Solution of linear equations

Let the system of m equations having n unknowns $x_1, x_2 -- x_n$ are

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 --- a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 --- a_{2n}x_n &= b_2 \\ - - - - - & \\ - - - - & \\ a_{m1}x_1 + a_{m2}x_2 --- a_{mn}x_n &= b_m \end{aligned}\right\} \quad (1)$$

where $a_{11} -- a_{1n}, a_{21} --- a_{2n} -- a_{m1} -- a_{mn}$ and $b_1, b_2 -- b_m$ are Constants.

(1) Can be written in the matrix form as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & -- & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Composite Matrix $[A : B]$ is called Augmented Matrix and can be written as

$$M = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} & b_1 \\ a_{21} & a_{22} & -- & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & -- & a_{mn} & b_m \end{bmatrix}$$

Note!- These linear system of equations are associated with many Engineering problems.

If the number of simultaneous equations is small (3 or 4) then Crammer's Rule can be used to solve the sytem of equations but if the system is large then above Rule can not be applied. So we adopt some other methods to solve the system of linear equations. There are two types of method to solve system of linear equations

- Direct Method
  - Gauss Elimination Method
  - Gauss Jordan Method
- Iterative Method
  - Gauss Jacobi Method
  - Gauss Seidal Method

Direct Method:

Gauss elimination Method

steps are as follows!

(1) System of equations are written in augmented Matrix.

(2) Augmented Matrix is transformed into

* upper traingular Matrix [A matrix in which all elements below the diagonal elements are zero.

(3) Use Back substitution method to find solution

* upper traingular Matrix Conversion

Process of partial pivoting Method:

(a) The first column in the augmented Matrix is called first pivot column, second column second pivot column and so on.

(b) Select the largest absolute value from the first pivot column which is called pivot

Ex|- If $A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right]$

$\uparrow$ first pivot column $\quad \underbrace{\uparrow_{2nd} \quad \uparrow_{3rd}}_{\text{pivot column}}$

Also $|-3| > 1, 2$

(c) Rearrange the rows so that pivot will be on top of first column.

Ex1-

$$\begin{array}{cccccc} \left[\begin{array}{ccc:c} -3 & 2 & -3 & -6 \\ 1 & -1 & 1 & 1 \\ 2 & -5 & 4 & 5 \end{array}\right] & \begin{array}{l} \longrightarrow R_1 \\ \longrightarrow R_2 \\ \longrightarrow R_3 \end{array} \\ \quad \downarrow C_1 \quad \downarrow C_2 \quad \downarrow C_3 & \end{array}$$

(d) Make the pivot (a_{11}) as 1 by dividing the first row by the pivot.

$$\begin{array}{c} \\ 0\downarrow \\ 0\downarrow \end{array} \left[\begin{array}{ccc:c} 1 & -\frac{2}{3} & 1 & 2 \\ 1 & -1 & 1 & 1 \\ 2 & -5 & 4 & 5 \end{array}\right]$$

(e) Below the pivot make all components as zero by using elementry row operation. and with the help of pivot a_{11}

Hence $R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 2R_1$ gives

$$\left[\begin{array}{ccc:c} 1 & -2/3 & 1 & 2 \\ 0 & -1/3 & 0 & -1 \\ 0 & -11/3 & 2 & 1 \end{array}\right]$$

(f) Now in the second pivot column $|-11/3| > |-\frac{1}{3}|$

Hence interchange R_3 and R_2 to move pivot on a_{22}

$$\left[\begin{array}{ccc|c} 1 & -2/3 & 1 & 2 \\ 0 & -11/3 & 2 & 1 \\ 0 & -1/3 & 0 & -1 \end{array}\right]$$

(g) ~~divide the pivot $a_{22} = -11/3$ by~~

Divide R_2 by $-11/3$ to get pivot a_{22} as 1

$$\left[\begin{array}{ccc|c} 1 & -2/3 & 1 & 2 \\ 0 & 1 & -\frac{6}{11} & -\frac{3}{11} \\ 0 & -1/3 & 0 & -1 \end{array}\right]$$

