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Fixed point iteration / Method of successive approximation / Iteration Method

Let $f(x) = 0$ be the given equation. Let $x = \alpha$ be the root of $f(x) = 0$, which belong to interval I.

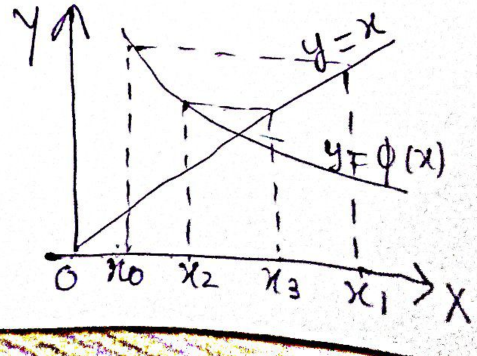
Let $f(x) = 0$ Can be written as
$$x = \phi(x) \text{ such that}$$

$\boxed{|\phi'(x)| < 1}$ for all $\boxed{x \in I}$ then first approximation x_1 is given by.

$$\left.\boxed{\begin{aligned} x_1 &= \phi(x_0). \\ x_2 &= \phi(x_1) \\ &- - \\ x_n &= \phi(x_{n-1}) \end{aligned}}\right\} \text{ Continue up to Convergent solution}$$

where $\boxed{x_0}$ ~~is~~ the initial approximation chosen in interval $I = (a, b)$ such that
$$f(a) \cdot f(b) < 0.$$

In this method the roots of $f(x) = 0$ are same as point of intersection of stright line $y = x$ and curve $y = \phi(x)$ which gives $x = \phi(x)$



Hence x_1 is given by $x_1 = \phi(x_0)$
$$x_2 = \phi(x_1)$$
$= =$ and so on

where x_0 is initial root

Theory.

Some important points / Conditions.

① The iteration process Converges faster if $|\phi'(x)| < 1$, where $x = \phi(x)$ [Arranging given equation]

② $|\phi'(x)| > 1$ and hence method will not give convergent solution.

③ Hence the chocie of $x = \phi(x)$ is very important according as Condition ① & ②

Example :- $f(x) = x^3 + x - 1 = 0$

Sol :- Write the function $f(x) = 0$ in the form of $x = \phi(x)$ such that

$|\phi'(x)| < 1$ for all $x \in I$ and $|\phi'(x)| \not> 1, x \in I$

Here $x^3 + x - 1 = 0 \Rightarrow x(x^2+1) - 1 = 0$

OR $x = \frac{1}{x^2+1} = \phi(x)$, Hence $\phi(x) = \frac{1}{x^2+1}$

$$\boxed{x = \frac{1}{x^2+1} = \phi(x)} \quad (A)$$

$\because$ $f(x) = x^3 + x - 1 = 0$.
$f(0) = -ve$ and $f(1) = +ve$;
Root lies in between 0 and 1 $\therefore I = (0, 1)$

$\therefore\ x = \dfrac{1}{1+x^2} = \phi(x)$

Also $\phi'(x) = \dfrac{-2x}{(1+x^2)^2}$

And $|\phi'(x)| = \dfrac{2x}{(1+x^2)^2}$

$|\phi'(x)| < 1$ for each $x \in (0,1)$

Example if $x = 0.1$ $|\phi'(0.1)| = \dfrac{2\times(0.1)}{(1+0.1^2)^2} < 1$

$|\phi'(0.2)| < 1$ $|\phi'(0.3)| < 1$ Also $|\phi'(0.7)| = \dfrac{2\times(0.7)}{(1+(0.7)^2)^2}$

$= \dfrac{1.4}{(1.49)^2} < 1$

Hence $\boxed{x = \phi(x) = \dfrac{1}{1+x^2}}$ A

which shows that iteration will give better result.

But if we write $f(x)=0$ as $x^3+x-1=0$

or $x = 1-x^3 = \phi(x)$

and $\phi'(x) = -3x^2$, $|\phi'(x)| = 3x^2 > 1$ for any $x \in (0,1)$

ex:- if $x = 0.7$ $|\phi'(x)| = 3\times(0.7)^2 > 1$

$\therefore$ $f(x)=0 \Rightarrow x = 1-x^3 = \phi(x) \longrightarrow$ wrong

Hence iteration will not work if we choose $\phi(x) = 1-x^3$, here in $f(x) = x^3+x-1 = 0$

A (4) The Convergent of iteration Method is <u>linear</u>.

Ques 1 : find a real root of the equation $\cos x = 3x - 1$ correct to three decimal places using <u>Iteration method</u>.

Solution Here

Step 1 find roots lying.

