Multi-Objective Optimization Problems with Well-Conditioned Solutions

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Certificate

*This is to certify that the work contained in this thesis entitled "**Multi-Objective Optimization Problems with Well-Conditioned Solutions**" is a bonafide work of **Harshal Bharat Dupare, Roll no. 18MA20015**, carried out in the Department of Mathematics, Indian Institute of Technology Kharagpur under my supervision and that it has not been submitted elsewhere for a degree.*

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Abstract

This thesis studies the methodologies to solve Functional Multi-Objective Optimizations Problems (fMOOP) in which we are looking for functions as solutions of Multi-Objective Optimization Problems, where one of the objective/constraint is to minimize/bound the condition number of such a solution function found. First we will see why such problems are important. Then we will survey the different methodologies for solving MOOP in general and then summarize few results on condition number of matrices and of function. Then We will analyze different methods in regards to their potential to be used for finding well condition solutions of fMOOP. Based on the analysis we will Select few methods and compare them with experimental results.

Will be completed when other contents are added

- what is my research problem and objectives/goal [Done]
- what methods did/can I used [Done]
- what were the key results and arguments
- conclusion

Contents

Chapter 1

Introduction

Optimization problems are of great interest academically along with being core to the functioning of today's society by having application in variety of fields such as social studies, economics, finance, agriculture, automotive, engineering, computer science, networking and many others.

In general there can be more than one objective and constraint in the formulation of optimization problems. Optimization problems are classified into 2 broad categories namely *Single Objective Optimization Problems*(SOOP) and *Multi-Objective Optimization Problems* (MOOP), the latter can be considered a super-set of the former. Our focus will be on a special subset of MOOP where the *Unknown/Decision Variable* is a *function* lets call them fMOOP. fMOOP are of great interest because of their vast applications in modeling the relation between 2 sets, for e.g. all of the Neural Network so heavily researched today fall into this category with there myriads of applications.

Generally when we have a function, we want it to have some good properties like being correct, easy to compute, easy to store, behave nicely over the input set, or any other desirable properties based on the context. One such very important property for most of such functions of interest is being *Well-Conditioned*, meaning small changes in the input should result in too big changes in the output. We focus on this problem namely the problem to find function solutions of fMOOP which are well-conditioned.

Goal of this study is to analyze the implications of restricting the condition number for solutions of fMOOP, and study how the existing methodologies perform on such problems and device new methodologies for finding well-conditioned solutions to fMOOP.

1.1 Preliminaries

1.1.1 Definitions

Aggregation Scheme: Given a value function $value(\cdot)$ over x, for $x \in X$, a method to aggregate those values to one value. Denoted as follows:

$$\underset{\forall x \in X}{Agg}\ value(x) \tag{1.1}$$

e.g. Expectation over given probability distribution $p(\cdot)$ over X.

$$\underset{\forall x \in X}{Agg}\ value(x) = \int_{\forall x \in X} value(x)p(x)\ dx \tag{1.2}$$

e.g. Exponential decaying average, given a index-function $t : X \to [|X|]$, where $[n] = \{0, 1, 2, ..., n-1\}$.

$$\underset{\forall x \in X}{Agg}\ value(x) = \sum_{\forall x \in X} \lambda^{t(x)} value(x) \tag{1.3}$$

Absolute Condition Number: *Absolute Condition Number* is defined for a function $f : X \to Y$, where X and Y have a norm $||.||$ is defined over them. Denoted as follows:

$$cond_{abs}(f) = \lim_{\epsilon \to 0} \sup_{||\delta x|| \leq \epsilon} \frac{||\delta f(x)||}{||\delta x||} \tag{1.4}$$

Relative Condition Number: *Relative Condition Number* is defined for a function $f : X \to Y$, where X and Y have a norm $||.||$ is defined over them. Denoted as follows:

$$cond_x(f) = \lim_{\epsilon \to 0} \sup_{||\delta x|| \leq \epsilon} \frac{||\delta f(x)||/||f(x)||}{||\delta x||/||x||} \tag{1.5}$$

Matrix Transform Norm: Given a Matrix Transform $T^{m \times k} : \mathbb{R}^{n \times m} \to \mathbb{R}^{n \times k}$ and a norm $||\cdot||$ defined over $\mathbb{R}^{n \times m}$ and $\mathbb{R}^{n \times k}$ we define norm of the matrix transform as

$$||T^{m \times k}|| = \sup_{X \in \mathbb{R}^{n \times m} / \{\bar{0}\}} \frac{||X \times T^{m \times k}||}{||X||} \tag{1.6}$$

Note: Whenever the dimensions of the matrix in the context are already defined or are obvious from the context we will omit the superscript denoting its dimension.

