Spline means obtain a single polynomial which fits the curve (and not smooth)

Cubic Spline interpolation :-

(From cubic spline we obtain a smooth curve, which we do not obtain in spline interpolation.

Consider the problem of interpolation between (x_0, y_0), (x_1, y_1), (x_2, y_2) - - - by spline fitting then cubic spline $f(x)$ (polynomial) is such that

(1) $f(x)$ is linear polynomial outside the interval (x_0, x_n)

(2) $f(x)$ is cubic polynomial in each of the subintervals

(3) $f'(x)$ and $f''(x)$ are continuous at each point. Also $f''(x)$ is linear

$s_i'' = y''(x_i) = M_i$

CUBIC SPLINE INTERPOLATION

cubic spline interpolation is given by:
a function which is spline. spline means drawing a smooth curve.

Let we have $(n+1)$ data points (x_i, y_i) where $i = 0, 1, 2 - - n-1$ where x_i is not equally spaced and $x_0 = a$ and $x_n = b$

Then cubic spline function $S(x)$ is given by:

$$\left.\begin{array}{l} S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \\ \text{in each interval } (x_i, x_{i+1}),\ i = 0, 1, 2, --(n-1) \end{array}\right\} \quad A$$

where
$$a_i = \frac{M_{i+1} - M_i}{6h_i}, \quad b_i = \frac{M_i}{2}$$

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i M_i + h_i M_{i+1}}{6} \quad B$$

Here $\boxed{h_i = x_{i+1} - x_i}$

$$d_i = y_i$$

~~Also $S_0 = S_n = 0$ which means $M_0 = 0, M_n = 0$~~

Now two cases arise to find the unknown values M_i

<u>Case I</u> If $y''(x_0) = 0$ & $y''(x_n) = 0$

i.e $M_0 = 0$ and $M_n = 0$ } Note $y''(x_i) = S_i'' = M_i$

Remember:- $y''(x_0) = S''(x_0) = M_0$

$y''(x_n) = S''(x_n) = M_n$.

following system of equations in Matrix Notation

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & \vdots & \vdots & \vdots \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-1} \end{bmatrix}$$

$$= 6 \begin{bmatrix} \frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0} \\ \frac{y_3-y_2}{h_2} - \frac{y_2-y_1}{h_1} \\ \vdots \\ \frac{y_n-y_{n-1}}{h_{n-1}} - \frac{y_{n-1}-y_{n-2}}{y_{n-2}} \end{bmatrix} \quad \text{(C)}$$

Ex! - 1. Fit a cubic spline curve that passes through (0,1), (1,4), (2,0), (3,-2) with boundary Conditions $S''(0) = S''(3) = 0$

Sol^n! — Here given that

	x_0	x_1	x_2	x_3
x:	0	1	2	3
	y_0	y_1	y_2	y_3
y:	1	4	0	-2

$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ y_0 = 1\ \ y_1 = 4\ \ y_2 = 0\ \ y_3 = -2$

we have $i = 0, 1, 2, 3$. and data sets are (x_0, x_3) with $n = 4$ Hence $i = 0$ to $(n-1)$

Apply Case I.

$$\left.\begin{aligned} \text{Here}\ h_0 &= x_1 - x_0 = 1, \\ h_1 &= x_2 - x_1 = 1 \\ h_2 &= x_3 - x_2 = 1 \end{aligned}\right\} \text{from (B)}$$

Also given $S''(0) = S''(3) = 0$ Means.

$$\left.\begin{aligned} &S''(x_0) = 0 \text{ and } S''(x_3) = 0 \text{ OR} \\ &M_0 = 0 \text{ and } M_3 = 0 \text{ find} \\ &M_1 \text{ and } M_2. \end{aligned}\right\}$$

put the values in (C)

$$\begin{bmatrix} 2(2) & 1 \\ 1 & 2(2) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = 6\begin{bmatrix} \left(\frac{0-4}{1}\right) - \left(\frac{4-1}{1}\right) \\ \left(\frac{-2-0}{1}\right) - \left(\frac{0-4}{1}\right) \end{bmatrix} = 6\begin{bmatrix} -4-(3) \\ -2+4 \end{bmatrix}$$

$$= \begin{bmatrix} -42 \\ 12 \end{bmatrix}$$

OR. $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} -42 \\ 12 \end{bmatrix}$

