MODULE-2

Random variables
And
Special Probability distribution

CONTENT

Introduction to Random variables, - One dimensional Random Variables, Discrete and Continuous RV- Density and Distribution function of RV, Expectation, Variance, and its properties, Covariance, and Moments. Moment Generating function

Special Distributions

Binomial and Poisson distributions – Normal distribution, Exponential distributions, Weibull distribution

RANDOM VARIABLE

- Stochastic variable
- Chance variable

TYPES OF R.V

1. Discrete random variable
2. Continuous random variable

DISCRETE RANDOM VARIABLE

It takes only finite and countable number of values

Probability function of a discrete Random Variable.

QUESTION

Let X be a discrete random variable with

$$P(X = j) = C.\left(\frac{1}{2}\right)^{j}, j = 1,2, \ldots .$$

Find C, E(X), Var (X)

QUESTION

The probability function of a discrete random variable is as follows:

$X=x$	0	1	2	3	4	5	6	7
$P(X=x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	K^2+k

Find (i) k

(ii) $p(X<6)$

(iii) $p(0<x<5)$

(iv) distribution function of x

MOMENTS

Moments in statistics measure something relative to the center of the values.

MOMENTS

Interpretation of Moment Statistics

Mean (M_1) - 1^{st} moment about the origin - central tendency measure.

Variance (m_2) - 2^{nd} moment about the mean - dispersion measure.

Skewness (a_3) - 3^{rd} standardize moment - skewness measure.

$a_3 = 0 \rightarrow$ symmetrical

$a_3 > 0 \rightarrow$ positively skewed

$a_3 < 0 \rightarrow$ negatively skewed

for a_3 between $\pm$ 0.2, the distribution can be assumed to be normal with respect to skewness.

Kurtosis (a_4-3.0) - (4^{th} standardized moment - 3) - kurtosis measure.

$a_4 - 3 = 0 \rightarrow$ same peakedness as normal curve.

$a_4 - 3 > 0 \rightarrow$ more peakedness than normal curve.

$a_4 - 3 < 0 \rightarrow$ flattter than normal curve.

for $a_4 - 3$ between $\pm$ 0.5, the curve can be considered normal with respect to kurtosis.

Moments about the origin $\alpha = 0$, X_i = raw score

First Moment (M_1) - Mean = $\bar{X}$

$$M_1 = \frac{1}{N}\sum_{i=1}^{N} X_i = \frac{1}{N}\sum_{i=1}^{N}(freq * X)$$

Second Moment (M_2)

$$M_2 = \frac{1}{N}\sum_{i=1}^{N} X_i^2 = \frac{1}{N}\sum_{i=1}^{N}(freq * X^2)$$

Third Moment (M_3)

$$M_3 = \frac{1}{N}\sum_{i=1}^{N} X_i^3 = \frac{1}{N}\sum_{i=1}^{N}(freq * X^3)$$

Fourth Moment (M_4)

$$M_4 = \frac{1}{N}\sum_{i=1}^{N} X_i^4 = \frac{1}{N}\sum_{i=1}^{N}(freq * X^4)$$

Example 3·9. *Calculate the first four moments of the following distribution about the mean and hence find* β_1 and β_2.

x:	0	1	2	3	4	5	6	7	8
f:	1	8	28	56	70	56	28	8	1

Solution. CALCULATION OF MOMENTS

x	f	$d = x - 4$	fd	fd^2	fd^3	fd^4
0	1	–4	–4	16	–64	256
1	8	–3	–24	72	–216	648
2	28	–2	–56	112	–224	448
3	56	–1	–56	56	–56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
Total	256	0	0	512	0	2,816

Moments about the points $x = 4$ are

$$\mu_1' = \frac{1}{N}\Sigma fd = 0, \ \mu_2' = \frac{1}{N}\Sigma fd^2 = \frac{512}{256} = 2,$$

$$\mu_3' = \frac{1}{N}\Sigma fd^3 = 0 \text{ and } \mu_4' = \frac{1}{N}\Sigma fd^4 = \frac{2816}{256} = 11$$

Moments about mean are :

$$\mu_1 = 0, \ \mu_2 = \mu_2' - \mu_1'^2 = 2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 11$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \ \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{4} = 2{\cdot}75$$

Moment Generating Function.

The moment generating function

(m.g.f.) of a random variable X (about origin) having the probability function $f(x)$ is given by

$$M_X(t) = E(e^{tX}) = \begin{cases} \int e^{tx} f(x)\, dx, & \text{(for } \textit{continuous probability distribution}\text{)} \\ \sum_x e^{tx} f(x), & \text{(for } \textit{discrete probability distribution}\text{)} \end{cases}$$

Example 6·37. *Let the random variable X assume the value 'r' with the probability law :*

$$P(X=r)=q^{r-1}p\,;\ r=1,2,3,\ldots$$

Find the m.g.f. of X and hence its mean and variance.

Solution. $M_X(t) = E(e^{tX})$

$$= \sum_{r=1}^{\infty} e^{tr} \quad q^{r-1} p = \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$

$$= \frac{p}{q} qe^t \sum_{r=1}^{\infty} (qe^t)^{r-1} = pe^t \left[1 + qe^t + (qe^t)^2 + \ldots\right]$$

$$= \left(\frac{pe^t}{1 - qe^t}\right)$$

If dash (′) denotes the differentiation w.r.t. t, then we have

$$M_X'(t) = \frac{pe^t}{(1 - qe^t)^2}, \; M_X''(t) = pe^t \frac{(1 + qe^t)}{(1 - qe^t)^3}$$

$\therefore$ $$\mu_1' \text{ (about origin)} = M_X'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$\mu_2' \text{ (about origin)} = M_X''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}.$$

Hence $$\text{mean} = \mu_1' \text{ (about origin)} = \frac{1}{p}$$

and $$\text{variance} = \mu_2 = \mu_2' - \mu_1'^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

TYPES OF PROBABILITY DISTRIBUTION

Binomial

Poisson

Normal

BINOMIAL DISTRIBUTION

Binomial distribution is a discrete probability distribution.

