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Eigen Value Problem

Syllabus

Module No.	Topic	Hours/ Lectures
1	**Matrix methods to linear differential equations** Eigen value problem - Eigen values and eigen vectors – properties of eigen values and eigen vectors – diagonalization using similarity and orthogonal transformations – solution of system of linear first order differential equations with constant coefficients	6
2	**Fourier series** Introduction to Fourier series - Euler formulas for the Fourier coefficients - Dirichlet's conditions – Fourier convergence Theorem (Statement) – Fourier series over a shifted interval - Half range Fourier series – Parseval's identity – Frequency and amplitude spectra of a function - computation of harmonics	6
	Before CAT-1	

Syllabus

Module No.	Topic	Hours/ Lectures
3	**Fourier Transform** Fourier Integral Theorem (statement only), Fourier Transform of a function, Fourier Sine and Cosine Integral Theorem (statement only), Fourier Cosine & Sine Transforms of elementary functions. Properties of Fourier Transform: Linearity, Shifting, Change of scale. Examples. Fourier Transform of Derivatives. Examples. Convolution Theorem (statement only), Inverse of Fourier Transform, Examples. Application of Fourier Transform in Real Life.	6
4	**Z transform** Definition of Z-transform - Z-transform of discrete sequences and functions – Relation between Z transform and Laplace transform – Definition of inverse Z transform – Inverse Z transform by partial fraction and convolution methods	5

Syllabus

Module No.	Topic	Hours/ Lectures
5	**Difference Equations** Introduction to Difference equation – order and linear - Fibonacci sequence - solution of linear difference equations with constant coefficients - complementary functions - particular integrals by the method of undetermined coefficients - solution of simple difference equations using Z-transforms	5
After CAT-2		

TYPES OF MATRICES

Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Square matrix

The number of rows is equal to the number of columns

(a square matrix **A** has an order of m) m x m

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i = j$

$a_{ij} = 0$ for some or all $i = j$

Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0$$ For all *i,j*

Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

Definitions

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A ***real*** square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged,

(1) $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$, thus $a_{kj} = a_{jk}$,

skew-symmetric if transposition gives the negative of **A**,

(2) $\mathbf{A}^{\mathsf{T}} = -\mathbf{A}$, thus $a_{kj} = -a_{jk}$,

orthogonal if transposition gives the inverse of **A**,

(3) $\mathbf{A}^{\mathsf{T}} = \mathbf{A}^{-1}$.

Linear Equations

Linear equations are common and important for survey problems

Matrices can be used to express these linear equations and aid in the computation of unknown values

Example

n equations in *n* unknowns, the a_{ij} are numerical coefficients, the b_i are constants and the x_j are unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Linear Equations

The equations may be expressed in the form

AX = B

where

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n1} \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

n x n   n x1   n x1

Number of unknowns = number of equations = n

Eigen values and Eigen vectors

- A matrix eigenvalue problem considers the vector equation

$$AX = \lambda X \qquad (1)$$

Here **A** is a given square matrix, λ an unknown scalar, and **x** an unknown vector. In a matrix eigenvalue problem, the task is to determine λ's and **x**'s that satisfy (1). Since $X = 0$ is always a solution for any λ and thus not interesting, we only admit solutions with $X \neq 0$. The solutions to (1) are given the following names: The λ's that satisfy (1) are called **eigenvalues or *characteristic value* of A** and the corresponding nonzero **x**'s that also satisfy (1) are called **eigenvectors or *characteristic vectors* of A**.

The set of all the eigenvalues of **A** is called the **spectrum** of **A**. We shall see that the spectrum consists of at least one eigenvalue and at most of *n* numerically different eigenvalues. The largest of the absolute values of the eigenvalues of **A** is called the *spectral radius* of **A.**

APPLICATION OF EIGEN VALUE PROBLEM

From this rather innocent looking vector equation flows an amazing amount of relevant theory and an incredible richness of applications. Indeed, eigenvalue problems come up all the time in engineering, physics, geometry, numeric, theoretical mathematics, biology, environmental science, urban planning, economics, psychology, and other areas. Thus, in your career you are likely to encounter eigenvalue problems. In applications ranging from mass–spring systems of physics to population control models of environmental science.

What is Eigen Value Problem

- Let us consider multiplying nonzero vectors by a given square matrix, such as

$$\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}=\begin{bmatrix} 33 \\ 27 \end{bmatrix}, \qquad \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}=\begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

We can see the influence of the multiplication of the given matrix on the vectors.

