Example 5

The joint pdf of a two-dimensional RV (X, Y) is given by $f(x, y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$.

Compute $P(X > 1)$, $P\left(Y < \frac{1}{2}\right)$, $P(X > 1/Y < 1/2)$

$P\left(Y < \frac{1}{2}/X > 1\right)$, $P(X < Y)$ and $P(X + Y \le 1)$.

Here the rectangle defined by $0 \le x \le 2$, $0 \le y \le 1$ is the range space R. $R_1, R_2 \ldots$ are event spaces.

(i) $P(X > 1) = \underset{\substack{R_1 \\ (x>1)}}{\int\int} f(x, y)\, dx\, dy$

$$= \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx dy = \frac{19}{24}$$

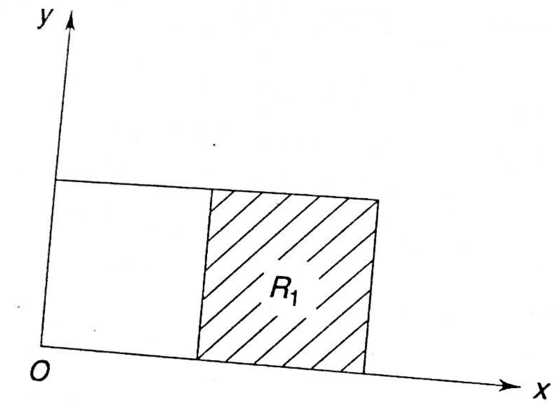


Fig. 2.1

(ii) $P(Y < 1/2) = \underset{\substack{R_2 \\ \left(y<\frac{1}{2}\right)}}{\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^{1/2} \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \frac{1}{4}$$

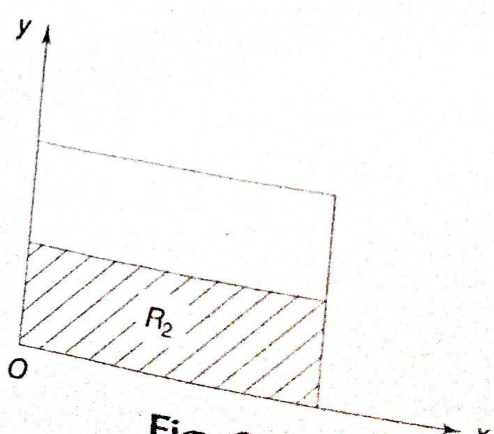


Fig...

(iii) $$P(X > 1, Y < 1/2) = \underset{\substack{R_3 \\ \left(x>1\ \&\ y<\frac{1}{2}\right)}}{\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^{1/2}\int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \frac{5}{24}$$

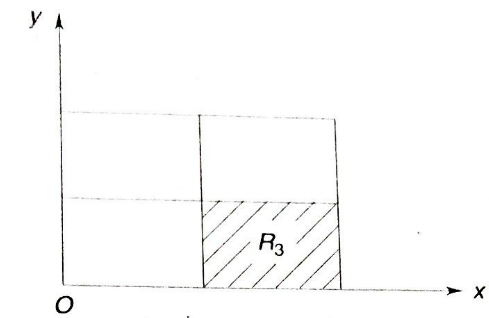


Fig. 2.3

(iv) $$P(X > 1/Y < \frac{1}{2}) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)} = \frac{5/24}{1/4} = \frac{5}{6}$$

(v) $$P(Y < \frac{1}{2}/X > 1) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P\,(X > 1)} = \frac{5/24}{19/24} = \frac{5}{19}$$

(vi) $$P(X < Y) = \underset{\substack{R_4 \\ (x<y)}}{\int\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1\int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{53}{480}$$

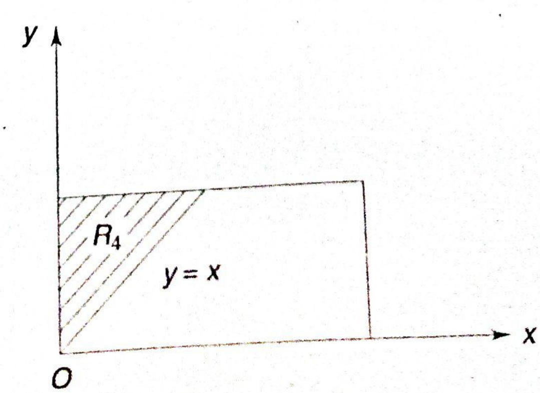


Fig. 2.4

(vii) $P(X+Y \le 1) = \underset{\substack{R_5 \\ (x+y \le 1)}}{\int\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{13}{480}$$

