Signals

Harsh Agrawal

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Abstract

This notebook compiles content from Signals lectures delivered by Prof. Dario Farina as part of the BIOE50005 Module (2nd Year Imperial MBE). A total of 9 lectures are covered in these notes. Please suggest any relevant changes/improvements at ha1822@ic.ac.uk.

Introduction and Types of Signals

Mathematically, signals are functions of one or more independent variables. Eg: The instantaneous velocity of a moving body at every given time interval can be represented as a signal: $v(t)$. Based on the type of input, there are two types of signals:

1. **Continuous-time**: Signals where the domain of the input variable is continuous. An analog signal is a continuous-time signal. The notation used for continuous-time signals is:

$$x(t), \quad t \in \mathbb{R}$$

2. **Discrete-time**: Signals where the domain of the input signal is not a continuum and belongs to a discrete set. Discrete-time signals can be generated by sampling points from a continuous signal. The commonly used notation for discrete-time signals is:

$$x[n], \quad n \in \mathbb{Z}$$

Discrete-time signals can also be interpreted as follows:

- Sequence of numbers: A signal $x[n]$ can be interpreted as a sequence of numbers $x[n]$ where n is the index of each number.

$$x = \{x[n]\}, \quad -\infty < n < \infty$$

- Vector: A signal $s[n]$ with defined over an input set of length x can be interpreted as a vector of x dimensions.