Case-I: Here we get a totally new vector with a different direction and different length as compared to the original vector. Which usually happens and is of no interest

Case-II: The multiplication produces a vector $[30 \quad 40]^T = 10[3 \quad 4]^T$ which means the new vector has the same direction as the original vector. The scale constant, which we denote by λ is 10. *The problem of systematically finding such* λ's *and nonzero vectors for a given square matrix* is called the *matrix eigenvalue* problem or, more commonly, the *eigenvalue* problem.

The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

This example illustrates the general case as follows. Equation (1) written in components is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ \cdots\cdots&\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n. \end{aligned}$$

Transferring the terms on the right side to the left side, we have

(2)
$$\begin{aligned} (a_{11}-\lambda)x_1 + \quad a_{12}x_2 + \cdots + \quad a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + \quad a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{n1}x_1 + \quad a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0. \end{aligned}$$

In matrix notation,

(3) $$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}.$$

this homogeneous linear system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero:

(4) $$D(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

$\mathbf{A} - \lambda\mathbf{I}$ is called the **characteristic matrix** and $D(\lambda)$ the **characteristic determinant** of **A**. Equation (4) is called the **characteristic equation** of **A**. By developing $D(\lambda)$ we obtain a polynomial of nth degree in λ. This is called the **characteristic polynomial** of **A**.

Properties of Eigen Values and Eigen Vectors

- Any square matrix and its transpose have same Eigen values
- Sum of elements of principal diagonal of a matrix is called `Trace of A’.
- Sum of Eigen values of a matrix is equal to `Trace of A’.
- Product of Eigen values of a matrix is equal to `determinant of A’.
- For Upper triangular, Lower triangular and diagonal matrices Eigen values are given by the diagonal elements.

- If λ_1 , λ_2 are the Eigen values of a matrix A then the Eigen values of

$$kA \text{ are } \mathrm{k}\lambda_1 , k\lambda_2$$

$$A^m \text{ are } {\lambda_1}^m , {\lambda_2}^m$$

$$A^{-1} \text{ are } \frac{1}{\lambda_1}, \frac{1}{\lambda_2}$$

$$adjA \text{ are } \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}$$

- Eigen vector X of a matrix is not unique.
- Two Eigen vector X_1 and X_2 are called orthogonal if $X_1.X'_2 = 0$.
- The order of an eigenvalue λ as a root of the characteristic polynomial is called the **algebraic multiplicity** of λ .
- The number of free variables in eigenvectors corresponding to λ is called the **geometric multiplicity**

Similarity of Matrices. Diagonalization

DEFINITION

Similar Matrices. Similarity Transformation

An $n \times n$ matrix $\mathbf{B}$ is called **similar** to an $n \times n$ matrix $\mathbf{A}$ if

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

for some (nonsingular!) $n \times n$ matrix $\mathbf{P}$. This transformation, which gives $\mathbf{B}$ from $\mathbf{A}$, is called a **similarity transformation**.

Theorem

Eigenvalues and Eigenvectors of Similar Matrices

If $\mathbf{B}$ *is similar to* $\mathbf{A}$, *then* $\mathbf{B}$ *has the same eigenvalues as* $\mathbf{A}$. *Furthermore, if* $\mathbf{x}$ *is an eigenvector of* $\mathbf{A}$, *then* $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ *is an eigenvector of* $\mathbf{B}$ *corresponding to the same eigenvalue.*

Theorem

Diagonalization of a Matrix

If an $n \times n$ *matrix* $\mathbf{A}$ *has* n *eigenvectors, then*

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is diagonal, with the eigenvalues of $\mathbf{A}$ *as the entries on the main diagonal. Here* $\mathbf{X}$ *is the matrix with these eigenvectors as column vectors. Also,*

$$\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X} \qquad (m = 2, 3, \dots).$$

EXAMPLE Diagonalization

Diagonalize

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}.$$

Solution.

The characteristic determinant gives the characteristic equation $-\lambda^3 - \lambda^2 + 12\lambda = 0$. The roots (eigenvalues of $\mathbf{A}$) are $\lambda_1 = 3$, $\lambda_2 = -4$, $\lambda_3 = 0$. By the Gauss elimination applied to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = \lambda_1, \lambda_2, \lambda_3$ we find eigenvectors and then $\mathbf{X}^{-1}$.