1.1.2 Problem

Given:

Domain $X_1 \cup X_2 \subseteq X$ and their associated Ranges $Y_1 \cup Y_2 \subseteq Y$, where $X_i \equiv Y_i$ such that for each $x \in X_i$ we have a corresponding $y \in Y_i$ i.e. $y \equiv x$.

Aggregation schemes $\underset{\forall x \in X_1}{Agg_1}$ defined over $\mathcal{L} : Y \times Y \to \mathbb{R}$ call $\mathcal{L}(\cdot, \cdot)$ the *Loss* function and $\underset{\forall x \in X_2}{Agg_2}$ defined over $\mathcal{R} : Y \times Y \to \mathbb{R}$ call $\mathcal{R}(\cdot, \cdot)$ the *Reward* function.

A scalar $\alpha \in \mathbb{R}^+$ and subset $X_3 \subseteq X$.

For a given $\mathcal{F}$ a family of functions $f : X \to Y$ we need to find a function $f \in \mathcal{F}$ which solves the following

Functional Multi-Objective Optimization Problem:

$$\min_{f \in \mathcal{F}} \underset{\forall x \in X_1, y \in Y_1, y \equiv x}{Agg_1} \mathcal{L}(y, f(x)) \tag{1.7}$$

$$\max_{f \in \mathcal{F}} \underset{\forall x \in X_2, y \in Y_2, y \equiv x}{Agg_2} \mathcal{R}(y, f(x)) \tag{1.8}$$

subject to one of the following

$$cond_{abs}(f) \leq \alpha \tag{1.9}$$

$$\max_{x \in X_3} cond_x(f) \leq \alpha \tag{1.10}$$

$$\min_{f \in \mathcal{F}} cond_{abs}(f) \tag{1.11}$$

$$\min_{f \in \mathcal{F}} \max_{x \in X_3} cond_x(f) \tag{1.12}$$

It is very important to mention that the above problem formulation highly depends on the nature of the mapping f and the family $\mathcal{F}$ that it belongs to. For example if f models a function for which we expect smooth change in its value w.r.t. its domain then its a reasonable formulation, and if there are no reasons to believe that it should be smooth over its domain then its not a reasonable formulation. Similarly for a chosen family $\mathcal{F}$ to model f, if $\mathcal{F}$ doesn't have smooth functions belonging to it then its not a reasonable formulations irrespective of the nature of the function that f is trying to model.

Note: Here we restricted the problem to 2 objectives and only 1 condition which majorly characterizes this problem, but in general we can have arbitrary number of objectives and other conditions along with at least 1 of the 1.9, 1.10 conditions.

What we mean by minimizing over a functions $f \in \mathcal{F}$ of a family of functions $\mathcal{F}$ is that we minimize over the parameters of that family of function $\mathcal{F}$.

1.2 Motivation and Applications

1.2.1 Linear Factor Model

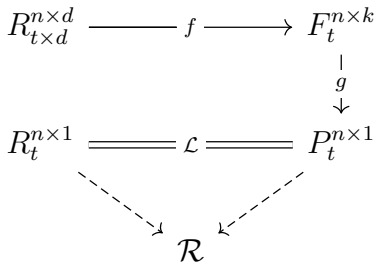
The problems defined in the former section was motivated from the problem on linear factor models with added constraint for the returns-to-factor matrix be orthonormal columns. We describe the problem below.

Given:

Returns data for n time series $R_t^{n\times 1} \in \mathbb{R}^{n\times 1}$, for $t \in [T]$

and denote $R_{t\times d}^{n\times 1} = [R_{t-1}^{n\times 1}, R_{t-2}^{n\times 1}, ..., R_{t-d}^{n\times 1}] \in \mathbb{R}^{n\times d}$

We need to design $F_t^{n\times k} \in \mathbb{R}^{n\times k}$ as a function of $R_{t\times d}^{n\times d}$, i.e. $F_t^{n\times k} = f(R_{t\times d}^{n\times d})$ and from that we can derive $P_t^{n\times 1} = g(F_t^{n\times k})$ such that $P_t^{n\times 1} \approx R_t^{n\times 1}$ which we can measure by *Average* aggregation scheme over the loss function $\mathcal{L}(y, \hat{y}) = ||y - \hat{y}||_2$ which we need to minimize and another *Average* aggregation scheme over the return function $\mathcal{R}(y, \hat{y}) = \frac{\langle y, \hat{y}\rangle}{||y||_2||\hat{y}||_2}$ defined over the testing period of $t \in [T, T+S] = \{T, T+1, ..., T+S-1\}$ which we need to maximize.



