Example 5

A line of length a units is divided into two parts. If the first part is of length X, find $E(X)$, var(X) and $E\{X(a - X)\}$.

Since the positions of the point of division are equally likely, X is uniformly distributed in $(0, a)$.

$$\therefore \quad f(x) = \frac{1}{a}$$

$$E(X) = \int_0^a x f(x)\,dx = \frac{1}{a}\int_0^a x\,dx = \frac{a}{2}$$

$$E(X^2) = \int_0^a x^2 f(x)\,dx = \frac{a^2}{3}$$

$$\therefore \quad \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

$$E\{X(a - X)\} = a\,E(X) - E(X^2) = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

Example 6

If X is a continuous RV, prove that

$$E(X) = \int_0^\infty [1 - F(x)]\,dx - \int_{-\infty}^0 F(x)\,dx.$$

$$E(X) = \int_{-\infty}^\infty x f(x)\,dx = \int_{-\infty}^0 x\,dF(x) - \int_0^\infty x\,d\{1 - F(x)\}$$

[since $F'(x) = f(x)$]

$$= [xF(x)]_{-\infty}^0 - \int_{-\infty}^0 F(x)\,dx - [x\{1 - F(x)\}]_0^\infty + \int_0^\infty \{1 - F(x)\}\,dx$$

$$= \int_0^\infty \{1 - F(x)\}\,dx - \int_{-\infty}^0 F(x)\,dx$$

[since $F(-\infty) = 0$ and $F(\infty) = 1$]

Example 7

If the random variable X follows $N(0, 2)$ and $Y = 3X^2$, find the mean and variance of Y.

Since X follows $N(0, 2)$, $E(X) = 0$ and var$(X) = 4$

$$E(X^2) = \text{var}(X) + \{E(X)\}^2 = 4$$

Now

$$E(Y) = E(3X^2) = 3 \times 4 = 12$$

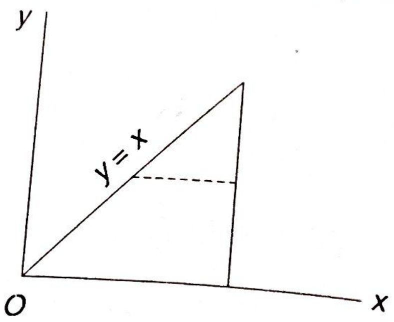
$$E(Y^2) = E(9X^4) = 9 \times 3 \times 2^4$$

[since for the normal distribution $N(0, \sigma)$, $E(X^{2r}) \ldots$

$$\text{Var}(Y) = E(Y^2) - \{E(Y)\}^2$$
$$= 27 \times 2^4 - 12^2 = 288$$

Example 8

If the joint pdf of (X, Y) is given by $f(x, y) = 24y(1-x)$, $0 \leq y \leq x \leq 1$, find ...



$$E(XY) = \int_0^1 \int_y^1 xyf(x, y)\, dx\, dy$$

$$= 24 \int_0^1 \int_y^1 xy^2(1-x)\, dx\, dy$$

$$= 24 \int_0^1 y^2\left(\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right) dy$$

$$= \frac{4}{15}$$

Example 9

If X and Y are two independent RVs with $f_X(x) = e^{-x}U(x)$ and $f_Y(y) = e^{\ldots}$ and $Z = (X - Y)\, U(X - Y)$, prove that $E(Z) = 1/2$.

$$U(X-Y) = \begin{cases} 1 & \text{if } X > Y \\ 0 & \text{if } X < Y \end{cases}$$

$\therefore$

$$Z = \begin{cases} X - Y & \text{if } X > Y \\ 0 & \text{if } X < Y \end{cases}$$

$$E(Z) = \int_0^\infty \int_0^\infty z\, e^{-(x+y)}\, dx\, dy$$

[since X and Y are independent, $f(x, y) = f_X(x) \times f_Y(y)$]

$$= \int_0^\infty \int_y^\infty (x - y)\, e^{-(x+y)}\, dx\, dy$$

$$= \int_0^\infty e^{-y} [(x - y)(-e^{-x}) - e^{-x}]_y^\infty\, dy$$

$$= \int_0^\infty e^{-2y}\, dy = \frac{1}{2}$$

Example 10

The joint pdf of (X, Y) is given by $f(x, y) = 24xy$; $x > 0$, $y > 0$, $x + y \le 1$, and $f(x, y) = 0$, elsewhere, find the conditional mean and variance of Y, given X.

