PROBABILITY

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Probability is a concept which numerically measure the degree of uncertainity and therefore, of certainity of the occurrence of events.

If an event A can happen in m ways, and fail in n ways, all these ways being equally likely to occur, then the probability of the happening of A is

$$= \frac{\text{Number of favourable cases}}{\text{Total number of mutually exclusive and equally likely cases}} = \frac{m}{m+n}$$

and that of its failing is defined as $\frac{n}{m+n}$

If the probability of the happening $= p$

and the probability of not happening $= q$

then $$p+q = \frac{m}{m+n} + \frac{n}{m+n} = \frac{m+n}{m+n} = 1 \text{ or } p+q = 1$$

For instance, on tossing a coin, the probability of getting a head is $\frac{1}{2}$.

DEFINITIONS

1. **Die :** It is a small cube. Dots are . .. ... :: ::. ::: marked on its faces. Plural of the die is dice. On throwing a die, the outcome is the number of dots on its upper face.
2. **Cards :** A pack of cards consists of four suits *i.e.* Spades, Hearts, Diamonds and Clubs. Each suit consists of 13 cards, nine cards numbered 2, 3, 4, ..., 10, and Ace, a King, a Queen and a Jack or Knave. Colour of Spades and Clubs is black and that of Hearts and Diamonds is red. Kings, Queens, and Jacks are known as *face* cards.
3. **Exhaustive Events or Sample Space :** The set of all possible outcomes of a single performance of an experiment is exhaustive events or sample space. Each outcome is called a sample point. In case of tossing a coin once, $S = (H,T)$ is the *sample space*. Two outcomes Head and Tail constitute an exhaustive event because no other outcome is possible.
4. **Random Experiment :** There are experiments, in which results may be altogether different, even though they are performed under identical conditions. They are known as random experiments. Tossing a coin or throwing a die is random experiment.
5. **Trail and Event :** Performing a random experiment is called a trial and outcome is termed as event. Tossing of a coin is a trial and the turning up of head or tail is an event.
6. **Equally likely events:** Two events are said to be '*equally likely*', if one of them cannot be expected in preference to the other. For instance, if we draw a card from well-shuffled pack, we may get any card, then the 52 different cases are equally likely.
7. **Independent event :** Two events may be *independent*, when the actual happening of one does not influence in any way the probability of the happening of the other.

   Example. The event of getting head on first coin and the event of getting tail on the second coin in a simultaneous throw of two coins are independent.

8. **Mutually Exclusive events:** Two events are known as *mutually exclusive*, when the occurrence of one of them excludes the occurrence of the other. For example, on tossing of a coin, either we get head or tail, but not both.
9. **Compound Event :** When two or more events occur in composition with each other, the simultaneous occurrence is called a compound event. When a die is thrown, getting a 5 or 6 is a compound event.
10. **Favorable Events :** The events, which ensure the required happening, are said to be favourable events. For example, in throwing a die, to have the even numbers, 2, 4 and 6 are favourable cases.
11. **Conditional Probability :** The probability of happening an event A, such that event B has already happened, is called the conditional probability of happening of A on the condition that B has already happened. It is usually denoted by $P(A/B)$.
12. **Odds in favour of an event and odds against an event**
    If number of favourable ways = m, number of not favourable events = n

    (*i*) Odds in favour of the event = $\frac{m}{n}$, Odds against the event = $\frac{n}{m}$.
13. **Classical Definition of Probability.** If there are N equally likely, mutually, exclusive and exhaustive of events of an experiment and m of these are favourable, then the probability of the happening of the event is defined as $\frac{m}{N}$.
14. **Expected value.** If $p_1, p_2, p_3, \ldots, p_n$ of the probabilities of the events $x_1, x_2, x_3 \ldots x_n$ respectively the expected value

$$E(x) = p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_n x_n = \sum_{r=1}^{n} p_r x_r$$

Example 1. *Find the probability of throwing*
(a) 5, (b) an even number with an ordinary six faced die.

Solution. (*a*) There are 6 possible ways in which the die can fall and there is only one way of throwing 5.

$$\text{Probability} = \frac{\text{Number of favourable ways}}{\text{Total number of equally likely ways}} = \frac{1}{6}$$ **Ans.**

(*b*) Total number of ways of throwing a die = 6
Number of ways falling 2, 4, 6 = 3

The required probability = $\frac{3}{6} = \frac{1}{2}$ **Ans.**

Example 2. *Find the probability of throwing 9 with two dice.*

Solution. Total number of possible ways of throwing two dice
$= 6 \times 6 = 36$

Number of ways getting 9. *i.e.*, $(3 + 6), (4 + 5), (5 + 4), (6 + 3) = 4$.

