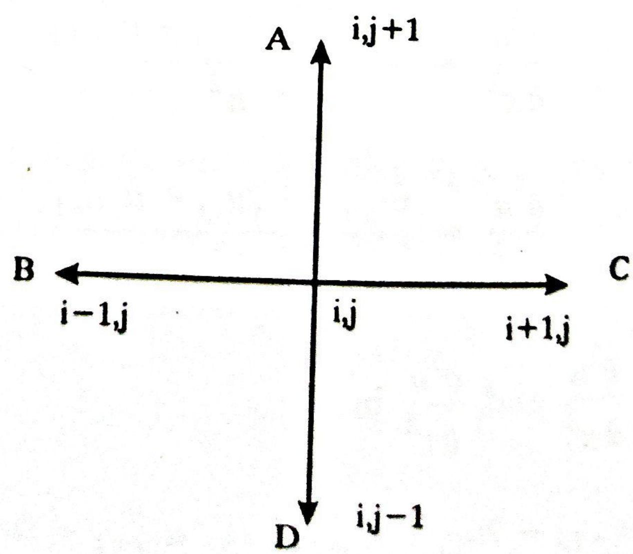
$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \lambda^2 a^2 u_{i+1,j} - 2\lambda^2 a^2 u_{i,j} + \lambda^2 a^2 u_{i-1,j}$$

$$\therefore \quad u_{i,j+1} = \lambda^2 a^2 u_{i+1,j} - 2\lambda^2 a^2 u_{i,j} + \lambda^2 a^2 u_{i-1,j} + 2u_{i,j} - u_{i,j-1}$$

$$u_{i,j+1} = \lambda^2 a^2 u_{i+1,j} + 2u_{i,j} (1 - \lambda^2 a^2) + \lambda^2 a^2 u_{i-1,j} - u_{i,j-1}$$

Taking $\lambda^2 a^2 = 1$ we get $\boxed{\mathbf{u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}}}$

Diagramatically



A = B + C − D

In problems one extra condition $\frac{\partial u}{\partial t} = 0$ will be given at $t = 0$

$$\therefore \frac{u_{i,j+1} - u_{i,j}}{k} = 0$$

$$\therefore u_{i,j+1} - u_{i,j} = 0$$

$\therefore u_{i,j+1} = u_{i,j} \Rightarrow$ First row and second row are same.

Note :

We can also use Central difference formula for $\frac{\partial u}{\partial t}$ which is given by

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \text{ at } t = 0$$

The answer will be different, in this case.

WORKED EXAMPLES

1. Solve $u_{tt} = u_{xx}$ given $u(0, t) = u(4, t) = 0$, $u(x, 0) = \frac{1}{2}x(4 - x)$ and $u_t(x, 0) = 0$. Take $h = 1$, find the solution upto 5 steps in 't' direction.

Solution : The hyperbolic equation is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

The given problem is $u_{tt} = u_{xx}$

i.e.,
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\therefore a^2 = 1$$

$$\therefore a = 1$$

For hyperbolic equation

$$\lambda^2 a^2 = 1 \text{ and } \lambda = \frac{k}{h}$$

$$\therefore \lambda^2 1^2 = 1 \qquad \Rightarrow \lambda^2 = 1$$

$$\lambda = 1$$

$$1 = \frac{k}{h}$$

$$k = h$$

Since $h = 1$, $\quad k = 1$

$\therefore$ For $h = 1$, $k = 1$, $a = 1$

$$\lambda^2 a^2 = 1$$

We can use the difference scheme for solving this problem.

Using the condition $\quad u(0, t) = 0$

$\Rightarrow$ all values in the first column are 0

The second condition is $\quad u(4, t) = 0$

$\Rightarrow$ All the values in the last column are zeros.

Now to find the I row, the given condition is

$$u(x, 0) = \frac{1}{2} x (4 - x)$$

Here $h = 1$, $\quad \therefore$ increase the values of x by 1.