Example : Number of defectives in a lot of size ‘n’

Probability function

$$^{n}C_{r}\,p^{r}q^{n-r}$$

Constants

Mean = np

Variance = npq

A COIN IS TOSSED FOUR TIMES. WHAT IS THE PROBABILITY OF GETTING-

No head

Exactly one head

Exactly two heads

Exactly three heads

Exactly four heads

At least two heads

More than two heads

At most 2 heads

Less than three heads

SOLUTION

Number of trails (n) = 4

Probability of getting a head (p) = ½

Probability of not getting a head (q) = 1- (1/2) = ½

Binomial distribution =

$$P(r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

NO HEADS P(R=0)

$$P(0) = {}^4C_0 p^0 q^{4-0}$$

$$P(0) = \frac{1}{16}$$

EXACTLY ONE HEAD

$$P(1) = {}^{4}C_{1}p^{1}q^{4-1}$$

$$P(1) = \frac{1}{4}$$

Exactly two heads P(2) = 3/8
Exactly three heads P(3) = ¼
Exactly four heads P(4) = 1/16

AT LEAST 2 HEADS

$P(r=2 \text{ or more}) = P(2)+P(3)+P(4) = 11/16$

More than 2 heads

$P(r = 3 \text{ or more}) = P(3) + P(4) = 5/16$

At most 2 heads

$P(r = 0 \text{ or } 1 \text{ or } 2) = P(0) + P(1) + P(2) = 11/16$

Less than 3 heads

$P(r = 0 \text{ or } 1 \text{ or } 2) = P(0) + P(1) + P(2) = 11/16$

AN UNBIASED CUBIC DIE IS THROWN FOUR TIMES. WHAT IS THE PROBABILITY OF OBTAINING

No six

At least one six

At least one even digit

A multiple of 3

A multiple of 2 or 3

AN UNBIASED CUBIC DIE IS THROWN FOUR TIMES. WHAT IS THE PROBABILITY OF OBTAINING

No six

= 625/1296

At least one six

= 671/1296

At least one even digit

= 15/16

A multiple of 3

= 2/3

A multiple of 2 or 3

= 8/81

CALCULATE MEAN OF THE BINOMIAL DIST.

No. of trails = 6, probability of success = 1/3

No. of trails = 9, probability of failure = 1/3

Probability of failure = 2/3, standard deviation = 2

Probability of failure = 5/6, variance = 25

CALCULATE MEAN OF THE BINOMIAL DIST.

No. of trails = 6, probability of success = 1/3

2

No. of trails = 9, probability of failure = 1/3

6

Probability of failure = 2/3, standard deviation = 2

6

Probability of failure = 5/6, variance = 25

30

POISSON DISTRIBUTION

The Poisson distribution is used as a limiting form of binomial distribution.

It is a discrete prob. Dist.

It applies in a situations where the prob. of success (p) is very small and that of failure (q) is very high, almost equal to 1.

Example

The number of cars arriving per minute at a service station

The number of persons born blind per year in a city

PROBABILITY FUNCTION

$$P(r) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where

$x = 0,1,2,........$

$e = 2.7183$

$\lambda = mean$

Constants

Mean = Variance = np (λ)

IF A RANDOM VARIABLE X FOLLOWS POISSON DISTRIBUTION SUCH THAT $P(X = 1) = P(X = 2)$, FIND THE MEAN AS WELL AS VARIANCE OF THE DISTRIBUTION

IF A RANDOM VARIABLE X FOLLOWS POISSON DISTRIBUTION SUCH THAT P(X = 1) = P(X = 2), FIND THE MEAN AS WELL AS VARIANCE OF THE DISTRIBUTION

$$P(X=1)=P(X=2)$$

$$\frac{e^{\lambda}\lambda^{1}}{1!}=\frac{e^{\lambda}\lambda^{2}}{2!}$$

$$\lambda=2$$

$$mean=\mathrm{var}=2$$

QUESTION

In a town 10 accidents take place in span of 50 days. Find the probability that there will be 3 or more accidents in a day.

$$(\square\, e^{-0.2} = 0.8187)$$

QUESTION

In a town 10 accidents take place in span of 50 days. Find the probability that there will be 3 or more accidents in a day.

$$(\because e^{-0.2} = 0.8187)$$

Solution

Average no. of accidents per day = 10/50 = 0.2

Prob. Of 3 or more accidents would be = 1-[p(x= 0)+p(x=1)+p(x=2)]

0.0012

QUESTIONS

In a radio manufacturing factory, average number of defective is 1 in 10 radios. Find the probability of getting exactly 2 defective radios in a random sample of 10 radios using Poisson distribution

In a certain manufacturing process, 5% of the tools produced turn out to be defective. Find the probability that in a sample of 40 tools, utmost 2 will be defective

A manufacturer of lenses knows that on an average 5% of his product is defective. He sells lenses in boxes of 100 and gives guarantee to consumer that not more than 4 lenses will be defective in a box. What is the probability that each box will meet the guaranteed quality?