Module – 3

Numerical Differentiation and Integration

* Derivatives using Newton's forward and Backward interpolations.
* Newton-Cotes quadrature rule
* Trapezoidal rule
* Simpson's 1/3 Rule
* Simpson's 3/8 Rules
* Gaussian quadrature rule
  Two point & three point Gauss-Legendre
* quadratic rules
  Composite quadrature rules
* Romberg method.

* Derivatives using Newton's forward and Backward interpolations.

To find the derivative at a given point (x_0) we have to apply Newton's forward difference formula or Backward formula according to the point at which the derivative is required.

Calculation is same as in interpolation difference table.

Method and derivation of formula

Let $y = f(x)$ be given for the independent variable $x = x_0 + ph$ where $p = 0, 1, 2, 3 \ldots$

To find the derivative of such tabular function

Case I :- using forward difference operator Δ

$$\because \quad hD = \log E$$

$$hD = \log(1+\Delta)$$

$$hD = \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \cdots\right)$$

$$\therefore \quad D = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} - \cdots\right)$$

or $$D f(x_0) = \frac{1}{h}\left(\Delta f(x_0) - \frac{\Delta^2 f(x_0)}{2} + \frac{\Delta^3 f(x_0)}{3} - \frac{\Delta^4 f(x_0)}{4} + \cdots\right)$$

$$\therefore \quad D = f'(x_0) = \frac{d}{dx} f(x_0) = \frac{1}{h}\left(\cancel{\Delta f(x_0)}\ \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} - \cdots\right)$$

(1)

$\therefore \; y = f(x)$

$\therefore \; x = x_0 + ph$

$\Rightarrow \quad x_0 = x - ph$

$$\boxed{\begin{array}{c} \text{at } x = x_0 \\ p = 0 \\ \text{at } x = x_1 \\ p = \dfrac{x_1 - x_0}{h} \end{array}}$$

$\cancel{\frac{d}{dx}(x-ph)}$

Alternate proof!-

$$\because \; f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

where $x = x_0 + ph$ & $p = \dfrac{x - x_0}{h}$.

$$\therefore \; f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

diff w.r.to p.

$$h f'(x_0 + ph) = \Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2 - 6p + 2)}{6}\Delta^3 y_0 + \frac{(2p^3 - 9p^2 + 11p - 3)}{12}\Delta^4 y_0 + \cdots \quad \text{(A)}$$

Also

$$h^2 f''(x_0 + ph) = \Delta^2 y_0 + \frac{\cancel{6}(p-1)}{\cancel{6}}\Delta^3 y_0 + \frac{(6p^2 - 18p + 11)}{12}\Delta^4 y_0 + \cdots \quad \text{(B)}$$

$$h^3 f'''(x_0 + ph) = \Delta^3 y_0 + \frac{(12p - 18)}{12}\Delta^4 y_0 + \cdots \quad \text{(C)}$$

$\because \; p = \dfrac{x - x_0}{h} \; \therefore$, at $\boxed{x = x_0}$. $p = \dfrac{x_0 - x_0}{h} = 0$

$$\therefore f'(x_0) = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 - - -\right) \quad (D)$$

$$f''(x_0) = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - - - -\right) \quad (E)$$

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + - -\right] \text{ and so on} \quad (F)$$

Backward formula :-

$$\boxed{\begin{aligned} &\because x = x_n + ph \\ &\text{at } x = x_n \quad p = 0 \text{ at} \\ &x = x_1, \quad p = \frac{x_1 - x_n}{h} \end{aligned}}$$

$$\because f(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{\lfloor 4}\nabla^4 y_n + - - -$$

diff w.r.to p.

$$h f'(x_n + ph) = \nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2 + 6p + 2}{6}\right)\nabla^3 y_n + \frac{2p^3 + 9p^2 + 11p + 3}{12}\nabla^4 y_n + - - \quad (G)$$

(H)

Again diff w.r.to p

$$h^2 f''(x_n + ph) = \nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2 + 18p + 11}{12}\nabla^4 y_n + - - \quad (I)$$

$$h^3 f'''(x_n + ph) = \nabla^3 y_n + \frac{2p+3}{2}\nabla^4 y_n$$

At $x = x_n$

$$f'(x_n) = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + - -\right) \quad (J)$$

$$f''(x_n) = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + - -\right) \quad (K)$$

$$f'''(x_n) = \frac{1}{h^3}\left(\nabla^3 y_n + 3/2\,\nabla^4 y_n + - - -\right) \text{ and so on} \quad (L)$$

[By Newton's forward interpolation]

Example 1 find the first, second and third derivatives of the function tabulated at the point $x = 1.5$

Soln :-

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.0	13.625	24.0	38.875	59.0

① prepare difference table first.

