Difference operators

Let $y = f(x)$ be a given function and h is fixed value and interval

Let $x = x_0, x_1, x_2 \ldots x_n$ are equidistant

Such that

$$x_1 = x_0 + h, \ x_2 = x_1 + h = x_0 + 2h, \ x_3 = x_2 + h = x_0 + 3h$$

$$\ldots \ x_n = x_{n-1} + h = x_0 + nh$$

Also $y_0 = f(x_0), \ y_1 = f(x_1), \ y_2 = f(x_2) \ldots y_n = f(x_n)$

Operators :—

(1) Shifting operator (E)

It defined as $E f(x) = f(x+h)$

$$E^2 f(x) = f(x+2h)$$

$$\left[As \ E^2 f(x) = E\left[E(f(x))\right] = E\left[f(x+h)\right]\right.$$
$$= f(x+2h)$$

Also $E^{-1} f(x) = f(x-h)$

$$E^{-2} f(x) = f(x-2h) \qquad E^{1/2} f(x) = f(x + \tfrac{1}{2}$$

$$E y_0 = E f(x_0) = f(x_0+h) = f(x_1) = y_1$$

$$\boxed{E y_1 = y_2}$$

$$\boxed{E^{-1} y_1 = y_0}$$ and so on.

(2) forward difference operators (Δ) defined as

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$
$$= \Delta[f(x+h) - f(x)]$$
$$= \Delta f(x+h) - \Delta f(x)$$
$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$
$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

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$$\Delta y_0 = \Delta f(x_0) = f(x_1+h) - f(x_0)$$
$$= f(x_1) - f(x_0)$$
$$\boxed{\Delta y_0 = y_1 - y_0}$$

$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2 \quad ----$$

2nd order

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$
$$= \Delta y_1 - \Delta y_0$$
$$= y_2 - y_1 - y_1 + y_0$$
$$= y_2 - 2y_1 + y_0.$$

③ Backward difference operator (∇)

define $\nabla f(x) = f(x) - f(x-h)$

$$\nabla y_1 = \nabla f(x_1)$$
$$= f(x_1) - f(x_1 - h)$$
$$= f(x_1) - f(x_0 + h - h)$$
$$= f(x_1) - f(x_0)$$

$$\boxed{\nabla y_1 = y_1 - y_0}$$

$$\nabla y_2 = y_2 - y_1$$
$$\nabla^2 y_2 = \nabla(\nabla y_2)$$
$$= \nabla(y_2 - y_1)$$
$$= \nabla y_2 - \nabla y_1$$
$$= y_2 - y_1 - (y_1 - y_0)$$
$$\nabla^2 y_2 = y_2 - 2y_1 + y_0 \quad ----$$

④ central difference operator: (δ)

$$\boxed{\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}$$

⑤ Averaging operator: (μ)

defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right]$$

(A) Relation between operators.
(B) Ques based on operators.
(C) How the operator works on function.

(A) Relation:-

1. $\Delta = E - 1$
2. $\nabla = 1 - E^{-1}$
3. $\delta = E^{1/2} - E^{-1/2}$
4. $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$
5. $\left(E^{1/2} + E^{-1/2}\right)(1+\Delta)^{1/2} = 2 + \Delta$
6. $E = e^{hD}$ where $D = \frac{d}{dx}$
7. $hD = \log(1+\Delta) = -\log(1-\nabla)$
8. $(1+\Delta)(1-\nabla) = 1$

(1) $\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x) = E f(x) - f(x)$

or $\boxed{\Delta = E - 1}$

(2) $\nabla f(x) = f(x) - f(x-h)$

$\nabla f(x) = f(x) - E^{-1} f(x)$

$\boxed{\nabla = 1 - E^{-1}}$

(3) $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

$\delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$

$\delta f(x) = (E^{1/2} - E^{-1/2}) f(x)$

$\Rightarrow \boxed{\delta = E^{1/2} - E^{-1/2}}$

(4) $\mu f(x) = \frac{1}{2}\left[f(x+\frac{h}{2} - f(x-\frac{h}{2})\right]$

$= \frac{1}{2}\left[E^{1/2} f(x) - E^{-1/2} f(x)\right]$

$\mu f(x) = \frac{1}{2}(E^{1/2} - E^{-1/2}] f(x)$

$\boxed{\therefore \mu = \frac{1}{2}(E^{1/2} - E^{-1/2}]}$

(5) $\left[E^{1/2} + E^{-1/2}\right](1+\Delta)^{1/2}$

$= \left[E^{1/2} + E^{-1/2}\right][E]^{1/2}$ as $1+\Delta = E$

$= E + 1$

$= 1 + \Delta + 1$

$= 2 + \Delta$

$\therefore \boxed{(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta)}$

(8) $(1+\Delta)(1-\nabla)$

$= E\,E^{-1}$

$= 1$

$$\boxed{(1+\Delta)(1-\nabla) = 1}$$

(6) $E = e^{hD}$

By definition of shifting operator

$E\,f(x) = f(x+h)$

$E\,f(x) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) - - -$

(Taylors theorem)

