CS 240

Transitive closure

Strong Connectivity Decomposition

$C = \{a, b, c\}$ $\quad$ $\{d, e\}$

$A \cup B$ = Have some way that A can reach B.

Strong Conn. Decomp. Graph

$E(C_1, C_2) = \{(a, d)\}$

$E(C_2, C_1) = \emptyset$

$E(C_1 \to C_2)$ $\quad$ $E(C_2 \to C_1)$

Strong Connectivity

Transitive Closure

$a \to b$, $b \to c$ $\quad$ $a \to c$

$a \to b$, $b \to c$, $c \to d$ $\quad$ $a \to d$

$a \to b$, $b \to c$, $c \to d$, $d \to e$ $\quad$ $a \to e$

$a \to b$, $b \to c$, $c \to d$, $d \to e$, $e \to f$ $\quad$ $a \to f$

$a \to b$, $b \to c$, $c \to d$, $d \to e$, $e \to f$, $f \to g$ $\quad$ $a \to g$

$a \to b$, $b \to c$, $c \to d$, $d \to e$, $e \to f$, $f \to g$, $g \to h$ $\quad$ $a \to h$

$\frac{|E|^2}{|V|} \geq 2$ $\quad$ $\frac{|E|}{|V|} \geq 1$

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Assume for induction $\frac{|E|}{|V|} \geq 1$

Assume for induction $\frac{|E|}{|V|} \geq 2$

$\frac{|E|}{|V|} \geq 2 \Rightarrow \frac{|E|}{|V|} \geq 1$

$\Rightarrow \frac{|E|}{|V|} \geq 1 \Rightarrow \frac{|E|}{|V|} \geq 2$

Induction on $|V|$

$\left(\frac{|E|}{|V|} \geq 1\right) \Rightarrow \frac{|E|}{|V|} \geq 2$

$\Rightarrow \frac{|E|}{|V|} \geq 1 \Rightarrow \frac{|E|}{|V|} \geq 2$

Assume for induction $\frac{|E|}{|V|} \geq 2$

$\Rightarrow \frac{|E|}{|V|} \geq 1 \Rightarrow \frac{|E|}{|V|} \geq 2$

Induction on $|V|$

$\Rightarrow \frac{|E|}{|V|} \geq 1 \Rightarrow \frac{|E|}{|V|} \geq 2$

Transitive closure algorithm:

Step 1: Find strong connectivity decomposition

Step 2: Find transitive closure of strong connectivity decomposition

Step 3: Find transitive closure of original graph

Step 4: Use transitive closure to answer queries

Step 1: Find strong connectivity decomposition

$\Rightarrow$ If the input is strongly connected, $\Rightarrow$ $O(|V| + |E|)$ reachability of the entire graph

$O(|V| + |E|)$

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$\Rightarrow$ Step 3 takes $O(|V|^2)$ time.

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