54. A continuous RVX that can assume values between $x = 2$ and $x = 5$ has a density function given by $f(x) = 2(1 + x)/27$. Find $P(3 < X < 4)$.
(MU — Nov. 96)
55. A continuous RV has the pdf $f(x) = kx^4$, $-1 < x < 0$. Find the value of k and also $P(X > -1/2 \,/\, X < -1/4)$.
56. Suppose that the life length of a certain radio tube (in hours) is a continuous RV X with pdf $f(x) = \dfrac{100}{x^2}$ $x > 100$ and $= 0$, elsewhere.
   (a) What is the probability that a tube will last less than 200 h, if it is known that the tube is still functioning after 150 h of service?
   (b) What is the probability that if 3 such tubes are installed in a set, exactly 1 will have to be replaced after 150 h of service?
   (c) What is the maximum number of tubes that may be inserted into a set so that there is a probability of 0.1 that after 150 h of service all of them are still functioning?
57. If the cdf of a continuous RV X is given by $F(x) = \dfrac{1}{2}e^{kx}$, $x \le 0$, and $F(x) = 1 - \dfrac{1}{2}e^{-kx}$, $x > 0$, find $P(|x| \le 1/k)$. Prove that the density function of X is
$$f(x) = \frac{k}{2}\, e^{-k|x|} \quad -\infty < x < \infty, \text{ given that } k > 0.$$
58. If the distribution function of a continuous RV X is given by $F(x) = 0$, when $x < 0$; $= x$, when $0 \le x < 1$ and $= 1$, when $1 \le x$, find the pdf of X. Also find $P(1/3 < X < 1/2)$ and $P(1/2 < X < 2)$ using the cdf of X.
59. A point is chosen on a line of length a at random. What is the probability that the ratio of the shorter to the longer segment is less than 1/4?
60. If the RV k is uniformly distributed over (1, 7) what is the probability that the roots of the equation $x^2 + 2kx + (2k + 3) = 0$ are real?
61. If $f(t)$ is the unconditional density of time to failure T of a system and $h(t)$ is the conditional density of T, given $T > t$, find $h(t)$ when (i) $f(t) = \lambda e^{-\lambda t}$ (ii) $f(t) = \lambda^2 te^{-\lambda t}$, $t > 0$. Prove also that $h(t)$ is not a density function.
62. If the continuous RV X follows $N(1000, 20)$, find
   (i) $P(X < 1024)$, (ii) $P(X < 1024 \,/\, X > 961)$ and
   (iii) $P(31 < \sqrt{X} \le 32)$.

Two-Dimensional Random Variables

So far we have considered only the one-dimensional RV, i.e., we have considered such random experiments, the outcome of which had only one characteristic and hence was assigned a single real value. In many situations, we will be interested in recording 2 or more characteristics (numerically) of the outcome of a random experiment. For example, both voltage and current might be of interest in a certain experiment.

Definitions: Let S be the sample space associated with a random experiment E. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcomes $s \in S$. Then (X, Y) is called a *two-dimensional random variable.*

If the possible values of (X, Y) are finite or countably infinite, (X, Y) is called a *two-dimensional discrete* RV. When (X, Y) is a two-dimensional discrete RV, the possible values of (X, Y) may be represented as (x_i, y_j), $i = 1, 2, \ldots, m, \ldots$, $j = 1, 2, \ldots, n, \ldots$.

If (X, Y) can assume all values in a specified region R in the xy plane, (X, Y) is called a *two-dimensional continuous* RV.

Probability Function of (X, Y)

If (X, Y) is a two-dimensional discrete RV such that $P(x = x_i, y = y_j) = p_{ij}$, then p_{ij} is called the *probability mass function* or simply the *probability function* of (X, Y) provided the following conditions are satisfied.

(i) $p_{ij} \geq 0$, for all i and j

(ii) $\sum_j \sum_i p_{ij} = 1$

The set of triples $\{x_i, y_j, p_{ij}\}$, $i = 1, 2, \ldots, m, \ldots, j = 1, 2, \ldots, n, \ldots$, is called the *joint probability distribution of* (X, Y).