$= f(x) + h D f(x) + \frac{h^2}{\lfloor 2} D^2 f(x) + \frac{h^3}{\lfloor 3} D^3 f(x) - - - -$

$E\,f(x) = (1 + hD + \frac{h^2}{\lfloor 2} D^2 - - - - -) f(x)$

$$\boxed{\therefore E = e^{hD}}$$

(7) $hD = \log(1+\Delta) = -\log(1-\nabla)$

we know that $E = e^{hD}$ — (1)

Taking log $\log E = hD$ — (2)

$\Rightarrow \log(1+\Delta) = hD$ (3) as $E = 1+\Delta$

By (2) $-\log E = -hD$

$\log E^{-1} = -hD$

$\Rightarrow \log(1-\nabla) = -hD$ $\Rightarrow -\log(1-\nabla) = hD$ — (4)

From (3) & (4) $\boxed{hD = \log(1+\Delta) = -\log(1-\nabla)}$

(B) Qus based on operators

(1) Prove that

$$\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E e^x}{\Delta^2 e^x} = e^x$$

LHS :- $\frac{(E-1)^2}{E} e^x \quad \frac{e^{x+h}}{(E-1)^2 e^x}$ $\quad \because E f(x) = f(x+h)$, $\Delta = E-1$

$$= \left(\frac{E^2-2E+1}{E}\right)\cancel{e}^x \cdot \frac{e^{x+h}}{(E^2-2E+1)\cancel{e}^x}$$

$$= (E-2+E^{-1})\, e^x \quad \frac{e^{x+h}}{(E^2 e^x - 2E e^x + e^x)}$$ $\quad \because E^{-1} f(x) = f(x-h)$

$$= (e^{x+h} - 2e^x + e^{x-h}) \frac{e^{x+h}}{(e^{x+2h} - 2e^{x+h} + e^x)}$$

$$= \frac{(e^{x+h} - 2e^x + e^{x-h})\, e^{(x+h)}}{(e^{x+2h} - 2e^{x+h} + e^x)}$$

$$= \frac{(e^{x+h} - 2e^x + e^{x-h})\, e^x e^h}{(e^{x+2h} - 2e^{x+h} + e^x)}.$$

$$= \frac{\cancel{(e^{x+2h} - 2e^{x+h} + e^x)}\, e^x}{\cancel{(e^{x+2h} - 2e^{x+h} + e^x)}} = e^x.$$

Ques 2 Evaluate.

$$\left(\frac{\Delta^2}{E}\right)\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E \sin(x+h)}$$

$$= \frac{(E-1)^2}{E}\sin(x+h) + \frac{(E-1)^2 \sin(x+h)}{\sin(x+2h)}$$

$$= \frac{(E^2-2E+1)}{E}\sin(x+h) + \frac{(E^2-2E+1)\sin(x+h)}{\sin(x+2h)}$$

$$= (E-2+E^{-1})\sin(x+h) + \frac{\sin(x+3h) - 2\sin(x+2h) + \sin(x+h)}{\sin(x+2h)}$$

$$= \sin(x+2h) - 2\sin(x+h) + \sin x + \frac{2\sin(x+2h)\cos h - 2\sin(x+2h)}{\sin(x+2h)}$$

$$= 2\sin(x+h)\cos h - 2\sin(x+h) + \frac{2\cancel{\sin(x+2h)}[\cos h - 1]}{\cancel{\sin(x+2h)}}$$

$$= 2\sin(x+h)[\cos h - 1] + 2(\cos h - 1)$$

$$= (\cos h - 1)[2\sin(x+h) + 2] \rightarrow \text{Ans}$$

Note :- (Simplify operator first)

Qus

(1) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right)$

(2) find the second difference of x^2-5x+6, the interval of difference being 1 (i.e $h=1$)

