FINDING CUTS AND SEPARATORS

8.1 minimum cuts

Harshal Gawai
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Graph cutting

Problems related to cutting a graph into parts certain properties or related to separating different parts of the graph from each other of graph theory and combinatorial optimization

- Eg. As applied in the field of computer vision, graph cut optimization can be employed to efficiently solve a wide variety of low-level computer vision problems(image smoothing)
- Solving a maximum flow problem in a graph[max-flow min-cut theorem]

Graph cutting

Many different versions:- Removing sets of edges or vertices in a directed or undirected graph.

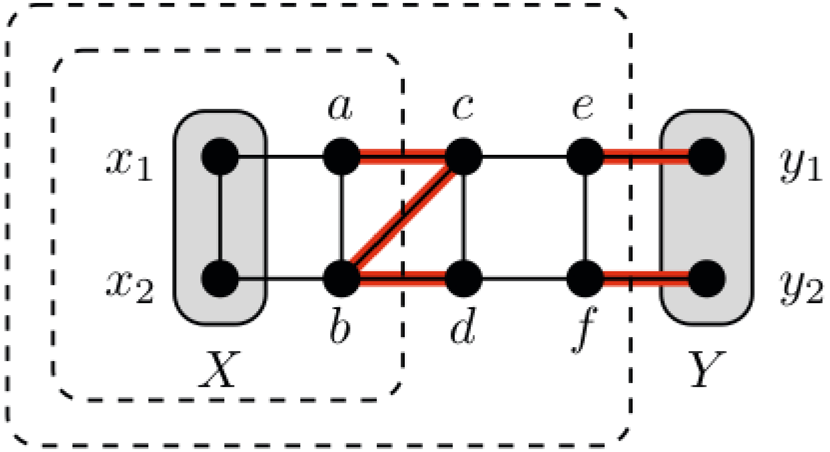
- Most of the problems are NP hard.
- Except:- minimum s−t cut, minimum multiway cut in

  planar graphs with fixed number of terminals
- We'll see the fixed-parameter tractability of some of these problems parameterized by the size of the solution.
- There are many variants: we can delete vertices or edges, the graph can be directed or undirected, we may add weights.

- We’ll be looking at the most general form of these results, we mostly focus on the undirected edge versions, as they are the most intuitive and notationally cleanest.
- **A cut** is a partition of the vertices of a graph into two disjoint subsets. Any cut determines a **cut-set**, the set of edges that have one endpoint in each subset of the partition.
- These edges are said to **cross** the cut.

Minimum cuts

- An (X, Y)-cut is a set S of edges that separates X and Y from each other, that is, $G \backslash S$ has no $X-Y$ path.
- Notions of minimality:
- An (X, Y)-cut S is a minimum (X, Y)-cut if there is no (X, Y)-cut S' with $|S'| < |S|$.
- An (X, Y)-cut is (inclusion-wise) minimal if there is no (X, Y)-cut S' with $S' \subset S$.



a
c
e
x_1
y_1
x_2
y_2
b
d
f
X
Y

- If G is an undirected graph and $R \subseteq V(G)$ is a set of vertices, then we denote by $\Delta_G(R)$ the set of edges with exactly one endpoint in R, and we denote $d_G(R) = |\Delta_G(R)|$
- Let S be a minimal (X, Y)-cut in G and let R be the set of vertices reachable from X in $G \setminus S$

  $\Rightarrow X \subseteq R \subseteq V(G) \setminus Y$
- Then it is easy to see that S is precisely $\Delta_G(R)$. **(PROOF)**

- Every outgoing edge has to be in S (otherwise a vertex of V(G)\R would be **reachable** from X)

$$S \subset \Delta_G(R)$$

and

- S cannot have an edge with both endpoints in R or both endpoints in V (G)\R, as omitting any such edge would not change the fact that the set is an (X, Y)-cut, contradicting **minimality**.

$$\Delta_G(R) \subset S$$

Proposition 8.1.