(1.13)

What makes this model linear is that the function $f(\cdot)$ and $g(\cdot)$ are linear w.r.t. their argument. In this case it can be written as

$$F_t^{n\times k} = f(R_{t\times d}^{n\times d}) = R_{t\times d}^{n\times d} \times A^{d\times k} \tag{1.14}$$

$$P_t^{n\times 1} = g(F_t^{n\times k}) = F_t^{n\times k} \times \beta^{k\times 1} \tag{1.15}$$

And the orthonormal condition requires

$$A^{d\times k}(A^{d\times k})^\top = I^{d\times d} \tag{1.16}$$

From the 1.16 we can infer that $(A^{d\times k})^T$ belongs to the set of right inverses of $A^{d\times k}$.

Further if we relax the 1.16 to bounds on condition number by $\alpha \geq 1$

$$\max_{R_{t\times d}^{n\times d}\in\mathbb{R}^{n\times d}} cond_{R_{t\times d}^{n\times d}}(f) = \lim_{\epsilon\to 0} \sup_{||\delta R_{t\times d}^{n\times d}||\leq\epsilon} \frac{||\delta R_{t\times d}^{n\times d} \times A^{d\times k}||}{||\delta R_{t\times d}^{n\times d}||} \frac{||F_t^{n\times k} \times (A^{d\times k})^\top||}{||F_t^{n\times k}||} \leq \alpha \quad (1.17)$$

which implies

$$\max_{R_{t\times d}^{n\times d}\in\mathbb{R}^{n\times d}} cond_{R_{t\times d}^{n\times d}}(f) \leq ||A^{d\times k}||||(A^{d\times k})^\top|| \leq \alpha \quad (1.18)$$

Note that $\alpha = 1$ contains the set which satisfies 1.16 equation since

$$1 = ||I^{d\times d}|| = ||A^{d\times k}(A^{d\times k})^\top|| \leq ||A^{d\times k}||||(A^{d\times k})^\top|| \leq \alpha \quad (1.19)$$

So any $\alpha \geq 1$ will contain the set specififed by the condition 1.16
Combining all of that together gives us the **fMOOP** formulation of this problem as

$$\min_{\beta^{k\times 1}\in\mathbb{R}^{k\times 1}, A^{d\times k}\in\mathbb{R}^{d\times k}} \frac{1}{T-d+1} \sum_{\forall t\in[d,T]} ||R_t^{n\times 1} - R_{t\times d}^{n\times d} \times A^{d\times k} \times \beta^{k\times 1}||_2 \quad (1.20)$$

$$\max_{\beta^{k\times 1}\in\mathbb{R}^{k\times 1}, A^{d\times k}\in\mathbb{R}^{d\times k}} \frac{1}{S} \sum_{\forall t\in[T,T+S]} \frac{\langle R_t^{n\times 1}, R_{t\times d}^{n\times d} \times A^{d\times k} \times \beta^{k\times 1}\rangle}{||R_t^{n\times 1}||_2||R_{t\times d}^{n\times d} \times A^{d\times k} \times \beta^{k\times 1}||_2} \quad (1.21)$$

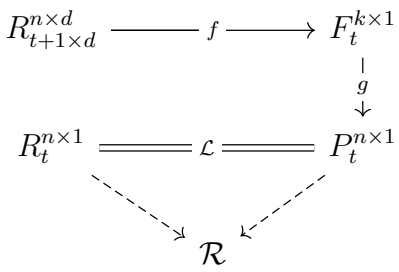
subject to

$$\max_{R_{t\times d}^{n\times d}\in\mathbb{R}^{n\times d}} cond_{R_{t\times d}^{n\times d}}(f) \leq \alpha \quad (1.22)$$