(Hint) $\Delta^2(x^2-5x+6)$

$$= \Delta\left[\Delta(\underbrace{x^2-5x+6}_{f(x)})\right]$$

$$= \Delta\cdot\left[(x+1)^2-5(x+1)+6\right]-\left[(x^2-5x+6)\right]$$

$= = =$ Simplify

(1) $$\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right) = \Delta^2\frac{(5x+12)}{(x+2)(x+3)}$$

$$= \Delta^2\left(\frac{A}{x+2}+\frac{B}{x+3}\right)$$

$$= \Delta\left(\Delta\frac{A}{x+2}+\Delta\frac{B}{x+3}\right)$$

Take $h=1$ if not give

= Simplify ?? (yourself)

How operator works on function

Interpolation

Interpolation is a process of computing intermediate values of y from a given set of values of function.

Let we have set of values x and y as follows:

x : x_0 x_1 x_2 .... x_n

y : y_0 y_1 y_2 .... y_n

which are equally distributed by interval h such that

$x_1 = x_0 + h$, $x_2 = x_0 + 2h$, --- $x_n = x_0 + nh$

Suppose we want to find out y_i for a value of x_i different from given values (X) then we use interpolation technique.

Here $x_i \in (x_0, x_n)$

For example:-

x:	20	23	26	29
y:	0.3420	0.3907	0.4384	0.4848

Find $f(21)$

Solution:- Here we will find out $f(21)$. Means we have to find value of y at $x = 21$.

Since $x = 21$ is nearer to start value we use

(1) Newton forward formula (Interpolation)

$$y_n(x) = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{\lfloor 2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)\Delta^3 f(x_0)}{\lfloor 3} + \text{-----}$$

where $p = \frac{x - x_0}{h}$

x_0 = Begning value of x

h = difference in the class (equal)

This formulla can be written in easy form

$$y_n(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \text{-----} \quad \text{—(1)}$$

Here $\Delta y_0 = y_1 - y_0$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$

$$= \Delta y_1 - \Delta y_0$$

$$= y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$= = =$$

In tabular form values can be written as :

x_k	y_k	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$	$\Delta^4 y_k$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				
⋮	⋮	⋮	⋮	⋮	⋮

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0 20	y_0 0.3420			
		$0.0487 = \Delta y_0$		
x_1 23	y_1 0.3907		$\Delta^2 y_0 = -0.0010$	
		$0.0477 = \Delta y_1$		$\Delta^3 y_0 = -0.0003$
x_2 26	y_2 0.4384		$\Delta^2 y_1 = -0.0013$	
		$0.0464 = \Delta y_2$		
x_3 29	y_3 0.4848			

In this Example $x_0 = 20$, $h = 3$ and $x = 21$

$$\therefore\ p = \frac{x - x_0}{h} = \frac{21-20}{3} = 0.3333$$

Substitute in equation ①

$$y_n(21) = 0.3420 + (0.3333)(0.0487) + \frac{(0.3333)(0.3333-1)}{2}(-0.001$$

$$\frac{}{}(-0.0003)$$

$$+ \frac{(0.3333)(0.3333-1)(0.333-2)}{6}(-0.0003)$$

$$\boxed{y_n(21) = 0.3583} \rightarrow \text{Ans}$$

Another type Ques. (Missing term)

Ex!-

x:	1	2	3	4	5	6	7
	x_0	x_1	x_2	x_3	x_4	x_5	x_6
y:	2	4	8	—	32	64	128
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Solution!- There are 6 given values of y $\therefore$ we use 6th degree polynomial formula which satisfy

$$\Delta^6 y_0 = 0. \quad \text{--- (1)}$$

$$\Delta = (E - 1)$$

As $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = E f(x) - f(x)$$

$$\Rightarrow \Delta f(x) = (E - 1) f(x)$$

Hence $\boxed{\Delta = E - 1}$

write in (1)

$$(E-1)^6 y_0 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) y_0 = 0$$

$$E^6 y_0 - 6E^5 y_0 + 15 E^4 y_0 - 20 E^3 y_0 + 15 E^2 y_0 - 6 E y_0 + y_0$$

$$y_6 - 6y_5 + 15 y_4 - 20 y_3 + 15 y_2 - 6 y_1 + y_0 = 0$$

$$128 - 6(64) + 15(32) - 20 y_3 + 15(8) - 6(4) + 2 = 0$$

$$\Rightarrow 20 y_3 = 322$$

$$y_3 = 16.1 \Rightarrow \text{Missing value}$$

well lane for dine-in customers

WHAT WILL REMAIN SHUT

Pradesh