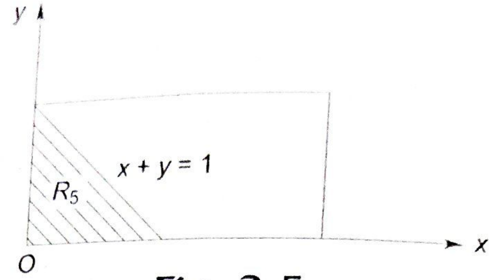


Fig. 2.5

Example 6

If the joint pdf of the RV (X, Y) is given by $f(x, y)$
$= \frac{1}{2\pi\sigma^2} \exp\{-(x^2+y^2)/2\sigma^2\}, -\infty < x, y < \infty$, find $P(X^2 + Y^2 \le a^2)$.

Here the entire xy-plane is the range space R and the event-space D is the interior of the circle $x^2 + y^2 = a^2$.

$$P(X^2 + Y^2 \le a^2) = \iint_{x^2+y^2 \le a^2} f(x, y)\, dx\, dy$$

Transform from cartesian system to polar system, i.e., put $x = r\cos\theta$ and $y = r\sin\theta$.

Then $dx\, dy = r dr d\theta$.

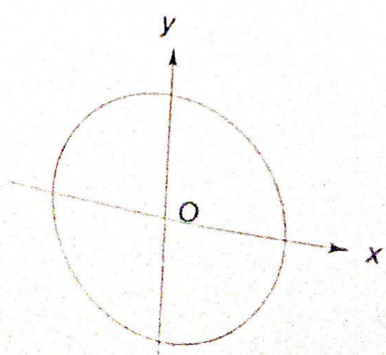
The domain of integration becomes $r \le a$.

Then $P(X^2 + Y^2 \le a^2) = \int_0^{2\pi} \int_0^a \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, r dr d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(-e^{-r^2/2\sigma^2}\right)_0^a d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(1 - e^{-a^2/2\sigma^2}\right) d\theta$$

$$= 1 - e^{-a^2/2\sigma^2}$$



Now $$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\, dy$$

$$= \frac{2}{\pi R^2}\sqrt{R^2-x^2} = \frac{2}{\pi R}\sqrt{1-\left(\frac{x}{R}\right)^2} \quad -R \le x \le R$$

Note *Whenever we are required to find the marginal and conditional density functions, the ranges of the concerned variables should also be specified.*

Example 8

The joint pdf of the RV (X, Y) is given by $f(x, y) = k\,x\,y\,e^{-(x^2+y^2)}$, $x > 0$, $y > 0$. Find the value of k and prove also that X and Y are independent.

Here the range space is the entire first quadrant of the xy-plane.

By the property of the joint pdf

$$\iint_{x>0,\, y>0} kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

i.e., $$k\int_0^\infty ye^{-y^2}\, dy \int_0^\infty xe^{-x^2}\, dx = 1$$

i.e., $$\frac{k}{4} = 1$$

$\therefore$ $$k = 4$$

Now $$f_X(x) = \int_0^\infty 4x\, e^{-x^2} \times ye^{-y^2}\, dy = 2x\, e^{-x^2},\; x > 0$$

Similarly, $f_Y(y) = 2ye^{-y^2}$, $y > 0$.

Now $f_X(x) \times f_Y(y) = 4\,x\,y\,e^{-(x^2+y^2)} = f(x, y)$

$\therefore$ The RVs x and y are independent.

Note *If $f(x, y)$ can be factorised as $f_1(x) \times f_2(y)$ then X and Y will be independent.*

Example 9

Given $f_{XY}(x, y) = cx\,(x - y)$, $0 < x < 2$, $-x <, y < x$, and 0 elsewhere, (a) evaluate c, (b) find $f_X(x)$, (c) $f_{Y/X}(y/x)$ and (d) $f_Y(y)$.

(BDU — Apr. ...)

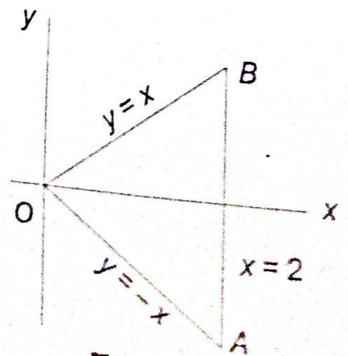


Fig. 2.8

Here the range space is the area within the triangle OAB (shown in the figure), defined by $0 < x < 2$ and $-x < y < x$.

