Newton's Cotes Quadrature formula:

Let $I = \int_a^b f(x)\,dx$ is a definite integral in which a and b are Constants and $f(x)$ is a function given. Then I is the Area under the curve of $f(x)$ between $x = a$ and $x = b$.

Importance!- In engineering problem we apply the numerical integration of the function of a single variable given by a table. In this case we just use $f(x) = y$ by interpolation formula and then integrate this formula between the limits a and b. Hence in this way we derive the Cote's Quadrature formula.

Let $$I = \int_{x_0}^{x_0+nh} y(x)\,dx \quad \text{— (1)}$$ where h is width and n is equidistant intervals

Consider the Newton's forward formula.

$$y(x) = y(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots \quad \text{— (2)}$$

Now

$$I=\int_{x_0}^{x_0+nh} y(x)\,dx=\int_{x_0}^{x_0+nh}\left[y_0+p\Delta y_0+\frac{p(p-1)}{\lfloor 2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0+\cdots\right]dx$$

Let $x = x_0 + ph$

$\Rightarrow$ $dx = h\,dp$.

Limits when $x = x_0$; $\boxed{p=0}$ (as $x_0 = x_0 + ph$ $\Rightarrow p = 0$.

when $x = x_0 + nh$

then $x = x_0 + ph$ gives $\cancel{x_0} + nh = \cancel{x_0} + ph$

$\Rightarrow \boxed{p=n}$

$$\therefore\ I = h\int_0^n\left[y_0+p\Delta y_0+\frac{p(p-1)}{\lfloor 2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0+\cdots\right]dp.$$

$$=h\left[(p\,y_0)_0^n+\left(\frac{p^2}{2}\Delta y_0\right)_0^n+\left(\frac{p^3}{6}-\frac{p^2}{4}\right)_0^n\Delta^2 y_0+\left(\frac{p^4}{24}-\frac{p^3}{6}+\frac{p^2}{6}\right)_0^n\Delta^3 y_0+\cdots\right]$$

$$=h\left[n y_0+\frac{n^2}{2}\Delta y_0+\frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0+\frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0+\cdots\right]$$

$$\therefore\ \int_{x_0}^{x_0+nh} y(x)\,dx = h\left[n y_0+\frac{n^2}{2}\Delta y_0+\frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0+\frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0+\cdots\right]$$

(A)

Trapezoidal Rule :- $(n = 1)$

put $n = 1$ in (1).

$$\int_{x_0}^{x_1 = x_0 + 1h} y(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$ neglect higher derivative

$$= \frac{h}{2}(2y_0 + \Delta y_0)$$

$$= \frac{h}{2}(2y_0 + y_1 - y_0)$$

$$= \frac{h}{2}(y_0 + y_1) \quad \text{— (1)}$$

Now

$$\int_{x_1 = x_0 + h}^{x_2 = x_0 + 2h} y(x)\,dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right]$$

$$= \frac{h}{2}\left[2y_1 + \Delta y_1\right]$$

$$= \frac{h}{2}\left[2y_1 + y_2 - y_1\right]$$

$$= \frac{h}{2}\left[y_1 + y_2\right] \quad \text{— (2)}$$

= =

Hence,

$$\int_{x_0 + (n-1)h}^{x_0 + nh} y(x)\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right] \quad \text{— (3)}$$

Adding (1), (2) and (3)

$$\int_{x_0}^{x_0 + nh} y(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots y_{n-1})\right]$$

$$\int_{x_0+(n-2)h}^{x_0+nh} y(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right] \quad -(4)$$

Adding (1), (2) and (3) we get

$$\int_{x_0}^{x_0+nh} y(x)\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots) + y_n\right]$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 4(\text{sum of odd terms}) + 2(\text{even terms})\right]$$

This is called Simpson's 1/3 Rule.

*** Simpson's 3/8 Rule formula (n = 3)

$$\int_{x_0}^{x_0+nh} y(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots y_{n-1}) + 2(y_3 + y_6 + \cdots y_{n-3})\right]$$

(Proof Yourself)

Note: The Trapezoidal Rule is simplest formula for numerical integration but it is not accurate. The accuracy can be improved by increasing the interval h.

Simpson's 1/3rd Rule $(n=2)$

put $n=2$ in the above formula

(A). $$\int_{x_0}^{x_0+2h} y(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$ neglecting higher differences above 2nd order

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[\frac{6y_0 + 6\Delta y_0 + \Delta^2 y_0}{6}\right]$$

$$= 2h\left[\frac{6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)}{6}\right]$$

$$\int_{x_0}^{x_0+2h} y(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \quad -(1)$$

Now.

$$\int_{x_0+2h}^{x_0+4h} = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] \quad -(2)$$

Example 1 :- Using Trapezoidal Rule with $h = 0.2$ find $\int_0^1 \frac{dx}{1+x^2}$. Hence determine π

Soln :-

	x_0	x_1	x_2	x_3	x_4	x_5
x :	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$:	1	0.9615	0.8621	0.7353	0.6098	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}\left[1.5 + 2(3.1687)\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.78374$$

$$\Rightarrow \left[\tan^{-1} x\right]_0^1 = 0.78374$$

$$\Rightarrow \frac{\pi}{4} = (0.78374)$$

$$\Rightarrow \pi = 4 \times (0.78374)$$

$$\boxed{\pi = 3.13496}$$

Example 2. Using Trapezoidal Rule to find $\int_{0.6}^{2} y\, dx$ from the following

x :	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y :	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45

By Trapezoidal Rule.

$$\int_{0.6}^{2} y\,dx = \frac{h}{2}\left[(y_0+y_7)+2(y_1+y_2+y_3+y_4+y_5+y_6)\right]$$

$$= \frac{0.2}{2}\left[(13.68)+2(1.58+2.03+4.32+6.25+8.36+10.23)\right]$$

$$= 0.1\,(79.22)$$

$$\int_{0.6}^{2} y\,dx = 7.922 \quad \therefore \text{Ans}$$

<u>Example</u>:- find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3}dx$ using $1/3^{rd}$ Rule with $h=0.25$

<u>Sol^n</u>:- Here $h=0.25$ $\quad y(x)=\frac{x^2}{1+x^3}$.

x :	0	0.25	0.5	0.75	1
$y=\frac{x^2}{1+x^3}$:	0	0.06154	0.22222	0.39560	0.5000
	y_0	y_1	y_2	y_3	y_4

$1/3^{rd}$ Rule $\int_0^1 \frac{x^2}{1+x^3}dx = \frac{h}{3}\left[(y_0+y_4)+2y_2+4(y_1+y_3)\right]$

$$= \frac{0.25}{3}\left[(0+0.5)+2(0.22222)+4(0.06154+0.39560)\right]$$

$$= \frac{0.25}{3}(0.5 + 0.44444 + 1.82856)$$

$$\int_0^1 \frac{x^2}{1+x^3}dx = 0.231083 \quad \text{Ans.}$$

Now,

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{1}{3}\log(1+x^3)_0^1$$

$$= \frac{1}{3}\left[\log 2 - \log 1\right]$$

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{1}{3}(\log 2 - 0)$$

$$= \frac{1}{3}\log_e 2$$

$$\int_0^1 \frac{x^2}{1+x^3}dx = \log_e 2^{1/3}$$

$$\boxed{\text{Hence } \log_e 2^{1/3} = 0.231083} \quad \therefore \text{Ans}$$