

Signals

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Abstract

This notebook compiles content from Signals lectures delivered by Prof. Dario Farina as part of the BIOE50005 Module (2nd Year Imperial MBE). A total of 9 lectures are covered in these notes. Please suggest any relevant changes/improvements at ha1822@ic.ac.uk.

Introduction and Types of Signals

Mathematically, signals are functions of one or more independent variables. Eg: The instantaneous velocity of a moving body at every given time interval can be represented as a signal: $v(t)$. Based on the type of input, there are two types of signals:

1. **Continuous-time:** Signals where the domain of the input variable is continuous. An analog signal is a continuous-time signal. The notation used for continuous-time signals is:

$$x(t), \quad t \in \mathbb{R}$$

2. **Discrete-time:** Signals where the domain of the input signal is not a continuum and belongs to a discrete set. Discrete-time signals can be generated by sampling points from a continuous signal. The commonly used notation for discrete-time signals is:

$$x[n], \quad n \in \mathbb{Z}$$

Discrete-time signals can also be interpreted as follows:

- Sequence of numbers: A signal $x[n]$ can be interpreted as a sequence of numbers $x[n]$ where n is the index of each number.

$$x = \{x[n]\}, \quad -\infty < n < \infty$$

- Vector: A signal $s[n]$ with defined over an input set of length x can be interpreted as a vector of x dimensions.

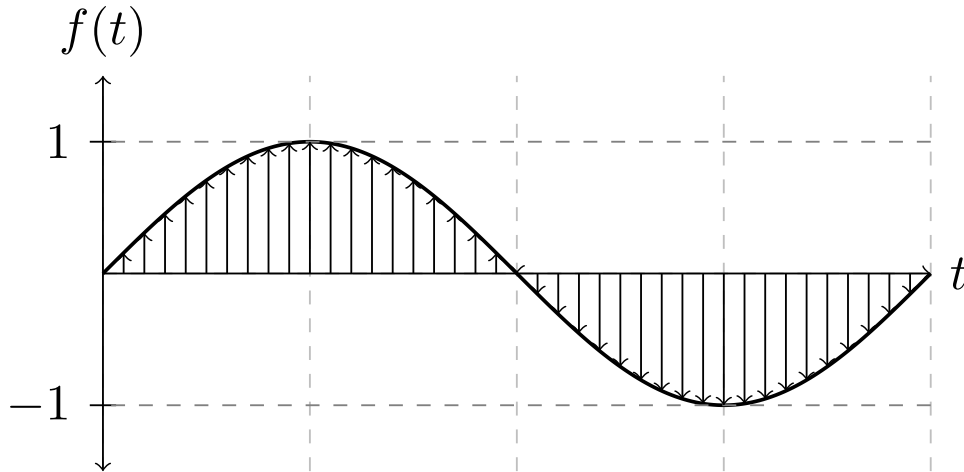


Figure 1: Sampling of a continuous signal $f(t) = \sin(t)$

Digital Signals & Quantization

Digital Signal: A discrete time signal that is (often) sampled from a continuous analog signal such as figure 2. Digital signals are also often quantized.

Quantization: When sampling a continuous signal, the amplitude of the signal is rounded to the nearest quantized level.

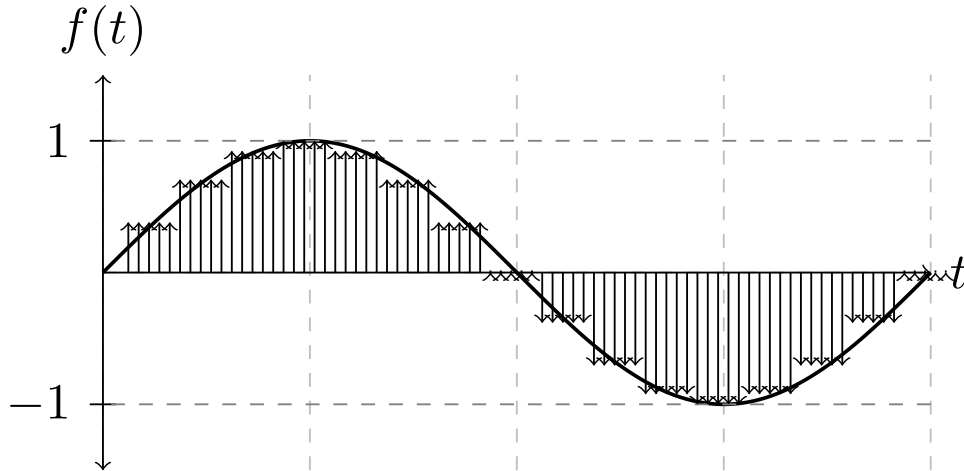


Figure 2: Quantization of sin wave. This figure isn't a mathematically rigorous depiction as the quantization levels might not be equally spaced.

