Worked Example 3(A)

Example 1

Out of 800 families with 4 children each, how many families would be expec[ted] to have (i) 2 boys and 2 girls, (ii) at least 1 boy, (iii) at most 2 girls and [having] children of both sexes. Assume equal probabilities for boys and girls.

Considering each child as a trial, $n = 4$. Assuming that birth of a boy [is a] success, $p = \frac{1}{2}$ and $q = \frac{1}{2}$. Let X denote the number of successes (boys).

(i) $P(\text{2 boys and 2 girls}) = P(X = 2)$

$$= 4C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{4-2}$$

$$= 6 \times \left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

$\therefore$ No. of families having 2 boys and 2 girls

$= N \cdot (P(X = 2)$ (where N is the total no. of families considered)

$$= 800 \times \frac{3}{8}$$

$= 300.$

(ii) $P(\text{at least 1 boy}) = P(X \geq 1)$

$= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= 1 - P(X = 0)$

$$= 1 - 4C_0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

$\therefore$ No. of families having at least 1 boy

$$= 800 \times \frac{15}{16} = 750.$$

(iii) $P(\text{at most 2 girls}) = P(\text{exactly 0 girl, 1 girl or 2 girls})$

$= P(X = 4, X = 3 \text{ or } X = 2)$

$= 1 - \{P(X = 0) + P(X = 1)\}$

$$= 1 - \left\{ 4C_0 \cdot \left(\frac{1}{2}\right)^4 + 4C_1 \cdot \left(\frac{1}{2}\right)^4 \right\}$$

$$= \frac{11}{16}$$

∴ No. of families having at most 2 girls

$$= 800 \times \frac{11}{16} = 550.$$

(iv) P(children of both sexes)

$$= 1 - P(\text{children of the same sex})$$
$$= 1 - \{P(\text{all are boys}) + P(\text{all are girls})\}$$
$$= 1 - \{P(X = 4) + P(X = 0)\}$$
$$= 1 - \left\{ 4C_4 \cdot \left(\frac{1}{2}\right)^4 + 4C_0 \cdot \left(\frac{1}{2}\right)^4 \right\}$$
$$= 1 - \frac{1}{8} = \frac{7}{8}$$

∴ No. of families having children of both sexes

$$= 800 \times \frac{7}{8} = 700.$$

Example 2

An irregular 6-faced die is such that the probability that it gives 3 even numbers in 5 throws is twice the probability that it gives 2 even numbers in 5 throws. How many sets of exactly 5 trials can be expected to give no even number out of 2500 sets?

Let the probability of getting an even number with the unfair die be p.

Let X denote the number of even numbers obtained in 5 trials (throws).

Given: $P(X = 3) = 2 \times P(X = 2)$

i.e., $5C_3\, p^3 q^2 = 2 \times 5C_2\, p^2 q^3$

i.e., $p = 2q = 2(1 - p)$

∴ $3p = 2$ or $p = \frac{2}{3}$ and $q = \frac{1}{3}$

Now P(getting no even number)

$$= P(X = 0)$$
$$= 5C_0 \cdot p^0 \cdot q^5 = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

∴ Number of sets having no success (even number) out of N sets $= N \times P(X = 0)$

∴ Required number of sets $= 2500 \times \frac{1}{243}$

$= 10$, nearly

Example 3

The probability of a successful rocket launching is p. If launching attempts are made until 3 successful launchings have occurred, what is the probability that exactly 5 attempts will be necessary? What is the probability that fewer than 5 attempts will be necessary?

If launching attempts are made until 3 consecutive successful launchings occur, what are the probabilities? [See example (1) in section 1 (c)]

(i) Exactly 5 attempts will be required to get 3 successes (successful launchings of rockets), if 2 successes occur in the first 4 attempts and the success occurs in the fifth attempt.

$\therefore$ P(exactly 5 attempts are required)

$$= P(\text{2 successes in 4 attempts}) \times P(\text{success in the single 5th attempt})$$

$$= 4C_2\, p^2 q^2 \times p$$

[$\because$ The no. of successes in the 4 independent attempts follow $B(4, p)$]

$$= 6\, p^3 q^2$$

(ii) P\{fewer than 5 attemps are required\}

$$= P\{\text{exactly 3 or 4 attempts are required}\}$$

$$= [P\{\text{2 successes in the first 2 attempts}\} \times P(\text{success in the 3rd attempt})]$$
$$+ [P\{\text{2 successes in the first 3 attempts}\} \times P(\text{success in the 4th attempt})]$$

$$= 2C_2\, p^2 q^0 \times p + 3C_2\, p^2 q^1 \times p$$

$$= p^3 + 3p^3 q = p^3 (1 + 3q)$$

(iii) Five attempts will be required to get 3 consecutive successes, if the first 2 attempts result in failures and the last 3 attempts result in successes.

