Classification of (PDE) ①

The general second order partial differential eq. is:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = \phi(x,y) \text{ or } = 0$$

Find A, B and C — (1)

If $B^2 - 4AC < 0$ (1) is elliptic

If $B^2 - 4AC = 0$ (1) is parabolic

If $B^2 - 4AC > 0$ (1) is Hyperbolic.

Stability analysis :- The solution of PDE is said to be stable if the difference between exact solution and numerical solution is very small.

Find the nature of following Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$

Here $A = 1,\ B = 0,\ C = 1$

$\therefore\ B^2 - 4AC = 0 - 4\times1\times1 = -4 < 0$

Hence equation (Laplace is elliptic)

Classify the PDE

① $\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t}$

$\because$ $A = 1$, $B = 0$, $C = 0$

$B^2 - 4AC = 0$ $\therefore$ eq is parabolic type

② $\frac{\partial^2 w}{\partial x^2} = a \frac{\partial^2 w}{\partial t^2}$ where $a > 0$.

$A = 1$, $B = 0$, $C = -a$

$\therefore$ $B^2 - 4AC = 0 - 4 \times 1(-a) = 4a > 0$

$\therefore$ Hyperbolic

③ $x^2 f_{xx} + (1-y^2) f_{yy} = 0$ for $-1 < y < 1$; $-\infty < x < \infty$.

Here $A = x^2$, $B = 0$, $C = 1 - y^2$

$B^2 - 4AC = 0 - 4x^2(1-y^2)$

$= 4x^2(y^2-1)$

$\because$ x^2 Always +ve in $-\infty < x < \infty$

~~the~~ (y^2-1) is -ve $-1 < y < 1$ (Alwa

$\therefore$ $B^2 - 4AC = -$ve when $x \neq 0$.

$\therefore$ The eq is ellipse.

Qus for practice

1. $x f_{xx} + y f_{yy} = 0 \qquad x > 0,\ y > 0.$

2. $f_{xx} - 2 f_{xy} = 0 \qquad x > 0,\ y > 0$

3. $U_{xx} - 2U_{xy} + U_{yy} = 0.$

4. $f_{xx} + 2 f_{xy} + 4 f_{yy} = 0 \qquad x > 0,\ y > 0.$

5. $x f_{xx} + f_{yy} = 0.$

6. $x^2 f_{xx} + (1-y)^2 f_{yy} = 0$

Solution of Laplace eq. (Elliptic eq.) [Leibmann iteration process]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad -(1)$$

$$\because \frac{\partial^2 u}{\partial x^2} = \frac{U_{i+1,J} - 2U_{i,J} + U_{i-1,J}}{h^2}$$

$$\& \quad \frac{\partial^2 u}{\partial y^2} = \frac{U_{i,J+1} - 2U_{i,J} + U_{i,J-1}}{K^2}$$

put in (1)

$$\frac{U_{i+1,J} - 2U_{i,J} + U_{i-1,J}}{h^2} + \frac{U_{i,J-1} - 2U_{i,J} + U_{i,J-1}}{K^2} = 0$$

for a square mesh always $h = K$

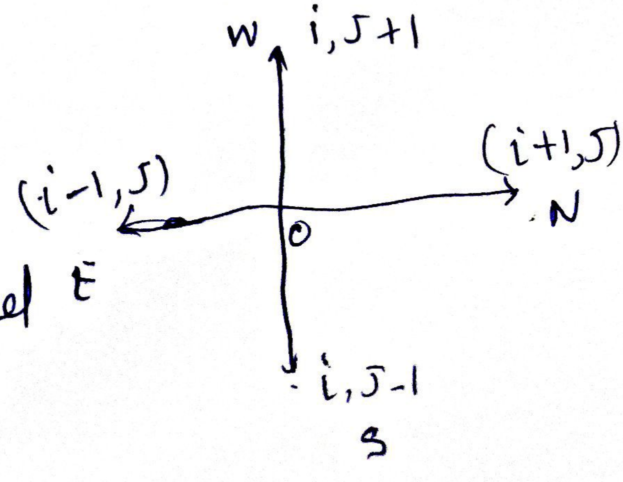
$$\therefore \frac{U_{i+1,J} - 2U_{i,J} + U_{i-1,J}}{h^2} + \frac{U_{i,J-1} - 2U_{i,J} + U_{i,J-1}}{h^2} = 0$$