(h) Make the component zero below the pivot $a_{22} = 1$

$\therefore \quad R_3 \to R_3 + \frac{1}{3} R_2$

$$\left[\begin{array}{ccc|c} 1 & -2/3 & 1 & 2 \\ 0 & 1 & -6/11 & -3/11 \\ 0 & 0 & -2/11 & -12/11 \end{array}\right] \leftarrow \text{New pivot}$$

(i) Divide R_3 by $-2/11$ to get pivot a_{33} as 1

$$\left[\begin{array}{ccc|c} 1 & -2/3 & 1 & 2 \\ 0 & 1 & -6/11 & -3/11 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

6.

(5) Use back substitution process by changing the Matrix again into $AX = B$.

$$\begin{bmatrix} 1 & -2/3 & 1 \\ 0 & 1 & -6/11 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3/11 \\ 6 \end{bmatrix}$$

$x_3 = 6$

$x_2 - 6/11\, x_3 = -3/11$

$x_2 = \frac{-3}{11} + \frac{6}{11} \times (x_3) = \frac{-3}{11} + \frac{6}{11} \times (6)$

$= 3.$

$x_1 - \frac{2}{3} x_2 + x_3 = 2$

$x_1 - \frac{2}{3} \times 3 + 6 = 2$

$x_1 = -2$

Hence solu - $x_1 = -2$, $x_2 = 3$, $x_3 = 6$

Checking $x_1 - x_2 + x_3 = 1$

$-2 - 3 + 6 = 1$ Correct

Ques 2 Solve by

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

Soln :- Write the equations in Augmented form.

Step (1)
$$\left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{array}\right]$$

First pivot Column

Since $|10| > |2|$ & $|3|$

∴ pivot element $a_{11} = 10$

Step (2)
$$\begin{array}{c} \\ 0\downarrow \\ 0\downarrow \end{array}\left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{array}\right] \overset{R_1/10}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{-2}{10} & \frac{3}{10} & \frac{23}{10} \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{array}\right]$$

Step (3) $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$ gives

$$\left[\begin{array}{c|ccc} 1 & -2/10 & 3/10 & 23/10 \\ \hline 0 & 52/5 & -28/5 & -188/5 \\ 0 & -17/5 & 91/10 & 341/10 \end{array}\right]$$

2nd pivot Column

step (4) $\because \left|\frac{52}{5}\right| > \left|-\frac{17}{5}\right|$

$\therefore$ pivot $a_{22} = \frac{52}{5}$

Divide R_2 by $52/5$ i.e $R_2 \to R_2 \div \left(\frac{52}{5}\right)$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{10} & 3/10 & 23/10 \\ 0 & 1 & -7/13 & -47/13 \\ 0 & -17/5 & 91/10 & 341/10 \end{array}\right]$$

step (5) Make below component of a_{22} as zero.

$R_3 \to R_3 + \left(\frac{17}{5}\right) R_2$

$$\begin{bmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 1 & \frac{-7}{13} & \frac{-47}{13} \\ 0 & 0 & \frac{189}{26} & \frac{567}{26} \end{bmatrix}$$

Step (6) Make a_{33} as 1. So divide R_3 by $\frac{189}{26}$.

$$\begin{bmatrix} 1 & -1/5 & 3/10 & 23/10 \\ 0 & 1 & -7/3 & -47/13 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step (7) use Back substitution Method

Ans! -
$z = 3$
$y = -2$
$x = 1$

check $2x + 10y - 5z = -33$
$2\times(1) + 10(2) - 5(1) = -33$
$-33 = -33.$

practice ques. (Gauss elimination)

Solve. $3x + y - z = 3$ 1.72.

$2x - 8y + z = -5$

$x - 2y + 9z = 8$.