Energy Methods

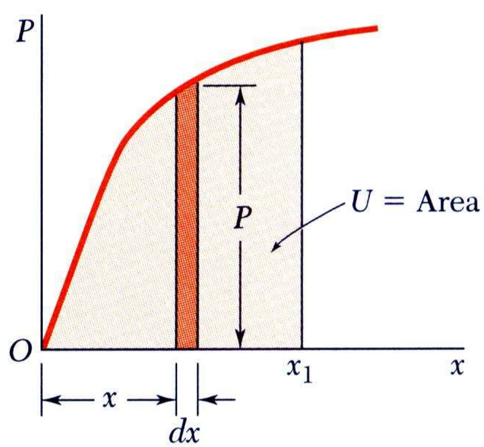
- We have seen how to find the strain in axial loaded bar, deflection of points on a beam etc.
- Same can be done using energy methods
- We have to develop a relation between work done by externally applied forces and strain energy of the deforming body

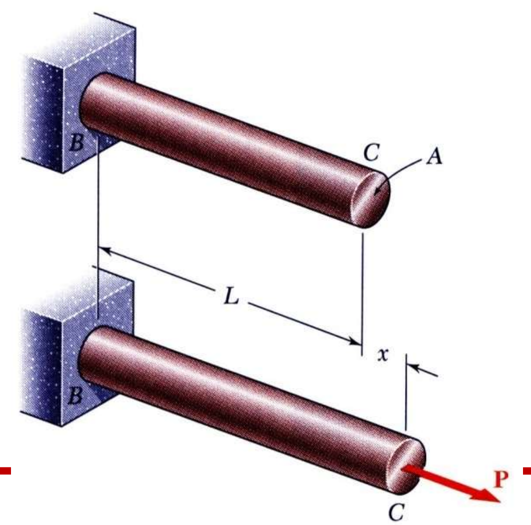
Work done

- A uniform rod is subjected to a slowly increasing load
- The elementary work done by the load *P* as the rod elongates by a small *dx* is

$$dU = P\,dx = elementary\ work$$

which is equal to the area of width *dx* under the load-deformation diagram





Work done

- The total work done by the load for a deformation x1,

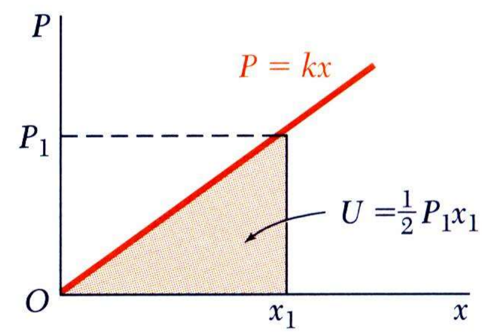
$$U = \int_0^{x_1} P\,dx = total\ work = strain\ energy$$

which results in an increase of *strain energy* in the rod.

- For linear elastic deformation

$$U = \int_0^{x_1} kx\,dx = \frac{1}{2}kx_1^2 = \frac{1}{2}P_1 x_1$$

k= ??



Strain energy

- When loads are applied to a body, they will deform the material. Provided no energy is lost in the form of heat, the external work done by the loads will be converted into internal work called ***strain energy.***

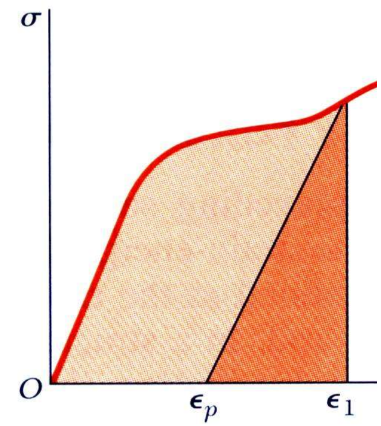
- This energy, which is always positive, is stored in the body and is caused by the action of either normal or shear stress.

