Runge - Kutta method :- (R-K) (1)

(1) Runge kutta method of order 2.
(2) Runge kutta method of order 4.

(1) Runge kutta method of order 2.

Let $\frac{dy}{dx} = f(x,y)$; where $y(x_0) = y_0$.

then the value of better approximation of y_1 is found by Euler's and Euler's modified method as follows:

$$\text{Here } y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

where $K_1 = h\, f(x_0, y_0)$
& $K_2 = h\, f(x_0 + h, y_0 + K_1)$

—x—

Proof!- Since we know that by Euler's Modified method that

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] \quad -(1)$$

& By Euler's method

$$y_1 = y_0 + h\, f(x_0, y_0) \quad -(2)$$

put (2) in (1)

$$y_r = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right]$$

$$= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_0 + hf(x_0, y_0))\right]$$

$$= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_0 + hf(x_0, y_0))\right]$$

$$= y_0 + \frac{\cancel{h}}{2}\left[hf(x_0, y_0) + h f(x_0+h, y_0 + hf(x_0, y_0))\right]$$

$$= y_0 + \frac{1}{2}\left[k_1 + k_2\right]$$

where $k_1 = hf(x_0, y_0)$ &

$$k_2 = h f(x_0+h, y_0 + hf(x_0, y_0)) \quad (3)$$

or. $k_2 = h f(x_0+h, y_0+k_1)$ (4)

Hence $\boxed{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)}$ where

k_1 and k_2 are given by (3) & (4)

=p=

Runga kutta method of order 4

formula

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = h\, f(x_0, y_0)$ — 1

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$ — 2

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$ — 3

$k_4 = h\, f(x_0 + h, y_0 + k_3)$ — 4

Runga kutta method formula as follows

let the given differential equation is

$$\frac{dy}{dx} = f(x, y) \text{ . then}$$

$$y_1 = y_0 + \Delta y .$$

where $\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where k_1, k_2, k_3, k_4 are obtained

by 1, 2, 3 & 4 respectively.

Similary

$$y_2 = y_1 + \Delta y$$

where $\Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$

$$k_1 = h f(x_1, y_1)$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

Similaly we can find y_3, y_4 for the better approximation of y.

Exampeles :—

$$3\frac{dy}{dx} + 5y^2 = \sin x \qquad y(0.3) = 5$$

find $y(0.6)$

Sol^n :- Here $x_0 = 0.3$, $y_0 = 5$ & $h = 0.6$

$$f(x, y) = \frac{1}{3}\left(\sin x - 5y^2\right)$$

$$k_1 = h f(x_0, y_0) = f(0.3, 5)(0.6)$$

$$= \frac{1}{3}\left[\sin(0.3), -5(5)^2\right](0$$

$= -41.5682$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= f[0.3 + 0.3, 5 + (-41.5682)] \ (0.6)$

$= f(0.6,$

$\therefore \cancel{\Delta y} =$

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Ans

2)

Using R-K method solve $y' = x + y$ where $y(0) = 1$, to find $y(0.1)$ and $y(0.2)$

Solution :- given $y(0) = 1$

i.e $x_0 = 0, \quad y_0 = 1 \quad , x_1 = 0.1$

and $\frac{dy}{dx} = (x+y) \qquad x_2 = 0.2$

$\therefore \quad f(x,y) = (x+y) \quad — \quad (1)$

Also $h = x_1 - x_0 = x_2 - x_1 = 0.1$

To find y_1 at x_1 and y_2 at x_2

$y_1 = y(0.1) = y_0 + \Delta y$ (By R-K method of fourth order)

where $\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$k_1 = h\, f(x_0, y_0)$

$= 0.1 \; \cancel{f} \left[x_0 + y_0\right]$ By (1)

$= 0.1\,(0 + 1) = 0.1$

$k_2 = h\, f\left(x_0 + \frac{h}{2}, \; y_0 + \frac{k_1}{2}\right)$

$= 0.1\, f\left(0 + \frac{0.1}{2}, \; 1 + \frac{0.1}{2}\right)$

$= 0.1\, f(0.05, 1.05)$

$= 0.1\, (0.05 + 1.05)$ as $f(x, y) = x + y$ given

$= 0.1\, (1.10) = 0.110$

$K_3 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= 0.1\, f\left[0 + \frac{0.1}{2}, 1 + \frac{0.110}{2}\right]$

$= 0.1\, f\,(0.050, 1.055)$

$= 0.1105$

$K_4 = h\, f\,(x_0 + h, y_0 + K_3)$

$= 0.1\, f\,(0 + 0.1, 1 + 0.1105)$

$= 0.1\, f\,(0.1, 1.1105)$

$= 0.12105$

$\therefore \Delta y = \frac{1}{6}\left[0.1 + 2(0.110) + 2(0.1105) + 0.12105\right]$

$= 0.1103$

$\therefore y_1 = y_0 + \Delta y = 1 + 0.1103 = 1.1103$ at x_1

$\boxed{\therefore y_1 = 1.1103 \text{ at } x_1 = 0.1}$

To find y_2 at $x_2 = 0.2$

$$y_2 = y_1 + \Delta y$$

where $\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

here, $K_1 = h\, f(x_1, y_1)$

$$= 0.1\, f(0.1,\ 1.1103)$$

$$= 0.1\,(1.2103) = 0.12163$$

$$K_2 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{K_1}{2}\right)$$

$$= 0.1\, f\left(0.1 + \frac{0.1}{2},\ 1.1103 + \frac{0.12163}{2}\right)$$

$$= 0.1\, f(0.150,\ 1.17108)$$

$$= 0.1\,(1.3208) = 0.13208$$

$$K_3 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{K_2}{2}\right)$$

$$= 0.1\, f\left(0.1 + \frac{0.1}{2},\ 1.1103 + \frac{0.13208}{2}\right)$$

$$= 0.1\, f(0.150,\ 1.1763)$$

$$= 0.1\,(1.3263)$$

$$K_3 = 0.13263$$

$K_4 = h f(x_1 + h, y_1 + k_3)$

$= 0.1 f(0.1 + 0.1, 1.1103 + 0.13263)$

$= 0.1 f(0.2, 1.2429)$

$= 0.1 (1.4429$

$= 0.14429$

$\therefore \Delta y = \frac{1}{6}\left[0.12103 + 2(0.13208) + 2(0.13263) + 0.14429\right]$

$= \frac{1}{6}[0.794]$

$= 0.13245$

$\therefore y_2 = y_1 + \Delta y$

$= 1.1103 + 0.13245$

Ans: $\boxed{\therefore y_2 = 1.24275 \quad \text{at} \quad x_2 = 0.2}$