Starting value of x is 0

When $x = 0$

$x = 1$

$x = 2$

$x = 3$

Final value of x is 4

$x = 4$

Last condition is

$\Rightarrow$ I row

Starting value of x is 0

When $x = 0$ $\quad u = \frac{1}{2} 0 (4 - 0) = 0$

$x = 1$ $\quad u = \frac{1}{2} 1 (4 - 1) = \frac{3}{2} = 1.5$

$x = 2$ $\quad u = \frac{1}{2} 2 (4 - 2) = 2$

$x = 3$ $\quad u = \frac{1}{2} \times 3 (4 - 3) = \frac{1}{2} \times 3 = 1.5$

Final value of x is 4

$x = 4$ $\quad u = \frac{1}{2} . 4 (4 - 4) = 0$

Last condition is $\quad u_t (x, 0) = 0$

But $u_t = \frac{\partial u}{\partial t}$

$$\therefore \frac{\partial u}{\partial t} = 0$$

$$\Rightarrow \frac{u_{i,j+1} - u_{i,j}}{k} = 0$$

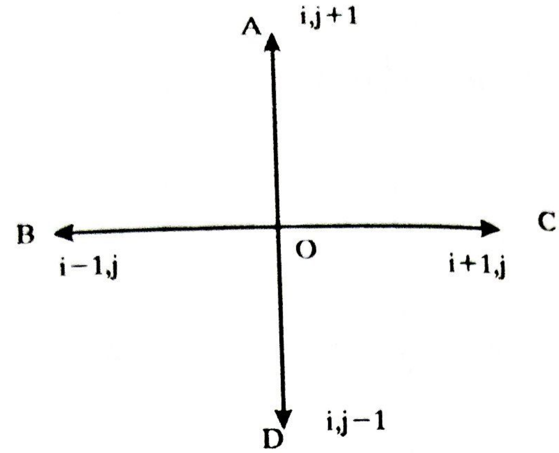
$$u_{i,j+1} - u_{i,j} = 0$$

$$u_{i,j+1} = u_{i,j}$$

$\Rightarrow$ I row and II row are same.

problem.

Other values can be obtained using the following figure.



$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

0	−1.5	−2	−1.5	0
0	−1.5	−2	−1.5	0
0	−0.5	−1	−0.5	0
0	0.5	1	0.5	0
0	1.5	2	1.5	0
0	1.5	2	1.5	0

2. The function u satisfies the equation

ASK

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

and the condition. $u(x, 0) = \frac{1}{8} \sin \pi x$, $u_t(x, 0) = 0$, **for** $0 \le x \le 1$ **$u(0, t) = u(1, t) = 0$ for $t \ge 0$. Use the explicit scheme to calculate u for $x = 0(0.1)\ 0.5$ and $t = 0(0.1)\ 0.5$**

Solution : The hyperbolic equation is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

...re.

	0
	0
	0
	0
	0
	0

...), for $0 \le x \le 1$

...eme to calculate

The problem is $u_{tt} = u_{xx}$

$$\text{i.e.,}\quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$a^2 = 1$$

$$a = 1$$

For hyperbolic equation

$$\lambda^2 a^2 = 1 \text{ and } \lambda = \frac{k}{h}$$

here $h = 0.1;\ k = 0.1$

$$\lambda^2 1^2 = 1$$

$$\therefore \quad \lambda = 1$$

Also $h = 0.1,\quad k = 0.1$ given

$$\therefore \lambda = \frac{k}{h} = \frac{0.1}{0.1} = 1$$

For $h = 0.1,\quad k = 0.1,\quad a = 1$

$$\lambda^2 a^2 = 1$$

We can use the difference scheme for solving this problem.

Using the condition

$$u(0, t) = 0$$

➡ all the values in the first column are zero.

