Module - 4.

Initial Value problems for ordinary differential equations

* Numerical methods for differential eqs are of great importance and challange to the Engineers and Scientist as Model problems often lead to diffential equations that cannot be solved sometimes easily analytically. We have already seen that the algebraic and transcendental equations System of equations, ~~Numeri~~ differentiation, Integration etc. can be solved easily by Numerical Methods which we have studied in previous chapters. Here we are discussing <u>various methods</u> of finding the solution of ordinary diff. eq with initial conditions to any desired degree of accuracy.

Suppose the first order diff eq. is

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x_0) = y_0.$$

Here we apply step-by-step method to find the solution y numerically

Step 1 :- Start with $y(x_0) = y_0$

i.e $y = y_0$ when $x = x_0$

Step 2 :- Compute y_1 by taking h as a fixed step size (h is called step size)

$\therefore$ $x_1 = x_0 + h$. where $h = |x_1 - x_0|$ small as much possible

Step 3 :- Compute y_2 by x_2 where

$x_2 = x_1 + h$ or $x_0 + 2h$.

$= - -$

Continue till we get two same approximation in steps.

There are several methods for solving diff equations numerically and the most important methods are :

(1) Taylor's series Method.
(2) Euler's Method (Modified)
(3) Runge-Kutta Method
(4) Milne's predictor - Corrector Method

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(1) Taylor's series Method :-

We know that if $y = f(x)$ be the function of x then Taylor's Series about the point $x = x_0$ is given by

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{\lfloor 2} f''(x_0) + \frac{(x-x_0)^3}{\lfloor 3} f'''(x_0) + \cdots$$

This can also be written as:

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{\lfloor 2} y_0'' + \frac{(x-x_0)^3}{\lfloor 3} y_0''' + \cdots$$

where x_0 and y_0 are initial value of x and y. i.e $x = x_0$ & $y = y_0$.

Now find

y_1 for $x_1 = x_0 + h$

where, it is desirable to keep $|x - x_0|$ numerically small in order to get desired convergent solution (y).

Hence $y_1 = y(x_1) = y(x_0 + h)$

$y_2 = y(x_2) = y(x_0 + 2h)$

$y_3 = y(x_3) = y(x_0 + 3h)$

$= \; =$ and so on.

Hence
$$y_1 = y_0 + (x_1 - x_0) y_0' + \frac{(x_1 - x_0)^2}{\lfloor 2} y_0'' + \frac{(x_1 - x_0)^3}{\lfloor 3} y_0''' + \cdots$$

& $x_1 = x_0 + h$ gives $|x_1 - x_0| = h$

$$\therefore \; y_1 = y_0 + h y_0' + \frac{h^2}{\lfloor 2} y_0'' + \frac{h^3}{\lfloor 3} y_0''' + \cdots$$

Similarly
$$y_2 = y_1 + h y_1' + \frac{h^2}{\lfloor 2} y_1'' + \frac{h^3}{\lfloor 3} y_1''' + \cdots$$

In general
$$\boxed{y_{n+1} = y_n + h y_n' + \frac{h^2}{\lfloor 2} y_n'' + \frac{h^3}{\lfloor 3} y_n''' + \cdots}$$

★ **Note :-**

It is convenient for us to find the higher order derivatives say y'', y''', y^{iv}, y^{v}, --- before applying Taylor formula.

Q 1.

Find the value of $y(0.1)$ correct to 4 decimal places from

$$\frac{dy}{dx} = x^2 - y, \quad y(0) = 1 \quad \text{with } h = 0.1$$

Using Taylor's Series method.

Sol^n:- $\because \frac{dy}{dx} = x^2 - y$ and $\boxed{x_0 = 0,\ y_0 = 1 \quad h = 0.1}$

$\therefore \quad y' = x^2 - y$

We know that

$$y_1 = y_0 + h\, y_0' + \frac{h^2}{\lfloor 2} y_0'' + \frac{h^3}{\lfloor 3} y_0''' + \text{---} \quad \text{— (1)}$$

To find y', y'', y''', y^{iv} -- first and substitute in these in equation (1) at $x = x_0$.

$x_0 = 0, \; y_0 = 1$ ⑥

$y' = x^2 - y$	$y_0' = x_0^2 - y_0$	$= 0 - 1 = -1$
$y'' = 2x - y'$	$y_0'' = 2x_0 - y_0'$	$= 2(0) - (-1) = 1$
$y''' = 2 - y''$	$y_0''' = 2 - y_0''$	$y_0''' = 2 - 1 = 1$
$y^{iv} = -y'''$	$y_0^{iv} = -y_0'''$	$y_0^{iv} = -1$
$= =$	$= =$	$- -$

put these value in ①
when $h = 0.1$

$$y_1 = 1 + \frac{(0.1)}{\underline{1}}(-1) + \frac{(0.1)^2}{\underline{2}}(1) + \frac{(0.1)^3}{\underline{3}}(1) + \frac{(0.1)^4}{\underline{4}}(-1) + \cdots$$

$$= 1 - 0.1 + \frac{0.01}{2} + \frac{0.001}{6} - \frac{0.0001}{24} + \cdots$$

$$y_1 = 0.905163.$$

$$\boxed{\text{or } y(0.1) = 0.905163.}$$

as. $x_1 = x_0 + h$
$x_1 = 0 + (0.1)$
$= 0.1$

or $y(x_1) = y_1 = \underline{\underline{0.905163}}$

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Q 2. Using Taylor's Series Compute $y(0.2)$ Correct to 3 decimal places.
where $\frac{dy}{dx} = 1 - 2xy$ given that $\underline{\underline{y(0) = 0}}$