Strain energy density

- To remove the size effects due to specimen geometry, we calculate the energy density

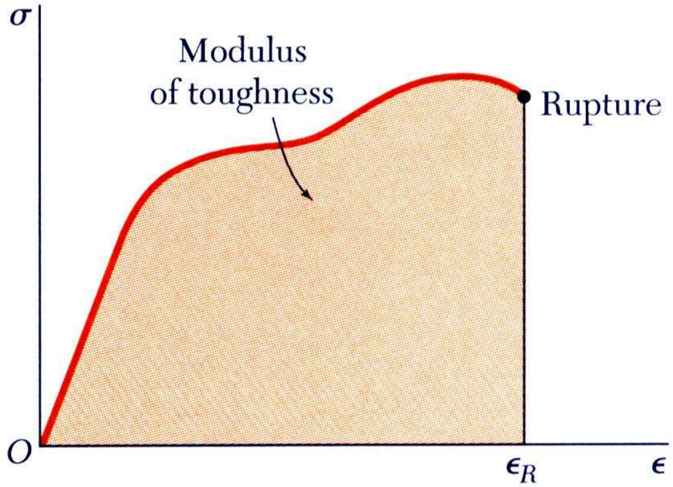
$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L}$$

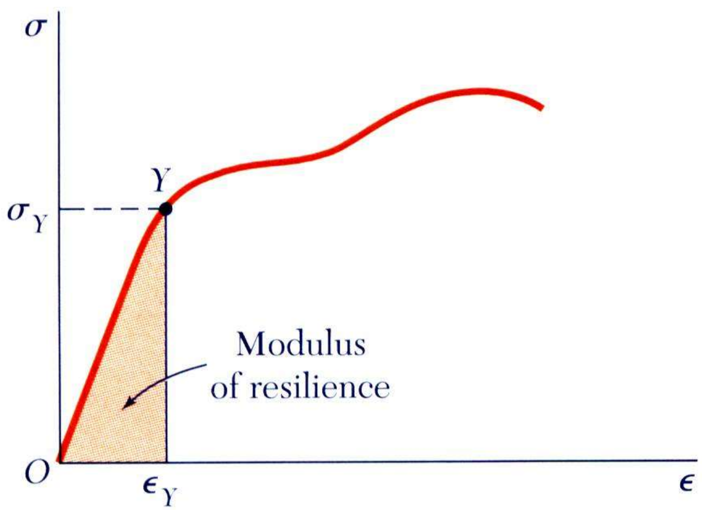
$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon = strain\ energy\ density$$



- The total strain energy density resulting from the deformation is equal to the area under the curve up to ε_1.
- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.
- Remainder of the energy spent in deforming the material is dissipated as heat.

Strain energy density





- The strain energy density resulting from setting $\varepsilon_1 = \varepsilon_R$ is the *modulus of toughness*.
- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.
- If the stress remains within the proportional limit,

$$u = \int_0^{\varepsilon_1} E\varepsilon_1 \, d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

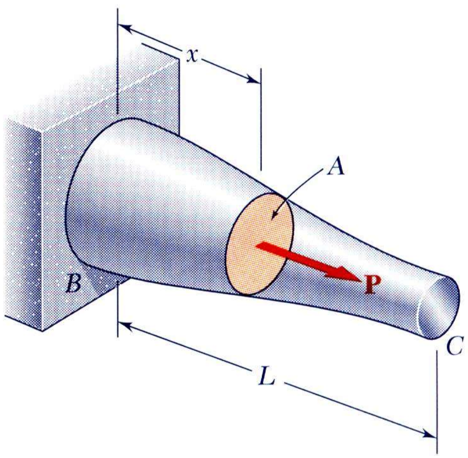
- The strain energy density resulting from setting $\sigma_1 = \sigma_Y$ is the *modulus of resilience*.

$$u_Y = \frac{\sigma_Y^2}{2E} = modulus\ of\ resilience$$

Elastic Strain Energy for Normal Stresses

- In an element with a nonuniform stress distribution,

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

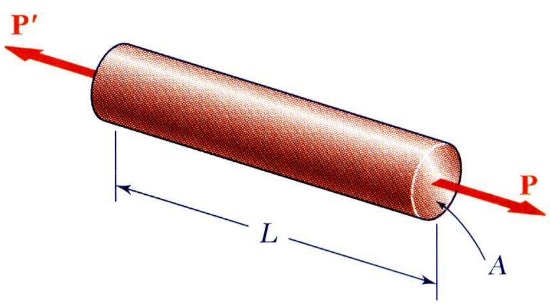


- For values of $u < u_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} dV = \textit{elastic strain energy}$$

