Gauss Quadrature formula

In this method instead of finding the area under the given curve Gauss tried to find it at under some other points.

* Gauss changed the interval (a, b) to $(-1, 1)$ by using the transformation

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) t$$

* Hence the variable x changed to t.

* He used to find interpolation at two points say t_1 and t_2

Hence

$$\boxed{\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt = \omega_1 f(t_1) + \omega_2 f(\omega_2)} \quad (1)$$

Here ω_1, ω_2, t_1 and t_2 are Unknown quantities. To find it

Let $f(t) = 1$, $f(t) = t$, $f(t) = t^2$, $f(t) = t$

(2)

When $f(t) = 1$ Hence $f(t_1) = 1$ & $f(t_2) = 1$

$$\int_{-1}^{1} 1\, dt = 2 = w_1 + w_2$$

$$\therefore \quad w_1 + w_2 = 2 \quad \text{— (2)}$$

When $f(t) = t$ $f(t_1) = t_1$ & $f(t_2) = t_2$

$$\therefore \int_{-1}^{1} t\, dt = \left(\frac{t^2}{2}\right)_{-1}^{1} = 0 = w_1 t_1 + w_2 t_2$$

$$\therefore \quad w_1 t_1 + w_2 t_2 = 0 \quad \text{— (3)}$$

When $f(t) = t^2$ $f(t_1) = t_1^2$, $f(t_2) = t_2^2$

$$\int_{-1}^{1} t^2\, dt = \left(\frac{t^3}{3}\right)_{-1}^{1} = 2/3 = w_1 t_1^2 + w_2 t_2^2$$

$$\therefore \quad w_1 t_1^2 + w_2 t_2^2 = 2/3 \quad \text{— (4)}$$

When $f(t) = t^3$. $f(t_1) = t_1^3$, $f(t_2) = t_2^3$

$$\therefore \int_{-1}^{1} t^3\, dt = \left(\frac{t^4}{4}\right)_{-1}^{1} = 0 = w_1 t_1^3 + w_2 t_2^3$$

$$\therefore \quad w_1 t_1^3 + w_2 t_2^3 = 0 \quad \text{— (5)}$$

Solve (2), (3), (4) & (5)

From (3)

$w_1 t_1 + w_2 t_2 = 0$ (3)

$\therefore w_1 t_1 = -w_2 t_2$ (6)

from (5)

$w_1 t_1^3 = -w_2 t_2^3$ (7)

From (6) & (7)

$t_1 = -t_2$

$\frac{w_1 t_1}{w_1 t_1^3} = \frac{-w_2 t_2}{-w_2 t_2^3}$

$t_1^2 = t_2^2$

$t_1^2 - t_2^2 = 0$

$t_1 = t_2$ &

$t_1 = -t_2$

hence $\underline{w_1 = w_2 = 1}$

From (4)

$t_1^2 + t_2^2 = 2/3$

$t_1 = \frac{1}{\sqrt{3}}$ and $t_2 = -\frac{1}{\sqrt{3}}$

Hence from (1)

$$\boxed{\begin{aligned} I &= \int_{-1}^{1} f(t)\,dt = w_1 f(t_1) + w_2 f(t_2) \\ I &= f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) \end{aligned}}$$

This is known as Gauss

Two point formula

Ans. 1 Evaluate

$\int_1^2 \frac{dx}{x}$ using Gauss 2 point formula.

Sol^n:- Let $x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) t$

& $t = -1$ to 1. & $a = 1$, $b = 2$.

$dx = \frac{dt}{2}$.

$$I = \frac{1}{2}\int_{-1}^{1} \frac{dt}{\frac{(1+2)}{2} + \left(\frac{2-1}{2}\right) t}$$

$$= \frac{1}{\cancel{2}}\int_{-1}^{1} \frac{\cancel{2}\cdot dt}{(3+t)} = \int_{-1}^{1} \frac{dt}{3+t}$$

$$f(t) = \frac{1}{(3+t)}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3+\frac{1}{\sqrt{3}}} = 0.2795$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{3-\frac{1}{\sqrt{3}}} = 0.41288$$

$\therefore I = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$ where $f(t) = \frac{1}{3+t}$

$= 0.2795 + 0.41288$

$\boxed{I = 0.6923}$

Ques 2 $\int_1^2 \frac{dx}{1+x^3}$ using Gaussian 2 point formula. (5)

Soln:- $x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) t$

$\therefore dx = \frac{dt}{2}$, $a = 1$, $b = 2$

$x = \frac{3+t}{2}$

$$I = \int_1^2 \frac{dx}{1+x^3} = \frac{1}{2}\int_{-1}^{1} \frac{dt}{1+\left(\frac{3+t}{2}\right)^3}$$

$$= 4\int_{-1}^{1} \frac{dt}{8+(3+t)^3}$$

$$= 4 \times \left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$\because f(t) = \frac{1}{8+(3+t)^3}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{8+\left(3+\frac{1}{\sqrt{3}}\right)^3} = 0.0185$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{8+\left(3-\frac{1}{\sqrt{3}}\right)^3} = 0.045$$

$$\therefore I = \int_1^2 \frac{dx}{1+x^3} = 4\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= 4\left[0.0185 + 0.045\right]$$

Ans: $\boxed{I = 0.254}$

Qus Try !-

3.68 (1) $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss 2 point

3.69 (2) $\int_{-1}^{1} (3x^2 + 5x^4)\, dx$.