Note:

1. For characteristic equation of order 3 × 3 $\lambda^3 - (tracA)\lambda^2 + (\boldsymbol{A_{11}} + \boldsymbol{A_{22}} + \boldsymbol{A_{33}})\lambda - |A| = 0.$

2.For characteristic equation of order 2 × 2

$\lambda^2 + (tracA)\lambda - |A| = 0.$

The results are

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix},$$

$$\mathbf{X}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Calculating **AX** and multiplying by $\mathbf{X}^{-1}$ from the left, we thus obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Ex : Eigenvalue problems and diagonalization programs

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{The eigenvalues}: \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\ \lambda=4\Rightarrow \text{ the eigenvector } \mathbf{p}_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$(2)\ \lambda = -2 \Rightarrow \text{the eigenvector } \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \ \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Note: If $P = [\mathbf{p}_2 \quad \mathbf{p}_1 \quad \mathbf{p}_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Ex : A matrix that is not diagonalizable

Show that the following matrix is not diagonalizable

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

The eigenvalue $\lambda_1 = 1,$ and then solve $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ for eigenvectors

$$\lambda_1 I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{eigenvector } \mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since A does not have two linearly independent eigenvectors, A is not diagonalizable

- Ex : Diagonalizing a matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+2)(\lambda-3) = 0$$

The eigenvalues : $\lambda_1 = 2,\ \lambda_2 = -2,\ \lambda_3 = 3$

$$\lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \Rightarrow \text{eigenvector } \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2 \Rightarrow \lambda_2 I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} \Rightarrow \text{eigenvector } \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 3 \Rightarrow \lambda_3 I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \Rightarrow \text{ eigenvector } \mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \text{ and it follows that}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Note: a quick way to calculate A^k based on the diagonalization technique

$$(1)\ D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

$$(2)\ D = P^{-1}AP \ \Rightarrow \ D^k = \underbrace{\underbrace{P^{-1}AP}\ \underbrace{P^{-1}AP} \cdots \underbrace{P^{-1}AP}}_{\text{repeat } k \text{ times}} = P^{-1}A^kP$$

$$A^k = PD^kP^{-1}, \text{ where } D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

- Ex : Determining whether a matrix is diagonalizable

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Normalised form of vector $\mathrm{X} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is

$$\mathrm{X} = \begin{bmatrix} a/\sqrt{a^2 + b^2 + c^2} \\ b/\sqrt{a^2 + b^2 + c^2} \\ c/\sqrt{a^2 + b^2 + c^2} \end{bmatrix}.$$

Stretching of an Elastic Membrane

An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point P: (x_1, x_2) goes over into the point Q: (y_1, y_2) given by

$$(1) \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \begin{aligned} y_1 &= 5x_1 + 3x_2 \\ y_2 &= 3x_1 + 5x_2. \end{aligned}$$

in components,

Find the **principal directions**, that is, the directions of the position vector **x** of P for which the direction of the position vector **y** of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Solution.
We are looking for vectors **x** such that $\mathbf{y} = \lambda\mathbf{x}$. Since $\mathbf{y} = \mathbf{A}\mathbf{x}$, this gives $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, the equation of an eigenvalue problem. In components, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is

$$(2) \qquad \begin{aligned} 5x_1 + 3x_2 &= \lambda x_1 \\ 3x_1 + 5x_2 &= \lambda x_2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} (5-\lambda)x_1 + \qquad 3x_2 &= 0 \\ 3x_1 + (5-\lambda)x_2 &= 0. \end{aligned}$$

The characteristic equation is

$$(3) \qquad \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0.$$

Its solutions are $\lambda_1 = 8$ and $\lambda_2 = 2$. These are the eigenvalues of our problem. For $\lambda_1 = 8$, our system (2) becomes

