NEWTON'S RAPHSON METHOD (N-R Method)

* Discovered by Newton (17th century)
* It is used in both Algebraic and transcendental equations (for real and Complex root both)
* It is used to improve the result obtained by previous method
* It is faster provided initial approximation x_0 is chosen sufficiently close to the root. Initial choice (x_0) is important.
* If it is not near the root, procedure may be an endless ~~cycle~~ cycle.
* Best way is to ~~use~~ take $x_0 = \frac{a+b}{2}$ as an initial approximation such that

$$\boxed{f(a) \cdot f(b) < 0}$$

* The Convergence of N-R method is quadratic (Second Order Convergence) as error in each step (approximation) is proportional to the square of the previous error. While Bisection method is linear order convergence

formula :-

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2 \ldots)}$$

PROOF :-

Let x_0 be the approximate root of $f(x)=0$ and $x_1 = x_0 + h$ be the exact root, then x_1 satisfies $f(x)=0$

$\therefore f(x_1) = 0$.

$\Rightarrow \quad f(x_0 + h) = 0$

Expand by Taylor's theorem (Series)

$$f(x_0) + h\, f'(x_0) + \frac{h^2}{\lfloor 2}\, f''(x_0) + \ldots\ldots = 0 \quad -(1)$$

Since h is small neglecting higher power of h we get from (1)

$$f(x_0) + h\, f'(x_0) = 0 \qquad -(2)$$

from (2) we have $h = -\dfrac{f(x_0)}{f'(x_0)}$

$\because \quad x_1 = x_0 + h$

$$\therefore \quad \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

starting with this x_1 we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

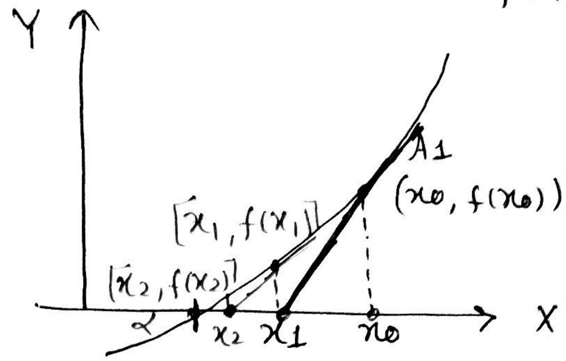
In general $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$ $(n = 0, 1, 2 \ldots)$

This is known as N-R iteration formula.

Geometrical Interpretation of N-R

Let x_0 be the approximate root of $f(x)=0$

Draw the figure of the curve $f(x)$



Equation of the tangent at A_1 is

$$(y - y_0) = \frac{dy}{dx}(x - x_0)$$

$$\boxed{\because \quad y = f(x), \quad \frac{dy}{dx} = f'(x)}$$

OR

$$(y - f(x_0)) = f'(x_0)\,[x - x_0]$$

$$y - f(x_0) = x f'(x_0) - x_0 f'(x_0) \quad \text{— ①}$$

Tangent A_1 cut at x-axis where $y=0$ and $x=x_1$

put in ① $y=0$ and $x=x_1$

$$0 - f(x_0) = x_1 f'(x_0) - x_0 f'(x_0)$$

$$x_0 f'(x_0) - f(x_0) = x_1 f'(x_0)$$

$$\Rightarrow \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

which is first approximation of $f(x)=0$

Tangent A_2 cut at x-axis where $y=0$ and $x=x_2$

$x_0=x_1$, put in ① we get

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

Repeating this process we approach to the root α rapidly.

Steps for N-R Method :-

① Write $f(x) = 0$, find closer root x_0 for this

② Select a and b so that

$$\boxed{f(a) \cdot f(b) < 0}$$

③ for Convience take $a = 0$, $b = 1$ or -1

such that $f(a) \cdot f(b) < 0$

find $x_0 = \frac{a+b}{2}$ is first approximation or closer root

④ 1st approximation

$h=0$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\quad$ [put $n=0$ in $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$]

2nd approximation

$h=1$ $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

3rd approximation

$h=2$ $\quad x_3 = x_2 - \frac{f(x_2)}{f(x_2)}$

= =

Continue till we get two same values if $x_3 = x_2$ then stop at 3rd approximation and root is x_3. Otherwise Continue the process until two same value.

Ques Based on N-R Method.