$\therefore$ Required probability $= q \cdot q \cdot p \cdot p \cdot p = p^3 q^2$

(iv) Three attempts will be required to get 3 consecutive successes, if each attempt results in a success.

$\therefore$ Probability for this $= p^3$.

Four attempts will be required to get 3 consecutive successes, if the first attempt results in a failure and the remaining attempts result in a success each.

$\therefore$ Probability for this $= qp^3$

$\therefore$ P\{fewer than 5 attempts are required\}

$$= p^3 + qp^3 = p^3 (1 + q).$$

Example 4

A communication system consists of n components, each of which will independently function with probability p. The total system will be able to

$$= \sum_{r=1}^{n} \frac{r \cdot nC_r \, p^r q^{n-r}}{1 - P(X=0)}$$

$$= \frac{1}{1-q^n} \sum_{r=0}^{n} r \cdot nC_r \, p^r q^{n-r}$$

$$= \frac{np}{1-q^n}.$$

Example 6

Two dice are thrown 120 times. Find the average number of times in which the number on the first dice exceeds the number on the second dice.

The number on the first dice exceeds that on the second die, in the following combinations:

(2, 1); (3, 1), (3, 2); (4, 1), (4, 2), (4, 3); (5, 1), (5, 2), (5, 3); (5, 4); (6, 1), (6, 2), (6, 3), (6, 4), (6, 5),

where the numbers in the parentheses represent the numbers in the first and second dice respectively.

$\therefore$ P(success) = P(no. in the first die exceeds the no. in the second die)

$$= \frac{15}{36} = \frac{5}{12}$$

This probability remains the same in all the throws that are independent. If X is the no. of successes, then X follows a binomial distribution with parameters n (= 120) and $p\left(= \frac{5}{12}\right)$.

$$\therefore \quad E(X) = np = 120 \times \frac{5}{12} = 50.$$

Example 7

Fit a binomial distribution for the following data:

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80

Fitting a binomial distribution means assuming that the given distribution is approximately binomial and hence finding the probability mass function and then finding the theoretical frequencies.

To find the binomial frequency distribution $N(q + p)^n$, which fits the given data, we require N, n and p. We assume N = total frequency = 80 and n = no. of trials = 6 from the given data.

To find p, we compute the mean of the given frequency distribution and equate it to np (mean of the binomial distribution).

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80
fx:	0	18	56	36	28	30	24	192

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{192}{80} = 2.4$$

$np = 2.4$ or $6p = 2.4$

$p = 0.4$ and $q = 0.6$

If the given distribution is nearly binomial, the theoretical frequencies are given by the successive terms in the expansion of $80(0.6 + 0.4)^6$. Thus we get.

x:	0	1	2	3	4	5	6
Theoretical f:	3.73	14.93	24.88	22.12	11.06	2.95	0.33

Converting these values into whole numbers consistent with the condition that the total frequency is 80, the corresponding binomial frequency distribution is as follows:

x:	0	1	2	3	4	5	6	Total
f:	4	15	25	22	11	3	0	80

Example 8

The number of monthly breakdowns of a computer is a RV having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month

(a) without a breakdown,

(b) with only one breakdown and

(c) with atleast one breakdown.

Let X denote the number of breakdowns of the computer in a month. X follows a Poisson distribution with mean (parameter) $\lambda = 1.8$.

$$\therefore P\{X = r\} = \frac{e^{-\lambda}.\lambda^r}{r!} = \frac{e^{-1.8}.(1.8)^r}{r!}$$

(a) $P(X = 0) = e^{-1.8} = 0.1653$

(b) $P(X = 1) = e^{-1.8}(1.8) = 0.2975$

(c) $P(X \geq 1) = 1 - P(X = 0) = 0.8347$

Example 9

It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the number of packets containing at least, exactly and at most 2 defective items in a consignment of 1000 packets using (i) binomial distribution and (ii) Poisson approximation to binomial distribution.

$$\text{Variance} = npq = n \times \frac{1}{2} \times \frac{1}{2} = \frac{n}{4} \quad \text{...(2)}$$