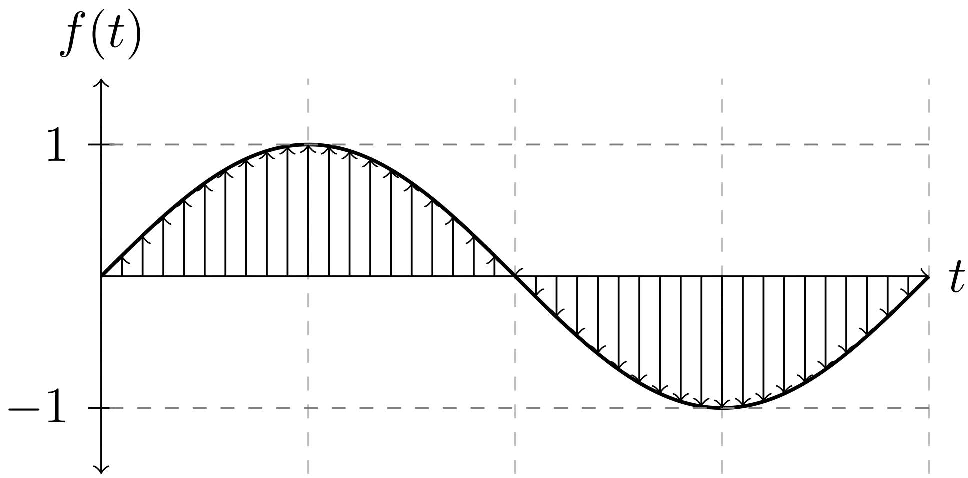

Figure 1: Sampling of a continuous signal $f(t) = \sin(t)$

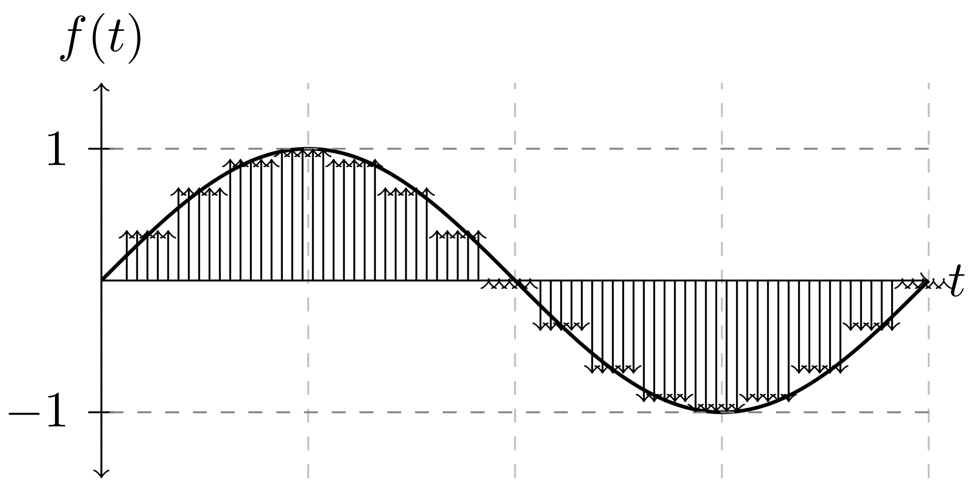

Figure 2: Quantization of sin wave. This figure isn't a mathematically rigorous depiction as the quantization levels might not be equally spaced.

Digital Signals & Quantization

Digital Signal: A discrete time signal that is (often) sampled from a continuous analog signal such as figure2. Digital signals are also often quantized.
Quantization: When sampling a continuous signal, the amplitude of the signal is rounded to the nearest quantized level.

*Quantization always implies **losing information** as the exact amplitude of the wave is lost while sampling at a given time interval.*

Other Types of Signals

Depending on the property of prediction of signal beforehand, there are two categories of signals:

1. **Deterministic**: These signals can be predicted exactly, and before they've been observed. This implies that a known mathematical formulation for the signal is already present. Eg: If we know a signal follows the function $x(t) = \sin(2\pi t)$, we already know all information about the signal.

2. **Stochastic**: These signals can't be predicted before they are observed. These signals convey new information when observed. Eg: EEG measurement, etc. If we already knew the analytical formulation of these signals then we wouldn't be required to observe these signals.

Periodic Signals: A signal is termed periodic if it follows this property:

$$x(t) = x(t+T) \quad \forall t \tag{1}$$

The signal should follow this for all values of t ($\forall$ *represents 'for all'*). Here, T = Period of the signal. Similarly, for a discrete signal: $x[n] = x[n+N]$.

Signal Energy & Power

For a continuous-time signal, $x(t)$ for $t_1 \leq t \leq t_2$, The Energy (E) of the signal and Power (P) are defined as:

$$E = \int_{t_1}^{t_2} |x(t)|^2 \cdot dt \tag{2}$$

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \cdot dt \tag{3}$$

Similarly, for a discrete-time signal $x[n]$ for $n_1 \leq n \leq n_2$, The Energy (E) of the signal and Power (P) are defined as:

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2 \tag{4}$$

$$P = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 \tag{5}$$

These definitions hold when $-\infty < t < \infty$ or $-\infty < n < \infty$

Complex Signals

Before understanding complex signals, it's helpful to recap complex numbers.

The complex plane is quite useful when dealing with two dimensions: the real dimension (x-axis) and the imaginary dimension (y-axis). Complex numbers can be represented in multiple ways:

- **Vector Form**: $\vec{z} = x + j \cdot y$ where $j = \sqrt{-1}$. The magnitude of this vector is given by:

$$|\vec{z}|^2 = \sqrt{x^2 + y^2}$$

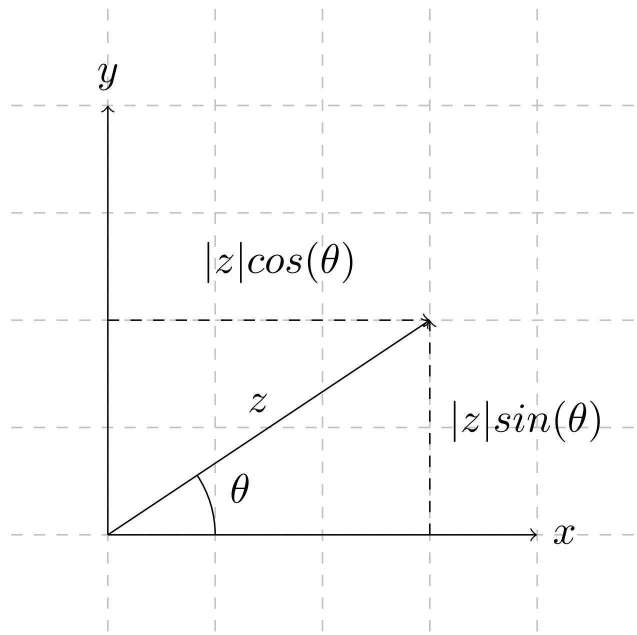

Figure 3: Vector representation of the generic complex number $\vec{z} = x + j \cdot y$.