- Under axial loading, $\sigma_x = P/A \qquad dV = A\,dx$

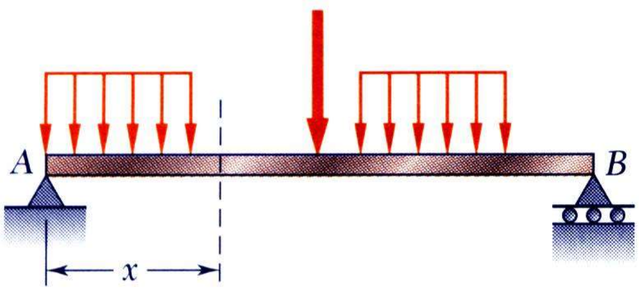
$$U = \int_0^L \frac{P^2}{2AE} dx$$



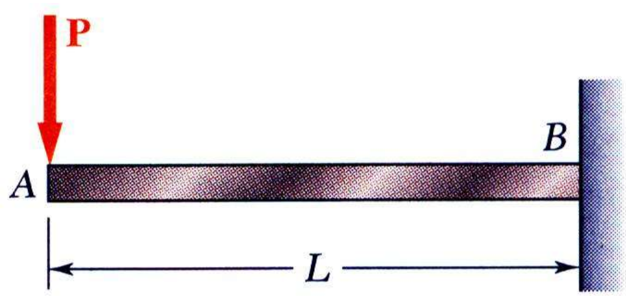
- For a rod of uniform cross-section,

$$U = \frac{P^2 L}{2AE}$$

Elastic Strain Energy for Normal Stresses



$$\sigma_x = \frac{My}{I}$$



- For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting *dV = dA dx,*

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\, dx = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx$$

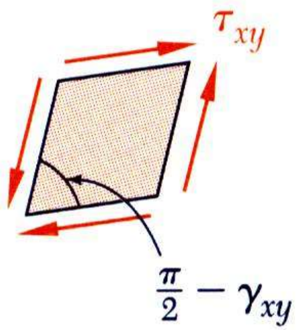
$$= \int_0^L \frac{M^2}{2EI} dx$$

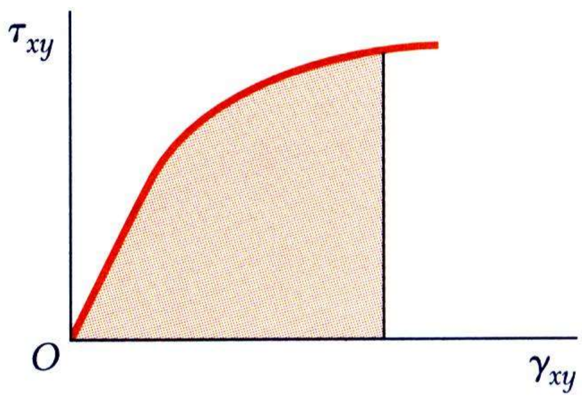
- For an end-loaded cantilever beam,

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

Strain Energy For Shearing Stresses





- For a material subjected to plane shearing stresses,

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}$$

- For values of τ_{xy} within the proportional limit,

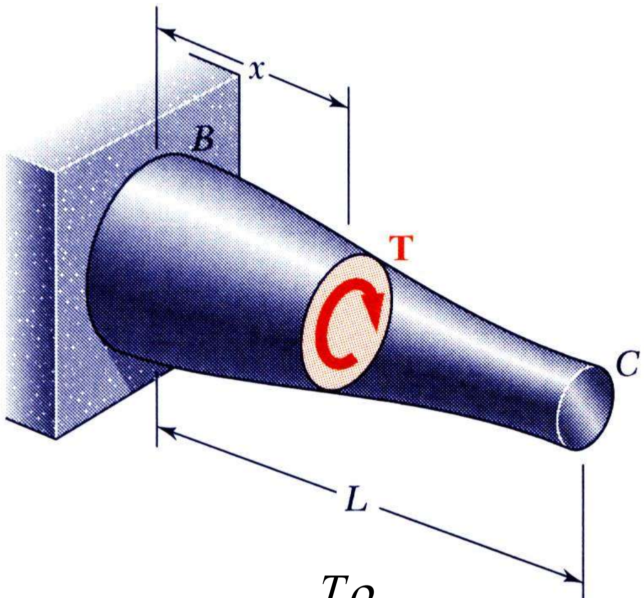
$$u = \frac{1}{2} G \gamma_{xy}^2 = \frac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