Fig. 4.1

$$f_X(x) = \int_0^{1-x} 24xy\, dy$$

2. $f_Y(y) = \int_0^\infty f(x,y)\, dx$

$$= 12x(1 - x)^2, 0 < x < 1$$

Now
$$f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{2y}{(1-x)^2}, 0 < y < 1 - x$$

$$E(Y/X = x) = \int_0^{1-x} y f(y/x)\, dy$$

$$= \int_0^{1-x} \frac{2y^2}{(1-x)^2}\, dy = \frac{2}{3}(1 - x)$$

$$E(Y^2/x) = \int_0^{1-x} y^2 \times f(y/x)\, dy = \frac{1}{2}(1 - x)^2$$

$$\mathrm{Var}(Y^2/x) = E(Y^2/x) - \{E(Y/x)\}^2$$

$$= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2$$

$$= \frac{1}{18}(1-x)^2$$

Example 11

If (X, Y) is uniformly distributed over the semicircle bounded by $y = \sqrt{1-x^2}$ and $y = 0$, find $E(X/Y)$ and $E(Y/X)$. Also verify the $E\{E(X/Y)\} = E(X)$ and $E\{E(Y/X)\} = E(Y)$.

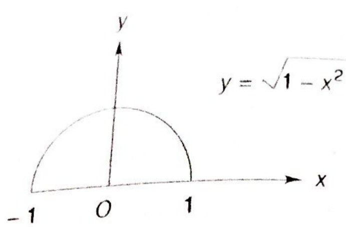


Fig. 4.2

$$f(x, y) = k$$

$$\iint f(x, y)\, dy\, dx = 1$$

i.e.,
$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} k\, dy\, dx = 1$$

i.e.,
$$2k \int_{0}^{1} \sqrt{1-x^2}\, dx = 1$$

$\therefore$
$$k = \frac{2}{\pi}$$

$$f_X(x) = \int_{0}^{\sqrt{1-x^2}} \frac{2}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2}, \; -1 \le x \le 1$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}\, dx = \frac{4}{\pi}\sqrt{1-y^2}, \; 0 \le y \le 1$$

$$f(x/y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{2\sqrt{1-y^2}}, \; -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$

$$f(y/x) = \frac{1}{\sqrt{1-x^2}}, \; 0 \le y \le \sqrt{1-x^2}$$

$$E(X) = \int_{-1}^{1} x f_X(x)\, dx = \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}\, dx = 0$$

(since the integrand is od

$$E(X/Y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x f(x/y)\, dx$$

$$= \frac{1}{2\sqrt{1-y^2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\, dx = 0$$

(since the integrand is odd)

$\therefore \quad E\{E(X/Y)\} = E\{0\} = 0 = E(X)$

$$E(Y) = \int_0^1 y f_Y(y)\, dy = \frac{4}{\pi} \int_0^1 y\sqrt{1-y^2}\, dy = \frac{4}{3\pi}$$

$$E(Y/X) = \int_0^{\sqrt{1-x^2}} y f(y/x)\, dy = \frac{1}{\sqrt{1-x^2}} \cdot \left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} = \frac{1}{2}\sqrt{1-x^2}$$

$\therefore \quad E\{E(Y/X)\} = E\left\{\frac{1}{2}\sqrt{1-X^2}\right\}$

$$= \int_{-1}^{1} \frac{1}{2}\sqrt{1-x^2}\, f_X(x)\, dx$$

$$= \frac{2}{\pi}\int_0^1 (1-x^2)\, dx = \frac{4}{3\pi}$$

$\therefore \quad E\{E(Y/X)\} = E(Y)$

Example 12

If (X, Y) follows a bivariate normal distribution $N(0, 0; \sigma_X, \sigma_Y; r)$, find $E(Y/X)$, $E(Y^2/X)$, $E(XY)$ and $E(X^2Y^2)$.

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{y}{\sigma_y} - \frac{rx}{\sigma_x}\right)^2 - \frac{x^2}{2\sigma_x^2}\right\}$$

$$f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp(-x^2/2\sigma_X^2)$$

[refer to the worked Example 12 in Chapter 2 on two-dimensional RVs]

$$\therefore \quad f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{\sigma_y\sqrt{1-r^2}\sqrt{2\pi}}$$