(3) $\int_{0}^{\pi/2} \sin x\, dx$ by Gauss 2 point.

(4) $\int_{0}^{\pi/2} \log(1+x)\, dx$ by Gauss 2 point

Gauss Quadrature (3 point)

$$\int_a^b f(x)\, dx = \int_{-1}^{1} f(t)\, dt$$

Let $x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) t$.

$$\int_{-1}^{1} f(t)\, dt = A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)$$

where
$A_1 = 0.5555$
$A_2 = 0.8888$
$t_1 = -0.7745$
$t_2 = 0$
$t_3 = 0.7745$

Q) Evaluate

$\int_1^2 \frac{dx}{x}$ using Gauss 3-point

Sol^n :- Let $x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) t$

$x = \frac{3+t}{2}$, $dx = \frac{dt}{2}$

$I = \int_1^2 dx = \int_{-1}^{1} f(t)\, dt$ $= \frac{dt}{2}\int_{-1}^{1} \times \frac{2\,1}{(3+t)}$ $\therefore f(t) = \frac{1}{3+t}$

$= A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)$

Here $A_1 = 0.5555 = A_3$
$A_2 = 0.8888$ (1)
$t_1 = (-0.7745)$, $t_2 = 0$, $t_3 = (0.7745)$

$f(t_1) = f(-0.7745)$ $\therefore f(t) = \frac{1}{3+t}$

$\therefore f(t_1) = \frac{1}{3 - 0.7745} = 0.4493$

$f(t_2) = f(0) = \frac{1}{3} = 0.3333$

$f(t_3) = f(0.7745) = \frac{1}{3+0.7745} = 0.2649$

$\therefore I = 0.5555(0.4493) + 0.8888(0.3333) + (0.2649)(0.5555)$

$\boxed{I = 0.6929}$ Ans

Qus: $\int_{0.2}^{1.5} e^{-x^2}\,dx$ using three point Gaussian Quadrature.

Sol^n! - Change x by t by using

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$

$a = 0.2$ & $b = 1.5$, $dx = \frac{dt}{2}$.

$$x = \frac{(1.7) + (1.3)t}{2}, \quad dx = \frac{(1.3)t}{2} = 0.65\,dt$$

$$\therefore I = \int_{0.2}^{1.5} e^{-x^2}\,dx = \int_{-1}^{1} e^{-\left(\frac{1.7+1.3t}{2}\right)^2} (0.65)\,dt$$

$$= 0.65 \int_{-1}^{1} e^{-\left(\frac{1.7+1.3t}{2}\right)^2} dt \quad - (1)$$

$$I = 0.65\left[A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)\right] - (2)$$

$$f(t) = e^{-\left(\frac{1.7+1.3t}{2}\right)^2}$$

$A_1 = A_3 = 0.5555$, $A_2 = 0.8888$, $t_1 = -0.7745$
$t_2 = 0$, $t_3 = 0.7745$

$$f(t_1) = (-0.7745) = e^{-\left(\frac{1.7+1.3(-0.7745)}{2}\right)^2}$$

$$\left.\begin{aligned} &= 0.8868 \\ f(t_2) &= 0.48555 \\ f(t_3) &= 0.16013 \end{aligned}\right\} (3).$$

Substituting in 2 we get,

$I = 0.5555(0.8868) + 0.8888(0.4855) + (0.5555)(0.16013)$

$I = 0.4926 + 0.4315 + 0.08895$

$\boxed{I = 1.01307}$: Ans

Q $\int_0^1 \frac{1}{1+t}\,dt$ by Gaussian form (3 point)

Let $t = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)x.$

$a = 0, \quad b = 1$

$t = \frac{1}{2} + \frac{x}{2} \quad ; \quad dt = \frac{dx}{2}$

$\therefore t = \frac{x+1}{2}$

$t = 0, \quad x = -1$

$t = 1, \quad x = 1$

$\therefore I = \int_0^1 \frac{dt}{1+t} = \frac{1}{2}\int_{-1}^{1} \frac{dx}{1+\left(\frac{x+1}{2}\right)} = \int_{-1}^{1} \frac{dx}{2+x+1}$

$I = A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)$

$f(x) = \frac{1}{x+3}.$

$A_1 = 0.5555 \quad A_2 = 0.8888 \quad A_3 = 0.5555$

$x_1 = -0.7745$, $x_2 = 0$, $x_3 = 0.7745$.

$\therefore f(x_1) = f(-0.7745) = \frac{1}{(-0.7745+3)} = 0.4493$

& $f(x_2) = f(0) = \frac{1}{3}$

$f(x_3) = f(0.7745) = \frac{1}{0.7745+3} = 0.2649$

Hence

$I = [(0.5555 \times 0.4493) + (0.8888 \times 0.333) + (0.5555 \times 0.2649)]$

$= 0.2495 + 0.2962 + 0.14715$

$\boxed{I = 0.69285}$ $\therefore$ Ans

Q Evaluate following by Gaussian 3 point formula:

3.80. (a) $\int_1^2 \frac{dx}{1+x^3}$

3.79 (b) $\int_0^1 \frac{dx}{1+x^2}$.