$\therefore$ The required probability = $\frac{4}{36} = \frac{1}{9}$. **Ans.**

Example 3. *From a pack of 52 cards, one is drawn at random. Find the probability of getting a king.*

Solution. A king can be chosen in 4 ways.
But a card can be drawn in 52 ways.

$\therefore$ the required probability $= \frac{4}{52} = \frac{1}{13}$ **Ans.**

EXERCISE 61.1

1. In a class of 12 students, 5 are boys and the rest are girls. Find the probability that a student selected will be a girl. **Ans.** $\frac{7}{12}$

2. A bag contains 7 red and 8 black balls. Find the probability of drawing a red ball. **Ans.** $\frac{7}{15}$

3. Three of the six vertices of a regular hexagon are chosen at random. Find the probability that the triangle with three vertices is equilateral.

Ans. $\frac{1}{10}$

4. What is the probability that a leap year, selected at random, will contain 53 Sundays.

Ans. $\frac{2}{7}$

Fill in the blanks with appropriate correct answer

5. Chance of throwing 6 at least once in four throws with single dice is ....

Ans. $\frac{671}{1296}$

6. A pair of fair dice is thrown and one die shows a four. The probability that the other die shows 5 is ....

Ans. $\frac{1}{36}$

ADDITION LAW OF PROBABILITY

If A and B are two events associated with an experiment; then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof. Let m_1, m_2, and m be the number of favourable outcomes to the events A, B and $A \cap B$ respectively. The mutually exclusive outcomes in the sample space of the experiment be n.

$$P(A) = \frac{m_1}{n},\ P(B) = \frac{m_2}{n},\ P(A \cap B) = \frac{m}{n}$$

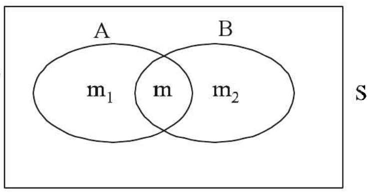
The favourable outcomes to the event A only $= m_1 - m$

The favourable outcomes to the event B only $= m_2 - m$

The favourable outcomes to the event $A \cap B$ only $= m$.

The favourable outcomes to the events A or B or both i.e.,

$$A \cup B = m_1 - m) + (m_2 - m) + m$$
$$= m_1 + m_2 - m$$



So, $$P(A \cup B) = \frac{m_1 + m_2 - m}{n}$$

$$= \frac{m_1}{n} + \frac{m_2}{n} - \frac{m}{n}$$

$$= P(A) + P(B) - P(A \cap B)$$

Theorem. If A and B are any two events prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and hence prove that if A, B and C are any three events.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Note: Mutually Exclusive Events

Consider the case where two events A and B are not mutually exclusive. The probability of the event that either A or B or both occur is given as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 4. *An urn contains 10 black and 10 white balls. Find the probability of drawing two balls of the same colour.*

Solution. Probability of drawing two black balls = $\dfrac{{}^{10}C_2}{{}^{20}C_2}$

$\therefore$ Probability of drawing two white balls = $\dfrac{{}^{10}C_2}{{}^{20}C_2}$

$\therefore$ Probability of drawing two balls of the same colour

$$= \frac{{}^{10}C_2}{{}^{20}C_2} + \frac{{}^{10}C_2}{{}^{20}C_2} = 2.\frac{{}^{10}C_2}{{}^{20}C_2} = 2.\frac{\dfrac{10\times 9}{2\times 1}}{\dfrac{20\times 19}{2\times 1}} = \frac{9}{19}$$ **Ans.**

Example 5. *A bag contains four white and two black balls and a second bag contains three of each colour. A bag is selected at random, and a ball is then drawn at random from the bag chosen. What is the probability that the ball drawn is white ?.*

Solution. There are two mutually exclusive cases,
(i) when the first bag is chosen, (ii) when the second bag is chosen.