$f(x) = \cos x - 3x + 1 = 0 \quad \text{--- (1)}$

$f(0) = \cos 0 - 3 \times 0 + 1 = 2 = +ve$

$f(\pi/2) = \cos \pi/2 - 3\pi/2 + 1 = -\frac{3\pi}{2} + 1 = -ve$

$\therefore$ Root in between 0 and $\pi/2$ $\therefore I = (0, \pi/2)$

Step 2 write the $f(x) = 0$ in the form of $x = \phi(x)$ so that $|\phi'(x)| < 1$ for $x \in (0, \pi/2)$

(1) gives $\cos x + 1 = 3x$

or $3x = \cos x + 1$

$$\boxed{x = \frac{1}{3}(\cos x + 1) = \phi(x)}$$

Step 3

Now $\phi'(x) = \frac{1}{3}[-\sin x]$ Also $|\phi'(x)| = \frac{1}{3} \sin x$

$|\phi'(x)| = \frac{\sin x}{3} < 1$ for all $x \in (0, \pi/2)$

$\therefore x = \phi(x) = \frac{\sin x}{3}$.

Step 4 Hence iteration method can be applied. Choose x_0 nearer to root as follows:

$$\boxed{x_0 = 0}$$

✓ $x_1 = \phi(x_0) = \frac{1}{3}(\cos x_0 + 1) = \frac{1}{3}(\cos 0 + 1) = 0.667$

$\checkmark\; x_2 = \phi(x_1) = \frac{1}{3}(\cos x_1 + 1) = \frac{1}{3}(\cos 0.6667 + 1)$

$= 0.5953$

$\checkmark\; x_3 = \phi(x_2) = \phi(0.5953)$

$= \frac{1}{3}(\cos(0.5953) + 1) = 0.6093$

$\checkmark\; x_4 = \phi(x_3) = \phi(0.6093)$

$= \frac{1}{3}(\cos(0.6093) + 1) = 0.6067$

$\checkmark\; x_5 = \phi(x_4) = \phi(0.6067)$

$= \frac{1}{3}(\cos(0.6067) + 1) = 0.6072$

$\checkmark\; x_6 = \phi(x_5) = \frac{1}{3}(\cos(0.6072) + 1) = 0.6071$

Hence $x_5 = x_6$ Correct to 3 decimal places.

$\therefore$ the root is $\underline{0.607}$ Correct to 3 decimal places.

Ans |- $\boxed{0.607}$ $\checkmark$

Q. 2 Find a real root of $2x - \log_{10} x = 7$ Correct to 4 decimal places by approximation method

Soln |- $f(x) = 2x - \log_{10} x - 7$

$\because f(3) = 6 - \log_{10} 3 - 7 = 6 - 0.4771 - 7$

$= -1.4471 = -ve$

$f(4) = 8 - \log_{10} 4 - 7 = 8 - 0.602 - 7 = 0.398 = +ve$

$\therefore$ A root lies in betn 3 & 4

$\because$ $f(4) = 0.398$, $f(3) = -1.4471$

$|f(4)| = 0.398$, $|f(3)| = 1.4471$

Also $|f(4)| < |f(3)|$

$\therefore$ the root is nearer to 4 $\boxed{\therefore \text{Let } x_0 = 3.6}$

$$\boxed{\therefore\ x_0 = 3.6}$$

Also $f(x) = \cancel{2x} + 2x - \log_{10} x - 7 = 0$

$\Rightarrow$ $2x - \log_{10} x - 7 = 0$

$\Rightarrow$ $2x = \log_{10} x + 7$

$x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$

As $\phi'(x) = \frac{1}{2}\left[\frac{1}{x}\log_{10} e\right]$

And $|\phi'(x)| = \frac{1}{2}\left[\frac{1}{x}\log_{10} e\right] < 1$ when $3 < x < 4$

$[$ As $\log_{10} e = 0.4343]$

$$\boxed{\therefore\ x = \phi(x) = \frac{1}{2}(\log_{10} x + 7)}$$

Iteration I Now $x_1 = \frac{1}{2}(\log_{10} x_0 + 7) = \frac{1}{2}(\log_{10} 3.6 + 7)$

$= 3.77815$

$x_2 = \frac{1}{2}(\log_{10} x_1 + 7) = \frac{1}{2}(\log_{10} 3.77815 + 7)$

$= 3.78863$

$x_3 = \frac{1}{2}(\log_{10} x_2 + 7) = \frac{1}{2}(\log_{10} 3.78863 + 7)$

$= \boxed{3.78924}$

$x_4 = \frac{1}{2}(\log_{10} x_3 + 7) = \frac{1}{2}(\log_{10} 3.78924 + 7)$

$= \boxed{3.78924}$

$\therefore$ $x_3 = x_4$ And root is 3.7892 (up to 4 places)

Note :- This method of approximation is very useful to find the root of an equation given the form of an infinite series.