$$\theta = \text{Phase of the complex number}$$
$$x = |z|\cos(\theta)$$
$$y = |z|\sin(\theta)$$

By putting the values of x and y into the vector form of complex numbers, we can obtain:

$$\vec{z} = x + j \cdot y = |z|(\cos(\theta) + j \cdot \sin(\theta))$$

- **Exponential Form**: We can use Euler's relation, $e^{j \cdot \theta} = \cos(\theta) + j \cdot \sin(\theta)$, to obtain the following equality from the above equation:

$$\vec{z} = x + j \cdot y = |z|(\cos(\theta) + j \cdot \sin(\theta)) = |z| \cdot e^{j \cdot \theta} \tag{6}$$

Examples

Proof of a Periodic Signal

We can use the condition for periodicity we defined above (eqn.1) to prove whether a given signal is periodic.

Let's use the complex exponential as an example:

$$x(t) = e^{j\cdot\omega_0\cdot t}$$

This equation is periodic only if $x(t) = x(t+T)$, thus, the following needs to be true:

$$\begin{aligned} e^{j\cdot\omega_0\cdot t} &= e^{j\cdot\omega_0\cdot(t+T)} \\ &= e^{(j\cdot\omega_0\cdot t)+(j\cdot\omega_0\cdot T)} \\ &= e^{(j\cdot\omega_0\cdot t)} \cdot e^{(j\cdot\omega_0\cdot T)} \end{aligned}$$

We know from Euler, that:

$$e^{(j\cdot\omega_0\cdot T)} = \cos(\omega_0 \cdot T) + j \cdot \sin(\omega_0 \cdot T)$$

To ensure LHS=RHS, and prove that the above signal is periodic, we need to find a value of $\omega_0 \cdot t$ such that $e^{j\cdot\omega_0\cdot T} = 1$. Cleverly, if we set $\omega_0 \cdot T = \pm 2\pi$, then $\cos(\omega_0 \cdot T) = 1$, $\sin(\omega_0 \cdot T) = 0$, and ultimately, $e^{j\cdot\omega_0\cdot T} = 1$.

To set a generic condition for periodicity:

$$T \;=\; \frac{2\pi}{|\omega_0|}$$

The reason we only take the absolute value of ω_0 is because T is always positive and only ω_0 can take negative values.

A similar is true for discrete-time signals:

$$N = \frac{2\pi}{|\omega_0|}$$

However, here N can only be an integer, thus ω_0 also needs to be a fraction of 2π or:

$$\omega_0 = \frac{2\pi}{k}, \quad k \in Integer$$

This also tells us that in the discrete-time domain, frequencies can't be arbitrarily increased from $0 \to +\infty$ to increase the rotation speed. They can only be sampled from a given domain. If we go any faster, we won't be able to distinguish the additional speed. Eg: when the wheel of a car goes too fast, we can't identify its speed.

Example Calculation of Energy & Power

Let's calculate the energy for a complex signal $x(t) = e^{j\omega_0 t}$ and $\omega_0 =$ angular frequency for the range $-\infty < t < \infty$. We can use the mathematical definition of energy for a continuous-time signal from eq. 2 as follows:

$$\text{Energy} \;=\; \int_{-\infty}^{\infty} |x\,(t)|^2 \cdot dt$$

Using Euler's relation, we can separately evaluate $|x(t)|^2$:

$$|x(t)|^2 = |e^{j\omega_0 t}|^2$$

Complex numbers have the following property:

$$|z|^2 = |z| \cdot |\bar{z}|; \quad \bar{z}\text{: complex conjugate } (a - j \cdot b)$$