- Proposition 8.1.: If S is a minimal (X, Y)-cut in G, then S = $\Delta_G(R)$, where R is the set of vertices reachable from X in G\S.

- Therefore, we may always characterize a minimal (X, Y)-cut S as $\Delta(R)$ for some set X, s.t. $X \subseteq R \subseteq V(G) \setminus Y$.

- Also note that $\Delta(R)$ is an (X, Y)-cut for every such set R with
- $X \subseteq R \subseteq V(G) \setminus Y$, but not necessarily a minimal (X, Y)-cut.

Maximum flow and minimum cut duality

- The size of the minimum (X, Y)-cut is the same as the maximum number of pairwise edge-disjoint X−Y paths.

- Classical maximum flow algorithms can be used to find a minimum cut and a corresponding collection of edge disjoint X − Y paths of the same size.(**PROOF**)

- Each round of the algorithm of Ford and Fulkerson takes linear time, and k rounds are sufficient to decide if there is an (X, Y)-cut of size at most k.

- 1> Every path on pairwise edge-disjoint paths(pEDP) should have exactly one edge from min-cut

  Because if **pEDP** > 1 edge in S (minimumness)

  if = 0 (there is no cut)

- 2> For any two edges in S, there exist P_1, P_2 connecting X and Y
- $P_1 \cap P_2 = \phi$

Theorem 8.2.

Given a graph G with n vertices and m edges, disjoint sets

$X, Y \subseteq V(G)$, and an integer k, there is an $O(k(n + m))$-time algorithm that either:

- correctly concludes that there is no (X, Y)-cut of size at most k,

  or

- returns a minimum (X,Y)-cut $\Delta(R)$ **and** a collection of $|\Delta(R)|$ pairwise edge-disjoint X−Y paths.

Submodular Inequality

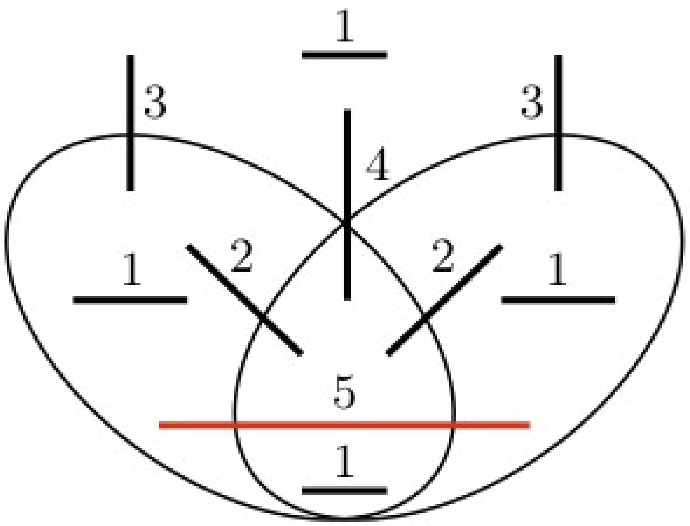
Let $f : 2^{V(G)} \rightarrow R$ be a set function assigning a real number to each subset of vertices of a graph G. We say that f is submodular if it satisfies the following inequality for every $A, B \subseteq V(G)$:

- $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$

- This function $\mathbf{d_G(X) = |\Delta_G(X)|}$ is sub-modular.

Theorem 8.3.

- Theorem: The function d_G is submodular for every undirected graph G.



The different types of edges in the proof of Theorem 8.3

Classify each edge e according to the location of its endpoints:

- 1. If both endpoints of e are in $A \cap B$, in $A \setminus B$, in $B \setminus A$, or in $V(G) \setminus (A \cup B)$, then e contributes 0 to both sides.

2. If one endpoint of e is in $A \cap B$, and the other is either in $A \setminus B$ or i $B \setminus A$, then e contributes 1 to both sides.