Joint Probability Density Function

If (X, Y) is a two-dimensional continuous RV such that.

$$P\left\{x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2} \text{ and } y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right\} = f(x, y)\, dx\, dy,$$

then $f(x, y)$ is called *the joint pdf* of (X, Y), provided $f(x, y)$ satisfies the following conditions.

(i) $f(x, y) \geq 0$, for all $(x, y) \in R$, where R is the range space.

(ii) $\iint_R f(x, y)\, dx\, dy = 1$.

Moreover if D is a subspace of the range space R, $P\{(X, Y) \in D\}$ is defined as

$$P\{(X, Y) \in D\} = \iint_D f(x, y) dx\, dy. \text{ In particular}$$

$$P\{a \leq X \leq b, c \leq Y \leq d\} = \int_c^d \int_a^b f(x, y) dx\, dy$$

Cumulative Distribution Function

If (X, Y) is a two-dimensional RV (discrete or continuous), then $F(x, y) = P\{X \leq x \text{ and } Y \leq y\}$ is called *the cdf of* (X, Y).

In the discrete case,

$$F(x, y) = \sum_{\substack{j \\ y_j \leq y}} \sum_{\substack{i \\ x_i \leq x}} p_{ij}$$

In the continuous case,

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y) dx\, dy$$

Properties of $F(x, y)$

(i) $F(-\infty, y) = 0 = F(x, -\infty)$ and $F(\infty, \infty) = 1$
(ii) $P\{a < X < b, Y \leq y\} = F(b, y) - F(a, y)$
(iii) $P\{X \leq x, c < Y < d\} = F(x, d) - F(x, c)$
(iv) $P\{a < X < b, c < Y < d\} = F(b, d) - F(a, d) - F(b, c) + F(a, c)$
(v) At points of continuity of $f(x, y)$

$$\frac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$$

Marginal Probability Distribution

$$P(X = x_i) = P\{(X = x_i \text{ and } Y = y_1) \text{ or } (X = x_i \text{ and } Y = y_2) \text{ or etc.}\}$$

$$= p_{i1} + p_{i2} + \ldots = \sum_j p_{ij}$$

$P(X = x_i) = \sum_j p_{ij}$ is called *the marginal probability function of X*. It is defined for $X = x_1, x_2, \ldots$ and denoted as P_{i*}. The collection of pairs $\{x_i, p_{i*}\}, i = 1, 2, 3, \ldots$ is called the *marginal probability distribution of X*.
Similarly the collection of pairs $\{y_j, p_{*j}\}, j = 1, 2, 3, \ldots$ is called *the marginal probability distribution of Y*, where $p_{*j} = \sum_i P_{ij} = P(Y = y_j)$.

In the continuous case,

$$P\{x - \frac{1}{2}\text{d}x \leq X \leq x + \frac{1}{2}\text{d}x, -\infty < Y < \infty\}$$

$$= \int_{-\infty}^{\infty} \int_{x-\frac{1}{2}\text{d}x}^{x+\frac{1}{2}\text{d}x} f(x, y)\, \text{d}x\, \text{d}y$$

$$= \left[\int_{-\infty}^{\infty} f(x, y)\, \text{d}y\right] \text{d}x$$ [since $f(x, y)$ may be treated a constant in $(x - 1/2\text{d}x, x + 1/2\,\text{d}x)$]

$$= f_X(x)\text{d}x, \text{ say}$$

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\text{d}y$ is called *the marginal density of X*.

Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,\text{d}x$ is called *the marginal density of Y*.

Note $P(a \le X \le b) = P(a \le X \le b, -\infty < Y < \infty)$

$$= \int_{-\infty}^{\infty}\int_{a}^{b} f(x, y)\,dx\,dy$$

$$= \int_{a}^{b}\left[\int_{-\infty}^{\infty} f(x, y)\,dy\right] dx = \int_{a}^{b} f_X(x)\,dx$$

Similarly, $P(c \le Y \le d) = \int_{c}^{d} f_Y(y)\,dy$

Conditional Probability Distribution

$P\{X = x_i / Y = y_j\} = \dfrac{P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}} = \dfrac{p_{ij}}{p_{*j}}$ is called *the conditional probab...*
function of X, given that $Y = y_j$.

