Mathemania 2025
Round 1

Problem 1

Deep in the heart of the Laboratory of Randomness, there exists a mysterious machine, affectionately called **Machine 'X'**. This ingenious device has one job: to paint grids of size $m \times n$, randomly assigning each cell a color, either **black** or **white**, with equal probability. After it finishes its work, it outputs the final grid for us to analyze.

Example Grid:

Let's say $m = 3$ and $n = 4$. Below is a possible grid that Machine 'X' might generate:

W	B	W	W
B	W	W	W
B	W	B	B

Machine 'X' has been working overtime, and we're curious about the probabilities of finding such a rectangle under different grid dimensions. Which of the following statements are correct: **Important Rule:** The corners of any rectangle within the grid must all be unique. Rectangles that are just a single row or a single column ($1 \times n$ or $m \times 1$) are too simple and do not count!

1. If $m = 6$ and $n = 6$, the probability that there exists a rectangle whose four corners are all of the same color lies in the closed interval $[0.5, 1]$.
2. If $m = 6$ and $n = 5$, the probability that there exists a rectangle whose four corners are all of the same color lies in the closed interval $[0, 0.5]$.
3. If $m = 6$ and $n = 4$, the probability that there exists a rectangle whose four corners are all of the same color is less than 1.
4. The exact probability stated in Option 1 remains unchanged even if Machine 'X' assigns colors with unequal probabilities.

Problem 2

Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ such that:

$$g(f(x+y)) = f(x) + (2x+y)g(y) \forall x, y \in \mathbb{R}, f(x) \neq 0, g(x) \neq 0$$

Then select all that follow-

1. $f(x)$ belongs to a family of Parabolas.
2. $g(x)$ can be a straight line passing through the origin.
3. $f(x)$ belongs to a family of Hyperbolas.
4. $g(x)$ belongs to a family of Ellipses.

Problem 3

For all $h \geq 2024^{2024}$ (where $h \in \mathbb{N}$)

$$\sqrt{((h-24)5^{-9h} - 16(5^{3-9h}))} + \sqrt{((h-22)5^{-9h} - 16(5^{3-9h}))} \equiv 0 \pmod{3^{5h}}$$

Here, $a \equiv b \pmod{m}$ means that $a - b$ is divisible by m. Then which of the following is true?

1. Infinitely many values of h satisfy the equation.
2. Exactly 2025 different values of h satisfy the equation.
3. No value of h satisfies the equation.
4. Exactly one value of h satisfy the equation.

Problem 4

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and differentiable function that satisfies:

$$f(a+b)[1+f(a)] = f(a^2+ab+a+b),$$
$$(c+d)f(c) + f(f(c)) = f(c^2+cd+c).$$

Then which of the following is true?

1. Infinitely many functions satisfy the equation.
2. At least two functions satisfy the equation.
3. There exists an even function that satisfies the equation.
4. Exactly four functions satisfy the equation.

Problem 5

Evaluate the following limit:

$$\lim_{n\to\infty} n \int_0^1 x^n \ln(1+x)\, dx$$

Problem 6

Raghuvir plays a game on the number line. He starts at zero. At each second, he either stays where he is or jumps some distance left or right. The distance he can jump at second s is 3^s units (he does not have to jump every second, but when he does, he jumps exactly 3^s units left or right). His goal is to reach 2025. He has devised a path to do the same in the minimum amount of time.
Find the sum of all values of s when he will jump, where $s < 10$, such that he reaches 2025 in minimum amount of time.

Problem 7

ϵ represents the **number of positive integral solutions** of the equation:

$$2 \cdot w \cdot y + x = |x|\sqrt{1 + 4 \cdot w^2}.$$

Using ϵ, a summation α is defined as:

$$\alpha = \sum_{i=0, i \neq 4k+3\cdot(\epsilon+1), k \in \mathbb{N}\cup\{0\}}^{8104} \left(i^6 + 2i^4 + 3i^3 + 6i + p\right).$$

Determine the least non-negative value of p such that α is divisible by 16. Using this p, compute the value of the final expression:

$$p^2 + 3^\epsilon.$$

Problem 8

In the vast realm of numbers, a dark secret lies buried beneath layers of complexity. A mysterious number, n, waits to be discovered—but its location is veiled in a cryptic equation:

$$(x+1)^2 - xy(2x - xy + 2y) + (y+1)^2 = n.$$

The task is simple yet daunting: find the smallest positive integer n for which the number of ordered pairs (x, y), where both x and y are positive integers, satisfying the equation equals 2027. Since n can be too large , your answer would be remainder of n when divided by 7.

Problem 9

For how many integers n between 1 and 10^{2007} inclusive, are the last 2007 digits of n and n^3 the same? (If n or n^3 has fewer than 2007 digits, treat it as if it had zeros on the left to compare the last 2007 digits.)

Problem 10

Rahul's house has a 1012×1012 size of floor fully consisting of square tiles, all of which are colored white. Rahul wants to color some tiles blue. In one move, Rahul can select one row or column whose every tile is white, choose exactly 500 tiles in this row or column, and color all of them blue. Find the maximum number of tiles that Rahul can color blue in a finite number of moves.

Problem 11

Let $f(x)$, $g(x)$, and $h(x)$ be quadratic equations. Consider the composite function $f(g(h(x)))$.

Task

Can $f(g(h(x)))$ can have roots as $1, 2, 3, 4, 5, 6, 7, 8$?

Upon some inspection, it can be observed that it cannot. Your task is to **make the least "change(s)"** to the array $\{1, 2, 3, 4, 5, 6, 7, 8\}$ so that the resulting set represents possible roots of $f(g(h(x)))$. Also don't worry about the ordering.

You can **ONLY** use **INTEGERS** from 1 to 8(including both 1 and 8).

Examples

- $\{1, 4, 3, 6, 8, 2, 7, 5\}$ and $\{4, 1, 3, 6, 8, 7, 2, 5\}$ and $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are all **same**(i.e. you **DON'T NEED ANY "CHANGES** to go from one to another
- $\{1, 2, 3, 4, 1, 2, 3, 4\}$ is a **VALID** "changed" array using 4 changes.
- $\{1, 2, 3, 4, 5, 5, 5, 5\}$ is a **VALID** "changed" array using 3 changes.
- $\{1, 2, 3, 4, 10, 9, 9, 8\}$ is **NOT VALID** as some entries are more than 8.
- $\{1, 2, 3, 4\}$ is **NOT VALID** as there are fewer than 8 entries.

What is the minimum number of changes required?

Problem 12

We are given $n = 2025$ sequences of real numbers, each containing n elements. Let the i-th sequence be represented as $(a_{i1}, a_{i2}, \ldots, a_{in})$. These sequences satisfy the following properties:

1. $\mathbf{a}_{ij}^2 \neq 1$ **for all** i, j
2. **Unit Norm of Each Sequence**:

$$\sum_{j=1}^{n} a_{ij}^2 = 1 \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

3. **Orthogonality Between Sequences**:

$$\sum_{k=1}^{n} a_{ik} a_{jk} = 0 \quad \text{for all } i \neq j,\ i, j \in \{1, 2, \ldots, n\}.$$

Find the minimum value of the double summation:

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \right)^2$$