*Quantization always implies **losing information** as the exact amplitude of the wave is lost while sampling at a given time interval.*

Other Types of Signals

Depending on the property of prediction of signal beforehand, there are two categories of signals:

1. **Deterministic:** These signals can be predicted exactly, and before they've been observed. This implies that a known mathematical formulation for the signal is already present. Eg: If we know a signal follows the function $x(t) = \sin(2\pi t)$, we already know all information about the signal.
2. **Stochastic:** These signals can't be predicted before they are observed. These signals convey new information when observed. Eg: EEG measurement, etc. If we already knew the analytical formulation of these signals then we wouldn't be required to observe these signals.

Periodic Signals: A signal is termed periodic if it follows this property:

$$x(t) = x(t + T) \quad \forall t \quad (1)$$

The signal should follow this for all values of t (\forall represents 'for all'). Here, T = Period of the signal. Similarly, for a discrete signal: $x[n] = x[n + N]$.

Signal Energy & Power

For a continuous-time signal, $x(t)$ for $t_1 \leq t \leq t_2$, The Energy (E) of the signal and Power (P) are defined as:

$$E = \int_{t_1}^{t_2} |x(t)|^2 \cdot dt \quad (2)$$

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \cdot dt \quad (3)$$

Similarly, for a discrete-time signal $x[n]$ for $n_1 \leq n \leq n_2$, The Energy (E) of the signal and Power (P) are defined as:

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2 \quad (4)$$

$$P = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 \quad (5)$$

These definitions hold when $-\infty < t < \infty$ or $-\infty < n < \infty$

Complex Signals

Before understanding complex signals, it's helpful to recap complex numbers.

The complex plane is quite useful when dealing with two dimensions: the real dimension (x-axis) and the imaginary dimension (y-axis). Complex numbers can be represented in multiple ways:

- **Vector Form:** $\vec{z} = x + j \cdot y$ where $j = \sqrt{-1}$. The magnitude of this vector is given by:

$$|\vec{z}|^2 = \sqrt{x^2 + y^2}$$

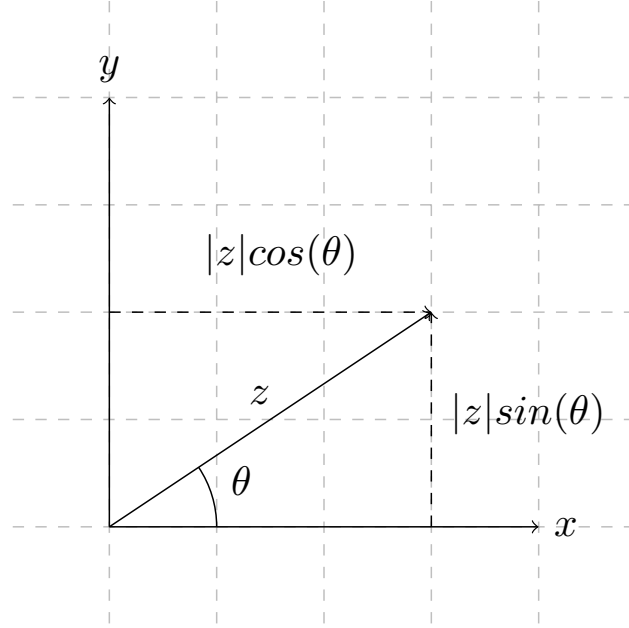


Figure 3: Vector representation of the generic complex number $\vec{z} = x + j \cdot y$.

θ = Phase of the complex number

$$x = |z| \cos(\theta)$$

$$y = |z| \sin(\theta)$$

By putting the values of x and y into the vector form of complex numbers, we can obtain:

$$\vec{z} = x + j \cdot y = |z|(\cos(\theta) + j \cdot \sin(\theta))$$

- **Exponential Form:** We can use Euler's relation, $e^{j \cdot \theta} = \cos(\theta) + j \cdot \sin(\theta)$, to obtain the following equality from the above equation:

$$\vec{z} = x + j \cdot y = |z|(\cos(\theta) + j \cdot \sin(\theta)) = |z| \cdot e^{j \cdot \theta} \quad (6)$$

Examples

Proof of a Periodic Signal

We can use the condition for periodicity we defined above (eqn.1) to prove whether a given signal is periodic.