The collection of pairs, $\left\{x_i, \dfrac{p_{ij}}{p_{*j}}\right\}$ $i = 1, 2, 3, \ldots,$

is called *the conditional probability distribution of X, given* $Y = y_j$.

Similarly, the collection of pairs, $\left\{Y_j, \dfrac{p_{ij}}{p_{i*}}\right\}$, $j = 1, 2, 3, \ldots,$ is called the *con...*
tional probability distribution of Y given $X = x_i$. In the continuous case,

$$P\left\{x - \frac{1}{2}dx \le X < x + \frac{1}{2}dx / Y = y\right\}$$

$$= P\left\{x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx / y - \frac{1}{2}dy \le Y \le y + \frac{1}{2}dy\right\}$$

$$= \frac{f(x, y)\,dx\,dy}{f_Y(y)dy} = \left\{\frac{f(x, y)}{f_Y(y)}\right\} dx.$$

$\dfrac{f(x, y)}{f_Y(y)}$ is called *the conditional density of X, given Y*, and is denoted by $f(x/$...

Similarly, $\dfrac{f(x, y)}{f_X(y)}$ is called *the conditional density of Y, given X*, and is deno...
by $f(y/x)$.

Independent RVs

If (X, Y) is a two-dimensional discrete RV such that $P\{X = x_i / Y = y_j\} = P(X =$...
i.e., $\dfrac{p_{ij}}{p_{*j}} = p_{i*}$, i.e., $p_{ij} = p_{i*} \times p_{*j}$ for all i, j then X and Y are said to be independ...
RVs.

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$$f(x_1 / x_2, x_3) = \frac{f(x_1, x_2, x_3)}{f_{X_2, X_3}(x_2, x_3)}$$

Worked Example 2(B)

Example 1

Three balls are drawn at random without replacement from a box containin white, 3 red and 4 black balls. If X denotes the number of white balls drawn Y denotes the number of red balls drawn, find the joint probability distributio (X, Y).

As there are only 2 white balls in the box, X can take the values 0, 1 and 2 an can take the values 0, 1, 2 and 3.

$P(X = 0, Y = 0)$ $= P$(drawing 3 balls none of which is white or red)

$= P$(all the 3 balls drawn are black)

$$= 4C_3/9C_3 = \frac{1}{21}$$

$P(X = 0, Y = 1)$ $= P$(drawing 1 red and 2 black balls)

$$= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

Similarly, $P(X = 0, Y = 2) = \dfrac{3C_2 \times 4C_1}{9C_3} = \dfrac{1}{7}$; $P(X = 0, Y = 3) = \dfrac{1}{84}$

$P(X = 1, Y = 0) = \dfrac{1}{7}$; $P(X = 1, Y = 1) = \dfrac{2}{7}$; $P(X = 1, Y = 2) = \dfrac{1}{14}$;

$P(X = 1, Y = 3) = 0$ (since only 3 balls are drawn)

$P(X = 2, Y = 0) = \dfrac{1}{21}$; $P(X = 2, Y = 1) = \dfrac{1}{28}$; $P(X = 2, Y = 2) = 0$;

$P(X = 2, Y = 3) = 0$

The joint probability distribution of (X, Y) may be represented in the form of table as given below:

X	Y			
	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Note *Sum of all the cell probabilities = 1.*

Example 2

For the bivariate probability distribution of (X, Y) given below, find $P(X \le 1)$, $P(Y \le 3)$, $P(X \le 1, Y \le 3)$, $P(X \le 1/Y \le 3)$, $P(Y \le 3/X \le 1)$ and $P(X + Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

$$P(X \le 1) = P(X = 0) + P(X = 1)$$

$$= \sum_{j=1}^{6} P(X = 0, Y = j) + \sum_{j=1}^{6} P(X = 1, Y = j)$$

$$= \left(0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$$

$$P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= \sum_{i=0}^{2} P(X = i, Y = 1) + \sum_{i=0}^{2} P(X = i, Y = 2)$$

$$+ \sum_{i=0}^{2} P(X = i, Y = 3)$$

$$= \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(\frac{1}{32} + \frac{1}{8} + \frac{1}{64}\right)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