- The total strain energy is found from

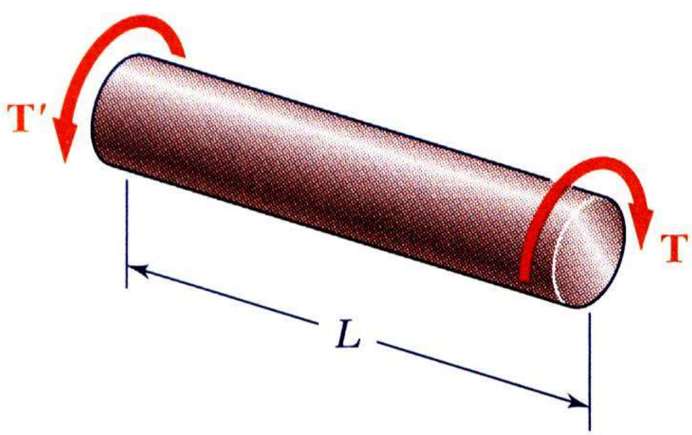
$$U = \int u\, dV$$

$$= \int \frac{\tau_{xy}^2}{2G} dV$$

Strain Energy For Shearing Stresses



$$\tau_{xy} = \frac{T\rho}{J}$$



- For a shaft subjected to a torsional load,

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GJ^2} dV$$

- Setting *dV = dA dx,*

$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} dA\, dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

$$= \int_0^L \frac{T^2}{2GJ} dx$$

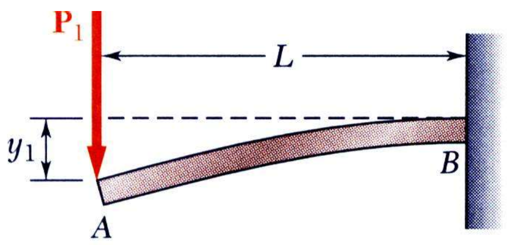
- In the case of a uniform shaft,

$$U = \frac{T^2 L}{2GJ}$$

Work and Energy Under a Single Load

- Strain energy may be found from the work of other types of single concentrated loads

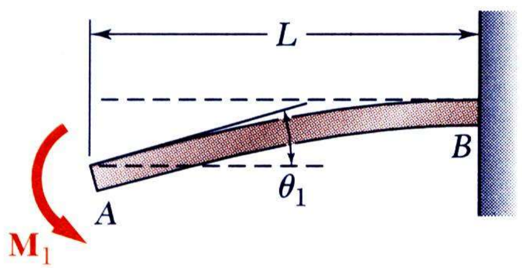
- Transverse load



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2}P_1 y_1$$

$$= \tfrac{1}{2}P_1\left(\frac{P_1 L^3}{3EI}\right) = \frac{P_1^2 L^3}{6EI}$$

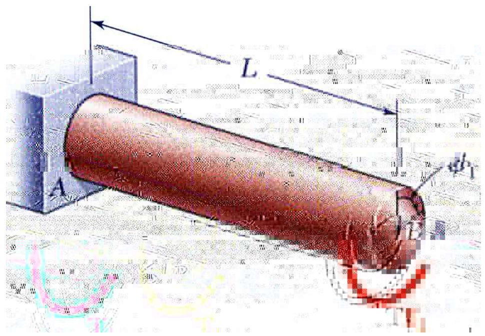
- Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2}M_1\theta_1$$

$$= \tfrac{1}{2}M_1\left(\frac{M_1 L}{EI}\right) = \frac{M_1^2 L}{2EI}$$

- Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2}T_1\phi_1$$

$$= \tfrac{1}{2}T_1\left(\frac{T_1 L}{JG}\right) = \frac{T_1^2 L}{2JG}$$

Strain Energy for a General State of Stress

- Previously found strain energy due to uniaxial stress and plane shearing stress. For a general state of stress,

$$u = \tfrac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

- With respect to the principal axes for an elastic, isotropic body,

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right]$$

$$= u_v + u_d$$

$$u_v = \frac{1-2\nu}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2 = \text{due to volume change}$$

$$u_d = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right] = \text{due to distortion}$$