1.2.2 Principle Time Series

Another task related to time series is to represent a set of n time series by less number of time series $k < n$. Which is some way is an application of the Linear Factor Models.
Given:
The definition of $R_t^{n\times 1}$ and $R_{t\times d}^{n\times 1}$ are same as before, but here we need to design latent time series denote them by $F_t^{k\times 1} \in \mathbb{R}^{k\times 1}$ as a function of $R_{t+1\times d}^{n\times d}$, i.e. $F_t^{k\times 1} = f(R_{t+1\times d}^{n\times d})$ which reduces the n time series observed from time $t-d$ to time t to set of k time series at time t such that its possible to sufficiently recover the n time series at time t by another

function $g(F_t^{k\times 1}) = P_t^{n\times 1}$ such that $P_t^{n\times 1} \approx R_t^{n\times 1}$ which we can measure by *Average* aggregation scheme over the loss function $\mathcal{L}(y, \hat{y}) = ||y - \hat{y}||_2$, which we need to minimize.



(1.23)

Further we can restrict the functions $f(\cdot)$ and $g(\cdot)$ to be linear w.r.t. their argument. In that case it can be written as

$$F_t^{k\times 1} = f(R_{t+1\times d}^{n\times d}) = A^{k\times n} \times R_{t+1\times d}^{n\times d} \times B^{d\times 1} \tag{1.24}$$

$$P_t^{n\times 1} = g(F_t^{k\times 1}) = C^{n\times k} \times F_t^{k\times 1} \tag{1.25}$$

For small changes in our time series data $\delta R_{t+1\times d}^{n\times d}$ we expect that there is small changes in the latent time series $\delta F_t^{k\times 1}$, which we can model by bounding the absolute condition number by a scalar α.

$$cond_{abs}(f) = \lim_{\epsilon\to 0} \sup_{||\delta R_{t+1\times d}^{n\times d}||\leq\epsilon} \frac{||A^{k\times n} \times \delta R_{t+1\times d}^{n\times d} \times B^{d\times 1}||}{||\delta R_{t+1\times d}^{n\times d}||} \tag{1.26}$$

Combining all of that together gives us the **fMOOP** formulation of this problem as

$$\min_{A\in\mathbb{R}^{k\times n}, B\in\mathbb{R}^{d\times 1}, C\in\mathbb{R}^{n\times k}} \frac{1}{T-d+1} \sum_{\forall t\in[d,T]} ||R_t^{n\times 1} - C^{n\times k} \times A^{k\times n} \times R_{t+1\times d}^{n\times d} \times B^{d\times 1}||_2 \tag{1.27}$$

subject to

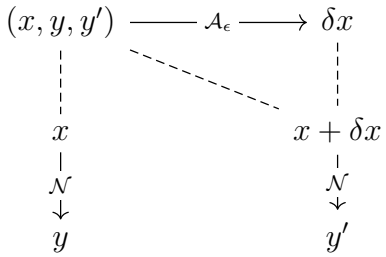
$$cond_{abs}(f) \leq \alpha \tag{1.28}$$

1.2.3 Defense Against Adversarial Attacks on Neural Networks

Say we are given a trained neural network $\mathcal{N} : X \to Y$ which has learnt a mapping from X, the input set to Y, the output set. If the underlining mapping that $\mathcal{N}$ was modeled to learn was smooth w.r.t. its input then we expect that for a well trained network $\mathcal{N}$ for any input $x \in X$ and for small enough $\delta x \in X$ δx the output of the network wont change much, lets say that it won't change more than a constant $\alpha > 0$ times the norm of x i.e. $||\mathcal{N}(x + \delta x) - \mathcal{N}(x)|| = ||\delta\mathcal{N}(x)|| \leq \alpha||\delta x||$. Which we can reformulate as follows

$$cond_{abs}(\mathcal{N}) = \lim_{\epsilon \to 0} \sup_{||\delta x|| \leq \epsilon} \frac{||\delta\mathcal{N}(x)||}{||\delta x||} \leq \alpha \tag{1.29}$$

Most of the networks which are being trained today don't account for such constraints giving the way to Adversarial attacks on them, which exploit this drawbacks of the network to force them to output unreasonably wrong value.



(1.30)

Here $\mathcal{A}_\epsilon$ the adversary which takes the input (x, y, y') being the input data x, the associate label y, and the expected forced wrong output y' and outputs the required perturbation δx, such that $||\delta x|| \leq \epsilon$. When δx is added to original input data the new malicious input $x_m = x + \delta x$ forces the network $\mathcal{N}$ to output y' instead of y whereas it would have given output of y on the input x. Note that the adversary prefers smaller values of ϵ, since that implies that it can generate malicious input with as little changes as possible.