(a) By the property of *jpdf*

$$\iint_{\Delta OAB} cx(x-y)\,dx\,dy = 1$$

$$\int_0^2 \int_{-x}^{x} cx(x-y)\,dy\,dx = 1$$

i.e., $8c = 1$

$\therefore$ $c = \dfrac{1}{8}$

(b) $$f_X(x) = \int_{-x}^{x} \frac{1}{8} x\,(x-y)\,dy$$

$$= \frac{x^3}{4}, \text{ in } 0 < x < 2$$

(c) $$f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{2x^2}(x-y), \; -x < y < x$$

(d) $$f_Y(y) = \int_{-y}^{2} \frac{1}{8} x\,(x-y)\,dx, \text{ in } -2 \le y \le 0$$

$$= \int_{y}^{2} \frac{1}{8} x\,(x-y)\,dx, \text{ in } 0 \le y \le 2$$

i.e., $$f_Y(y) = \begin{cases} \dfrac{1}{3} - \dfrac{y}{4} + \dfrac{5}{48} y^3, & \text{in } -2 \le y \le 0 \\ \dfrac{1}{3} - \dfrac{y}{4} + \dfrac{1}{48} y^3, & \text{in } 0 \le y \le 2 \end{cases}$$

Example 10

Train X arrives at a station at random in the time interval $(0, T)$ and stops for 'a' min. Train Y arrives independently in the same interval and stops for 'b' min.

(i) Find the probability P_1 that X will arrive before Y.

(ii) Find the probability P_2 that the two trains meet.

(iii) Assuming that they meet, find the probability P_3 that X arrived before Y.

(MSU — Nov. 96)

Let the trains X and Y arrive at the station at time instances X and Y respectively.

Then the lengths of the intervals $(0, X)$ and $(0, Y)$, namely X and Y are continuous RVs. Each of X and Y is uniformly distributed in $(0, T)$ (since the times of arrival are equally likely) with pdf $\dfrac{1}{T}$.

Averages

A discrete random variable (RV) is no doubt completely described by its probability mass function or probability distribution. Similarly, a continuous RV is completely described by its probability density function. For many purposes, this description is often considered to consist of too many details. It is sometimes simpler and more convenient to describe a RV or to characterise its distribution by a few parameters or summary measures that are representative of the distribution. These parameters or characteristic numbers are the various expected values or statistical averages of the RV.

Definitions: If X is a discrete RV, then ***the expected value*** or the mean value of $g(X)$ is defined as

$$E\{g(X)\} = \sum_i g(x_i)p_i,$$

where $\qquad p_i = P(X = x_i)$ is the probability mass function of X.

If X is a continuous RV with pdf $f(x)$, then

$$E\{g(X)\} = \int_{R_X} g(x)f(x)\,dx$$

Two expected values which are most commonly used for characterising a RV X are ***its mean*** μ_X and ***variance*** σ_X^2, which are defined as follows:

$$\begin{aligned}
\mu_X &= E(X) \\
&= \sum_i x_i\, p_i, \text{ if } X \text{ is discrete} \\
&= \int_{R_X} x f(x)\, dx, \text{ if } X \text{ is continuous}
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) = \sigma_X^2 &= E\{(X - \mu_X)^2\} \\
&= \sum_i (x_i - \mu_X)^2\, p_i, \text{ if } X \text{ is } \textit{discrete}
\end{aligned}$$

$$\int_{R_X} (x-\mu_X)^2 f(x)\,dx, \text{ if } X \text{ is continuous}$$

The square root of variance is called *the* **standard deviation**. The mean is its average value and the variance is a measure of the spread or dispersion of the values of the RV.

Note

$Var(X) = E(X^2) - \{E(X)\}^2$ (BU — Apr.)

$$\begin{aligned} Var(X) &= E\{(X-\mu_X)^2\} \\ &= E\{X^2 - 2\mu_X X + \mu_X^2\} \\ &= E(X^2) - 2\mu_X E(X) + \mu_X^2 \text{ (since } \mu_X \text{ is a constant)} \\ &= E(X^2) - \mu_X^2 \text{ [since } \mu_X = E(X)] \\ &= E(X^2) - E(X)\}^2 \end{aligned}$$

This modified formula for var(X) holds good for both discrete and continuous RVs.