Expanding $e^{j\omega_0 t}$ using Euler's relation and the above property:

$$\begin{aligned} e^{j\omega_0 t} &= (\cos(\omega_0 \cdot t) + j \cdot \sin(\omega_0 \cdot t)) \cdot (\cos(\omega_0 \cdot t) - j \cdot \sin(\omega_0 \cdot t)) \\ &= \cos^2(\omega_0 \cdot t) - j^2 \cdot \sin^2(\omega_0 \cdot t) \\ &= \cos^2(\omega_0 \cdot t) + \sin^2(\omega_0 \cdot t) \\ &= 1 \end{aligned}$$

Plugging this into the integral above, we obtain:

$$E \;=\; \int_{-\infty}^{+\infty} 1 \cdot dt = [t]_{-\infty}^{+\infty} = +\infty$$

We can say that complex exponentials are **signals with infinity energy**. We can also calculate the power of the complex exponential:

$$\begin{aligned} \text{Power} &= \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{+T} |x\,(t)|^2 \cdot dt \\ &= \frac{1}{2T} \cdot [t]_{-T}^{+T} \\ &= \frac{1}{2T} \cdot 2T \\ &= 1 \end{aligned}$$

From the above, we can say that complex exponentials are **signals with finite energy and equals to 1**.

Unit Impulse & Unit Step

Unit Impulse

In the discrete time domain, the unit impulse is the simplest signal, defined as follows:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

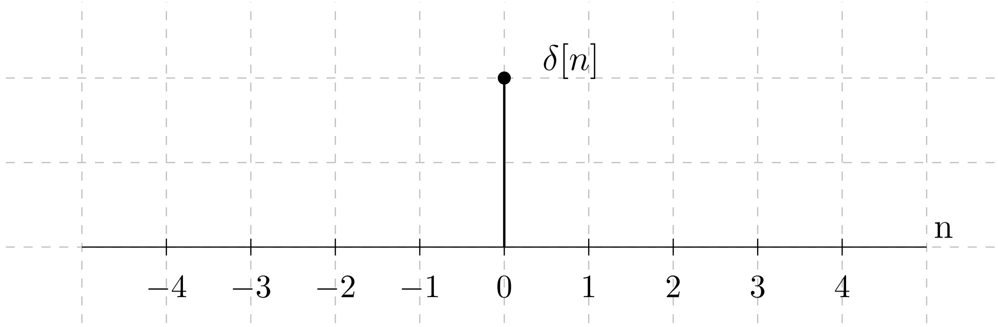

Figure 4: A unit impulse, $\delta[n]$ centered at n=0

Similarly, the discrete-time impulse delayed by the integer k is:

$$\delta[n-k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$

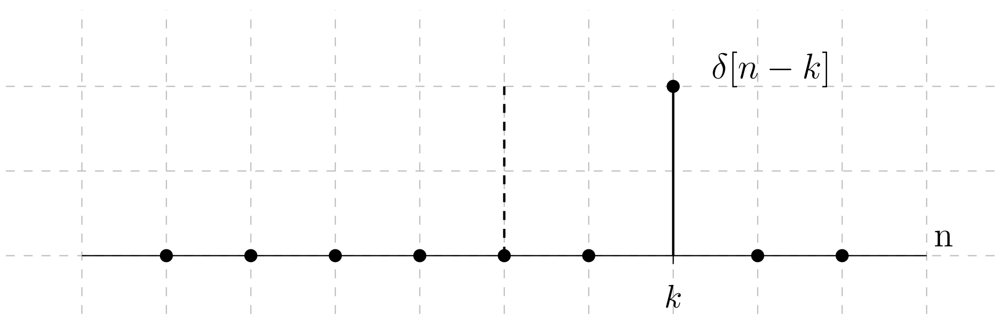

Figure 5: A unit impulse, $\delta[n-k]$ off-centered at k

If we multiply a generic discrete-time signal, $x[n]$, with a unit pulse centered at k, $\delta[n-k]$, we obtain:

$$x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k]$$

The reason this is true is because at $n \neq k$, the value of $\delta[n-k] = 0$. Thus, the output is simply the signal at the time step $n = k$ multiplied by the unit impulse at $n = k$. By definition, the unit impulse $= 1$ at $n = k$, so the equation now becomes:

$$x[n] \cdot \delta[n-k] = x[k] \cdot \delta[n-k] = x[k]$$

If we now compute:

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} x[n]\cdot\delta[n-k] &= \sum_{n=-\infty}^{+\infty} x[k]\cdot\delta[n-k] \\ &= x[k]\cdot\sum_{n=-\infty}^{+\infty}\delta[n-k] \\ &= x[k] \qquad (\because \sum_{n=-\infty}^{+\infty}\delta[n-k]=1)\end{aligned}$$

$$\therefore x[k] = \sum_{n=-\infty}^{+\infty} x[n]\cdot\delta[n-k]$$

If we arbitrarily change the variables and assume $n \rightleftharpoons k$, we obtain:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\cdot\delta[k-n]$$

But since the unit pulse = 1 at n=k (or k=n), $\delta[k-n]=\delta[n-k]$. In both situations, when k=n, $\delta[n-k]=\delta[n-k]$ (equal to 1 when n = k, equal to 0 when n $\neq$ k). So:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\cdot\delta[n-k] \tag{7}$$

What this equation intuitively tells us is that you can construct a discrete-time signal by taking the value of that discrete-time signal at different timesteps scaled by a unit impulse off-centered at that time step.

Another paraphrase: *Any arbitrary discrete-time signal can be expressed as the sum of scaled and delayed impulses.*

Unit Step

A unit step in the discrete-time domain is defined as:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = \sum_{k=0}^{+\infty} \delta[n-k]$$

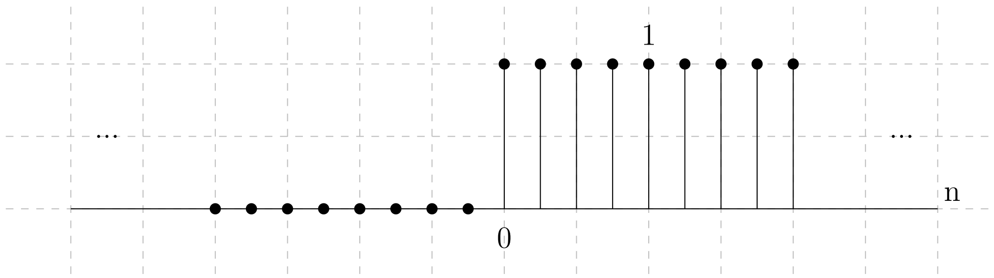

Figure 6: A unit step, $u[n]$

We can also note the following property:

$$\delta[n] = u[n] - u[n-1]$$

This means that the unit impulse is the discrete-time derivative of the unit step. Here, since 1 unit is the smallest increment in x, $dx = 1$, and $dy = u[n] - u[n] - 1$.

Similar to the unit step in the discrete time, we can define a unit step in the continuous time domain as:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

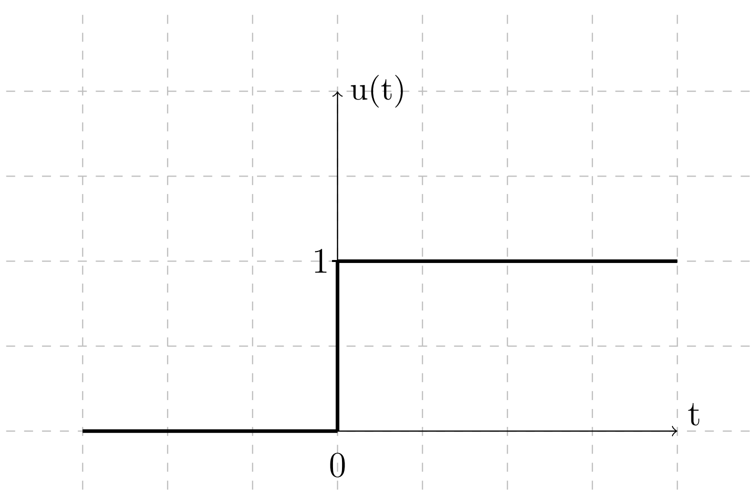

Figure 7: A unit step, $u(t)$ in continous-time

As we can see from figure 7, the unit step is discontinuous at $t = 0$. This discontunity is an issue in the continuous time domain as the unit step mentioned above is not **formally differentiable**.