$$P(X \le 1, Y \le 3) = \sum_{j=1}^{3} P(X = 0, Y = j) + \sum_{j=1}^{3} P(X = 1, Y = j)$$

$$= \left(0+0+\frac{1}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right)=\frac{9}{32}$$

$$P(X \le 1/Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)} = \frac{9/32}{23/64} = \frac{18}{23}$$

$$P(Y \le 3/X \le 1) = \frac{P(X \le 1, Y \le 3)}{P(X \le 1)} = \frac{9/32}{7/8} = \frac{9}{28}$$

$$P(X+Y \le 4) = \sum_{j=1}^{4} P(X=0, Y=j) + \sum_{j=1}^{3} P(X=1, Y=j) + \sum_{j=1}^{2} P(X=2, Y=j)$$

$$= \frac{3}{32}+\frac{1}{4}+\frac{1}{16}=\frac{13}{32}$$

Example 3

The joint probability mass function of (X, Y) is given by $p(x, y) = k(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $(X + Y)$.

The joint probability distribution of (X, Y) is given below. The relevant probabilities have been computed by using the given law.

X	Y		
	1	2	3
0	$3k$	$6k$	$9k$
1	$5k$	$8k$	$11k$
2	$7k$	$10k$	$13k$

$$\sum_{j=1}^{3} \sum_{i=0}^{2} p(x_i, y_j) = 1$$

i.e., the sum of all the probabilities in the table is equal to 1.

i.e., $72k = 1$.

$$\therefore \quad k = \frac{1}{72}$$

Marginal Probability Distribution of X: $\{i, p_{i\cdot}\}$

$X = i$	$p_{i\cdot} = \sum_{j=1}^{3} p_{ij}$
0	$p_{01} + p_{02} + p_{03} = \frac{18}{72}$
1	$p_{11} + p_{12} + p_{13} = \frac{24}{72}$
2	$p_{21} + p_{22} + p_{23} = \frac{30}{72}$
	Total $= 1$

Marginal Probability Distribution of Y: $\{j, p_{\cdot j}\}$

$Y = j$	$p_{*j} = \sum_{i=0}^{2} p_{ij}$
1	15/~~27~~ 72
2	24/72
3	33/72
	Total = 1

Conditional distribution of X, given $Y = 1$, is given by $\{i, P(X = i/Y = 1)\} = \{i, P(X = i, Y = 1)/P(Y = 1)\}$
$= \{i, p_{i1}/p_{*1}\}, i = 0, 1, 2.$
The tabular representation is given below:

$X = i$	p_{i1}/p_{*1}
0	$3k/15k = \frac{1}{5}$
1	$5k/15k = \frac{1}{3}$
2	$7k/15k = \frac{7}{15}$
	Total = 1

The other conditional distributions are given below:

C.P.D. of X, given $Y = 2$	
$X = i$	p_{i2}/p_{*2}
0	$\frac{6k}{24k} = \frac{1}{4}$
1	$\frac{8k}{24k} = \frac{1}{3}$
2	$\frac{10k}{24k} = \frac{5}{12}$
	Total = 1

C.P.D. of X, given $Y = 3$	
$X = i$	p_{i3}/p_{*3}
0	$\frac{9k}{33k} = \frac{3}{11}$
1	$\frac{11k}{33k} = \frac{1}{3}$
2	$\frac{13k}{33k} = \frac{13}{33}$
	Total = 1

C.P.D. of Y, given $X = 0$	
$Y = j$	p_{0j}/p_{0*}
1	$\frac{3k}{18k} = \frac{1}{6}$
2	$\frac{6k}{18k} = \frac{1}{3}$
3	$\frac{9k}{18k} = \frac{1}{2}$
	Total = 1

C.P.D. of Y, given $X = 1$	
$Y = j$	p_{1j}/p_{1*}
1	$\frac{5k}{24k} = \frac{5}{24}$
2	$\frac{8k}{24k} = \frac{1}{3}$
3	$\frac{11k}{24k} = \frac{11}{24}$
	Total = 1

