Poisson equation

An eq. of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ $\boxed{\nabla^2 u = f(x,y)}$ is Called poisson equation.

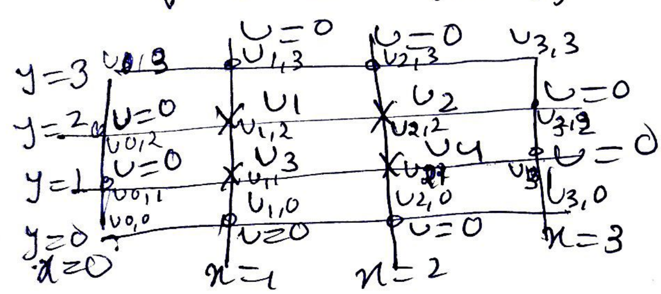
we use five point formula here.

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = f(ih, jk)$$

Q Solve following poisson eq. over the region bounded by lines $x=0, y=0, x=3, y=3$ given that $u=0$ by taking $h=1$ (throughout

$$\nabla^2 u = -(x+y)^2$$

Sol^n -



let u_1, u_2, u_3 and u_4 be the interior points

By five point formula we have

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -(ih + jk)^2$$

we have to find u_1, u_2, u_3 & u_4.

put $i=1, j=1$ and $x=1$ & $y=1$ ($\because h=1, k=1$)

$$u_{0,1} + u_{2,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = -(1+1)^2$$

$$0 + u_4 + u_1 + 0 - 4u_3 = -4 \quad (1)$$

put $i=2,\ j=1 \quad x=2,\ y=1$ (2)

$$u_{1,1}+u_{3,1}+u_{2,2}+u_{2,0}-4u_{2,1}=-(2^2+1^2+2(2)(1))$$

$$u_3+0+u_2+0-4u_4=-(4+1+4)$$

$$u_2+u_3-4u_4=-9 \quad (2)$$

put $i=1,\ j=2$ & $x=1,\ y=2.$

$$u_{0,2}+u_{2,2}+u_{1,3}+u_{1,1}-4u_{1,2}=-(1^2+2^2+(2)(1)(2))$$

$$0+u_2+0+u_3-4u_1=-(1+4+4)$$

$$-4u_1+u_2+u_3=-9 \quad (3)$$

put $i=2,\ j=2$ & $x=2,\ y=2$

$$u_{1,2}+u_{3,2}+u_{2,3}+u_{2,1}-4u_{2,2}=-(2^2+2^2+2(2)(2))$$

$$u_1+0+0+u_4-4u_2=-(4+4+8)$$

$$u_1-4u_2+u_4=-16 \quad \text{---} (4)$$

solve (1), (2), (3) & (4)

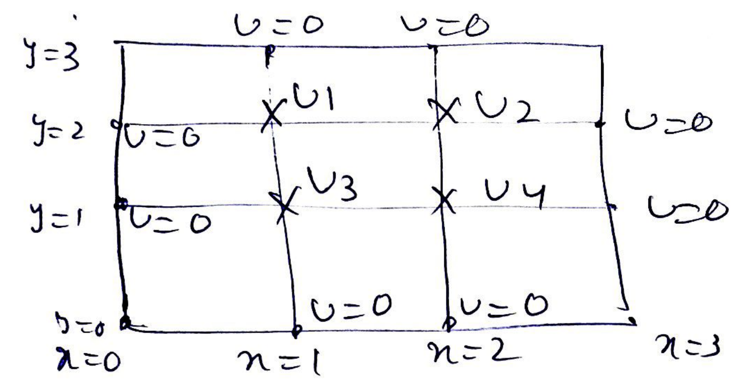
$$\left.\begin{aligned} u_1&=4.67\\ u_2&=6.33\\ u_3&=3.33\\ u_4&=4.65 \end{aligned}\right\} \text{Ans.}$$

Ques for practice ! -

Solve following poisson eq. bounded by $x=0$, $y=0$, $x=3$, $y=3$ given that $U=0$, throughout the boundaries take $h=1$, $\nabla^2 U = x^3+y^3$

Sol^n

Hint



five point formula

$$U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}$$

$$= f(ih, jk)$$

$$= \left[(i\cdot h)^3 + (jk)^3\right]$$

FDM finite difference Method:-

Finite difference Method can be used to solve ordinary second order differential equations also.

we write $\frac{d^2y}{dx^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$

Q Solve by FDM $y'' + y + 1 = 0$ with Boundary Conditions $y = 0$, when $x = 0$ and $y = 0$ when $x = 1$.

Sol^n:- $\because$ $y'' + y + 1 = 0$ — (1)

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i + 1 = 0$$ where $(y = y_i)$

OR $$\frac{y_{i-1} - 2y_i + y_{i+1} + h^2 y_i + h^2}{h^2} = 0$$

or $y_{i-1} - (2 - h^2) y_i + y_{i+1} = (-h^2)$

Let $h = 0.5$

$\therefore$ $y_{i-1} - (2 - 0.25) y_i + y_{i+1} = (-0.25)$ — (2)

$\therefore$ The values of x are $0, 0.5, 1$

$\therefore$ $x_0 = 0$ $x_1 = 0.5$ $x_2 = 1$ & Corresponding values of y be y_0, y_1, y_2

But $y = 0$. when $x = 0$

and $y = 0$ when $x = 1$

$\therefore$ $y_0 = 0$ & $y_2 = 0$

put $i = 1$ in eq (2)

$$y_0 - (2 - 0.25)\, y_1 + y_2 = (-0.25)$$

$$0 - 1.75\, y_1 + y_2 = -0.25$$

or $-1.75\, y_1 = -0.25$ as $y_2 = 0$

or $y_1 = 0.1429$

<u>Note</u> ! – We can also take $h = 0.25$ for better approximation.

<u>Q 2</u> Solve by FDM $\frac{d^2y}{dx^2} = y$ with $y(0) = 0$ and $y(2) = 3.627$

(Hint : take $h = 0.5$)

Ans $y_1 = 0.5262$
$y_2 = 1.1843$
$y_3 = 2.1382$

Q.3 Solve by FDM $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ (Hint take $h = 0.5$)