Using the condition

$$u(1, t) = 0$$

$\Rightarrow$ all the values in the last column are zeros

Now to find the I row the condition is

$$u(x, 0) = \frac{1}{8} \sin \pi x$$

When $x = 0$, $u = \frac{1}{8} \sin 0 = 0$ $[\pi = 180^\circ]$

$x = 0.1$, $u = \frac{1}{8} \sin 18 = \frac{1}{8} 0.31 = 0.04$

$x = 0.2$, $u = \frac{1}{8} \sin 36 = \frac{1}{8} (0.56) = 0.07$

$x = 0.3$, $u = \frac{1}{8} \sin 54 = \frac{1}{8} (0.81) = 0.10$

$x = 0.4$, $u = \frac{1}{8} \sin 72 = \frac{1}{8} (0.95) = 0.12$

$x = 0.5$, $u = \frac{1}{8} \sin 90 = \frac{1}{8} 1 = 0.13$

$x = 0.6$, $u = \frac{1}{8} \sin 108 = \frac{1}{8} 0.95 = 0.12$

$x = 0.7$, $u = \frac{1}{8} \sin 126 = \frac{1}{8} 0.81 = 0.10$

$x = 0.8$, $u = \frac{1}{8} \sin 144 = \frac{1}{8} (0.59) = 0.7$

$x = 0.9$, $\quad u = \frac{1}{8} \sin 162 = \frac{1}{8}(0.31) = 0.94$

$x = 1$, $\quad u = \frac{1}{8} \sin \pi = 0$

$[\pi = 180^\circ]$

... condition is $\quad u_t(x, 0) = 0$

$$\text{But } u_t = \frac{\partial u}{\partial t}$$

$$\therefore \frac{\partial u}{\partial t} = 0$$

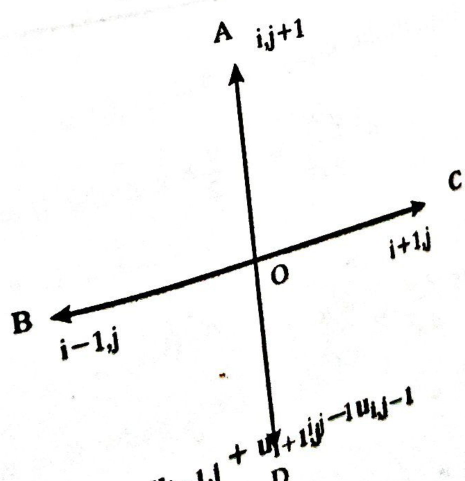
$$\Rightarrow \frac{u_{i,j+1} - u_{i,j}}{k} = 0$$

$$u_{i,j+1} - u_{i,j} = 0$$

$$u_{i,j+1} = u_{i,j}$$

$\Rightarrow$ I row and II row are same.

Other values can be obtained using the following figure.



$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

0	−0.02	0.59	0.23	0.16	−0.68	0.42	−0.04	−0.04	−0.02	0
0	−0.01	−0.01	0.62	0.26	0.18	−0.64	0.46	−0.01	−0.01	0
0	0.01	0.02	0.02	0.64	0.3	0.22	−0.61	0.49	0.01	0
0	0.03	0.04	0.04	0.06	0.68	0.33	0.25	−0.59	0.5	0
0	0.03	0.05	0.08	0.08	0.09	0.71	0.35	0.26	−0.6	0
0	0.03	0.07	0.09	0.11	0.11	0.11	0.72	0.34	−0.24	0
0	0.04	0.07	0.1	0.12	0.13	0.12	0.1	0.7	0.94	0
0	0.04	0.07	0.1	0.12	0.13	0.12	0.1	0.7	0.94	0

3. Solve $25\,u_{xx} = u_{tt}$ given $u_t(x, 0) = 0$, $u(0, t) = 0$, $u(5, t) = 0$ and

$$u(x, 0) = \begin{cases} 20x & \text{for } 0 \le x < 1 \\ 5(5 - x) & \text{for } 1 \le x \le 5 \end{cases}$$

Solution : The hyperbolic equation is

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

The problem is $\quad 25\,u_{xx} = u_{tt}$

i.e., $$25 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\therefore\ a^2 = 25$$

$$\therefore\ a = 5$$