Now the chance of choosing the first bag is $\frac{1}{2}$ and if this bag is chosen, the probability of drawing a white ball is 4/6. Hence the probability of drawing a white ball from first bag is

$$\frac{1}{2} \times \frac{4}{6} = \frac{1}{3}$$

Similarly the probability of drawing a white ball from second bag is $\frac{1}{2} \times \frac{3}{6} = \frac{1}{4}$

Since the events are mutually exclusive the required probability = $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ **Ans.**

MULTIPLICATION LAW OF PROBABILITY

If there are two independent events the respective probabilities of which are known, then the probability that both will happen is the product of the probabilities of their happening respectively.

$$P(AB) = P(A) \times P(B)$$

Proof. Suppose A and B are two independent events. Let A happen in m_1 ways and fail in n_1 ways.

$$\therefore \qquad P(A) = \frac{m_1}{m_1 + n_1}$$

Also let B happen in m_2 ways and fail in n_2 ways.

$$\therefore \qquad P(B) = \frac{m_2}{m_2 + n_2}$$

Now there are four possibilities

A and B both may happen, then the number of ways = $m_1 . m_2$.
A may happen and B may fail, then the number of ways = $m_1 . n_2$
A may fail and B may happen, then the number of ways = $n_1 . m_2$
A and B both may fail, then the number of ways = $n_1 . n_2$

Thus, the total number of ways $= m_1 m_2 + m_1 n_2 + n_1 m_2 + n_1 n_2 = (m_1 + n_1)(m_2 + n_2)$

Hence the probabilities of the happening of both A and B

$$P(AB) = \frac{m_1 m_2}{(m_1+n_1)(m_2+n_2)} = \frac{m_1}{m_1+n_1} \cdot \frac{m_2}{m_2+n_2} = P(A) \, . \, P(B)$$ **Proved.**

Example 6. *An article manufactured by a company consists of two parts A and B. In the process of manufacture of part A, 9 out of 100 are likely to be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part B. Calculate the probability that the assembled article will not be defective (assuming that the events of finding the part A non-defective and that of B are independent).*

Solution. Probability that part A will be defective $= \frac{9}{100}$

Probability that part A will not be defective $= \left(1 - \frac{9}{100}\right) = \frac{91}{100}$

Probability that part B will be defective $= \frac{5}{100}$

Probability that part B will not be defective $= \left(1 - \frac{5}{100}\right) = \frac{95}{100}$

Probability that the assembled article will not be defective = (Probability that part A will not be defective)× (Probability that part B will not be defective)

$$= \left(\frac{91}{100}\right) \times \left(\frac{95}{100}\right) = 0.8645$$ **Ans.**

Example 7. *The probability that machine A will be performing an usual function in 5 years' time is $\frac{1}{4}$, while the probability that machine B will still be operating usefully at the end of the same period, is $\frac{1}{3}$*

Find the probability in the following cases that in 5 years time :

(*i*) *Both machines will be performing an usual function.*

(*ii*) *Neither will be operating.*

(*iii*) *Only machine B will be operating.*

(*iv*) *At least one of the machines will be operating.*

Solution. P (A operating usefully) $= \frac{1}{4}$, so $q\ (A) = 1 - \frac{1}{4} = \frac{3}{4}$

P (B operating usefully) $= \frac{1}{3}$, so $q\ (B) = 1 - \frac{1}{3} = \frac{2}{3}$

(*i*) P (Both A and B will operate usefully) $= P(A). \, P(B) = \left(\frac{1}{4}\right) \times \left(\frac{1}{3}\right) = \frac{1}{12}$

(*ii*) P (Neither will be operating) $= q(A) \, . \, q(B) = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{2}$

(*iii*) P (Only B will be operating) $= \text{P (B)} \times q\text{ (A)} = \left(\frac{1}{3}\right) \times \left(\frac{3}{4}\right) = \frac{1}{4}$

(*iv*) P (At least one of the machines will be operating)

= 1 – P (none of them operates)

$$= 1 - \frac{1}{2} = \frac{1}{2}$$ **Ans.**

Example 8. *There are two groups of subjects one of which consists of 5 science and 3 engineering subjects and the other consists of 3 science and 5 engineering subjects. An unbiased* *die is cast. If number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately.*

Solution. Probability of turning up 3 or 5 = $\frac{2}{6} = \frac{1}{3}$

Probability of selecting engineering subject from first group = $\frac{3}{8}$

Now the probability of selecting engineering subject from first group on turning up 3 or 5

$$= \left(\frac{1}{3}\right) \times \left(\frac{3}{8}\right) = \frac{1}{8} \qquad \text{... (1)}$$

Probability of not turning up 3 or 5 = $1 - \frac{1}{3} = \frac{2}{3}$

Probability of selecting engineering subject from second group = $\frac{5}{8}$

Now probability of selecting engineering subject from second group on not turning up 3 or 5

$$= \frac{2}{3} \times \frac{5}{8} = \frac{5}{12} \qquad \text{... (2)}$$

Probability of the selection of engineering subject $= \frac{1}{8} + \frac{5}{12}$ [From (1) and (2)]

$$= \frac{13}{24}$$ **Ans.**

Example 9. *An urn contains nine balls, two of which are red, three blue and four black. Three balls are drawn from the urn at random. What is the probability that*

(*i*) *the three balls are of different colours?*

(*ii*) *the three balls are of the same colour?*

Solution.