This problem can be avoided if we can train the network to also minimize the absolute condition number, $cond_{abs}(\mathcal{N})$ or minimize the maximum relative condition number over a subset $X_I \subseteq X$, $\max_{x \in X_I} cond_x(\mathcal{N})$. We can also provide the constrains on their maximum values and model accordingly.

So if the network has been trained such that $||\delta\mathcal{N}(x)|| \leq \alpha||\delta x||$ holds, then for the adversary $\mathcal{A}_\epsilon$ to make the network's $\mathcal{N}$ output perturb by $||\delta\mathcal{N}(x)||$ it will have to change the input x value perturbation of norm more than $\frac{||\delta\mathcal{N}(x)||}{\alpha} \leq ||\delta x||$, which will not be possible for an adversary with $\epsilon < \frac{||\delta\mathcal{N}(x)||}{\alpha}$. Hence training with such constraints will force the adversary to make large changes to the input for designing an malicious input, which is unfavourable for the adversary.

Chapter 2

Related Works

Before we look for the related works to this problems it is important to understand the different dimensions of this problem and structure the related works accordingly. The problems stated in the section 1.1.2 has 1 major characteristic apart from being fMOOP is that it requires the solution to have bounded condition number (absolute or relative), and on top of that since such formulation has many applications it is desirable to have efficient algorithms to compute such solutions. So the following dimensions of the literate that interests us and can have impact of the problem are:

1. Methods to solve MOOP
2. Literature related to condition number
3. Efficiency in the computational aspects of both 1, and 2

As we will see that lot of research has been done in solving MOOP and we have fairly efficient methods to get the solutions. But when it comes to the implications of condition number most of the research is focused on the condition number of matrices compared to that of condition number of general functions.

2.1 Multi-Objective Optimization

2.1.1 Definitions

In MOOP since there are more than one objectives its unlikely that all of them achieve their optima at the same point, hence typically there is no single global solution. Which makes its necessary to rethink about the definition for an optimum and accordingly determine a set of points that can be deemed as the solutions. The most widely used concept concept in defining an optimal point is that of the *Pareto optimality* [6], which is defined as follows:
Below we will use $F : X \to \mathbb{R}^k$ as the objective function that we need to minimize over the input $x \in X$ and we will use minimization over all the objectives since maximization can be converted to minimization by negating the sign.

Pareto Optimal: A point, $x^* \in X$, is Pareto optimal iff there does not exist another point, $x \in X$, such that $F(x) \leq F(x)$, and $F_i(x) < F_i(x)$ for at least one function.

All Pareto optimal points lie on the boundary of the feasible criterion space [14]. Sometimes the pareto optamility is too strong of a requirement hence we define *weakly Pareto optimal* as follows:

Weakly Pareto Optimal: A point, $x^* \in X$, is weakly Pareto optimal iff there does not exist another point, $x \in X$, such that $F(x) < F(x)$.

Pareto optimal points are weakly Pareto optimal, but weakly Pareto optimal points are not Pareto optimal.
Alternatively we define the compromise solution, which minimizes the difference between the utopia point/ideal point, defined as follows [10]:

Utopia Point : A point $F^* \in \mathbb{R}^k$ is objective space, is a utopia point iff for each $i = 1, 2..., k$, $F_i^* = \underset{x \in X}{min}\{F_i(x) | x \in X\}$.

In general, F^* is infeasible due to other constrains. Then the next best solution that we can attain is a solution that is as close as possible to the utopia point. We call such solution a **compromise solution** and it is Pareto optimal. The meaning of close in this context needs to be clarified and generally we need to define norms to measure closeness of the solutions for example use of Euclidean norm [11]. Another problem with this ap-

proach is of different objective have different scales so generally we need to transform the objectives to a single scale for any meaningful use of the defined norm.

The methods to solve MOOP are classified into 4 classes [28] based on the availability and involvement of a external decision maker(DM) used to convey the preference over different pareto optimal solution.

1. **No preference methods**: No DM is available and a neutral compromise solution is identified without any specification of the preference information.

2. **A priori methods**: based on the preference information given by the DM the optimal solution is found.

3. **A posteriori methods**: a good representative set of Pareto optimal solutions is found, and from among them the DM must choose the best solution.