3. If one endpoint of e is in $V(G) \setminus (A \cup B)$, and the other is either in $A \setminus B$ or in $B \setminus A$, then e contributes 1 to both sides.

4. If e is between $A \cap B$ and $V(G) \setminus (A \cup B)$, then e contributes 2 to both sides.

5. If e is between $A \setminus B$ and $B \setminus A$, then e contributes 2 to the left-hand side and 0 to the right-hand side.

Why is submodularity so relevant here?

- If $\Delta(A)$ and $\Delta(B)$ are both (X, Y)-cuts, then $\Delta(A \cap B)$ and $\Delta(A \cup B)$ are both (X, Y)-cuts.

- Therefore, we can interpret **Theorem 8.3** as saying that if we have two (X, Y)-cuts $\Delta(A)$, $\Delta(B)$ of a certain size, then two new (X, Y)-cuts $\Delta(A \cap B)$, $\Delta(A \cup B)$ can be created and there is a <u>bound</u> on their total size.

Minimum cut

- The minimum (X, Y)-cut is not necessarily unique; in fact, a graph can have a large number of minimum (X, Y)-cuts. eg...

- There is a unique minimum (X, Y)-cut $\Delta(R_{min})$ that is closest to X.

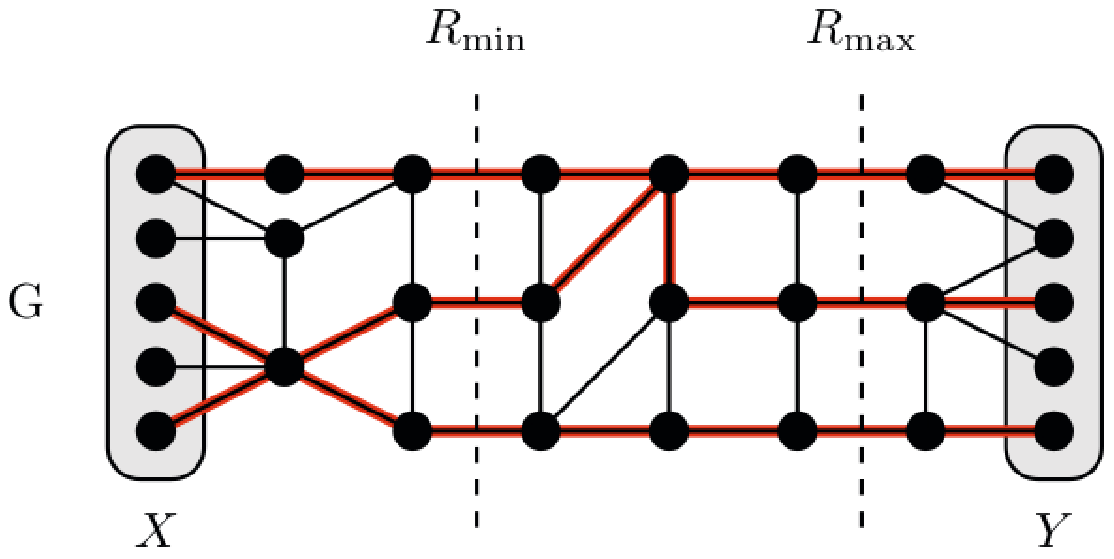
and

- A unique minimum (X, Y)-cut $\Delta(R_{max})$ closest to Y.

Theorem 8.4.

- **Theorem 8.4.:** Let G be a graph and X, Y $\subseteq$ V (G) two disjoint sets of vertices. There are two minimum (X, Y)-cuts $\Delta(R_{min})$ and $\Delta(R_{max})$ such that :

  if $\Delta(R)$ is a minimum (X, Y)-cut, then $R_{min} \subseteq R \subseteq R_{min}$.



- **PROOF HERE**

To prove: There is a unique inclusion-wise minimal set R_{min} and a unique inclusion-wise maximal set R_{max} in R.