Note

If X is a discrete RV and a is a constant, then (i) $E(aX) = a\,E(X)$, (ii) $Var(aX) = a^2 Var(X)$. (BU — Apr. 96)

(i) $E(aX) = \sum_j ax_j\,pi$
$\quad = a \sum_j x_j\,pi$
$\quad = aE(X)$

(ii) $Var(aX) = E(a^2X^2) - \{E(aX)\}^2$ (by Note 1)
$\quad = a^2\,E(X^2) - \{aE(X)\}^2$
$\quad = a^2[E(X^2) - \{E(X)^2]$
$\quad = a^2 Var(X)$

This result holds good for a continuous RV also.

Moments: If X is a discrete or continuous RV, $E(X^n)$ is called nth order moment of X about the origin and denoted by μ'_n.

$E\{(X-\mu_X)^n\}$ is called the nth order central moment of X and denoted by μ_n.

$E\{|X|^n\}$ and $E\{|X-\mu_X|^n\}$ are called absolute moments of X.

$E\{X-a)^n\}$ and $E\{|X-a|^n\}$ are called generalised moments of X.

Expected Values of a Two-Dimensional RV

If (X, Y) is a two-dimensional discrete RV with joint probability mass function p_{ij}, then $E\{g(X, Y)\} = \sum_j \sum_i g(x_i, y_i\}p_{ij}$.

If (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$, then

$$E\{g(X, Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f(x, y)\,dx\,dy$$

Properties of Expected V

We give below the proofs of the pr prove the properties for discrete RV

(i) $E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$, w

Proof

$E\{g(X)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) f$

$= \int_{-\infty}^{\infty} g(x) \Big[\int$

$= \int_{-\infty}^{\infty} g(x) f_X(x$

(ii) $E\{h(Y)\} = \int_{-\infty}^{\infty} h(y) f_Y(y)\,d$

where $f_Y(y)$ is the margina
(Proof is left as an exercis

(iii) $E(X + Y) = E(X) + E(Y)$

Proof

$E(X + Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f$

$= E(X) + E$

(iv) In general, $E(XY) \neq E($
$E(XY) = E(X) \times E(Y)$.

Proof

* $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$

(since X

for two variable

Properties of Expected Values

We give below the proofs of the properties for continuous RVs. Students can prove the properties for discrete RVs.

(i) $E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$, where $f_X(x)$ is the marginal density of X.

Proof

$$E\{g(X)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) f(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} g(x) \left[\int_{-\infty}^{\infty} f(x, y)\, dy\right] dx$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

(ii) $E\{h(Y)\} = \int_{-\infty}^{\infty} h(y) f_Y(y)\, dy$

where $f_Y(y)$ is the marginal density of Y.
(Proof is left as an exercise to the student.)

(iii) $E(X + Y) = E(X) + E(Y)$

Proof

$$E(X + Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x + y) f(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x, y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x, y)\, dx\, dy$$

$$= E(X) + E(Y)$$

(iv) In general, $E(XY) \neq E(X) \times E(Y)$, but if X and Y are independent RVs, $E(XY) = E(X) \times E(Y)$.

If $E(XY) = E(X) \times E(Y)$
$f(x,y) = f_X(x)\, f_Y(y)$
if X and Y are Independent

Proof

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x, y)\, dx\, dy$$

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\, dx\, dy$$

(since X and Y are independent)

$f(x,y) \neq f_X(x)\, f_Y(y)$
ie $E(XY) \neq E(X) \cdot E(Y)$
if X and Y are dependent

for two variable $E(XY) = \int\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx \times \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= E(X) \times E(Y)$$

In general, if X and Y are independent,

$$E\{g(X) \times h(Y)\} = E\{g(X)\} \times E\{h(Y)\}$$

Conditional Expected Values

If (X, Y) is a two-dimensional discrete RV with joint probability mass p_{ij}, then the conditional expectations of $g(X, Y)$ are defined as follows:

$$E\{g(X, Y)/Y = Y_j\} = \sum_i g(x_i, y_j) \times P(X = x_i / Y = y_j)$$

$$= \sum_i g(x_i, y_j) \frac{p_{ij}}{p_{*j}}$$

and $\quad E\{g(X, Y)/X = x_i\} = \sum_j g(x_i, y_j) p_{ij} / p_{i*}$

If (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$, then

$$E\{g(X, Y)/Y\} = \int_{-\infty}^{\infty} g(x, y) \times f(x/y)\, dx \text{ and}$$

