Q.1 Romberg's Method :-

Compute $\int_0^1 \frac{dx}{1+x^2}$ by using Trapezoidal Rule, taking $h=0.5$, $h=0.25$, Hence find the value of above by Romberg's method.

Sol^n/.-

Steps for Romberg's Method :

Apply Trapezoidal Rule several times by halving 'h' successively and find I'S as follows:

I_1 : $0 \quad \frac{h}{2} \quad h$ (Two parts)

1 2

I_2 ! $0 \quad \frac{h}{4} \quad \frac{h}{2} \quad \frac{3h}{2} \quad h$ (four parts)

1 2 3 4

I_3 ! $0 \quad \frac{h}{8} \quad \frac{h}{4} \quad \frac{3h}{8} \quad \frac{4h}{8} \quad \frac{5h}{8} \quad \frac{6h}{8} \quad \frac{7h}{8} \quad h$ (Eight parts)

1 2 3 4 5 6 7 8

I_4 ! $0 \quad \frac{h}{16} \quad \frac{2h}{16} \quad \frac{3h}{16} \quad \frac{4h}{16} \quad \frac{5h}{16} \quad \frac{6h}{16} \quad \frac{7h}{16} \quad \frac{8h}{16} \quad \frac{9h}{16} \quad \frac{10h}{16} \quad \frac{11h}{16} \quad \frac{12h}{16} \quad \frac{13h}{16} \quad \frac{14h}{16} \quad \frac{15h}{16} \quad \frac{16h}{16}$ (16 parts)

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Apply

$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$

$I = I_3 + \left(\frac{I_3 - I_2}{3}\right)$

$I = I_4 + \left(\frac{I_4 - I_3}{3}\right)$

...

The method may continue till we get two successive I's are equal

Sol^n Here $y = \dfrac{1}{1+x^2}$

& $h = 0.5$

I_1 :-

x	0	0.5	1.0
$y = \dfrac{1}{1+x^2}$	1	0.8	0.5
	y_0	y_1	y_2

By Trapezoidal Rule

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_2) + 2y_1\right]$$

$$= \frac{0.5}{2}\left[(1+0.5) + 2(0.8)\right]$$

$$= 0.775$$

I_2 :

x	0	0.25	0.5	0.75	1.00
$y = \dfrac{1}{1+x^2}$	1	0.9412	0.80	0.64	0.50
	y_0	y_1	y_2	y_3	y_4

$\therefore$ I_2 by Trapezoidal Rule

$$I_2 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{0.25}{2}\left[(1+0.5) + 2(0.9412 + 0.80 + 0.64)\right]$$

$$= 0.7828$$

$$\therefore I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.7828 + \frac{(0.7828 - 0.775)}{3}$$

Ans: $= 0.78$

Q.2 :- Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's Method. Hence obtain approximate value of π

Soln :- Here $y = \frac{1}{x^2+4}$.

I_1 : $h = 1$. $\left[\because h = \frac{0+2}{2} = 1\right]$

x	0	1	2
$y = \frac{1}{x^2+4}$	0.25	0.20	0.125
	y_0	y_1	y_2

By Trapezoidal Rule :

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_2) + 2y_1\right]$$

$$= \frac{1}{2}\left[(0.25+0.125) + 2(0.20)\right]$$

$$= 0.3875$$

I_2 : $h = 0.5$ $\left[\because h = \frac{0+2}{4} = 0.5\right]$

x	0	0.5	1	1.5	2
$y = \frac{1}{x^2+4}$	0.25	0.2353	0.20	0.160	0.125
	y_0	y_1	y_2	y_3	y_4

$$I_2 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{0.5}{2}\left[(0.25+0.125) + 2(0.2353+0.2+0.16)\right]$$

$$= 0.3914$$

I_3: $h = 0.25$ $\left[\text{as } \frac{0+2}{8} = 0.25\right]$ (4)

x:	0	0.25	0.5	0.75	1	1.25	1.50	1.75	2.00
$y=\frac{1}{x^2+4}$:	0.25	0.2462	0.2353	0.2192	0.20	0.1798	0.160	0.1416	0.125
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Trapezoidal Rule:

$$I_3 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_8) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \frac{0.25}{2}\left[(0.25)+(0.125) + 2(0.2462 + 0.2353 + 0.2192 + 0.20 + 0.1798 + 0.16 + 0.1416)\right]$$

$$\boxed{\therefore\ I_3 = 0.3924}$$

Using Romberg's formula for I, taking I_1 & I_2

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.3914 + \frac{(0.3914 - 0.3875)}{3}$$

$$= \underline{0.3927} \quad (1)$$

Using Romberg's formula for I, taking I_2 & I_3

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.3924 + \frac{(0.3924 - 0.3914)}{3}$$

$$= \underline{0.3927} \quad (2)$$

From (1) & (2) $I = \int_0^2 \frac{dx}{1+x^2} = 0.3927$ (3)

Exact Integration

$$I = \int_0^2 \frac{dx}{4+x^2} = \int_0^2 \frac{dx}{x^2+(\cancel{2})^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{1}{2}\left[\tan^{-1}\cancel{\tfrac{2}{2}} - \tan^{-1}0\right]$$

$$= \frac{1}{2} \times \frac{\pi}{4}$$

$$I = \frac{\pi}{8}$$

$\because$ from ③ $I = 0.3927$

$\therefore 0.3927 = \frac{\pi}{8}$

$\Rightarrow \pi = 8(0.3927)$

$$\boxed{\pi = 0.31416 \cdots \text{ (approx.)}}$$

P) 3.59.

Q Evaluate $I = \int_0^{1/2} \frac{x}{\sin x}\,dx$ Correct to 3 decimal places using Romberg's method.

(Apply Simpson's here to obtain $I_1, I_2, I_3 ---$

Romberg's Method :-

This method is used to remove the error occur in Trapezoidal Rule.

Since in Trapezoidal the Error occurs

$$E = -\frac{(b-a)}{12} h^2 \, y''(\xi)$$

$$= c h^2 \quad \text{where } c = -\frac{(b-a)}{12} y''(\xi).$$

Let us find the integral for Romberg's Method.

$$\because \; I = \int_a^b y \, dx \quad \text{where } y = f(x). \quad \text{—(1)}$$

Let us use Trapezoidal Rule to find the integral of (1) for two subintervals h_1 and h_2

Let I_1 be the integral corresponding to h_1 with Error E_1

and I_2 be the integral corresponding to h_2 with Error E_2

Then clearly

$$I = I_1 + E_1$$
$$= I_1 + c h_1^2 \quad - (2)$$

$$\& \quad I = I_2 + E_2$$
$$= I_2 + c h_2^2 \quad - (3)$$

From (2) & (3)

$$I_1 + c h_1^2 = I_2 + c h_2^2$$

$$\Rightarrow \quad c(h_1^2 - h_2^2) = I_2 - I_1$$

$$\text{or} \quad c = \frac{I_2 - I_1}{h_1^2 - h_2^2} \quad \text{or} \quad \frac{I_1 - I_2}{h_2^2 - h_1^2} \quad - (4)$$

From (2).

$$I = I_1 + \left(\frac{I_1 - I_2}{h_2^2 - h_1^2} \right) h_1^2$$

$$= \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \quad - (5)$$

Now let $h_1 = h$ & $h_2 = h/2$ we get

From (5) $I = \dfrac{I_1 \frac{h^2}{4} - I_2 h^2}{h^2 - h^2}$

$$= \frac{\cancel{h^2}\left(\frac{I_1}{4} - I_2\right)}{\cancel{h^2}\left(\frac{1}{4} - 1\right)}$$

$$= \frac{I_1 - 4I_2}{-3} = \frac{4I_2 - I_1}{3}$$

$$\boxed{I = I_2 + \frac{I_2 - I_1}{3}}$$

Let us apply Trapezoidal Rule several times by halving h successively and the find the value of I's. The method may continue till we get two successive values of I's are equal. The systemic refinement of the value's of I's are called Romberg's method:

<u>Method in short !-</u>

$I_1 \rightarrow$ Divide h in two two parts ($h/2$) $n=2$

$I_2 \rightarrow$ Divide h into 4 parts $\frac{h}{4}$)

$I_3 \rightarrow$ " " " 8 parts $h/8$)

and so on.