In order to calculate that we define another signal: $u_\Delta(t)$ as follows:

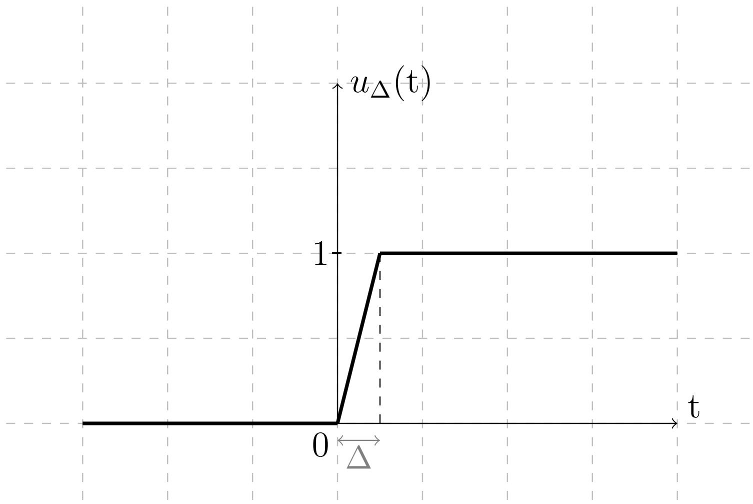

Figure 8: Modified unit step, $u_\Delta(t)$ in continuous-time

Now if we take the derivative of the above signal, we obtain:

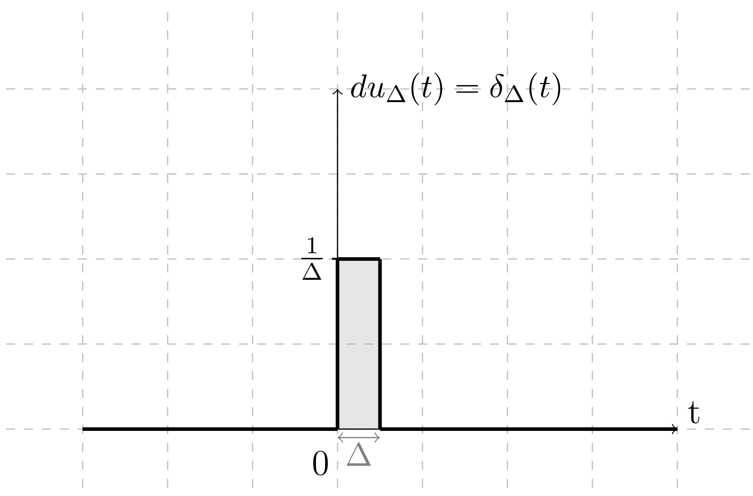

Figure 9: Derivative of modified unit step, $u_\Delta(t)$ in continuous-time

We can see that the derivative plot resembles a unit impulse ($\therefore du_\Delta(t) = \delta_\Delta(t)$). However, unlike the unit impulse in discrete time which is of infintesimally small duration, this unit impulse has a duration of size Δ. If Δ decreases, $\frac{1}{\Delta}$ increases; however, the area under the unit impulse remains constant and equals to 1. This can be mathematical represented by:

$$\int_{-\infty}^{+\infty} \delta_\Delta(t) \cdot dt = 1, \quad \forall \Delta$$

We can define the unit impulse in the continuous time domain ($\delta(t)$) as a limit of the unit impulse we obtained from the modified unit step ($\delta_\Delta(t)$) as $\Delta \to 0$:

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) = \begin{cases} 0, & \forall t \neq 0 \\ \neq 0, & t = 0 \end{cases}$$

Explaination: As we decrease Δ, $\frac{1}{\Delta}$ continues to increase till the point where $\delta_\Delta(t) \approx \infty$ and $t + dt \approx t$.

