CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Taylor's Theorem
- Worked Problem

Taylor's Theorem for a Function of Single Variables (Recall)

Assume that the function f has all derivatives up to the order $(n+1)$ in some interval containing the point $x = x_0$.

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) + R_n$$

$$R_n = \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \qquad x_0 < \xi < x_0 + h$$

Taylor's Theorem for a Function of Two Variables

Let a function be defined in some domain D in $\mathbb{R}^2$ and have continuous partial derivatives up to $(n+1)^{\text{th}}$ order in some neighborhood of a point $P(x_0, y_0)$ in D. Then

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) +$$

$$\dots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n$$

where the remainder is given by

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k), \qquad 0 < \theta < 1$$

Taylor's Theorem for a Function of Two Variables

Proof: For Simplicity, we take $n = 2$ (terms up to order 3)

Let $x = x_0 + th,\ y = y_0 + tk,$ where the parameter $t \in [0, 1]$.

Define $\phi(t) = f(x_0 + th, y_0 + tk)$

$$\phi'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0 + th, y_0 + tk)$$

$$\phi''(t) = h\left(\frac{\partial^2 f}{\partial x^2}h + \frac{\partial^2 f}{\partial y\,\partial x}k\right) + k\left(\frac{\partial^2 f}{\partial x\,\partial y}h + \frac{\partial^2 f}{\partial y^2}k\right)$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0 + th, y_0 + tk)$$

$$\phi'''(t) = h^2\left(\frac{\partial^3 f}{\partial x^3}h + \frac{\partial^3 f}{\partial y\,\partial x^2}k\right) + 2hk\left(\frac{\partial^3 f}{\partial x^2\,\partial y}h + \frac{\partial^3 f}{\partial x\,\partial y^2}k\right) + k^2\left(\frac{\partial^3 f}{\partial x\,\partial y^2}h + \frac{\partial^3 f}{\partial y^3}k\right)$$

$$= h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\,\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0 + th, y_0 + tk)$$

Using Taylor's Theorem for $\phi(t)$ about the point 0 as

$$\phi(t) = \phi(0) + t\,\phi'(0) + \frac{t^2}{2!}\,\phi''(0) + \frac{t^3}{3!}\,\phi'''(\theta t), \qquad 0 < \theta < 1$$

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\,\phi''(0) + \frac{1}{3!}\,\phi'''(\theta), \qquad 0 < \theta < 1$$

$$\phi(t) = f(x_0 + th, y_0 + tk)$$

$$\phi'(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0 + th, y_0 + tk)$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0 + th, y_0 + tk)$$

$$\phi'''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0 + th, y_0 + tk)$$

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \frac{1}{3!}\phi'''(\theta), \qquad 0 < \theta < 1$$

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0 + \theta h, y_0 + \theta k)$$

General Case:

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) +$$

$$\dots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k)$$

Alternatively,

$$f(x,y) = f(x_0, y_0) + \left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right) f(x_0, y_0) + \cdots$$

$$+ \frac{1}{(n+1)!}\left((x-x_0)\frac{\partial}{\partial x} + (x-x_0)\frac{\partial}{\partial y}\right)^{n+1} f\big(x_0+\theta(x-x_0), y_0+\theta(y-y_0)\big)$$

Problem - 1 Find the quadratic polynomial approximation to the function

$$f(x,y) = \frac{x-y}{x+y} \text{ about the point } (1,1)$$

$$f_x(x,y) = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2} \Rightarrow f_x(1,1) = \frac{1}{2}$$

$$f_y(x,y) = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2} \Rightarrow f_y(1,1) = -\frac{1}{2}$$

$$f_{xx}(x,y) = \frac{-4y}{(x+y)^3} \Rightarrow f_{xx}(1,1) = -\frac{1}{2} \qquad f_{yy}(x,y) = \frac{4x}{(x+y)^3}$$

$$\Rightarrow f_{yy}(1,1) = \frac{1}{2} \qquad f_{xy}(x,y) = \frac{2x-2y}{(x+y)^3} \Rightarrow f_{xy}(1,1) = 0$$

$$f_x(1,1) = \frac{1}{2} \qquad f_y(1,1) = -\frac{1}{2} \qquad f_{xx}(1,1) = -\frac{1}{2} \qquad f_{yy}(1,1) = \frac{1}{2} \qquad f_{xy}(1,1) = 0$$

$$P_2(x,y) = f(1,1) + f_x(1,1)\,(x-1) + f_y(1,1)(y-1) + \frac{1}{2} f_{xx}(1,1)(x-1)^2$$
$$+ f_{xy}(1,1)(x-1)(y-1) + \frac{1}{2} f_{yy}(1,1)(y-1)^2$$

$$P_2(x,y) = \frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 + \frac{1}{4}(y-1)^2$$

Problem - 2 Let $f(x,y) = x^2 + xy + y^2$ be linearly approximated by the Taylor's polynomial about the point $(1,1)$. Find out the maximum error in this approximation at a point in the square $|x-1| \leq 0.1, |y-1| \leq 0.1$.

$f_x(x,y) = 2x + y$ $\quad$ $f_{xx}(x,y) = 2$ $\quad$ $f_{xy}(x,y) = 1$

$f_y(x,y) = x + 2y$ $\quad$ $f_{yy}(x,y) = 2$

Remainder: $R_1 = \frac{1}{2}\left((x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right)^2 f\big(1+\theta(x-1), 1+\theta(y-1)\big)$

$$R_1 = \frac{1}{2}\left((x-1)^2 f_{xx} + 2(x-1)(y-1)f_{xy} + (y-1)^2 f_{yy}\right)$$

$$R_1 = (x-1)^2 + (x-1)(y-1) + (y-1)^2$$

Maximum Error: $R_1 \leq (0.1)^2 + (0.1)^2 + (0.1)^2 = 0.03$

Problem - 3 Obtain Taylor's formula about the point $(0,0)$ involving derivatives up to 3rd order for the function $f(x,y) = \cos(x+y)$.

Taylor's theorem:

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y)$$

$$0 < \theta < 1$$

- $f(0,0) = 1$
- First order derivatives: $f_x = -\sin(x+y) \quad \Rightarrow f_x(0,0) = 0$

  $f_y = -\sin(x+y) \quad \Rightarrow f_y(0,0) = 0$
- Second order derivatives: $f_{xx} = f_{yy} = f_{xy} = -\cos(x+y)$

$$\Rightarrow f_{xx}(0,0) = f_{yy}(0,0) = f_{xy}(0,0) = -1$$

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y)$$

$$0 < \theta < 1$$

- Third order derivatives: $f_{xxx} = f_{yyy} = f_{xxy} = f_{xyy} = \sin(x+y)$

$$f_{xxx}(\theta x, \theta y) = f_{yyy}(\theta x, \theta y) = f_{xxy}(\theta x, \theta y) = f_{xyy}(\theta x, \theta y) = \sin(\theta x + \theta y)$$

$$f(x,y) = 1 + 0 - \frac{1}{2!}(x^2 + 2xy + y^2) + \frac{1}{3!}(x^3 + 3x^2y + 3xy^2 + y^3)\sin(\theta x + \theta y)$$

$$f(x,y) = 1 - \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 \sin(\theta x + \theta y)$$

CONCLUSIONS

Taylor's Theorem for a Function of Two Variables

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots$$

$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k)$$