- Consider the collection R of every set $R \subseteq V(G)$ for which $\Delta(R)$ is a minimum (X, Y)-cut.
- Suppose for that $\Delta(R_1)$ and $\Delta(R_2)$ are minimum cuts for two inclusion-wise minimal sets $R_1 \neq R_2$ of R.

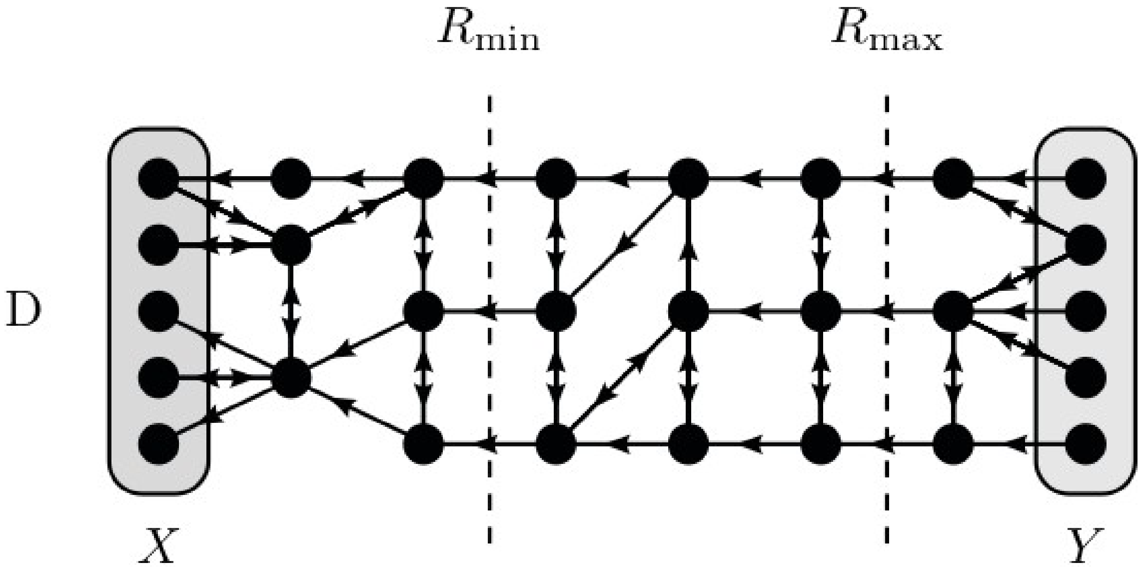
$$d_G(R_1) + d_G(R_2) \geq d_G(R_1 \cap R_2) + d_G(R_1 \cup R_2)$$

- If λ is the minimum (X, Y)-cut size, then LHS = 2λ
- Implies, $2\lambda \geq$ RHS (RHS is atmost 2λ)

- $\Delta(R_1 \cap R_2)$ and $\Delta(R_1 \cup R_2)$ are both (X, Y)-cuts.
- λ is the minimum (X,Y)-cut size, the right-hand side is also exactly 2λ, with both terms being exactly λ.
- i.e. $\Delta(R_1 \cap R_2)$ is a minimum (X, Y)-cut.
- Now, $R_1 \neq R_2$ implies that $(R_1 \cap R_2) \subset R_1 , R_2$
- Contradicting the assumption that both R_1 and R_2 are inclusion-wise minimal in R.
- The same argument gives a contradiction if $R_1 \neq R_1$ are inclusion-wise maximal sets of the collection: then we observe that $\Delta(R_1 \cup R_1)$ is also a minimum (X, Y)-cut.

Theorem 8.5.

- **Theorem 8.5.**: Let G be a graph with n vertices and m edges, and X, Y ⊆ V (G) be two disjoint sets of vertices. Let k be the size of the minimum (X, Y)-cut. The sets R_{min} and R_{max} of Theorem 8.4 can be found in time O(k(n +m)).



