Lecture - 3

Partial Derivatives ①

If z or $u = f(x, y)$ then

First - order partial derivatives are

$\frac{\partial u}{\partial x}$ or u_x or z_x with respect to x keeping y as a constant. we also write this by using first principle

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \left[\frac{f(x+h, y) - f(x, y)}{h} \right]$$

Similarly

$\frac{\partial u}{\partial y}$ or u_y or z_y with respect to y.

keeping as a constant is first partial derivative we write it

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \left[\frac{f(x, y+k) - f(x, y)}{k} \right]$$

Higher order derivatives ! -

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \text{ or } u_{xx}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \text{ or } u_{yy}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \text{ or } u_{xy}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \text{ or } u_{yx}$$

Always $\boxed{u_{xy} = u_{yx}.}$

This is called Commutative property.

Similarly we can write other Higher partial derivatives.

Ques 1 If $u = \log(\tan x + \tan y + \tan z)$ then Show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Soln/- $u = \log(\tan x + \tan y + \tan z)$

$$\frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y + \tan z)} (\sec^2 x)$$

$$\frac{\partial u}{\partial y} = \frac{1}{(\tan x + \tan y + \tan z)} (\sec^2 y)$$

$$\frac{\partial u}{\partial z} = \frac{1}{(\tan x + \tan y + \tan z)} \cdot \sec^2 z.$$

Hence

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= 2 \frac{\cancel{(\tan x + \tan y + \tan z)}}{\cancel{(\tan x + \tan y + \tan z)}} = 2.$$

Q.2 If $z = x^y + y^x$ then prove that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Soln:- $\because$ $z = x^y + y^x$

Or $z = e^{\log x^y} + e^{\log y^x}$

$z = e^{y \log x} + e^{x \log y}$ as $e^{\log} = 1$

$$\frac{\partial z}{\partial y} = \underset{\mathrm{I}}{e^{y \log x}} \times \underset{\mathrm{II}}{\log x} + \underset{\mathrm{I}}{e^{x \log y}} \cdot \underset{\mathrm{II}}{\frac{x}{y}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^{y \log x} \times \frac{1}{x} + \log x \cdot e^{y \log x} \cdot \frac{y}{x} +$$

$$e^{x \log y} \cdot \frac{1}{y} + e^{x \log y} \cdot \log y \, \frac{x}{y}$$

$$\text{Hence } \frac{\partial^2 z}{\partial x \partial y} = \frac{e^{y \log x}}{x}\left[y \log x + 1\right] + \frac{e^{x \log y}}{y}\left[1 + x \log y\right]$$

Similarly

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{e^{y\log x}}{x}(1+y\log x) + \frac{e^{x\log y}}{y}(x\log y+1)$$

Hence $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$

Some Ques. for practice

If $x^x y^y z^z = c$ show at $x=y=z$

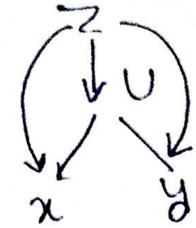
(I) $\dfrac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$

(II) $\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2(x^2-2)}{x(1+\log x)}$.

Chain Rule

Let $z = f(x, y) = f(u)$.

where $u = \phi(x, y)$



Chain Rule

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

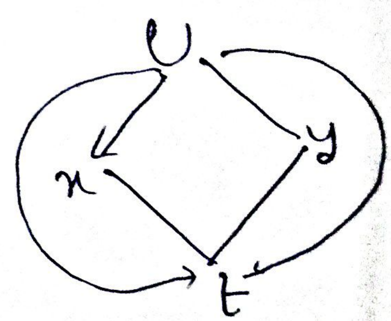
$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}.$$

Composite function of one variable

of $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$

then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$



Composite function of one variable

$\downarrow$

Called total derivative of u.

Ex 1- $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$

find $\frac{dz}{dt}$

Sol^n 1- $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$\frac{dz}{dt} = 8a^3t^3(1+t) + 2a^3t^3(4+t).$$
$$= 2a^3t^3(8+5t).$$

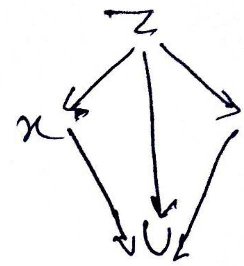
Composite function of two variables :-

Let $z = f(x, y)$

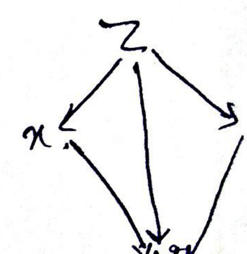
& $x = \phi(u, v)$

$y = \psi(u, v)$ then z is a function of u and v is called the Composite function of two variables u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$