By our original definition, the area under the unit impulse is still 1.

$$\int_{-\infty}^{+\infty} \delta(t) \cdot dt = 1$$

This is strange because the integral of a function whose base ≈ 0 should be 0 but in our case is 1. That's why the unit impulse we defined above is not a function but a **distribution** and should be graphically denoted with an arrow instead of a line as follows:

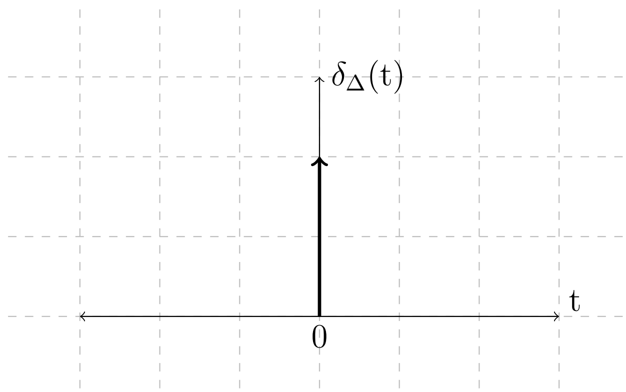

Figure 10: Unit Impulse Distribution representation in continous-time domain.

Similarly, a unit impulse in the continous time domain off-centered at $t = t_0$ looks like:

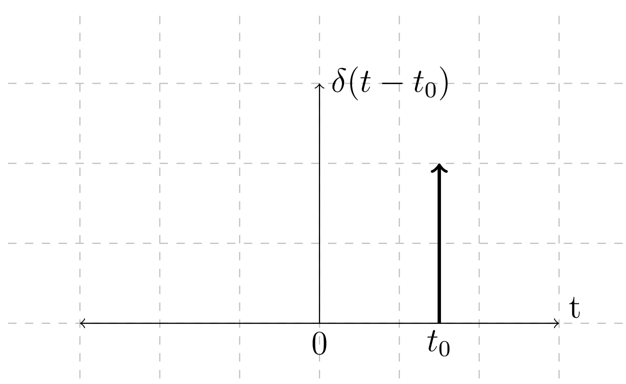

Figure 11: Unit Impulse Distribution representation in continous-time domain off-centered at $t = t_0$.

Now, if we try and obtain the product of the unit impulse, $\delta(t)$, with a generic signal, $x(t)$, in continous time, we can first calculate the product of the generic signal with our modified unit impulse, $\delta_\Delta(t)$, in continous time:

$$x(t) \cdot \delta_\Delta(t) \approx x(0) \cdot \delta_\Delta(t)$$

This is because the modified unit impulse is only non-zero at the region of Δ *which is* ≈ 0. To convert the modified unit impulse to a standard unit impulse, we can take the limit $\Delta \to 0$:

$$\lim_{\Delta \to 0} x(t) \cdot \delta_\Delta(t) = x(0) \cdot \lim_{\Delta \to 0} \delta_\Delta(t) = x(0) \cdot \delta(t)$$

Here, the amplitude of the $\delta(t)$ is infinite, but the product (or area) is still $= 1$. Mathematically,

$$\begin{aligned} \int_{t=-\infty}^{+\infty} x(0) \cdot \delta(t) \cdot dt &= x(0) \cdot \int_{t=-\infty}^{+\infty} \delta(t) \cdot dt \\ &= x(0) \end{aligned}$$

Note: Even though the amplitude of $\delta(t)$ *is infinite, when multiplied with a generic signal and integrated gives the area = 1 as per our definition.*

Interestingly, we have come to the same conclusion for the product of unit impulse with a generic signal in the continous time domain as we did in the discrete time domain.

$$x[n] \cdot \delta[n] = x[0] \cdot \delta[n], \quad \text{(Discrete Time)}$$

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t), \quad \text{(Continuos Time)}$$

We can continue the generalization for the product of a generic signal with a unit impulse off-centered at $t = t_0$ in the continous time domain:

$$x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$$