[1] Construction of D

- let $P_1, \ldots, P_k$ be the pairwise edge-disjoint X−Y paths returned by the algorithm(of theorem 8.2).
- We build the residual directed graph D as follows:
- 1)If edge *xy* of G is not used by any of the paths P_i , then we introduce directed edges both (x, y) and (y, x) into D.
- 2)If edge *xy* of G is used by some P_i in such a way that *x* is closer to X on path P_i , then we introduce the directed edge 1 (y, x) into D.

Simply, We have to show that R_{min} is the set of vertices reachable from X in the residual graph D and R_{max} is the set of vertices from which Y is not reachable in D.

- **[2]** Let $\Delta_G(R)$ be a minimum (X, Y)-cut of G.(all rechable parts of D from X)
- As $\Delta_G(R)$ is an (X, Y)-cut of size k, each of the k paths $P_1, \ldots, P_k$ uses exactly one edge of $\Delta_G(R)$.
- This means that after P_1 leaves R, it never returns to R. Therefore, if P_1 uses an edge $ab \in \Delta_G(R)$ with $a \in R$ and $b \notin R$, then a is closer to X on P_i .
- This implies that (a, b) is not an edge of D(athough (b,a) is edge of D). As this is true for every edge of the cut $\Delta_G(R)$, we get that **V(G)\R is not reachable from X in D**; in particular, **Y is not reachable**.

[3]

- Let R_{min} be the set of vertices reachable from X in D.
- Since R_{min} is a cut => $X \subseteq R_{min}$ ...[1]
- (from previous) Y is not reachable from X in D HOW?**(**Proof below) =>
- Suppose Y is rechable from X (in residual graph D)
- Then there exist a path **P** from X to Y in D
- Where **P** is a edge-disjoint path of {**P**, P1, P2...}
- This set of pairwise edge-disjoint paths have reverse edges in D(including **P**)

- Therefore **P** having reverse edge => There is no path from X to Y
- Hence, $R_{min} \subseteq V(G)\backslash Y$ ...[2]
- i.e. $X \subseteq R_{min} \subseteq V(G)\backslash Y$) ...[from 1, 2]
- Hence, $\underline{\Delta_G(R_{min})}$ <u>is an (X, Y)-cut of G</u>.

[4]

- Now, <u>we have to show that this cut is a minimum (X, Y)-cut</u>:

- We have shown before that if $\Delta_G(R)$ is a minimum (X, Y)-cut ...[2]
- then V (G)\R is not reachable from X in D
- $V(G)\setminus R_{min}$ is largest possible rechable set from X
- implying that $V(G) \setminus R \subseteq V(G) \setminus R_{min}$
- and hence, $R_{min} \subseteq R$.

[5] To prove: $\Delta_G(R_{min})$ have exactly k edges

- [5.a]There are atleast *k* edges in $\Delta_G(R_{min})$
- i,e. $\Delta_G(R_{min}) > k$
- We have 1 edge from cut per each pairwise Edge-Disjoint Path(pEDP)
- Let there exist an edge $\notin$ any pEDP
- This edge will be bidirectional in residual graph D
- => Vertex outside R_{min} is rechable from X
- => It should have been inside R_{min}

- [5.b] Every path P_i uses at least one edge of the (X, Y)-cut from $\Delta_G(R_{min})$.
- i.e. $k > \Delta_G(R_{min})$
- Suppose if P_i leaves R_{min} and later returns to R_{min} on an edge *ab* with $a \notin R_{min}$, $b \in R_{min}$ and a closer to X on P_i , then (*b, a*) is an edge of D and it follows that *a* is also reachable from X in D, contradicting $a \in R_{min}$.
- Therefore, P_i cannot use more than one edge of the cut ...[2]
- Therefore, $\Delta_G(R_{min})$ can have at most *k* edges

- [5.a] and [5.b] concludes $\Delta_G(R_{min})$ have exactly k edges.
- Therefore, R_{min} satisfies the requirements.

- A symmetrical argument shows that the set R_{max} containing all vertices from which Y is not reachable in D satisfies the requirements.

THANK YOU