Urn contains 2 Red balls, 3 Blue balls and 4 Black balls.

(*i*) Three balls will be of different colours if one ball is red, one blue and one black ball are drawn.

$$\text{Required probability } = \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{2 \times 3 \times 4}{84} = \frac{2}{7}$$ **Ans.**

(*ii*) Three balls will be of the same colour if either 3 blue balls or 3 black balls are drawn.

P (3 Blue balls or 3 Black balls) = P (3 Blue balls) + P (3 Black balls)

$$= \frac{{}^3C_3}{{}^9C_3} + \frac{{}^4C_3}{{}^9C_3} = \frac{1+4}{84} = \frac{5}{84}$$ **Ans.**

Example 10. *A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability that first is white and second is black ?*

Solution. Probability of drawing one white ball = $\frac{10}{25}$

Probability of drawing one black ball without replacement = 15/24

Required probability of drawing first white ball and second black ball

$$= \frac{10}{25} \times \frac{15}{24} = \frac{1}{4}$$ **Ans.**

Example 11. *A committee is to be formed by choosing two boys and four girls out of a group of five boys and six girls. What is the probability that a particular boy named A and a particular girl named B are selected in the committee?*

Solution. Two boys are to be selected out of 5 boys. A particular boy A is to be included in the committee. It means that only 1 boy is to be selected out of 4 boys.

Number of ways of selection = ${}^4C_1$

Similarly a girl B is to be included in the committee.

Then only 3 girls are to be selected out of 5 girls.

Number of ways of selection = ${}^5C_3$

Required probability = $\dfrac{{}^4C_1 \times {}^5C_3}{{}^5C_2 \times {}^6C_4} = \dfrac{4 \times 10}{10 \times 15} = \dfrac{4}{15}$ **Ans.**

Example 12. *Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child is selected at random from each group. Find the chance of selecting 1 girl and 2 boys.*

Solution. There are three ways of selecting 1 girl and two boys.

I way : Girl is selected from first group, boy from second group and second boy from third group.

Probability of the selection of (Girl + Boy + Boy) $= \dfrac{3}{4} \times \dfrac{2}{4} \times \dfrac{3}{4} = \dfrac{18}{64}$

II way : Boy is selected from first group, girl from second group and second boy from third group.

Probability of the selection of (Boy + Girl + Boy) $= \dfrac{1}{4} \times \dfrac{2}{4} \times \dfrac{3}{4} = \dfrac{6}{64}$

III way : Boy is selected from first group, second boy from second group and the girl from the third group.

Probability of selection of (Boy + Boy + Girl) $= \dfrac{1}{4} \times \dfrac{2}{4} \times \dfrac{1}{4} = \dfrac{2}{64}$

Total probability $= \dfrac{18}{64} + \dfrac{6}{64} + \dfrac{2}{64} = \dfrac{26}{64} = \dfrac{13}{32}$ **Ans.**

Example 13. *The number of children in a family in a region are either 0, 1 or 2 with probability 0.2, 0.3 and 0.5 respectively. The probability of each child being a boy or girl 0.5. Find the probability that a family has no boy.*

Solution. Here there are three types of families

(*i*) Probability of zero child (boys) = 0.2

(*ii*)

Boy	Girl
0	1
1	0

Probability of zero boy in case II = 0.3 × 0.5 = 0.15

(*iii*)

Boy	Girl
0	2
1	1
2	0

In this case probability of zero boy = $0.5 \times \dfrac{1}{3} = 0.167$

Considering all the three cases, the probability of zero boy

= 0.2 + 0.15 + 0.167 = 0.517 **Ans.**

Example 14. *A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that*

(i) both of them will be selected. (ii) only one of them will be selected, and (iii) none of them will be selected ?

Solution. P (husband's selection) = $\frac{1}{7}$, P (wife's selection) = $\frac{1}{5}$

(*i*) P (both selected) $=\frac{1}{7}\times\frac{1}{5}=\frac{1}{35}$

(*ii*) P (only one selected) = P (only husband's selection) + P (only wife's selection)\

$$=\frac{1}{7}\times\frac{4}{5}+\frac{1}{5}\times\frac{6}{7}=\frac{10}{35}=\frac{2}{7}$$

(*iii*) P (none of them will be selected) = $\frac{6}{7}\times\frac{4}{5}=\frac{24}{35}$ **Ans.**

Example 15. *A problem of statistics is given to three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved?*

Solution. The probability that A can solve the problem = $\frac{1}{2}$

The probability that A cannot solve the problem = $1-\frac{1}{2}$.