Note :- If x is initial value i.e $x = x_0$ Use formula D E F

Since Here $x = 1.5 = x_0$. hence we will apply formula D, E and F

$\boxed{h = 0.5}$

$$f'(1.5) = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 - - -\right) \quad ①$$

$$f''(1.5) = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - - -\right. \quad ②$$

$$f'''(1.5) = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + - - - -\right. \quad ③$$

Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0 1.5	$\boxed{y_0}$ 3.375				
		3.625 $\boxed{\Delta y_0}$			
x_1 2.0	y_1 7.0		3.0 $\boxed{\Delta^2 y_0}$		
		6.625		0.75 $\boxed{\Delta^3 y_0}$	
x_2 2.5	y_2 13.625		3.75		0 $\boxed{\Delta^4 y_0}$
		10.375		0.75	
x_3 3.0	y_3 24.0		4.50		0
		14.875		0.75	
x_4 3.5	y_4 38.875		5.25		
		20.125			
x_5 4.0	y_5 59.0				

$$\therefore f'(1.5) = \frac{1}{1.5}\left[3.625 - \frac{1}{2}(3.0) + \frac{1}{3}(0.75)\right] \quad \text{From ①}$$

$$= 4.75$$

by ②

$$f''(1.5) = \frac{1}{(1.5)^2}\left[3.0 - 0.75\right] = \underline{9.0}$$

by ③

$$f'''(1.5) = \frac{1}{(1.5)^3}\left[0.75 - 0\right] = \underline{6.0}$$

$\therefore$ Ans $f'(1.5) = 4.75$

$f''(1.5) = 9.0$

$f'''(1.5) = 6.0$

Example:- 2 find the value of $f'(0.04)$ using appropriate formula, (N.F.) for the given data:
(Use Newton's formula forward).

Sol^n:-

x:	0.01	0.02	0.03	0.04	0.05	0.06
y:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Sol Since Here $x = 0.04 \neq x_0 = 0.01$ apply formula (A) (B) and (C).

$$f'(x_0+ph) = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \frac{(2p^3-9p^2+11p-3)}{12}\Delta^4 y_0 + \cdots\right] \quad ①$$

$\because x_0 + ph = 0.04$

$(0.01) + p(0.01) = 0.04$

$\boxed{\therefore p = 3}$

Difference table

x		$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	0.01	(y_0) 0.1023					
x_1	0.02	y_1 0.1047	0.0024 (Δy_0)				
x_2	0.03	y_2 0.1071	0.0024	0.0000 ($\Delta^2 y_0$)			
x_3	0.04	y_3 0.1096	0.0025	0.0001	0.0001 ($\Delta^3 y_0$)		
x_4	0.05	y_4 0.1122	0.0026	0.0001	0.000	-0.0001 ($\Delta^4 y_0$)	
x_5	0.06	y_5 0.1148	0.0026	0.0000	-0.0001	-0.0001	0.0000 ($\Delta^5 y_0$)

$$y'(0.04) = \frac{1}{0.01}\left(.0024 + \left(\frac{6-1}{2}\right)(0.0000) + 0\right)$$

$$= \frac{1}{0.01}(0.0024)$$

$$\boxed{y'(0.04) = 0.24} \rightarrow \text{Ans}$$

Example 3. Based on forward formula

The population of a certain town is shown below:

Year	1931	1941	1951	1961	1971
population	40.6	60.8	79.9	103.6	132.7

find the rate of growth for 1961 and 1971

Sol^n :- Here $\boxed{h = 10}$ $\quad \because x_n + ph = x$

$x_n = 1971$ & $x = 1961$

$x_n + ph = 1961$

$1971 + p \times (10) = 1961 \Rightarrow p = \frac{1961 - 1971}{10} = -1$

$\boxed{p = -1}$

By. (ii) formula.

$$y'(1961) = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{(3p^2+6p+2)}{6}\nabla^3 y_n + \frac{(2p^3+9p^2+11p+3)}{12}\nabla^4 y_n - -\right]$$

prepare difference table

x (year)	y (population)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0 1931	y_0 40.6	20.2	-1.1	5.7	-4.9 $\nabla^4 y_n$
x_1 1941	y_1 60.8	19.1	4.6	0.8 $\boxed{\nabla^3 y_n}$	
x_2 1951	y_2 79.9	23.7	5.4 $\boxed{\nabla^2 y_n}$		
x_3 1961	y_3 103.6	29.1 $\boxed{\nabla y_n}$			
x_n 1971	$\boxed{y_n}$ 132.7				

$$\therefore y'(1961) = \frac{1}{10}\left[29.1 + \left(\frac{-1}{2}\right)(5.4) + \left(\frac{-1}{6}\right)(0.8) + \left(\frac{-1}{12}\right)(-4.9)\right]$$

$$= 2.6675$$

$\therefore$ The rate of growth of the population in the year 1961 is 2.6675 thousands.

= =

find $y'(1971)$ yourself

Using (I) formula.

$$f'(1971) = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + --\right)$$

Get the Result ??