Three types of Ques.

A. Ques Based on polynomial eq.

(1) Find the positive root of $x^4 = x + 10$ Correct to ~~three~~ two(2) decimal places using N-R Method.

Solution:- N-R formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \longrightarrow (A)$$

where $n = 0, 1, 2 \ldots$

Step 1:- Find roots

$f(x) = x^4 - x - 10 = 0$

put $x = 0 \Rightarrow f(0) = -10 = -ve$

$x = 1 \Rightarrow f(1) = 1 - 1 - 10 = -10 = -ve$

$x = 2 \Rightarrow f(2) = 16 - 2 - 10 = 4 = +ve$

$\therefore$ Root lies in between 1 and 2

as $f(1) \cdot f(2) = -10 \times 4 = -40 < 0$.

* Now since $|f(2)| < |f(1)| \therefore$ Root is closest to $x = 2$ $\therefore$ Take $\boxed{x_0 = 2}$ as first approximation.

put $n = 0$ in (A).

Step 2. Start iterations (approximations)

1st iteration :- when $n=0$.

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\because \; f(x) = x^4 - x - 10 \Rightarrow f(x_0) = x_0^4 - x_0 - 10$

$\therefore \; f'(x) = 4x^3 - 1 - 0 \Rightarrow f'(x_0) = 4x_0^3 - \cancel{x_0}\, 1$

Now $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{(x_n^4 - x_n - 10)}{4x_n^3 - 1}$ — (B)

put $n=0$ in (B) we get

$$x_1 = x_0 - \frac{(x_0^4 - x_0 - 10)}{4x_0^3 - 1} \quad \because x_0 = 2$$

$$\therefore x_1 = 2 - \left[\frac{2^4 - 2 - 10}{4(2)^3 - 1}\right] = 2 - \left[\frac{4}{32-1}\right]$$

$$x_1 = 2 - \frac{4}{31} = 2 - 0.1290 = 1.871$$

2nd iteration :- when $n=1$ & $\boxed{x_1 = 1.871}$

put $n=1$ in (B)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1^4 - x_1 - 10)}{4(x_1)^3 - 1}$$

$$= 1.871 - \left[\frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1}\right]$$

$$= 1.871 - \frac{0.3835}{25.199} = 1.871 - 0.0152$$

$$\boxed{x_2 = 1.8557} = \boxed{1.856}$$

3rd approximation when $n = 2$ & $x_2 = 1.856$

put $n=2$ in (B)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2^4 - x_2 - 10)}{4(x_2)^3 - 1}$$

$$= 1.856 - \left[\frac{(1.856)^4 - (1.856) - 10}{4 * (1.856)^3 - 1}\right]$$

$$= 1.856 - \frac{0.010}{24.574} = 1.856 - 0.000406$$

$$x_3 = 1.8555$$

$$\therefore x_3 = 1.856$$

Here $x_2 = x_3 = 1.856$ $\therefore$ +ve Root is 1.856

Table Method

value n	iteration	x_{n+1}	$f(x_{n+1})$
0	1	$x_1 = 1.871$	$f(x_1) = 0.3835$
1	2	$x_2 = 1.856$	$f(x_2) = 0.010$
2	3	$x_3 = 1.856$	$f(x_3) = 0.010$

Ans :- Root is $\boxed{1.856}$

② * Ques Based on Transcendental eq.

Find the root of the equation
$x^2 = -4 \sin x$.

Note :- If not mentioned decimal places take (By default) up to 4 decimal places

Let $f(x) = x^2 + 4 \sin x$
$f'(x) = 2x + 4 \cos x$.

Now $f(0) = 0^2 + 4 \sin 0 = 0$
$f(1) = 1 + 4 \sin(1) = 1 + 4(0.8414)$

Note :- Convert calculator into radian for sin(1)

$f(-1) = 1 + 4 \sin(-1) = 1 + 4(-0.8414)$
$f(-1) = 1 - 3.3660 = -2.366 = -ve$
$f(-2) = 4 + 4 \sin(-2) = 4 + 4(-0.9093)$
$f(-2) = 4 - 3.6372 = 0.3628 = +ve$

$\because f(-1) \cdot f(-2) < 0 \quad \therefore$ Roots lie between (-1) and (-2)

