Iterative Methods to find Solution of system of linear equations

* In iterative methods (Indirective Methods) we first develop a rule to find the best possible solution.

* We apply with an initial approximation and continue with this rule to get a better solution (Convergent) in many steps.

* When we apply the rule repetitively, then each successive calculation to determine the next approximation to the solution is called an iteration.

* Iteration Method is a self-correcting method, since the error made in any computation is corrected in the subsequent iterations.

* There are two iterative method to find solution of system of linear equations of any order.

(2) Gauss - Seidal Iterative Method

(1) Gauss - Jacobi Iterative Method

Advantage of Iterative Methods:

* ① For the large system of equations direct method can ~~be~~ not be applied, while iterative method may give the solution for any large system.

* ② These are Numerical Computational technique and may be faster than the direct method.

* ③ Round off errors in iterative methods are smaller.

* ④ Iterative method is a self correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

* ⑤ The method is very useful with less computation for the given system of equation whose augmented Matrix have a large no. of zero element.

* Disadvantage of iterative method

* ① For all system of equations, these method will not work (since convergent is not assured)

* ② It converges only for special system of equations in which condition of diagonal Dominance is satisfied.

Diagonal Dominant :-

A matrix is called diagonal dominant if the numerical value of the leading diagonal element in each row, is greater than or equal to the sum of numerical values of the other elements in that row.

Ex |- $\begin{bmatrix} 5 & 1 & -1 \\ 1 & 4 & 2 \\ 1 & -2 & 5 \end{bmatrix}$ whose equations are

$$5x + y - z = 1$$
$$x + 4y + 2z = 2$$
$$x - 2y + 5z = 3$$

is diagonal Dominent as

$$|5| > |1| + |-1| , \quad |4| > |1| + |2| , \quad |5| > |1| + |-2|$$

But $\begin{bmatrix} 5 & 1 & -1 \\ 5 & 2 & 3 \\ 1 & -2 & 5 \end{bmatrix}$ is not diagonal dominance

*③ For the iterative method to converge quickly, the coefficient matrix must be diagonally dominant. If it is not so, we have to arrange the equations in such a way that the coefficient matrix is diagonally dominant and then only we can apply these method.

STEPS:-

Steps are as follows:-

Consider the equations

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\}$$

step 1
Check diagonal domiance (If not rearrange the eq.)

$$|a_1| \geq |b_1| + |c_1|, \quad |b_2| > |a_2| + |c_2|, \quad |c_3| > |a_3| + |b_3|$$

step 2 If diagonal domienance is satisfied write the equations as

$$\left.\begin{aligned} x &= \frac{1}{a_1}\left[d_1 - b_1 y - c_1 z\right] \\ y &= \frac{1}{b_2}\left[d_2 - a_2 x - c_2 z\right] \\ z &= \frac{1}{c_3}\left[d_3 - a_3 x - b_3 y\right] \end{aligned}\right\}$$

step 3

Gauss Jacobi

In absence of better estimates x_0, y_0, z_0 we take $x_0=0$, $y_0=0$, $z_0=0$ in complete first iteration

first iteration

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

2nd iteration

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

— — — —

Iteration is stopped when the values of x, y, z start repeating with the required degree of accuracy.

Gauss-Seidal.

In absence of better estimates x_0, y_0, z_0. we take $y_0=0$, $z_0=0$ in first equation in first iteration only.

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

2nd iteration

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_2 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_2 - b_3 y_2)$$

— — — —

Iteration is stopped when the values of x, y, z starts almost same with the desired degree of decimal.

Examples :- (Practice)

Solve following by iterative method

$20x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

* **Step 1 :-** Check diagonal Dominance

$|20| > |1| + |-2|, \quad |20| > |3| + |-1|, \quad |20| > |2| + |-3|$

Satisfied

* **Step 2 :-** Rewrite the equations as

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\}$$

* **Step 3 :-** Start the iteration

Gauss Jacobi

$x_0 = 0,\ y_0 = 0,\ z_0 = 0$ (put in first iteration)

$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = \frac{17}{20} = 0.85$

$y_1 = \frac{1}{20}(-18 - 3x_0 + z_0) = \frac{-18}{20} = -0.9$

$z_1 = \frac{1}{20}(25 - 2x_0 + 3y_0) = \frac{25}{20} = 1.25$

Gauss Seidal

$y_0 = 0,\ z_0 = 0$ (put in first equation) of first iteration

$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$

$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0)$

$= \frac{1}{20}(-18 - 3 \times (.8500) + 0) = -1.0275$

5th approximation

$$x_5 = \frac{1}{20}(-17 - y_4 + 2z_4) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.000078$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.999956$$

Hence $x_5 = 0.999966$, $y_5 = -1.000078$, $z_5 = 0.999956$

6th approximation

$$x_6 = \frac{1}{20}(-17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -0.999997$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 0.999992$$

$x_6 = 1.0000$, $y_6 = -0.999997$, $z_6 = 0.999992$

The values in 5th and 6th iterations being practically the same, hence solⁿ is $x=1$, $y=-1$, $z=1$

IMP.

$\boxed{\text{Note :}}$- The convergence in the Gauss-Seidel method is twice as fast as in Jacobi Method as in the Gauss-Seidel most recent approximations of the unknowns are used while proceeding to the next step.

Now.

$x_1 = 0.85,\ y_1 = -0.9,\ z_1 = 1.25$

2nd approximation :-

$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$

$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$

$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.03$

now

$x_2 = 1.02,\ y_2 = -0.965,\ z_2 = 1.03$

3rd approximation :-

$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.00125$

$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -1.0015$

$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.00325$

4th approximation :-

$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0004$

$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.000025$

$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9965$

Continue ....

$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1)$

$= \frac{1}{20}\left[25 - 2\times(0.8500) + 3\times(-1.0275)\right]$

$= 1.0109$

$x_1 = 0.8500,\ y_1 = -1.0275,$
$z_1 = 1.0109$

2nd approximation

$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$

$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$

$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$

Now,

$x_2 = 1.0025,\ y_2 = -0.9998,\ z_2 = 0.9998$

3rd approximation

$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$

$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0001$

$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$

Note: The values in 2nd & 3rd iterations being practically the same we can stop

Hence soln is

$x = 1,\ y = -1,\ z = 1$

Activity 2: -

Solve the equations.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

by Gauss Jacobi and Gauss Seidal method

Tabular method.

Gauss Jacobi Tablular method

Gauss Seidal Tabular method.

Ques 2 Practice

(361) Theorem.

Solve the following system by Gauss-Jacobi and Gauss-Seidal methods:

$$\left.\begin{aligned} 10x - 5y - 2z &= 3 \\ 4x - 10y + 3z &= -3 \\ x + 6y + 10z &= -3 \end{aligned}\right\}$$

Ans: - $x = 0.342$
$y = 0.285$
$z = -0.505$

Also Make the table.

Iteration(i)	Gauss Jacobi x_i	y_i	z_i	Gauss Seidal x_i	y_i	z_i
1	—	—	—	—	—	—
2	—	—	—	—	—	—
3	—	—	—	—	—	—
				—	—	—