$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$



Ex :-

If $z = f(x, y)$, $x = u\cosh v$, $y = u\sinh v$ prove that

Composite function z of two variables u and v

$$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

Soln :-
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \boxed{\frac{\partial z}{\partial x}\cosh v + \frac{\partial z}{\partial y}\sinh v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \boxed{\frac{\partial z}{\partial x}\cdot u\sinh v + \frac{\partial z}{\partial y}u\cosh v}$$

$U \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \sinh v + \frac{\partial z}{\partial y} \cdot \cosh v$ — (2)

From (1) & (2)

$$\left(\frac{\partial z}{\partial U}\right)^2 - \frac{1}{U^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cosh^2 v + \left(\frac{\partial z}{\partial y}\right)^2 \sinh^2 v$$

$$+ 2 \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \cosh v \sinh v - \left(\frac{\partial z}{\partial x}\right)^2 \sinh^2 v$$

$$- \left(\frac{\partial z}{\partial y}\right)^2 \cosh^2 v - 2 \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \cosh v \sinh v$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \left(\cosh^2 v - \sinh^2 v\right) - \left(\frac{\partial z}{\partial y}\right)^2 \left(\cosh^2 v - \sinh^2 v\right)$$

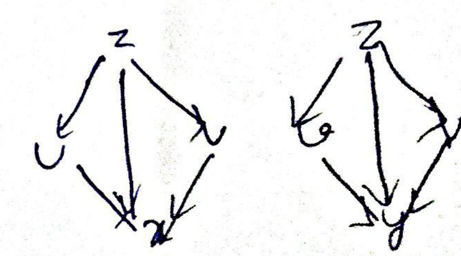
$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 \quad \text{proved}$$

Q. for practice

If $z = f(U, v)$ and $U = x^2 - y^2$, $v = 2xy$

Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left(U^2 + v^2\right)^{1/2}\left[\left(\frac{\partial z}{\partial U}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

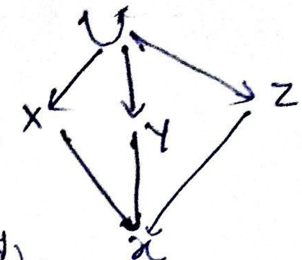


If $U = f\left(e^{y-z}, e^{z-x}, e^{x-y}\right)$ show that

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$$

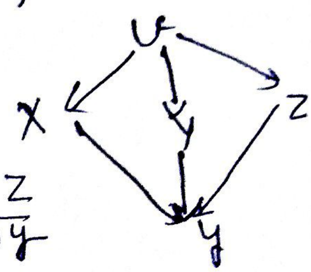
Let $X = e^{y-z}$, $Y = e^{z-x}$, $Z = e^{x-y}$

then $u = f(X, Y, Z)$



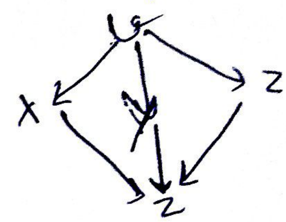
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}\, 0 + \frac{\partial u}{\partial Y}(-e^{z-x}) + \frac{\partial u}{\partial Z}(e^{x-y}).$$



$$= \frac{\partial u}{\partial Y}(-Y) + \frac{\partial u}{\partial Z}(Z) \quad -①$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y}$$



$$= X\frac{\partial u}{\partial X} - Z\frac{\partial u}{\partial Z} \quad -②$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z}$$

$u = f(X, Y, Z)$

$$= -X\frac{\partial u}{\partial X} + Y\frac{\partial u}{\partial Y} \quad -③$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$ Proved

Euler's Theorem for Homogenous functions :-

Homogenous functions ! -

A function $u = f(x, y)$ is said to be homogenous function if u can be written in the form $u = x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$ where n is the degree of homogenous function f

Euler's theorem :-

If u is a homogenous function of two variables x and y of degree n then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

Verify Euler's theorem for

$$v = \frac{x^3y^3}{x^3+y^3}$$

Sol^n- v is a Homogenous function of two variable x and y. Also v can be written as

$$v = \frac{x^6\left[\frac{y^3}{x^3}\right]}{x^3\left[1+\frac{y^3}{x^3}\right]} = x^3 F\left(\frac{y}{x}\right)$$

Hence v is a Homogenous function of degree 3 $\therefore$ applying Euler's theorem $\therefore$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v$$

Find $$\frac{\partial v}{\partial x} = \frac{(x^3+y^3)(3x^2y^3) - (x^3y^3)\cdot 3x^2}{(x^3+y^3)^2}$$

& $$\frac{\partial v}{\partial y} = \frac{(x^3+y^3)(3x^3y^2) - (x^3y^3)\cdot 3y^2}{(x^3+y^3)^2}$$

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3\frac{x^3y^3}{x^3+y^3} = 3v$$

proved