$$E\{g(X, Y)/X\} = \int_{-\infty}^{\infty} g(x, y) \times f(y/x)\, dy$$

In particular, the conditional means are defined as

$$\mu_{Y/X} = E(Y/X) = \int_{-\infty}^{\infty} y f(y/x)\, dy \text{ and}$$

$$\mu_{X/Y} = E(X/Y) = \int_{-\infty}^{\infty} x f(x/y)\, dx$$

The conditional variances are defined as

$$\sigma^2_{Y/X} = E\{(Y - \mu_{Y/X})^2\} = \int_{-\infty}^{\infty} (y - \mu_{Y/X})^2 f(y/x)\, dy \text{ and}$$

$$\sigma^2_{X/Y} = E\{(X - \mu_{X/Y})^2\} = \int_{-\infty}^{\infty} (x - \mu_{X/Y})^2\} f(x/y)\, dx$$

Properties

(1) If X and Y are independent RVs, then $E(Y/X) = E(Y)$ and $E(X/Y) =$

Proof

$$E(Y/X) = \int_{-\infty}^{\infty} y f(y/x)\, dy$$

$f(y/x) = \dfrac{f(x,y)}{f_X(x)}$

$$= \int_{-\infty}^{\infty} y \frac{f(x, y)}{f_X(x)}\, dy$$

$$= \int_{-\infty}^{\infty} y \frac{f_X(x) \times f_Y(y)}{f_X(x)}\, dy$$

(since X and Y are independent)

$$= \int_{-\infty}^{\infty} y f_Y(y)\, dy = E(Y)$$

A similar proof can be given for the other result.

(2) $E[E\{g(X, Y)/X\}] = E\{g(X, Y)\}$

Proof

$$E\{g(X, Y)/X\} = \int_{-\infty}^{\infty} g(x, y) f(y/x)\, dy$$

Since $E\{g(X, Y)/X\}$ is a function of the RV X.

$$E[E\{g(X, Y)/X\}] = \int_{-\infty}^{\infty} E\{g(X, Y)/X\} f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f(y/x) f_X(x)\, dx\, dy \text{ [from (1)]}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy$$

$$= E\{g(X, Y)\}$$

In particular,

$$E\{E(Y/X)\} = E(Y) \text{ and similarly}$$
$$E\{E(X/Y)\} = E(X).$$

(3) $E\{g_1(X) \times g_2(Y)\} = E[g_1(X) \times E\{g_2(Y)/X\}]$

Proof

$$E\{g_1(X) \times g_2(Y)\} = E[E\{g_1(X) \times g_2(Y)/X] \text{ (by Property (2))}$$
$$= E[g_1(X) \times E\{g_2(Y)/X\}] \text{ (since } X \text{ is given)}$$

In particular,

$$E(XY) = E[X \times E(Y/X)] \text{ and}$$
$$E(X^2Y^2) = E[X^2 \times E(Y^2/X)]$$

Worked Example 4(A)

Example 1

A lot is known to contain 2 defectives and 8 non-defective items. If these are inspected at random, one after another, what is the expected number of items that must be chosen in order to remove, both the defective ones?

Let the random variable X denote the number of items that must be drawn in order to remove both defective items.

Clearly X takes the values 2, 3, 4, ..., 10.

$P(X = r) = P(r$ items are to be drawn to remove both defectives)

$= P\{$the first $(r-1)$ items drawn should contain 1 defective and rth item drawn should be defective$\}$

$$= \frac{2C_1 \times 8C_{r-2}}{10C_{r-1}} \times \frac{1}{10-(r-1)} = \frac{2 \times 8C_{r-2}}{10C_{r-1}(11-r)}$$

$(r = 2, 3, ...$

The probability distribution of X will then be as follows:

$X = r$	2	3	4	5	6	7	8	9	10
p_r	1/45	2/45	3/45	4/45	5/45	6/45	7/45	8/45	9/45

$$E(X) = \sum_{r=2}^{10} r p_r = \frac{22}{3}$$

Example 2

A box contains 2^n tickets of which nC_r tickets bear the number r $(r = 0, 1, ..., n)$. Two tickets are drawn from the box. Find the expectation of the sum of the numbers.

Total number of tickets in the box.

$$\sum_{r=0}^{n} nC_r = nC_0 + nC_1 + \cdots + nC_n$$

$$= (1+1)^n = 2^n, \text{ as given.}$$

Let the RVs X and Y represent the numbers on the first and second tickets respectively.

Then $\quad E(X+Y) = E(X) + E(Y)$

X can take the values 0, 1, 2, ..., n with probabilities $\frac{nC_0}{2^n}, \frac{nC_1}{2^n}, ...$ respectively.