Let's use the complex exponential as an example:

$$x(t) = e^{j\omega_0 \cdot t}$$

This equation is periodic only if $x(t) = x(t + T)$, thus, the following needs to be true:

$$\begin{aligned} e^{j\omega_0 \cdot t} &= e^{j\omega_0 \cdot (t+T)} \\ &= e^{(j\omega_0 \cdot t) + (j\omega_0 \cdot T)} \\ &= e^{(j\omega_0 \cdot t)} \cdot e^{(j\omega_0 \cdot T)} \end{aligned}$$

We know from Euler, that:

$$e^{(j\omega_0 \cdot T)} = \cos(\omega_0 \cdot T) + j \cdot \sin(\omega_0 \cdot T)$$

To ensure LHS=RHS, and prove that the above signal is periodic, we need to find a value of $\omega_0 \cdot t$ such that $e^{j\omega_0 \cdot T} = 1$. Cleverly, if we set $\omega_0 \cdot T = \pm 2\pi$, then $\cos(\omega_0 \cdot T) = 1$, $\sin(\omega_0 \cdot T) = 0$, and ultimately, $e^{j\omega_0 \cdot T} = 1$.

To set a generic condition for periodicity:

$$T = \frac{2\pi}{|\omega_0|}$$

The reason we only take the absolute value of ω_0 is because T is always positive and only ω_0 can take negative values.

A similar is true for discrete-time signals:

$$N = \frac{2\pi}{|\omega_0|}$$

However, here N can only be an integer, thus ω_0 also needs to be a fraction of 2π or:

$$\omega_0 = \frac{2\pi}{k}, \quad k \in Integer$$

This also tells us that in the discrete-time domain, frequencies can't be arbitrarily increased from $0 \rightarrow +\infty$ to increase the rotation speed. They can only be sampled from a given domain. If we go any faster, we won't be able to distinguish the additional speed. Eg: when the wheel of a car goes too fast, we can't identify its speed.

Example Calculation of Energy & Power

Let's calculate the energy for a complex signal $x(t) = e^{j\omega_0 t}$ and ω_0 = angular frequency for the range $-\infty < t < \infty$. We can use the mathematical definition of energy for a continuous-time signal from eq.2 as follows:

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 \cdot dt$$

Using Euler's relation, we can separately evaluate $|x(t)|^2$:

$$|x(t)|^2 = |e^{j\omega_0 t}|^2$$

Complex numbers have the following property:

$$|z|^2 = |z| \cdot |\bar{z}|; \quad \bar{z}: \text{complex conjugate } (a - j \cdot b)$$

Expanding $e^{j\omega_0 t}$ using Euler's relation and the above property:

$$\begin{aligned} e^{j\omega_0 t} &= (\cos(\omega_0 \cdot t) + j \cdot \sin(\omega_0 \cdot t)) \cdot (\cos(\omega_0 \cdot t) - j \cdot \sin(\omega_0 \cdot t)) \\ &= \cos^2(\omega_0 \cdot t) - j^2 \cdot \sin^2(\omega_0 \cdot t) \\ &= \cos^2(\omega_0 \cdot t) + \sin^2(\omega_0 \cdot t) \\ &= 1 \end{aligned}$$

Plugging this into the integral above, we obtain:

$$E = \int_{-\infty}^{+\infty} 1 \cdot dt = [t]_{-\infty}^{+\infty} = +\infty$$

We can say that complex exponentials are **signals with infinity energy**. We can also calculate the power of the complex exponential:

$$\begin{aligned} \text{Power} &= \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 \cdot dt \\ &= \frac{1}{2T} \cdot [t]_{-T}^{+T} \\ &= \frac{1}{2T} \cdot 2T \\ &= 1 \end{aligned}$$

From the above, we can say that complex exponentials are **signals with finite energy and equals to 1**.

Unit Impulse & Unit Step

Unit Impulse

In the discrete time domain, the unit impulse is the simplest signal, defined as follows:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Similarly, the discrete-time impulse delayed by the integer k is:

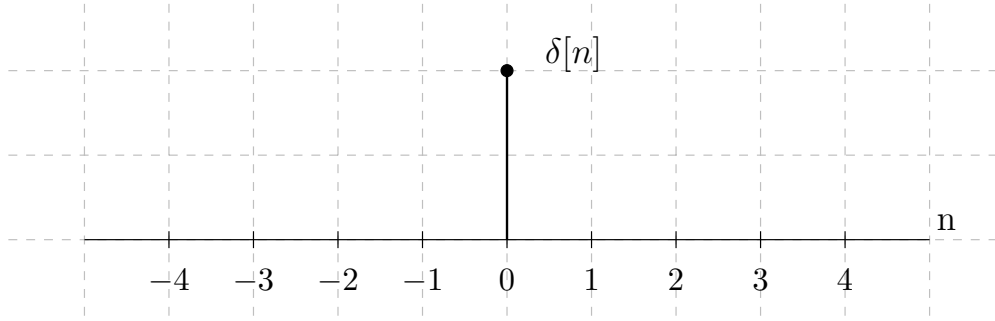


Figure 4: A unit impulse, $\delta[n]$ centered at $n=0$

$$\delta[n - k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$

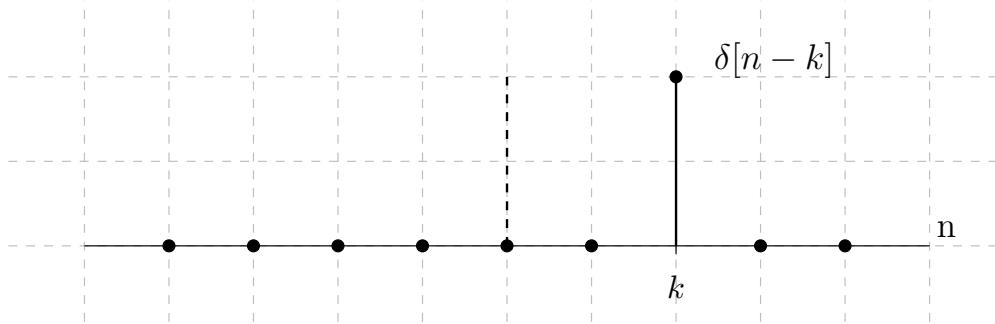


Figure 5: A unit impulse, $\delta[n - k]$ off-centered at k

If we multiply a generic discrete-time signal, $x[n]$, with a unit pulse centered at k , $\delta[n - k]$, we obtain:

$$x[n] \cdot \delta[n - k] = x[k] \cdot \delta[n - k]$$

The reason this is true is because at $n \neq k$, the value of $\delta[n - k] = 0$. Thus, the output is simply the signal at the time step $n = k$ multiplied by the unit impulse at $n = k$. By definition, the unit impulse = 1 at $n = k$, so the equation now becomes:

$$x[n] \cdot \delta[n - k] = x[k] \cdot \delta[n - k] = x[k]$$

If we now compute:

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} x[n] \cdot \delta[n - k] &= \sum_{n=-\infty}^{+\infty} x[k] \cdot \delta[n - k] \\
&= x[k] \cdot \sum_{n=-\infty}^{+\infty} \delta[n - k] \\
&= x[k] \quad (\because \sum_{n=-\infty}^{+\infty} \delta[n - k] = 1) \\
\therefore x[k] &= \sum_{n=-\infty}^{+\infty} x[n] \cdot \delta[n - k]
\end{aligned}$$

If we arbitrarily change the variables and assume $n \rightleftharpoons k$, we obtain:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot \delta[k - n]$$

But since the unit pulse = 1 at $n=k$ (or $k=n$), $\delta[k - n] = \delta[n - k]$. In both situations, when $k=n$, $\delta[n - k] = \delta[n - k]$ (equal to 1 when $n = k$, equal to 0 when $n \neq k$). So:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot \delta[n - k] \quad (7)$$

What this equation intuitively tells us is that you can construct a discrete-time signal by taking the value of that discrete-time signal at different timesteps scaled by a unit impulse off-centered at that time step.

Another paraphrase: *Any arbitrary discrete-time signal can be expressed as the sum of scaled and delayed impulses.*

Unit Step

A unit step in the discrete-time domain is defined as:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = \sum_{k=0}^{+\infty} \delta[n - k]$$

We can also note the following property:

$$\delta[n] = u[n] - u[n - 1]$$

This means that the unit impulse is the discrete-time derivative of the unit step. Here, since 1 unit is the smallest increment in x , $dx = 1$, and $dy = u[n] - u[n] - 1$.

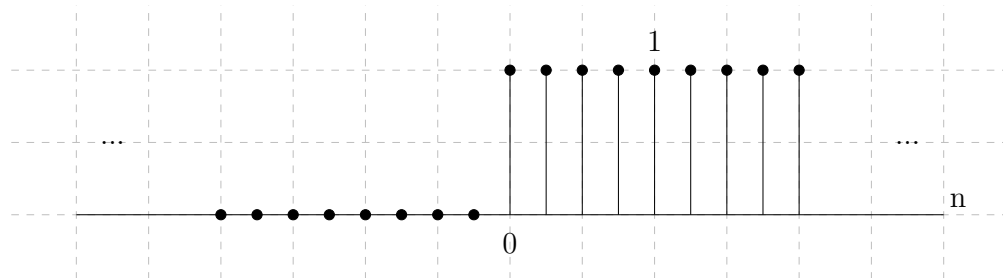


Figure 6: A unit step, $u[n]$