(1) and (2), we get

$$\text{Variance} = \frac{1}{2}\,(\text{mean}).$$

Example 4. *If the m.g.f. of a random variable X is*

$$\left(\frac{1}{3}+\frac{2}{3}e^t\right)^5, \text{ find } P\,(X=2).$$

Solution : Let

$$P\,(X=x) = {}^nC_x\, p^x q^{n-x},\ x = 0, 1, 2, \ldots, n$$

Then $$M_0\,(t) = (q + pe^t)^n \quad \text{...(1)}$$

Given $$M_0\,(t) = \left(\frac{1}{3}+\frac{2}{3}e^t\right)^5 \quad \text{...(2)}$$

Comparing (1) and (2), we get

$$q = \frac{1}{3},\ p = \frac{2}{3},\ n = 5$$

$\therefore$ $$P\,(X=x) = {}^5C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{5-x}$$

and $$P\,(X=2) = {}^5C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{5-2} = 10 \times \frac{2^2}{3^2} \times \frac{1}{3^3} = \frac{40}{243}.$$

Example 5. *The mean and variance of a Binomial distribution are* 4 *and* $\frac{4}{3}$ *respctively. Find* (i) *the probability of* 2 *successes,* (ii) *the probability of more than two successes,* (iii) *the probability of* 3 *or more than three successes.*

(Jiwaji 1992)

Solution : Let $P\,(X=x) = {}^nC_x\, p^x q^{n-x},\ x = 0, 1, 2, \ldots, n$

Then $$\text{mean} = np = 4$$

and $$\text{variance} = npq = \frac{4}{3}$$

$\Rightarrow$ $$\frac{npq}{np} = q = \frac{4/3}{4} = \frac{1}{3} \ \therefore\ p = \frac{2}{3} \text{ and } n = 6$$

(i) $$P\,(X=2) = {}^6C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^4 = \frac{15 \times 4}{3^6} = \frac{20}{243}$$

(ii)
$$P(X > 2) = 1 - P(x \le 2)$$
$$= 1 - \sum_{x=0}^{2} {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}$$
$$= 1 - \frac{73}{1029} = \frac{956}{1029}.$$

(iii)
$$P(X \ge 3) = P(X > 2) = \frac{956}{1029}.$$

Example 6. *A perfect cubical die is thrown a large number of times in sets of 8. The occurrence of 5 or 6 is called a success. In what proportional of the sets you expect 3 successes.* **(Jiwaji 1980)**

Solution : Here, $n = 8, p = \frac{1}{3}, q = \frac{2}{3}$.

Hence the Binomial distribution is given by
$$N\left(\frac{2}{3}+\frac{1}{3}\right)^8.$$

The number of sets in which 3 successes are expeced
$$= N\left\{{}^8C_3\left(\frac{1}{2}\right)^3\left(\frac{2}{5}\right)^5\right\} = N\,\frac{56\times 32}{243\times 27}.$$

$$\text{Percentage} = N\,\frac{56\times 32}{243\times 27}\times\frac{100}{N} = \frac{179200}{6591} = 27{\cdot}31\%.$$

Example 7. *Assuming that half the population are consumers of chocolate, so that the chance of an individual being a consumer is $\frac{1}{2}$, and assuming that 100 investigators each take 10 individuals to see whether they are consumers, how many investigators would you expect to report that three people or less were consumers ?*

Solution : Here, $p = \frac{1}{2}, n = 10, N = 100$

Binonial distribution is $100\left(\frac{1}{2}+\frac{1}{2}\right)^{10}$.

Number of investigators to report that no person is consumer, one person is consumer, 2 persons are consumers, 3 are consumers.
$$= 100\left[\left(\tfrac{1}{2}\right)^{10} + {}^{10}C_2\left(\tfrac{1}{2}\right)^{10}\ {}^{10}C_2 + \left(\tfrac{1}{2}\right)^{10} + {}^{10}C_3\left(\tfrac{1}{2}\right)^{10}\right]$$
$$= \frac{100}{2^{10}}(1 + 10 + 45 + 120) = \frac{17600}{1024} = 17 \text{ nearly.}$$

Example 8. *An irregular six–faced die is thrown, and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws would you expect it to give no even numbers ?*

Solution : Let p be the probability of getting an even number.

The probability of 5 even numbers in 10 thorws $= {}^{10}C_5\, p^5 q^5$.