4. **Interactive methods**: Pareto optimal solution(s) are shown to the DM, then the DM describes how the solution(s) could be improved. Then the next set of Pareto optimal solution(s) is generated based on the DM's feedback and iteratively the solutions are improved.

2.1.2 No Preference Methods

Since no preference information is provided the general one the approaches followed in [8] uses the definition of **utopia point** in 2.1.1 and then by properly scaling the objective function F to $\hat{F}$ and using a $||\cdot||$ norm defined over the objective space the following optimizations problem is formulated to minimize the following objective.

$$\min_{x \in X} ||\hat{F}(x) - \hat{F}^*|| \tag{2.1}$$

Other similar methods are described in [30].

2.1.3 A Priori Methods

Methods which come under this class can further divided into other smaller classes but all of them have a common feature that is,enough information is provided a priori to compare any candidate pareto optimal solutions.
Following are some of the Important methods under this class:

Utility Function Methods: Here we have a utility function $U : \mathbb{R}^k \to \mathbb{R}$, and the goal is to solve the following SOOP

$$\min_{x \in X} U(F(x)) \tag{2.2}$$

Notable methods which come under this utility model are

$$U(F(x)) = \sum_{\forall i \in [k]} w_i F_i(x) \tag{2.3}$$

Known as the Linear scalarization method, if all $\forall i \in [k], w_i > 0$ is a sufficient condition for the solution of 2.1.4 to be a pareto optima [3], but it is not a necessary condition [13].

$$U(F(x)) = \Big(\sum_{\forall i \in [k]} w_i (F_i(x) - F_i^*)^p \Big)^{1/p} \tag{2.4}$$

2.4 for $p > 0$,generally $p = 1, 2$ is another common extension of the formulation [36], for pareto optamility the conditions on w_i are same as they are for 2.1.4, along with that if any of the w_i is set to 0 then it can result in weak pareto optimality.

$$U(F(x)) = \max_{i \in [k]} \frac{F_i(x)}{w_i} \tag{2.5}$$

2.5 is known as Hypervolume/Chebyshev Scalarization method [37], and in this case if $\forall i \in [k]\ w_i > 0$, it is shown that the solution of 2.5 converges to the Pareto front even for non-convex pareto fronts.

ϵ-Constraint Method: In this method[7] we have a single most important objective function $F_s(x)$. and the remaining objective functions are used to form additional constraints $F_i(x) \leq \epsilon_i, \forall i \in [k]/\{s\}$.

$$\min_{x \in X} F_s(x) \tag{2.6}$$

subject to the constraints

$$F_i(x) \leq \epsilon_i, \ \ \forall i \in [k]/\{s\} \tag{2.7}$$

It is proven that the by a systematic variation of ϵ_i one can generate a set of Pareto

optimal solutions[27]. If the solution of 2.6,2.7 exists then it is a weakly Pareto optimal solution [31], and if the solution is unique, then it is Pareto optimal [31].

Lexicographic Method: As the name suggests, the Objective functions are ordered as per decreasing order of importance namely $F_i(x)$ is more important than $F_j(x)$ iff $i < j$. Then, the following optimization problems are solved starting from $i = 1, 2, ..., k$.

$$\min_{x \in X} F_i(x) \tag{2.8}$$

subject to

$$F_j(x) \leq F_j(x_j^*), \forall j \in \{1, 2, ...i-1\} \tag{2.9}$$

In 2.9 we can also have $=$ instead of $\leq$, [33]. Here x_j^* is the solution obtained at the j'th iteration, initially for $i = j = 1$ there are no constrains and $F_i(x)$ is minimized over $x \in X$.

Goal Programming Methods: Goal Programming method was developed by [2], [29], [5]. In this method we have been given goals g_j which are expected by the *DM* for the objective $F_j(x)$ respectively. To measure the deviations from the goal the sum of the absolute deviation is minimized.

$$\min_{x \in X} \sum_{\forall i \in [k]} |g_i - F_i(x)| \tag{2.10}$$

2.1.4 A Posteriori Methods

As the name suggest that the *DM* is involved a posteriori of finding solution, these apprpaches are also known as generate-first-choose-later approaches [17]. The goal is to produce a good enough representative subset of solutions which are Pareto optimal. In general they are classified into 2 classes.

1. **Mathematical programming methods**, which generally work by producing 1 pareto optimal solution per iteration/run of the algorithm.