gives

$$U_{i+1,J} - 4U_{i,J} + U_{i-1,J} + U_{i,J-1} + U_{i,J-1} = 0$$

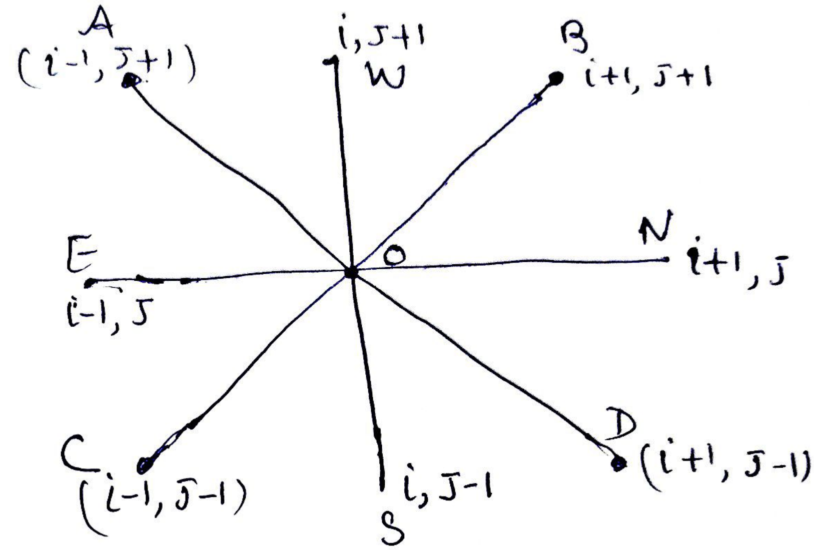
or $$U_{i,J} = \frac{1}{4}\left[U_{i+1,J} + U_{i-1,J} + U_{i,J+1} + U_{i,J-1}\right.$$

, $$U_{i,J} = \frac{1}{4}[N\ E\ W\ S]$$



This is called standard 5 point formula

Diagonal averaging formula.



$O = \frac{1}{4}\left[A + B + C + D\right]$ → Diagonal averaging formula.

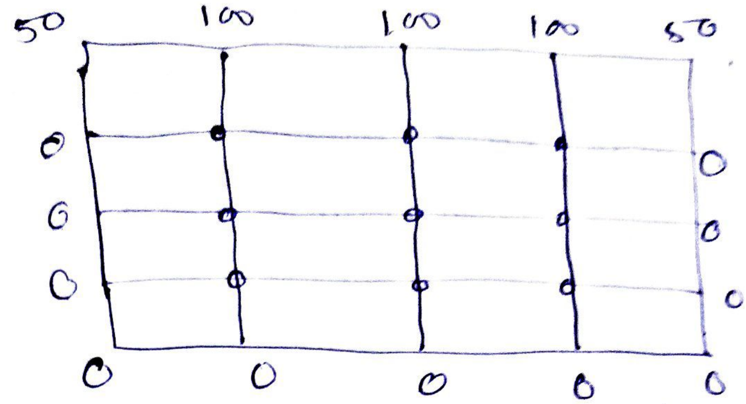
A. $U_{i,j}$ is the average of the sum of the 4 corner points.

i.e $U_{i,j} = \frac{1}{4}\left[\overset{N}{U_{i+1,j}} + \overset{E}{U_{i-1,j}} + \overset{W}{U_{i,j+1}}, \overset{S}{U_{i,j-1}}\right]$

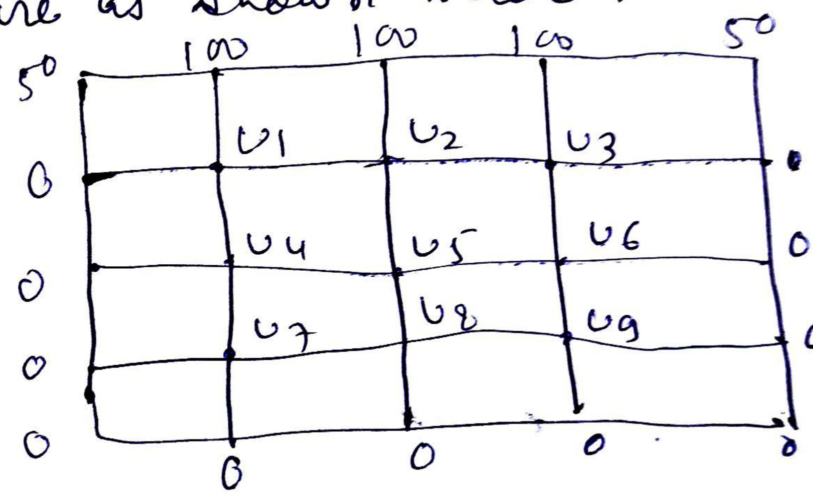
<u>Imp note</u>! -

If the cross averaging point are not available then we use diagonal averaging formula.

Q.1 Solve the elliptic equation using boundary Condition given.



Sol^n| - Let $U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9$ are interior points. Marking the point in the figure as shown below.



Since boundary values are same for

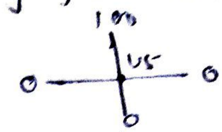
$$\left.\begin{aligned} U_1 &= U_3 \\ U_4 &= U_6 \\ U_7 &= U_9 \end{aligned}\right\} \text{By symmetry}$$

Find only $U_1, U_2, U_4, U_5, U_6, U_7, U_8$

I iteration| .