$\because |f(-2)| < |f(-1)| \quad \therefore x_0 = -2$

Now $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (A) gives

$$x_{n+1} = x_n - \frac{x_n^2 + 4 \sin x_n}{2x_n + 4 \cos x_n} \qquad (B)$$

First iteration put $n=0$ & $\boxed{x_0 = -2}$ in (B)

$$x_1 = x_0 - \left(\frac{x_0^2 + 4\sin x_0}{2x_0 + 4\cos x_0}\right)$$

$$= -2 - \left[\frac{4 + 4\sin(-2)}{2(-2) + 4\cos(-2)}\right]$$

$$= -2 \left[\frac{4 + 4(-0.9093)}{-4 + 4(-0.4161)}\right]$$

$$= -2 - \left(\frac{4 - 3.6372}{-4 - 1.6644}\right) = -2 - \frac{0.3628}{-5.6644}$$

$$= -2 + 0.0640$$

$$x_1 = -1.9359$$

Second iteration put $n=1$ & $x_1 = -1.9359$ in (B)

$$x_2 = x_1 - \left(\frac{x_1^2 + 4\sin x_1}{2x_1 + 4\cos x_1}\right)$$

$$= (-1.9359) - \left(\frac{(-1.9359)^2 + 4\sin(-1.9359)}{2(-1.9359) + 4\cos(-1.9359)}\right)$$

$$= (-1.9359) - \frac{3.7477 + 4(-0.9341)}{(-3.8718) + 4(-0.3570)}$$

$$= (-1.9359) - (0.00213)$$

$$\boxed{x_2 = -1.9338}$$

Similarly find x_3. by putting $n=2$ & $x_2 = (-1.9338)$ we get

$$\boxed{x_3 = -1.9338}$$

Since $x_2 = x_3 = -1.9338$ ∴ Ans Root is $= (-1.9338)$

Ans 3. Third type to find root, cube root etc. (13)
of any number.

find the $\sqrt{12}$ to four decimal places by N-R.

Sol:- Let $x = \sqrt{12}$
$x^2 = 12$
or $\boxed{f(x) = x^2 - 12 = 0}$, $f'(x) = 2x$.

Now $f(3) = 9 - 12 = -3$ (-ve)
$f(4) = 16 - 12 = 4$ (+ve)
$\therefore$ Root lies in between 3 & 4.

$\because |f(3)| < |f(4)|$ $\therefore$ Let $x_0 = 3$

So start iteration

formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{(x_n^2 - 12)}{2x_n}$. (A)

First iteration :- $n = 0$. $x_0 = 3$ From (A)

$\therefore x_1 = x_0 - \left[\dfrac{(x_0)^2 - 12}{2x_0}\right]$

$= 3 - \left(\dfrac{9-12}{6}\right) = 3 - \dfrac{(-3)}{6} = 3.5$

2nd iteration :- $n = 1$, $x_1 = 3.5$ gives

$x_2 = x_1 - \dfrac{(x_1)^2 - 12}{2x_1} = 3.5 - \left[\dfrac{(3.5)^2 - 12}{2(3.5)}\right] = 3.4643$

3rd iteration :- $n = 2$ $x_2 = 3.4643$

$x_3 = x_2 - \dfrac{(x_2)^2 - 12}{2x_2} = 3.4643 - \dfrac{(3.4643)^2 - 12}{2(3.4643)}$
$= 3.4641$.

4th iteration :- $n = 3$, $x_3 = 3.4641$

$$x_4 = x_3 - \frac{(x_3)^2 - 12}{2x_3} = 3.461 - \frac{(3.4641)^2 - 12}{2(3.4641)}$$

$$= 3.4641$$

$\because$ $x_3 = x_4 = 3.4641$ $\therefore$ Root is

$\boxed{3.4641}$

Ans $\sqrt{12} = 3.4641$ (Correct to 4 decimal places).

Table :-

Value n	iteration	x_{n+1}	$f(x_{n+1})$
0	1	$x_1 = 3.5$	$f(x_1) =$
1	2	$x_2 = 3.4643$	$f(x_2) =$
2	3	$x_3 = 3.4641$	$f(x_3) =$
3	4	$x_4 = 3.4641$	$f(x_4) =$

$\because$ $x_3 = x_4$ $\therefore$ Ans 3.461

Also $\boxed{f(x_3) = f(x_4)}$