MA 102: Linear Algebra, Integral Transforms & Special Functions
Tutorial Sheet - 6
Second Semester of the Academic Year 2023-2024

Notation : Field $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

---

1. Let $T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^3(\mathbb{R})$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$ is a basis for $\mathbb{R}^3$. Determine the matrix representation $[T]_\beta^\gamma$ of the linear transformation T. If $\alpha = \{(1,2), (2,3)\}$ is a given basis of $\mathbb{R}^2$, then find $[T]_\alpha^\gamma$.

2. Suppose $T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ be the linear transformation such that $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. Then find $T(2,3)$.

3. Prove that there exist a linear transformation $T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^3(\mathbb{R})$ such that $T(1,1) = (1,0,2)$, $T(2,3) = (1,-1,4)$. Then find the $T(8,11)$.

4. (a) Give an example of linear transformation that is one one but not onto.
   (b) Give an example of linear transformation that is onto but not one-one.

5. Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f'(x)$. Let β and γ be the standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively. Then find $[T]_\beta^\gamma$.

6. Let V and W be vector space over the field $\mathbb{F}$ and $T, U : V \to W$ be two linear transformations. Then prove that:
   (a) $T + U$ is a linear transformation.
   (b) αT is a linear transformation for any $\alpha \in \mathbb{F}$.

7. Using the operations of addition and scalar multiplication of linear transformations in the previous problem, show that the collection of all linear transformations $\mathcal{L}(V, W)$ from the vector space V to W is a vector space over $\mathbb{F}$.

8. Show that $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

9. Let V, W and Z be vector spaces over the same field $\mathbb{F}$, and let $T : V \to W$ and $U : W \to Z$ be linear. Then $U \circ T : V \to Z$ is linear.

10. Let T be a linear operator on $\mathbb{R}^3$, defined by $T(x, y, z) = (2y + z, x - 4z, 3x - 6z)$.
    (a) Find $[T]_B^B$, where $B = \{(1,1,0), (1,0,1), (0,1,1)\}$.
    (b) Verify that $[T]_B^B[v]_B = [T(v)]_B$ for any $v \in \mathbb{R}^3$.

11. Suppose V and W are finite-dimensional vector spaces over the field $\mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis β of V and a basis γ of W such that with respect to these bases, all entries of $[T]_\beta^\gamma$ equal to 1.

12. Suppose V is a finite-dimensional vector space, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

13. Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V, V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

14. For the following linear transformations T, determine whether T is invertible and justify your answer:
    (i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (3x - 2z, y, 3x + 4y)$.
    (ii) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2$.

********* END *********