Similarly the probability that B and C cannot solve the problem are

$$\left(1-\frac{3}{4}\right) \text{ and } \left(1-\frac{1}{4}\right)$$

$\therefore$ The probability that A, B, C cannot solve the problem

$$=\left(1-\frac{1}{2}\right)\times\left(1-\frac{3}{4}\right)\times\left(1-\frac{1}{4}\right)=\frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{32}$$

Hence, the probability that the problem can be solved $=1-\frac{3}{32}=\frac{29}{32}$ **Ans.**

Example 16. *A student takes his examination in four subjects α, β, γ, δ. He estimates his chances of passing in α as $\frac{4}{5}$, in β as $\frac{3}{4}$, in γ as $\frac{5}{6}$ and in δ as $\frac{2}{3}$. To qualify, he must pass in α and at least two other subjects. What is the probability that he qualifies?*

Solution. $P(\alpha)=\frac{4}{5},\quad P(\beta)=\frac{3}{4},\quad P(\gamma)=\frac{5}{6},\quad P(\delta)=\frac{2}{3}$

There are four possibilities of passing at least two subjects.

(*i*) Probability of passing β, γ and failing δ $=\frac{3}{4}\times\frac{5}{6}\times\left(1-\frac{2}{3}\right)=\frac{3}{4}\times\frac{5}{6}\times\frac{1}{3}=\frac{5}{24}$

(*ii*) Probability of passing γ, δ and failing β $=\frac{5}{6}\times\frac{2}{3}\times\left(1-\frac{3}{4}\right)=\frac{5}{6}\times\frac{2}{3}\times\frac{1}{4}=\frac{5}{36}$

(*iii*) Probability of passing δ, β and failing γ $=\frac{2}{3}\times\frac{3}{4}\times\left(1-\frac{5}{6}\right)=\frac{2}{3}\times\frac{3}{4}\times\frac{1}{6}=\frac{1}{12}$

(iv) Probability of passing $\beta, \gamma, \delta = \frac{3}{4}\times\frac{5}{6}\times\frac{2}{3}=\frac{5}{12}$

Probability of passing at least two subjects $=\frac{5}{24}+\frac{5}{36}+\frac{1}{12}+\frac{5}{12}=\frac{61}{72}$

Probability of passing α and at least two subjects. $=\frac{4}{5}\times\frac{61}{72}=\frac{61}{90}$ **Ans.**

Example 17. *There are 6 positive and 8 negative numbers. Four numbers are chosen at random, without replacement, and multiplied. What is the probability that the product is a positive number ?*

Solution. To get from the product of four numbers, a positive number, the possible combinations are as follows :

S. No.	*Out of 6 Positive Numbers*	*Out of 8 Negative Numbers*	*Positive Numbers*
1.	4	0	${}^6C_4 \times {}^8C_0 = \frac{6\times5}{1\times2}\times1=15$
2.	2	2	${}^6C_2 \times {}^8C_2 = \frac{6\times5}{1\times2}\times\frac{8\times7}{1\times2}=420$
3.	0	4	${}^6C_0 \times {}^8C_4 = 1\times\frac{8\times7\times6\times5}{1\times2\times3\times4}=70$
			Total = 505

Probability $=\dfrac{{}^6C_4\times{}^8C_0+{}^6C_2\times{}^8C_2+{}^6C_0\times{}^8C_4}{{}^{14}C_4}=\dfrac{15+420+70}{\frac{14\times13\times12\times11}{1\times2\times3\times4}}=\dfrac{505\times4\times3\times2\times1}{14\times13\times12\times11}=\dfrac{505}{1001}$ **Ans.**

Example 18. *A can hit a target 3 times in 5 shots, B 2 times in 5 shots and C three times in 4 shots. All of them fire one shot each simultaneously at the target. What is the probability that*

(i) 2 shots hit *(ii) At least two shots hit?*

Solution. Probability of A hitting the target $=\frac{3}{5}$

Probability of B hitting the target $=\frac{2}{5}$

Probability of C hitting the target $=\frac{3}{4}$

Probability that 2 shots hit the target

$= P(A)\,P(B)\,q(C) + P(A)\,P(C)\,q(B) + P(B)\,P(C)\,q(A)$

$=\frac{3}{5}\times\frac{2}{5}\times\left(1-\frac{3}{4}\right)+\frac{3}{5}\times\frac{3}{4}\times\left(1-\frac{2}{5}\right)+\frac{2}{5}\times\frac{3}{4}\times\left(1-\frac{3}{5}\right)=\frac{6}{25}\times\frac{1}{4}+\frac{9}{20}\times\frac{3}{5}+\frac{6}{20}\times\frac{2}{5}$