Find first U_5 by cross averaging formula

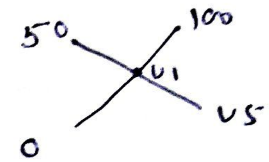
$U_5 = \frac{1}{4}[0+100+0+0] = 25$



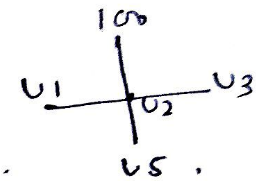
find U_1 by diagonal averaging formula

$U_1 = \frac{1}{4}[50+U_5+0+100]$

$= 43.75$



find U_2 by cross averaging formula

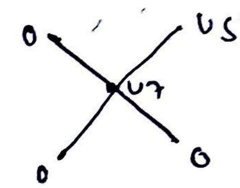


$U_2 = \frac{1}{4}[U_1+100+U_3+U_5]$

$= \frac{1}{4}[43.75+100+43.75+25]$ $\because U_1 = U_3$.

$= 53.125$

find U_7 by diagonal averaging formula

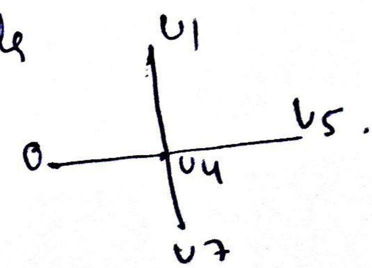


$U_7 = \frac{1}{4}[0+0+U_5+0]$

$= \frac{1}{4}[0+0+25+0]$

$= 12.5$

find U_4 by cross averaging formula

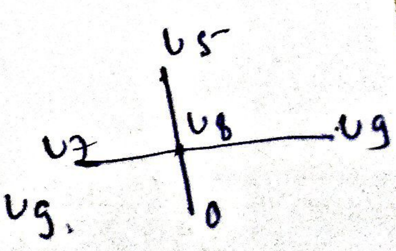


$U_4 = \frac{1}{4}[0+U_1+U_5+U_7]$

$= \frac{1}{4}[0+43.75+25+12.5]$

$= 20.31$

find U_8 by cross averaging formula



$U_8 = \frac{1}{4}[U_7+U_5+U_5+0]$ $\because U_7 = U_9$.

$$u_8 = \frac{1}{4}[12.5 + 25 + 12.5 + 0]$$
$$= 12.5$$

II iteration

From II iteration onwards, we have to use only cross averaging formula. We have to find $u_1, u_2, u_4, u_5, u_7, u_8$.

- The process is repeated as long as it produces any improvement in u's.

$$u_1 = \frac{1}{4}[0 + 100 + u_2 + u_4]$$
$$= \frac{1}{4}[0 + 100 + 53.125 + 20.31] = 43.358$$

$$u_2 = \frac{1}{4}[u_1 + 100 + u_3 + u_5]$$
$$= \frac{1}{4}[43.358 + 100 + u_1 + 25]$$
$$= 52.929..$$

$$u_4 = \frac{1}{4}[0 + u_1 + u_5 + u_7]$$
$$= \frac{1}{4}[0 + 43.358 + 25 + 12.5]$$
$$= 20.214$$

$$u_5 = \frac{1}{4}[u_4 + u_2 + u_6 + u_8] = 26.46$$

$$u_7 = \frac{1}{4}[0 + u_4 + u_8 + 0] = 8.1768$$

$$u_8 = \frac{1}{4}[0 + u_7 + u_5 + u_9] = \frac{1}{4}[0 + 8.178 + 26.46 + 8.178] = 10$$

III iteration I.

$u_1 = \frac{1}{4}[0 + 100 + u_2 + u_4]$

$= \frac{1}{4}[0 + 100 + 53.125 + 20.31]$

$= 43.358$

$u_2 = \frac{1}{4}(u_1 + 100 + u_3 + u_5) = 52.89$

$u_4 = \frac{1}{4}[0 + u_1 + u_5 + u_7] = 20.195$

$u_5 = \frac{1}{4}[u_4 + u_2 + u_6 + u_8] = \frac{1}{4}[20.195 + 52.89$

$+ u_4 + 12.5]$ $\because u_6 = u_4 = 20.214$ previous.

$= \frac{1}{4}[20.195 + 52.89 + 20.195 + 12.5]$

$u_5 = \frac{1}{4}(105.857) = 26.465$

$u_7 = \frac{1}{4}[0 + u_4 + u_8 + 0]$

$= \frac{1}{4}[0 + 20.195 + 12.5 + 0] = 8.178$

$u_8 = \frac{1}{4}(0 + u_7 + u_5 + u_9)$

$= \frac{1}{4}[0 + u_7 + u_5 + u_7)$ $\because u_7 = u_9$.

$= \frac{1}{4}[0 + 8.178 + 26.465 + 8.178] = 10.69$

IV Iteration

$u_1 = 43.27$

$u_2 = 53.2$

$u_4 = 19.472$

$u_5 = 25.708$

$u_7 = 7.540$

$u_8 = 10.197$

Can Complete