C.P.D. of Y, given $X = 2$	
$Y = j$	p_{2j}/p_{2*}
1	$\frac{7k}{30k} = \frac{7}{30}$
2	$\frac{10k}{30k} = \frac{1}{3}$
3	$\frac{13k}{30k} = \frac{13}{30}$
	Total = 1

Probability distribution of $(X + Y)$	
$(X + Y)$	P
1	$p_{01} = \frac{3}{72}$
2	$p_{02} + p_{11} = \frac{11}{72}$
3	$p_{03} + p_{12} + p_{21} = \frac{24}{72}$
4	$p_{13} + p_{22} = \frac{21}{72}$
5	$p_{23} = \frac{13}{72}$
	Total = 1

Example 4

A machine is used for a particular job in the forenoon and for a different job in the afternoon. The joint probability distribution of (X, Y), where X and Y represent the number of times the machine breaks down in the forenoon and in the afternoon respectively, is given in the following table. Examine if X and Y are independent RVs.

X	Y		
	0	1	2
0	0.1	0.04	0.06
1	0.2	0.08	0.12
2	0.2	0.08	0.12

X and Y are independent, if $P_{i*} \times P_{*j} = P_{ij}$ for all i and j. So, let us find P_{i*} P_{*j} for all i and j.

$$P_{0*} = 0.1 + 0.04 + 0.06 = 0.2;\ P_{1*} = 0.4;\ P_{2*} = 0.4$$

$$P_{*0} = 0.5;\ P_{*1} = 0.2;\ P_{*2} = 0.3$$

Now $P_{0*} \times P_{*0} = 0.2 \times 0.5 = 0.1 = P_{00}$

$$P_{0*} \times P_{*1} = 0.2 \times 0.2 = 0.04 = P_{01}$$

$$P_{0*} \times P_{*2} = 0.2 \times 0.3 = 0.06 = P_{02}$$

Similarly we can verify that

$$P_{1*} \times P_{*0} = P_{10};\ P_{1*} \times P_{*1} = P_{11};\ P_{1*} \times P_{*2} = P_{12};$$

$$P_{2*} \times P_{*0} = P_{20};\ P_{2*} \times P_{*1} = P_{21};\ P_{2*} \times P_{*2} = P_{22}$$

Hence the RVs X and Y are independent.

Now $\quad f_X(x) = \int\limits_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \frac{1}{\pi R^2}\, dy$

$$= \frac{2}{\pi R^2}\sqrt{R^2 - x^2} = \frac{2}{\pi R}\sqrt{1 - \left(\frac{x}{R}\right)^2} \quad -R \le x \le R$$

Note *Whenever we are required to find the marginal and conditional density function, the ranges of the concerned variables should also be specified.*

Example 8

The joint pdf of the RV (X, Y) is given by $f(x, y) = k\,x\,y\,e^{-(x^2 + y^2)}$ $x > 0, y >$ … Find the value of k and prove also that X and Y are independent.

Here the range space is the entire first quadrant of the xy-plane.

By the property of the joint pdf

$$\iint\limits_{x>0,\, y>0} kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

i.e., $\quad k\int\limits_0^\infty ye^{-y^2}\, dy \int\limits_0^\infty xe^{-x^2}\, dx = 1$

i.e., $\quad \frac{k}{4} = 1$

$\therefore \quad k = 4$

Now $f_X(x) = \int\limits_0^\infty 4x\, e^{-x^2} \times ye^{-y^2}\, dy = 2x\, e^{-x^2},\ x > 0$

Similarly, $f_Y(y) = 2ye^{-y^2},\ y > 0.$

Now $f_X(x) \times f_Y(y) = 4\,x\,y\,e^{-(x^2+y^2)} = f(x, y)$

$\therefore$ The RVs x and y are independent.

Note *If $f(x, y)$ can be factorised as $f_1(x) \times f_2(y)$ then X and Y will be independent.*

Example 9

Given $f_{XY}(x, y) = cx(x - y),\ 0 < x < 2,\ -x <, y < x$, and 0 elsewhere, (a) evaluate c, (b) find $f_X(x)$, (c) $f_{Y/X}(y/x)$ and (d) $f_Y(y)$.

(BDU — Apr. …)