$=\frac{6+27+12}{100}=\frac{45}{100}=\frac{9}{20}$ **Ans.**

(ii) Probability of at least two shots hitting the target

= Probability of 2 shots + probability of 3 shots hitting the target

$=\frac{9}{20}+P(A)\,P(B)\,P(C)=\frac{9}{20}+\frac{3}{5}\times\frac{2}{5}\times\frac{3}{4}=\frac{63}{100}$ **Ans.**

Example 19. *A and B take turns in throwing two dice, the first to throw 10 being awarded the prize. Show that if A has the first throw, their chances of winning are in the ratio 12: 11.*

Solution. The combinations of throwing 10 from two dice can be

(6 + 4), (4 + 6), (5 + 5).

The number of combinations is 3.

Total combinations from two dice = 6 × 6 = 36.

$\therefore$ The probability of throwing 10 = $p = \frac{3}{36} = \frac{1}{12}$

The probability of not getting 10 = $q = 1 - \left(\frac{1}{12}\right) = \frac{11}{12}$

If A is to win, he should throw 10 in either the first, the third, the fifth, ... throws.

Their respective probabilities are = $p,\ q^2 p,\ q^4 p,\ \ldots = \frac{1}{12}, \left(\frac{11}{12}\right)^2 \frac{1}{12}, \left(\frac{11}{12}\right)^4 \frac{1}{12} \ldots$

A's total probability of winning = $\frac{1}{12} + \left(\frac{11}{12}\right)^2 \cdot \frac{1}{12} + \left(\frac{11}{12}\right)^4 \cdot \frac{1}{12} + \ldots$

$$= \frac{\frac{1}{12}}{1 - \left(\frac{11}{12}\right)^2} = \frac{12}{23} \qquad \left[\text{This is infinte G.P. Its sum} = \frac{a}{1-r}\right]$$

B can win in either 2nd, 4th, 6th ... throws.

So B's total chance of winning = $qp + q^3 p + q^5 p + \ldots\ldots\ldots$

$$= \left(\frac{11}{12}\right)\left(\frac{1}{12}\right) + \left(\frac{11}{12}\right)^3 \left(\frac{1}{12}\right) + \left(\frac{11}{12}\right)^5 \left(\frac{1}{12}\right) + \ldots = \frac{\left(\frac{11}{12}\right)\left(\frac{1}{12}\right)}{1 - \left(\frac{11}{12}\right)^2} = \frac{11}{23}$$

Hence A's chance to B's chance = $\frac{12}{23} : \frac{11}{23} = 12 : 11$ **Proved.**

Example 20. *A and B throw alternatively a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. Find their respective chances of winning, if A begins.*

Solution. Number of ways of throwing 6

i.e. (1 + 5), (2 + 4), (3 + 3), (4 + 2), (5 + 1) = 5.

Probability of throwing $6 = \frac{5}{36} = p_1,\ q_1 = \frac{31}{36}$

Number of ways of throwing 7

i.e.; (1 + 6), (2 + 5), (3 + 4), (4 + 3), (5 + 2), (6 + 1) = 6

Probability of throwing $6 = \frac{6}{36} = \frac{1}{6} = P_2,\ q_2 = \frac{5}{6}$

$$P(A) = p_1 + q_1 q_2 p_1 + q_1^2 q_2^2 p_1 + \ldots.$$

$$P(B) = q_1 p_2 + q_1^2 q_2 p_2 + q_1^3 q_2^2 p_2 + \ldots.$$

Probability of A's winning = $p_1 + q_1 q_2 p_1 + q_1^2 q_2^2 p_1 + \ldots.$

$$= \frac{p_1}{1-q_1q_2} = \frac{\frac{5}{36}}{1-\frac{31}{36}\times\frac{5}{6}} = \frac{5}{36}\times\frac{36\times 6}{61} = \frac{30}{61}$$

Probability of B's winning $= q_1 p_2 + q_1^2 q_2 p_2 + q_1^3 q_2^2 p_2 + \ldots\ldots$

$$= \frac{q_1 p_2}{1-q_1q_2} = \frac{\frac{31}{36}\times\frac{1}{6}}{1-\left(\frac{31}{36}\right)\left(\frac{5}{6}\right)} = \frac{31}{36\times 6}\times\frac{36\times 6}{61} = \frac{31}{61}$$ **Ans.**

EXERCISE

1. The probability that Nirmal will solve a problem is $\frac{2}{3}$ and the probability that Satyajit will solve it is $\frac{3}{4}$. What is the probability that (*a*) the problem will be solved (*b*) neither can solve it.