2. **Evolutionary algorithms**, which produce a set of Pareto optimal solutions per iteration/run of the algorithm.

Mathematical Programming Methods

Few of the well known methods in this class are:

1. Normal Boundary Intersection Method
2. Modified Normal Boundary Intersection Method
3. Normal Constraint Method

Normal Boundary Intersection Method (NBI): As discussed the section the weighted sum method does provide a pareto optimal solution but its is very difficult to find evenly spread solution by varying the weights, to address these and other computational drawbacks Das and Dennis in their paper *Normal-boundary intersection: A new method for generating the Pareto surface in nonlinear multicriteria optimization problems* [15] presented the NBI method, whichis formulated as follows:

$$\underset{x \in X, t}{min}\ t \tag{2.11}$$

$$\Phi w + t\mu = F(x) - F^* \tag{2.12}$$

Where $\Phi \in \mathbb{R}^{k \times k}$ is the pay-off matrix whose Φ_{ij} entry measures the difference in the optimal value for j'th objective considering only the i'th objective to be minimized and that of the utopia point 2.1.1, i.e. $\Phi_{ij} = F_j(\underset{x \in X}{argmin}\, F_i(x)) - F_j^*$. Also $\mu = -\Phi e$, μ is known as the quasi-normal vector and $e^\top w = \sum_{i \in [k]} w_i = 1$ which is provided by the user. NBI method doesn't provide sufficient condition for finding pareto optimal solutions hence it is possible that the solutions obtained via this method are not pareto optimal and neither does this provide a necessary condition for pareto optimal solutions since for $k > 2$ for some problems it overlooks some of the pareto optimal solutions.

Modified Normal Boundary Intersection Method (NBIm): As stated in 2.1.4 the NBI method suffers from not even being a necessary condition for pareto optimal solutions, Das [26] and R de S Motta [24] proposed modified methods to cover these drawbacks

Normal Constraint Method (NC): Messac et al. [18] [19] proposed the NC method an alternative to NBI method, which provided some improvements. It uses the utopia point 2.1.1 to normalize the objective vector $F(x)$ to $\hat{F}(x)$ along with a pareto filter which removes the non-dominant solutions to keep only the dominant solutions.Also it always produces pareto optimal solutions along with it's performance being independent of design objective scales.

Evolutionary Algorithms (EA)

Evolutionary algorithms are one of the very actively researched methods [34] for solving MOOP and finding pareto optimal solutions. EA are subset of the paradigm which is inspired by nature and evolution in designing algorithms for various purposes including solving optimization problems. The general procedure that and EA follows is described below:

Generic EA

function EA($\mathcal{I}$) ▷ gets input parameters $\mathcal{I}$
  $\mathcal{P} \leftarrow initialize(\mathcal{I})$ ▷ initialize solution population $\mathcal{P}$
  while $converges(\mathcal{P}) \vee terminate(\mathcal{P})$ **do** ▷ till convergence or termination
    $\mathcal{P} \leftarrow evolve(\mathcal{P}, \mathcal{I})$ ▷ evolve the population to next generation
  return $\mathcal{P}$

Some of the notable methods in EA which are commonly used are:

1. Non-dominated Sorting Genetic Algorithm-II (NSGA-II)
2. Ant Colony Optimization (ACO)
3. Particle swarm optimization (PSO)

Non-dominated Sorting Genetic Algorithm-II (NSGA-II): Proposed by K. Deb et al. in the paper *A Fast and Elitist Multi-objective Genetic Algorithm: NSGA-II* [16], NSGA-II is based on elitist principle meaning only the elites of the populations survive to the next generation based on a partial-order sorting of the population, to decide the elites.

Ant Colony Optimization (ACO): it is based on idea of ant pheromone which ants use to communicate to form paths and explore more of the promising regions [25].

Particle Swarm Optimization (PSO): PSO is based on the flocking behaviour of the birds where each particle has a position (the solution) and a velocity (change in the solution) where the velocity is influenced by some neighbourhood of the particle [20].

The advantage the EA provides is that it can quickly provide a sets of solutions which even though are not guaranteed to be pareto optimal, but are non-dominant set and serve as good approximation to the entirety of the Pareto front. The disadvantages being that these algorithms are relatively slow and pareto optimality cant be guaranteed.