Q3. **Ex.** Find the smallest root of the eq. (By approximation method)

$$f(x) = 1 - x + \frac{x^2}{(\lfloor 2)^2} - \frac{x^3}{(\lfloor 3)^2} + \frac{x^4}{(\lfloor 4)^2} - \frac{x^5}{(\lfloor 5)^2} + \cdots = 0 \quad \text{--- (1)}$$

Sol^n/- writing $f(x) = 0$ in the form $x = \phi(x)$

so that $|\phi'(x)| < 1$ for all $x \in (a, b)$

From (1)

$$x = 1 + \frac{x^2}{(\lfloor 2)^2} - \frac{x^3}{(\lfloor 3)^2} + \frac{x^4}{(\lfloor 4)^2} - \frac{x^5}{(\lfloor 5)^2} + \cdots = \phi(x)$$

If we omit higher power of x we will get

$x = 1$ hence take $\boxed{x_0 = 1}$

Also $|\phi'(x)| = \left| \frac{2x}{(\lfloor 2)^2} - \frac{3x^2}{(\lfloor 3)^2} - \cdots \right| < 1$ at $x = 1$.

$$\therefore \quad x = \phi(x) = 1 + \frac{x^2}{(\lfloor 2)^2} - \frac{x^3}{(\lfloor 3)^2} + \frac{x^4}{(\lfloor 4)^2} - \frac{x^5}{(\lfloor 5)^2} \cdots$$

$$x_1 = \phi(x_0) = 1 + \frac{x_0^2}{(\lfloor 2)^2} - \frac{x_0^3}{(\lfloor 3)^2} + \frac{x_0^4}{(\lfloor 4)^2} - \frac{x_0^5}{(\lfloor 5)^2} - -$$

$$= 1 + \frac{1}{(\lfloor 2)^2} - \frac{1}{(\lfloor 3)^2} + \frac{1}{(\lfloor 4)^2} - \frac{1}{(\lfloor 5)^2} = 1.2239$$

Also $x_2 = \phi(x_1) \doteq 1.3263$, $x_3 = \phi(x_2) = 1.38$

$x_4 = \phi(x_3) = 1.409$ $\quad x_5 = \phi(x_4) = 1.425$, $x_6 = \phi(x_5) = 1.434$

$x_7 = \phi(x_6) = \boxed{1.439}$ $\quad x_8 = \boxed{1.442}$ $\therefore$ Root $= 1.44$ Correct to 3 decimal place

Q.4 Find the real root of the equation $x^3+x^2-1=0$ by iteration method

Soln:- $f(x) = x^3+x^2-1=0$

$f(0) = -1 = -ve$

$f(1) = +1 = +ve$

$\therefore$ root lies in between $(0,1)$ Hence $I=(0,1)$ Also $|f(0)| = |f(1)|$ $\therefore$ Let $x_0 = 0.5$

Now select $x = \phi(x)$ so that $|\phi'(x)| < 1, x \in (0,1)$

$$x^3+x^2-1=0$$

$$x^2(x+1)-1=0$$

$$\Rightarrow x^2 = \frac{1}{x+1} \quad \therefore x = \pm\sqrt{\frac{1}{x+1}}$$

Take $\boxed{x = \frac{1}{\sqrt{x+1}} = \phi(x) = (x+1)^{-1/2}}$

$$\phi'(x) = -\frac{1}{2}(x+1)^{-3/2} = -\frac{1}{2(x+1)^{3/2}}$$

$$|\phi'(x)| = \frac{1}{2(x+1)^{3/2}} < 1 \text{ in } (0,1)$$

$$\therefore x = \phi(x) = \frac{1}{\sqrt{(x+1)}} \quad \& \quad \boxed{x_0 = 0.5}$$

Iteration 1:- $x_1 = \phi(x_0) = \frac{1}{\sqrt{(x_0+1)}} = \frac{1}{\sqrt{(0.5+1)}} = 0.81649$

$x_2 = \phi(x_1) = \frac{1}{\sqrt{(x_1+1)}} = \frac{1}{\sqrt{(0.81649+1)}} = 0.74196$

$x_3 = \phi(x_2) = 0.75767$

$x_4 = \phi(x_3) = 0.75427$

$x_5 = \phi(x_4) = 0.75500$

$x_6 = \phi(x_5) = 0.75485$, $x_7 = \phi(x_6) = 0.75488$

Ans:- Root $= 0.75488$.

Practice work :-

Q.1 Solve the equation

$x^2 - 2x - 3 = 0$ by iterative method

for positive root.

Sol^n :- $x = \phi(x) = \sqrt{2x+3}$

Hint :- $f(-2) = +ve = 5$

$\left.\begin{array}{l} f(-1) = 0 \\ f(1) = -ve \\ \quad = -4 \end{array}\right\}$ $I = (-2, 1)$

but

$\because f(-1) = 0$ $\therefore$ -1 is a root of equation which is $-ve$ root. But we want to find the $+ve$ root by iteration method

Let $\boxed{x_0 = -1.2}$

Note :- we can take any value by guessing in the interval $(-2, 1)$

Find $x_1 = \phi(x_0)$, x_2, x_3 - - - -

Ans :- $2.9999 \approx 3.000$.

Q.2 Solve $2x - \log_{10} x = 7$ (By iteration method)

Ans :- 3.78927

Q.3 Solve $x = \frac{1}{2} + \sin x$ (By iteration method

Ans :- 1.4972

Q.4 Solve $1 - x + \frac{x^2}{(\underline{2})^2} - \frac{x^3}{(\underline{3})^2} + \frac{x^4}{(\underline{4})^2} - - = 0$ by iteration

$\boxed{\text{Ans : } 1.4391}$ 6

Q.5 Solve $e^x - 3x = 0$ (fixed point method) Ans $\boxed{0.6189}$