- Basis for the *maximum distortion energy* failure criteria,

$$u_d < \left(u_d\right)_Y = \frac{\sigma_Y^2}{6G} \quad \text{for a tensile test specimen}$$

Conservation of energy

- All energy methods used in mechanics are based on a balance of energy, often referred to as the conservation of energy
- Only mechanical energy will be considered in the energy balance
  - The energy developed by heat, chemical reactions, and electromagnetic effects will be neglected
- if a loading is applied *slowly* to a body, then physically the external loads tend to deform the body so that the loads do *external work* U_e as they are displaced
- This external work on the body is transformed into *internal work* U_i or strain energy which is stored in the body

Conservation of energy

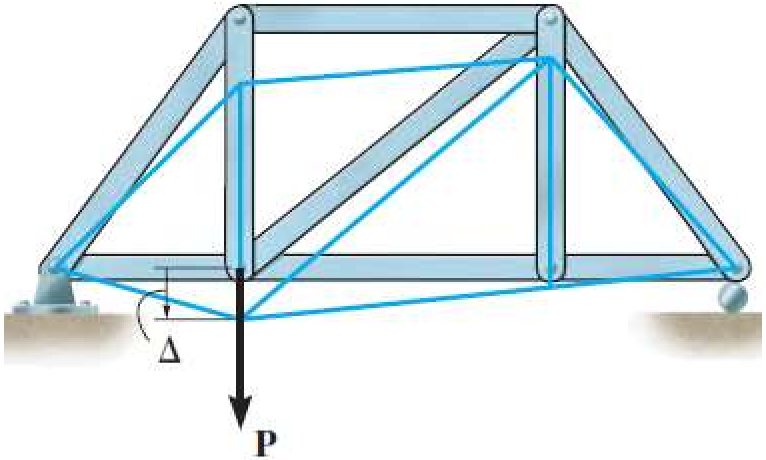
- When the loads are removed, the strain energy restores the body back to its original undeformed position, provided the material's elastic limit is not exceeded
- The conservation of energy for the body can therefore be stated mathematically as

$$\boxed{U_e = U_i}$$

- This equation can be applied to determine the displacement of a point on a deformable member or structure

Example

- Vertical displacement of truss at point P



External work done by P: $U_e = \frac{1}{2} P\Delta,$

If *N* is the axial force developed in truss, then strain energy stored,

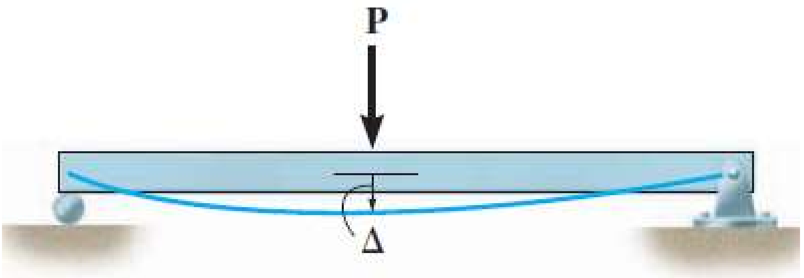
$$U_i = N^2 L / 2AE$$

Using conservation of energy: $\frac{1}{2} P\Delta = \sum \frac{N^2 L}{2AE}$

Example

- Vertical displacement at load point P

External work done by P: $U_e = \frac{1}{2} P \Delta,$



Strain energy due to internal bending moment *M*:

$$\int_0^L \frac{M^2}{2EI} dx$$

Using conservation of energy:

$$\frac{1}{2} P \Delta = \int_0^L \frac{M^2}{2EI} dx$$

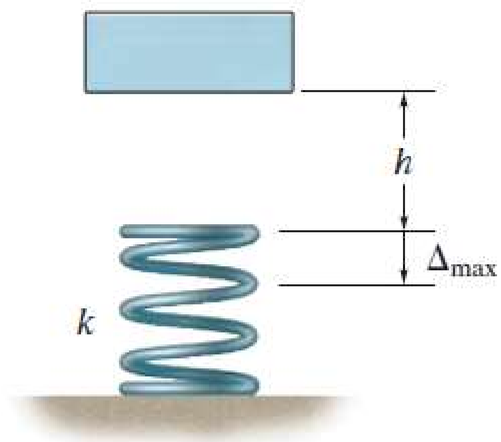
Impact loading

- So far we have considered all loadings to be applied to a body in a gradual manner
  - when they reach a maximum value the body remains static
- Some loading are dynamic – they vary with time – such as collision of objects
- This is called an impact loading
  - ***impact*** occurs when one object strikes another, such that large forces are developed between the objects during a very short period of time
  - If we assume no energy is lost during impact, due to heat, sound or localized plastic deformations, then we can study the mechanics of impact using the conservation of energy

Block and spring system

- When the block is released from rest, it falls a distance *h*, striking the spring and compressing it a distance $\Delta_{\max}$

If we neglect the mass of the spring and assume that the spring responds *elastically*, conservation of energy requires that the energy of the falling block be transformed into stored (strain) energy in the spring.



External work done by the block's weight: $W(h + \Delta_{\max})$

Force in the spring *F*: $F = k\Delta_{\max},$

Internal work done: $\frac{1}{2}(k\Delta_{\max})\,\Delta_{\max}$

Block and spring system

- According to conservation of energy,

$$U_e = U_i$$

$$W(h + \Delta_{\max}) = \frac{1}{2}(k\Delta_{\max})\,\Delta_{\max}$$

$$\Delta_{\max}^2 - \frac{2W}{k}\Delta_{\max} - 2\left(\frac{W}{k}\right)h = 0$$

$$\Delta_{\max} = \frac{W}{k} + \sqrt{\left(\frac{W}{k}\right)^2 + 2\left(\frac{W}{k}\right)h}$$

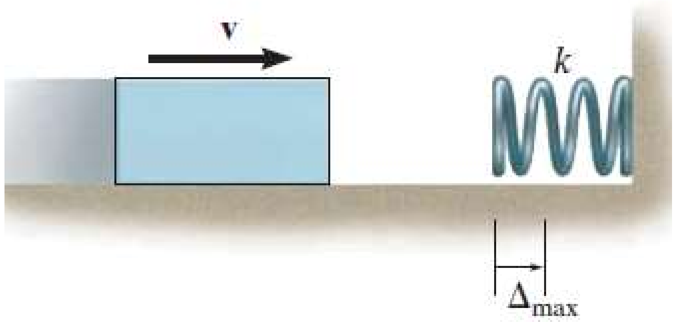
If the weight W is supported statically by the spring, then static displacement

$$\Delta_{st} = W/k$$

Block and spring system

- Determine the maximum displacement of the end of the spring if the block is sliding on a smooth horizontal surface with a known velocity **v** just before it collides with the spring

In this case the block's kinetic energy will be transformed into stored energy in the spring.



$$U_e = U_i$$

$$\frac{1}{2}\left(\frac{W}{g}\right)v^2 = \frac{1}{2}k\Delta_{max}^2$$

$$\Delta_{max} = \sqrt{\frac{Wv^2}{gk}}$$

$$\Delta_{st} = W/k,$$

$$\Delta_{max} = \boxed{\sqrt{\frac{\Delta_{st}v^2}{g}}}$$

Maximum stress

- Once Δ_{max} is determined, the maximum dynamic force can then be calculated from