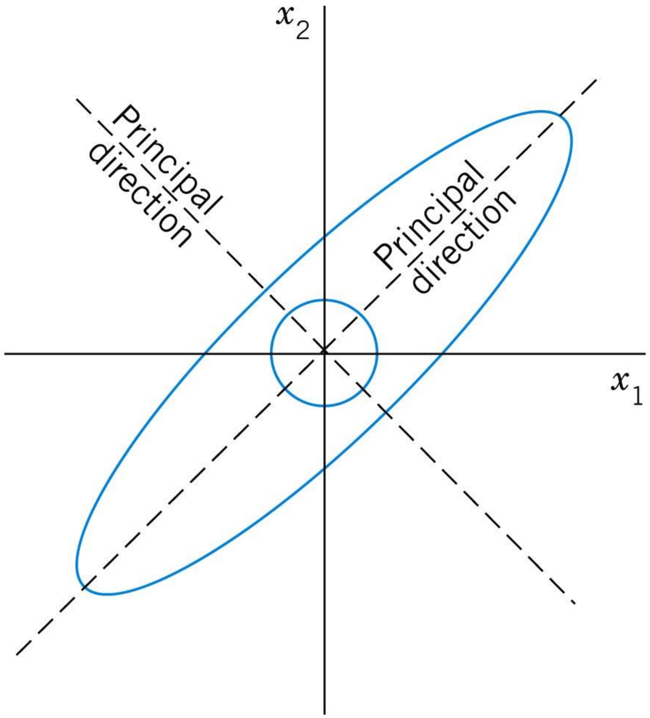
$$\begin{aligned} -3x_1 + 3x_2 &= 0, \\ 3x_1 - 3x_2 &= 0. \end{aligned} \quad \Bigg| \quad \begin{aligned} &\text{Solution } x_2 = x_1,\ x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = x_2 = 1. \end{aligned}$$

For $\lambda_2 = 2$, our system (2) becomes

$$\begin{aligned} 3x_1 + 3x_2 &= 0, \\ 3x_1 + 3x_2 &= 0. \end{aligned} \quad \Bigg| \quad \begin{aligned} &\text{Solution } x_2 = -x_1,\ x_1 \text{ arbitrary,} \\ &\text{for instance, } x_1 = 1,\ x_2 = -1. \end{aligned}$$

We thus obtain as eigenvectors of $\mathbf{A}$, for instance, $[1 \quad 1]^\mathsf{T}$ corresponding to λ_1 and $[1 \quad -1]^\mathsf{T}$ corresponding to λ_2 (or a nonzero scalar multiple of these). These vectors make 45° and 135° angles with the positive x_1-direction. They give the principal directions, the answer to our problem.

The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively



x_2
x_1
Principal direction
Principal direction

Accordingly, if we choose the principal directions as directions of a new Cartesian u_1u_2 coordinate system,
say, with the positive u_1-semi-axis in the first quadrant and the positive u_2-semi-axis in the second quadrant of
the x_1x_2-system, and if we set $u_1 = r\cos\varphi$, $u_2 = r\sin\varphi$, then a boundary point of the unstretched circular
membrane has coordinates $\cos\varphi$, $\sin\varphi$. Hence, after the stretch we have

$$z_1 = 8\cos\varphi\ ,\ z_2 = 2\sin\varphi\ .$$

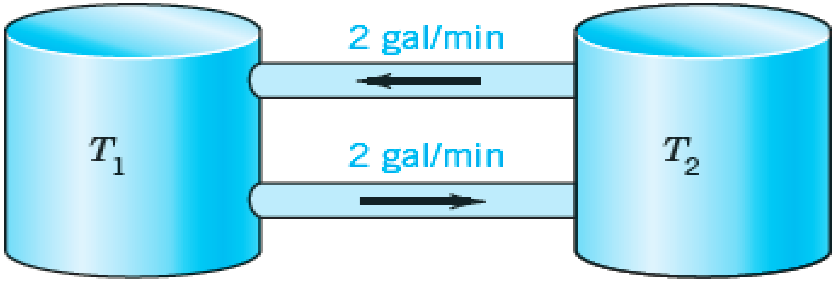
Since $\cos^2\varphi + \sin^2\varphi = 1$, this shows that the deformed boundary is an ellipse

(4)

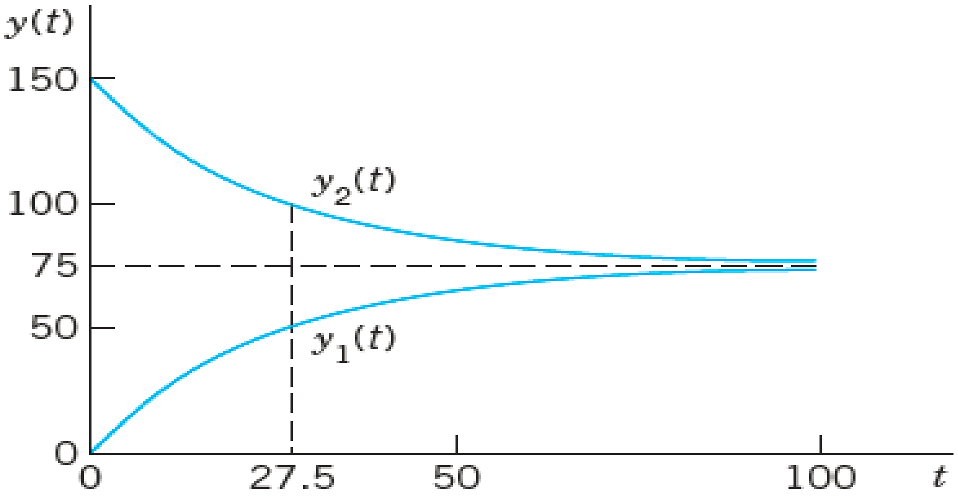
$$\frac{z_1^{\ 2}}{8^2} + \frac{z_2^{\ 2}}{2^2} = 1.$$