2.1.5 Interactive methods

Interactive methods require the *DM* to actively take part in pruning the candidates solutions suggested by the method, and based on the input of the *DM* the method adopts and suggest new solutions which are then again evaluated by the *DM* and the process is repeated to improve and adopt the solutions as per the needs of *DM*.

The generic structure of the Interactive methods is as follows [32]: The above method

Generic Interactive Method

$\mathcal{P} \leftarrow initialize(\mathcal{M})$ ▷ pareto optimal solution set: $\mathcal{P}$; no-preference method: $\mathcal{M}$

do

$\mathcal{R} \leftarrow getPreference(\mathcal{P}, DM)$ ▷ preference information: $\mathcal{R}$, decision maker: DM

$\mathcal{P} \leftarrow newSolutionSet(\mathcal{P}, \mathcal{R})$ ▷ update solution set based on $\mathcal{R}$

while $converges(\mathcal{P}, DM) \vee terminate(\mathcal{P}, DM)$ ▷ till convergence or termination

can be classified based on the which method is used as $\mathcal{M}$ and what type of preference information $\mathcal{R}$ is available/provided by the *DM*.

$\mathcal{M}$ can be chosen from the various options available in a posteriori methods/no preference method 2.1.4 based on the problem type.

Generally the choices of $\mathcal{R}$ are classified into 3 classes [32].

1. **Trade-off Between Objectives**: Here the *DM* is shown various trade-off scenarios and asked their preference in regards to those trade-off, and based on that the next solution set is generated [9].

2. **Reference Point**: Here the *DM* for a get set of $\mathcal{P}$ pareto optimal solutions needs to provide a reference point w.r.t. which the next iteration will generate the updated solution set $\mathcal{P}$ [12], [35].

3. **Classification of Objectives**: Here for a given $\mathcal{P}$ the *DM* classifies different objectives of the solutions such as to get more preferred solutions, and based on the classification the updates solution set $\mathcal{P}$ is generated.

2.2 Condition Number

The term ***Condition Number*** was first coined by Alan Turing in 1947 in his paper on Rounding-Off Errors in Matrix Processes [1]. He defined the condition number for matrices, further in 1966, John Rice in his paper *A Theory of Condition* [4] showed how to formulate the definition of condition number other classes of problems.

In our problem formulation may we not only require to compute the condition number of the function f to check if its bounded or not 1.9,1.10; but also to minimize the maximum over a set X_3 1.11,1.12.

Hence we need to methods which can either be used to

1. Bound the relative/absolute condition number
2. Compute relative/absolute condition number efficiently

Extensive amount of literature is available on condition number of matrices compared to that of general functions. Below we will see some results on condition number of matrix and also for general function.

2.2.1 Condition Number of Matrices

There are several papers which have proved various important properties of matrix condition number. Pierre Maréchal and Jane J. Ye in their paper *Optimizing Condition Numbers*[21] showed that for a symmetric positive semi-definite $n \times n$ matrix A minimizing the condition number $\kappa(A)$.

$$\kappa(A) = ||A||||A^{-1}|| \tag{2.13}$$

$$\min_{A} \kappa(A) \tag{2.14}$$

2.14 is euqivalent to minimizing the following objective

$$\min_{A} \lambda_1(A) - \kappa(\bar{A})\lambda_n(A) \tag{2.15}$$

where $\bar{A}$ is the optimal solution with minimum condition number, if such a solution's value of $\kappa(\bar{A})$ is not known, then one can use a desirable value of $\kappa(\bar{A})$ in its place. Here $\lambda_i(A)$ are the eigenvalues of the matrix A in decreasing order of their magnitude form $i = 1, 2, ..., n$. They also proved that that the problem of minimizing the condition number is non-smooth and non-convex optimization problem [21], which further increases the

difficulty of the problem. In their paper *Convexity Properties of the Condition Number* [22], C Beltrán et al. have studied the convexity properties of the condition number over norm over frobenius inner product. Further Xiaojun Chen et al. in their paper *Minimizing the Condition Number of a Gram Matrix* [23] analysed the same for A being a gram matrix of functions of a scalar x, namely $A(x) = V(x)^\top V(x)$ and derived formulas for generalized gradient of $\kappa(A(x))$, and using exponential smoothing function they also develop a globally convergent smoothing method to solve the 2.14 problem.

2.2.2 Condition Number of Functions

Chapter 3

Conclusion and Future Work

Write about the conclusion of your work and what you plan for the next semester.

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