$$P_{max} = k\Delta_{max}$$

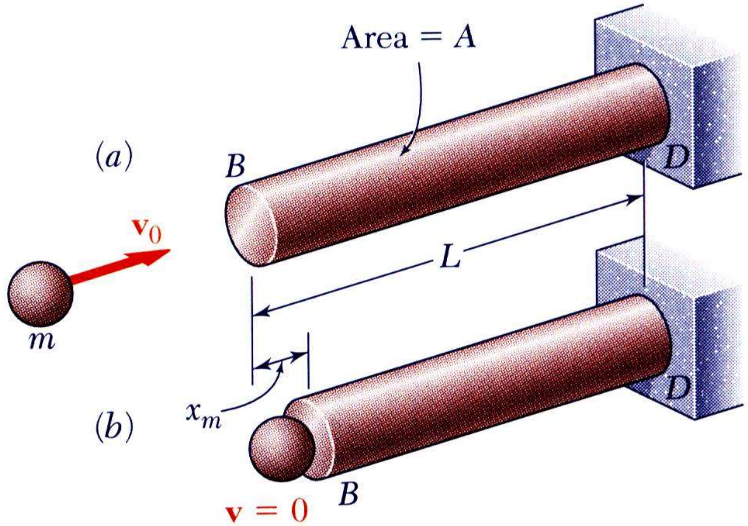
- If we consider P_{max} to be an equivalent load which produces same strain energy, then we can calculate maximum stress using statics
- The ratio of the equivalent static load P_{max} to the static load $P_{st} = W$ is called the *impact factor, n*

$$P_{max} = k\Delta_{max} \qquad P_{st} = k\Delta_{st},$$

$$\boxed{n = 1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}}$$

This factor represents the magnification of a statically applied load so that it can be treated dynamically.

Maximum stress



- Consider a rod which is hit at its end with a body of mass *m* moving with a velocity v_0.

- Rod deforms under impact. Stresses reach a maximum value σ_m and then disappear.

- To determine the maximum stress σ_m
  - Assume that the kinetic energy is transferred entirely to the structure,

$$U_m = \tfrac{1}{2} m v_0^2$$

  - Assume that the stress-strain diagram obtained from a static test is also valid under impact loading.

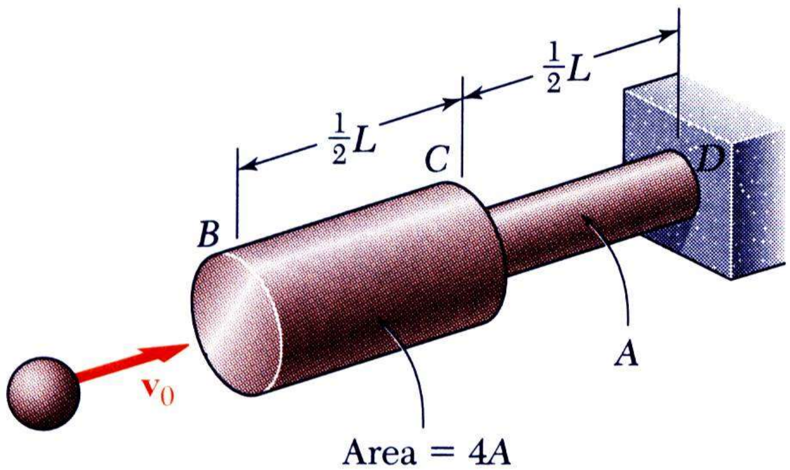
- Maximum value of the strain energy,

$$U_m = \int \frac{\sigma_m^2}{2E} dV$$

- For the case of a uniform rod,

$$\sigma_m = \sqrt{\frac{2U_m E}{V}} = \sqrt{\frac{m v_0^2 E}{V}}$$

Example



SOLUTION:

- Due to the change in diameter, the normal stress distribution is nonuniform.

$$U_m = \tfrac{1}{2} m v_0^2$$

$$= \int \frac{\sigma_m^2}{2E} dV \neq \frac{\sigma_m^2 V}{2E}$$

- Find the static load P_m which produces the same strain energy as the impact.

$$U_m = \frac{P_m^2 (L/2)}{AE} + \frac{P_m^2 (L/2)}{4AE} = \frac{5}{16}\frac{P_m^2 L}{AE}$$

$$P_m = \sqrt{\frac{16}{5}\frac{U_m AE}{L}}$$

- Evaluate the maximum stress resulting from the static load P_m

$$\sigma_m = \frac{P_m}{A}$$

$$= \sqrt{\frac{16}{5}\frac{U_m E}{AL}}$$

$$= \sqrt{\frac{8}{5}\frac{m v_0^2 E}{AL}}$$