The probability of 4 even numbers in 10 throws $= {}^{10}C_4\, p^4 q^6$.

$$\therefore \qquad {}^{10}C_5\, p^5 q^5 = 2 \times {}^{10}C_4\, p^4 q^6$$

$$i.e. \qquad \frac{3}{5}\, p = q \text{ i.e. } \frac{3}{5}\,(1-q) = q \text{ giving } q = \frac{3}{8}.$$

Hence the number of times in 10,000 throws where we get no even number

$$= 10{,}000 \left(\frac{3}{8}\right)^{10} = 1, \text{ nearly.}$$

Example 9. *Find the most probable number of success in a series of n independent trials; the probability of success in each trials being p.*

Solution : Here we are to find the number of successes which has a greater probability than any other.

The probability of r successes will be greater than or equal to that of $r-1$ or $r+1$ successes if

$${}^nC_{r-1}\, p^{r-1} q^{n-r+1} \le {}^nC_r\, p^r q^{n-r} \ge {}^nC_{r+1}\, p^{r+1} q^{n-r-1}$$

Simplifying we get

$$\frac{r}{n-r+1}\cdot\frac{q}{p} \le 1 \ge \frac{n-r}{r+1}\cdot\frac{p}{q}$$

i.e. if $\qquad rq \le np - rp + p$ and $rq + q > np - rp$.

i.e. if $\qquad np - q \le r \le np + p$

i.e. if $\qquad (n+1)\,p - 1 \le r \le np + p$, as $q = 1 - p$.

Case 1. *If $(n+1)\,p = k$ (an integer), then probability will increase till $r = k$ and it will be the same for $r = k - 1$ and after that it will begin to decrease.*

Case 2. *If np = an integer + a fraction, then probability is maximum when*

r = the integral part of $(np + p)$.

Note : The most probable number of success gives the mode of the Binomial distribution and its value is $np + p$.

Example 10. *Show that if np be a whole number, the mean of the binomial distribution*

Hence, $$f_p = \left(\frac{2}{\pi N}\right)^{1/2} e^{-2p^2/N}.$$

5.6. Fitting of Binomial Distribution

By fitting a theoretical distribution to a given observed distribution we mean the
...(1) determination of the theoretical frequencies to know how closely the observed distribution approximates a theoretical distribution.

To fit a Binomial distribution to the given data the first step is to estimate the parameter p if not given otherwise. The parameter p is estimated by $\hat{p} = \frac{\bar{x}}{n}$, where $\bar{x}$ = the mean of the
...(2) observed distribution.

The expected frequency of x successes is computed by

$$N \cdot {}^nC_x p^x (1-p)^{n-x}, \; x = 0, 1, 2, \ldots, n$$

The calculation of probabilities and hence theoretical frequencies is often simplified with the use of the recursion formula

$$p(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \, p(x).$$

Proof :

$$p(x) = {}^nC_x p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$p(x+1) = {}^nC_{x+1} p^{x+1} q^{n-x-1}$$

$$= \frac{n!}{(x+1)!(n-x-1)!} p^{x+1} q^{n-x-1}$$

$$\Rightarrow \quad \frac{p(x+1)}{p(x)} = \frac{p}{q} \cdot \frac{n-x}{x+1}$$

$$\Rightarrow \quad p(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot p(x).$$

Illustrative Examples

Example 1. *The number of males in each 106 eight pig litters was found and they are given by the following frequency distribution :*

Number of male per littter	0	1	2	3	4	5	6	7	8	*Total*
Frequency	0	5	9	22	25	26	14	4	1	106

Assuming that the probability of an animal being male or female is even i.e. $p = q = \frac{1}{2}$ and frequency distribution follows the Binomial law, calculate the expected frequencie

Solution : The Binomial distribution is $106\left(\frac{1}{2}+\frac{1}{2}\right)^8$.

The expected frequencies are the respective terms of this expansion which a respectively.

$$0{\cdot}4,\ 3{\cdot}3,\ 11{\cdot}6,\ 23{\cdot}2,\ 29,\ 23{\cdot}2,\ 11{\cdot}6,\ 3{\cdot}3,\ 0{\cdot}4.$$

Example 2. *In litters of 4 mice the number of littes which contained 0, 1, 2, 3, emales were noted. The figures are given in the table below :*

Number of female mice	0	1	2	3	4	*Total*
Number of litters	8	32	34	24	5	103

If the chance of obtaining a female in a single trial is assumed constant, estima this constant of unknown probability. Find also expected frequencies.

Solution : Mean of observations$= \dfrac{32+68+72+20}{103} = \dfrac{192}{103} = 1{\cdot}864$

Since $n = 4$ and np = mean = $1{\cdot}864$.