Ans. (*a*) $\frac{11}{12}$, (*b*) $\frac{1}{12}$

2. An urn contains 13 balls numbering 1 to 13. Find the probability that a ball selected at random is a ball with number that is a multiple of 3 or 4. **Ans.** $\frac{6}{13}$

3. Four persons are chosen at random from a group containing 3 men, 2 women, and 4 children. Show that the probability that exactly two of them will be children is $\frac{10}{21}$.

4. A five digit number is formed by using the digits 0, 1, 2, 3, 4 and 5 without repetition. Find the probability that the number is divisible by 6. **Ans.** $\frac{4}{25}$

5. The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chances of death by wrong diagnosis is 70%. A patient of doctor A, who had disease *X*, died, what is the chance that his disease was diagnosed correctly.

Ans. $\frac{6}{13}$

6. An anti-aircraft gun can take a maximum of four shots on enemy's plane moving from it. The probabilities of hitting the plane at first, second, third and fourth shots are 0.4, 0.3, 0.2 and 0.1 respectively. Find the probability that the gun hits the plane. **Ans.** 0.6976.

7. An electronic component consists of three parts. Each part has probability 0.99 of performing satisfactorily. The component fails if two or more parts do not perform satisfactorily. Assuming that the parts perform independently, determine the probability that the component does not perform satisfactorily.

Ans. 0.000298

8. The face cards are removed from a full pack. Out of the remaining 40 cards, 4 are drawn at random. What is the probability that they belong to different suits? **Ans.** $\frac{1000}{9139}$

9. Of the cigarette smoking population, 70% are men and 30% women, 10% of these men and 20% of these women smoke 'WILLS.' What is the probability that a person seen smoking a 'WILLS' will be a man.

Ans. $\frac{7}{13}$

10. A machine contains a component *C* that is vital to its operation. The reliability of component *C* is 80%. To improve the reliability of a machine, a similar component is used in parallel to form a system S. The machine will work provided that one of these components functions correctly. Calculate the reliability of the system *S*. **Ans.** 96%

11. The odds that a book will be favourably reviewed by three independent critics are 5 to 2, 4 to 3, 3 to 4 respectively. What is the probability that of the three reviews, a majority will be favourable?

Ans. $\frac{209}{343}$

12. A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of 11 steps, he is just one step away from the starting point.

Ans. 0.5263

13. A candidate is selected for interview for three posts. For the first post there are three candidates, for the second there are 4, and for the third are 2. What is the chance of getting at least one post?

Ans. $\frac{3}{4}$

14. The chance of hitting a target by a bomb is 50% when 4 bombs are dropped, what is the probability of destroying the target, if one bomb is just sufficient to destroy it.

Ans. $\frac{15}{16}$

15. Tick √ the correct answer :

(*i*) A, B, C in order toss a coin, the first to throw a head wins. Assuming if A begins and the game continues indefinitely their respective chances of winning are:

(*a*) $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$ (*b*) $\frac{1}{7}, \frac{4}{7}, \frac{2}{7}$ (*c*) $\frac{2}{7}, \frac{4}{7}, \frac{1}{7}$ (*d*) None of these

Ans. (*a*)

(*ii*) An unbiased die with faces marked 1, 2, 3, 4, 5, 6 is rolled 4 times, out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is then

(*a*) $\frac{16}{81}$ (*b*) $\frac{2}{9}$ (*c*) $\frac{80}{81}$ (*d*) $\frac{8}{9}$

Ans. (*a*)

(*iii*) India plays two matches each with England and Australia. In any match the probability of its getting points 0, 1 and 2 are 0.45, 0.05 and 0.5 respectively. Assuming the outcomes are independent, the probability that India gets at least seven points is

(*a*) 0.8750 (*b*) 0.0875 (*c*) 0.0625 (*d*) 0.0250

Ans. (*b*)

Fill up the blanks:

(*iv*) Probability of any event can not be greater than _____ and less than _____.