Solving this we get

$M_1 = -12, \quad M_2 = 6$ Also given $M_0 = 0$
$M_3 = 0$

Hence $M_0 = 0, \quad M_1 = -12, \quad M_2 = 6$ & $M_3 = 0$

$y_0 = 1 \quad y_1 = 4 \quad y_2 = 0 \quad y_3 = -2$ (given)

find cubic spline coefficients.

$$a_i = \frac{M_{i+1} - M_i}{6h_i}, \quad b_i = \frac{M_i}{2}$$

$$C_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_iM_i + h_iM_{i+1}}{6}$$

$$d_i = y_i$$

Let $i = 0$ first

$$a_0 = \frac{M_1 - M_0}{6h_0} = -2$$

$$b_0 = \frac{M_0}{2} = 0$$

$$C_0 = \frac{y_1 - y_0}{h_0} - \frac{2h_0M_0 + h_0M_1}{6} = 5$$

$$d_0 = y_0 = 1$$

$i = 1$	$i = 2$
$a_1 = 3$	$a_2 = -1$
$b_1 = -6$	$b_2 = 3$
$C_1 = -1$	$C_2 = -4$
$d_1 = 4$	$d_2 = 0.0$

Hence required spline is (Using 4)

$i=0$, $S_0(x) = a_0(x-x_0)^3 + b_0(x-x_0)^2 + c_0(x-x_0) + d_0$

$= -2x^3 + 5x + 1$ for $x_0 \le x \le x_1$ OR $0 \le x \le 1$)

$i=1$, $S_1(x) = a_1(x-x_1)^3 + b_1(x-x_1)^2 + c_1(x-x_1) + d_1$

$= 3(x-1)^3 - 6(x-1)^2 - (x-1) + 4$ for $1 \le x \le 2$ OR $x_1 \le x \le x_2$

$i=2$

$S_2(x) = a_2(x-x_2)^3 + b_2(x-x_2)^2 + c_2(x-x_2) + d_2$

$= -(x-2)^3 + 3(x-2)^2 - 4(x-2)$ for $2 \le x \le 3$ OR $x_2 \le x \le x_3$

= x =

Qus 2 find the cubic spline with

x	0	1	2	3
y	-5	-4	3	22

Also $f''(x_1) = f''(x_4) = 0$ (If not mention in the Qus take Case I only)

Sol^n:- Here $i = 1, 2, 3, 4$. $\boxed{\begin{matrix} n=4 \\ i = 1 \text{ to } n \end{matrix}}$

$x_1 = 0 \quad x_2 = 1 \quad x_3 = 2 \quad x_4 = 3$
$y_1 = -5 \quad y_2 = -4 \quad y_3 = 3 \quad y_4 = 22$

$i = 1, 2, 3, 4$.

Case I:- $h_1 = x_2 - x_1 = 1, \quad h_2 = 1 \quad h_3 = 1$
from (B)

Also given $f''(x_1) = S''(x_1) = S''(0) = M_1 = 0 = y''(x_1)$
$f''(x_4) = S''(x_4) = S''(3) = M_4 = 0 = y''(x_4)$.

We have to find M_2 and M_3 here.

$$\begin{bmatrix} 2(h_1+h_2) & h_2 \\ h_2 & 2(h_2+h_3) \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = 6 \begin{bmatrix} \dfrac{y_3-y_2}{h_2} - \dfrac{y_2-y_1}{h_1} \\ \dfrac{y_4-y_3}{h_3} - \dfrac{y_3-y_2}{h_2} \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = 6 \begin{bmatrix} \frac{7}{1} - \frac{1}{1} \\ 19 - 7 \end{bmatrix}$$

$$\left. \begin{aligned} 4 \cdot M_2 + M_3 &= 36 \\ M_2 + 4M_3 &= 72. \end{aligned} \right\}$$

Solve this