$\therefore$ We have $$p = \frac{1{\cdot}864}{4} = 0{\cdot}466.$$

Hence $$q = 1 - p = 1 - 0{\cdot}466 = {\cdot}534.$$

Therefore expected frequencies are the respective terms of the Binomial expansi of

$$103\,(0{\cdot}534 + 0{\cdot}466)^4.$$

Example 3. *The following data are the number of seeds germinating out of 10 damp filter for 80 sets of seeds. Fit a Binomial distribution to these data :*

x	0	1	2	3	4	5	6	7	8	9	10	*Total*
f	66	20	28	12	8	6	0	0	0	0	0	80

Solution : Here $n = 10, N = 80$, and total frequency $= 80$.

$$\therefore \quad \text{A.M.} = \frac{\Sigma fx}{\Sigma f} = \frac{1\times20+2\times28+3\times12+4\times8+5\times6}{80} = \frac{174}{80}$$

But $$\text{mean} = np = \frac{174}{80} \text{ whence } p = \frac{174}{800} \equiv 0{\cdot}2175.$$

$$\therefore \quad q = 1 - p = {\cdot}7825.$$

Hence the Binomial distribution to be fitted to the data is

$$80\,({\cdot}7825 + {\cdot}2175)^{10}.$$

om this expansion the successive frequencies of 0, 1, 2, ..., 10 successes a

$$6{\cdot}9,\ 19{\cdot}1,\ 24{\cdot}0,\ 17{\cdot}8,\ 8{\cdot}6,\ 2{\cdot}9,\ 0{\cdot}7,\ 0{\cdot}1,\ 0,\ 0,\ 0.$$

1. Compute mode of a Binomial distribution with $p = \frac{1}{4}$ and $n = 7$.

2. Show that a measure of skewness of the Binomial distribution is given by $\frac{q-p}{(npq)^{1/2}}$ and its kurtosis is $3 + \frac{1-6pq}{npq}$. (Measure of the skewness is γ_1 and of kurtosis is β_2)

3. Let X be a Binomially distributed random variable with mean 10 and variance 5. Show that

(a) $P(X > 6) = \left(\frac{1}{2}\right)^{20} \sum_{r=7}^{20} {}^{20}C_r$,

(b) $P(3 < X < 12) = \left(\frac{1}{2}\right)^{20} \sum_{r=4}^{11} {}^{20}C_r$.

4. The mean of a Binomial distribution is 20 and standard deviation is 4. Determine the distribution.

5. If $\beta_1 = \frac{1}{36}$ and $\beta_2 = \frac{35}{12}$, then find the corresponding Binomial distribution.

(Bhopal 1994; Vikram 89)

6. For a Binomial distribution

$$p(x) = {}^nC_x p^x q^{n-x}, \; x = 0, 1, 2, ..., n$$

Show that $$u_{r+1}' = npq\, u_r' + pq \frac{d\mu_r'}{dp}, \quad \text{where } u_r' = E(X^r).$$

7. What do you understand by Binomial distribution. Find mean and standard deviation and other constants (including β_1 and β_2) of a Binomial distribution.

(Bhopal 1993; Indore 89; Jabalpur 96; Jiwaji 80; Sagar 89; Vikram 94)

8. Calculate the value of p for a Binomial distribution if the ratio probability of an event occurring exactly r times n trials to the probability of the event occurring exactly $(n-r)$ times in n trials is independent of n. **(Bhopal 1991)**

9. Derive Binomial distribution and obtain β_1 and β_2. **(Bhopal 1998)**

10. The following results were obtained when 80 batches of seeds were allowed to germinate on damp filterpaper in a laboratory :

$$\beta_1 = \frac{2}{3} \text{ and } \beta_2 = 2\frac{2}{3}.$$

Determine the Binomial distribution. **(Bhopal 1993, 98)**

11. The incidence of an occupational disease in an industry is such that workmen has a 10% chance of suffering from it. What is the probability that in a group of 7 five or more will suffer from it? **(Indore 1997; Rewa 97)**

12. Find first four moments using m.g.f. otherwise of a Binomial distribution and with their help calculate first four central moments. *(Indore 1984)*

13. Using the m.g.f. about mean of a Binomial distribution, find u_1, u_2, sigma, u_3 and u_4.

14. In case of Binomial distribution comment on the following :
The mean of the Binomial distribution is 8 and variance is 10.