Ans. 0, 1

CONDITIONAL PROBABILITY

Let A and B be two events of a sample space S and let $P(B) \neq 0$. Then conditional probability of the event A, given B, denoted by $P(A/B)$, is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{... (1)}$$

Theorem. If the events A and B defined on a sample space S of a random experiment are independent, then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B)$$

Proof. A and B are given to be independent events,

$$P(A \text{ and } B) = P(A) \bullet P(B)$$

$$\Rightarrow \qquad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A).P(B)}{P(B)} = P(A)$$

$$\Rightarrow \qquad P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B).P(A)}{P(A)} = P(B)$$

BAYE'S THEOREM

If $B_1, B_2, B_3, \ldots\ldots. B_n$ are mutually exclusive events with $P(B_i) \neq 0$, $(i = 1, 2, \ldots n)$ of a random experiment then for any arbitrary event A of the sample space of the above experiment with $P(A) > 0$, we have

$$P(B_i / A) = \frac{P(B_i) P(A / B_i)}{\sum_{i=1}^{n} P(B_i) P(A / B_i)}$$

(for $n = 3$)

$$P(B_2 / A) = \frac{P(B_2) P(A / B_2)}{P(B_1) P(A / B_1) + P(B_2) P(A / B_2) + P(B_3) P(A / B_3)}$$

Proof. Let S be the sample space of the random experiment.

The events $B_1, B_2, \ldots, B_n$ being exhaustive

$$S = B_1 \cup B_2 \cup \ldots. \cup B_n$$

$$[\because A \subset S]$$

$$\therefore \qquad A = A \cap S = A \cap (B_1 \cup B_2 \cup \ldots. \cup B_n)$$

$$= (A \cap B_1) \cup (A \cap B_2) \cup \ldots. \cup (A \cap B_n) \qquad \text{[Distributive Law]}$$

$$\Rightarrow \qquad P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots. + P(A \cap B_n)$$

$$= P(B_1) P(A / B_1) + P(B_2) P(A / B_2) + \ldots. + P(B_n) P(A / B_n)$$

$$= \sum_{i=1}^{n} P(B_i) P(A / B_i) \qquad \ldots (1)$$

Now, $P(A \cap B_i) = P(A) P(B_i / A)$

$$\Rightarrow \qquad P(B_i / A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(B_i) P(A / B_i)}{\sum_{i=1}^{n} P(B_i) P(A / B_i)} \qquad \text{[Using (1)]}$$

Note. $P(B)$ is the probability of occurrence B. If we are told that the event A has already occurred. On knowing about the event A, $P(B)$ is changed to $P(B/A)$. With the help of Baye's theorem we can calculate $P(B / A)$.

Example 21. *An urn I contains 3 white and 4 red balls and an urn II contains 5 white and 6 red balls. One ball is drawn at random from one of the urns and is found to be white. Find the probability that it was drawn from urn I.*

Solution. Let U_1: the ball is drawn from urn I

U_2 : the ball is drawn from urn II

W : the ball is white.

We have to find $P(U_1/W)$

By Baye's Theorem

$$P(U_1/W)=\frac{P(U_1)P(W/U_1)}{P(U_1)P(W/U_1)+P(U_2)P(W/U_2)} \qquad \text{... (1)}$$

Since two urns are equally likely to be selected, $P(U_1) = P(U_2) = \frac{1}{2}$

$P(W/U_1) = P$ (a white ball is drawn from urn I) $= \frac{3}{7}$

$P(W/U_2) = P$ (a white ball is drawn from urn II) $= \frac{5}{11}$

$\therefore$ From (1), $$P(U_1/W)=\frac{\frac{1}{2}\times\frac{3}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}}=\frac{33}{68}$$ **Ans.**

Example 22. *Three urns contains 6 red, 4 black; 4 red, 6 black; 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red find the probability that it is drawn from the first urn.*

Solution. Let U_1: the ball is drawn from U_1.

U_2: the ball is drawn from U_2.

U_3: the ball is drawn from U_3.

R : the ball is red.

We have to find $P(U_1/R)$.

By Baye's Theorem,

$$P(U_1/R)=\frac{P(U_1)P(R/U_1)}{P(U_1)P(R/U_1)+P(U_2)P(R/U_2)+P(U_3)P(R/U_3)} \qquad \text{... (1)}$$

Since the three urns are equally likely to be selected $P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$

Also $P(R/U_1) = P$ (a red ball is drawn from urn I) $= \frac{6}{10}$

$P(R/U_2) = P$ (a red ball is drawn from urn II) $= \frac{4}{10}$

$P(R/U_3) = P$ (a red ball is drawn from urn III) $= \frac{5}{10}$

$\therefore$ From (1), we have $$P(U_1/R)=\frac{\frac{1}{3}\times\frac{6}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}}=\frac{2}{5}$$ **Ans.**

Example 23. *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total. If their output 5, 4 and 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B?*