Mixing Problem Involving Two Tanks

Tank T_1 and T_2 in Fig. 78 contain initially 100 gal of water each. In T_1 the water is pure, whereas 150 lb of fertilizer are dissolved in T_2. By circulating liquid at a rate of 2 gal/min and stirring (to keep the mixture uniform) the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t. How long should we let the liquid circulate so that T_1 will contain at least half as much fertilizer as there will be left in T_2?



System of tanks



Fertilizer content in Tanks T_1 (lower curve) and T_2

Solution. *Step 1.* **Setting up the model.** As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. From Fig. 78 we see that

$$y_1' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_2 - \frac{2}{100}y_1 \qquad \text{(Tank } T_1\text{)}$$

$$y_2' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_1 - \frac{2}{100}y_2 \qquad \text{(Tank } T_2\text{).}$$

Hence the mathematical model of our mixture problem is the system of first-order ODEs

$$\begin{aligned} y_1' &= -0.02y_1 + 0.02y_2 && \text{(Tank } T_1\text{)} \\ y_2' &= 0.02y_1 - 0.02y_2 && \text{(Tank } T_2\text{).} \end{aligned}$$

As a vector equation with column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and matrix $\mathbf{A}$ this becomes

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

we thus obtain a solution

$$\text{(3)} \qquad \mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-0.04t}$$

Step 3. **Use of initial conditions.** The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and (3) with $t = 0$ we obtain

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer

$$\mathbf{y} = 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)} e^{-0.04t} = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}.$$

In components,

$$y_1 = 75 - 75e^{-0.04t} \qquad \text{(Tank } T_1\text{, lower curve)}$$

$$y_2 = 75 + 75e^{-0.04t} \qquad \text{(Tank } T_2\text{, upper curve).}$$

Figure 78 shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 75 lb.

Step 4. **Answer.** T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50 lb. Thus

$$y_1 = 75 - 75e^{-0.04t} = 50, \qquad e^{-0.04t} = \tfrac{1}{3}, \qquad t = (\ln 3)/0.04 = 27.5.$$

Hence the fluid should circulate for at least about half an hour.

Mass on a Spring

To gain confidence in the conversion method, let us apply it to an old friend of ours, modeling the free motions of a mass on a spring (see Sec. 2.4)

$$my'' + cy' + ky = 0 \qquad \text{or} \qquad y'' = -\frac{c}{m}y' - \frac{k}{m}y.$$

For this ODE (8) the system (10) is linear and homogeneous,

$$y_1' = y_2$$

$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2.$$

Setting $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we get in matrix form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

It agrees with that in Sec. 2.4. For an illustrative computation, let $m = 1, c = 2$, and $k = 0.75$. Then

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + 0.5)(\lambda + 1.5) = 0.$$

This gives the eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$. Eigenvectors follow from the first equation in $\mathbf{A} - \lambda\mathbf{I} = \mathbf{0}$, which is $-\lambda x_1 + x_2 = 0$. For λ_1 this gives $0.5x_1 + x_2 = 0$, say, $x_1 = 2$, $x_2 = -1$. For $\lambda_2 = -1.5$ it gives $1.5x_1 + x_2 = 0$, say, $x_1 = 1, x_2 = -1.5$. These eigenvectors

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \qquad \text{give} \qquad \mathbf{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}.$$

This vector solution has the first component

$$y = y_1 = 2c_1e^{-0.5t} + c_2e^{-1.5t}$$

which is the expected solution. The second component is its derivative

$$y_2 = y_1' = y' = -c_1e^{-0.5t} - 1.5c_2e^{-1.5t}.$$