Q Find a particular soln of

$$y''-3y'-4y=2\sin t$$

$y(t)=A\sin t$, A is a constant to be determined

$y'(t)=A\cos t$, $y''(t)=-A\sin t$

$\therefore$ $-A\sin t-3A\cos t-4A\sin t=2\sin t$

$\Rightarrow (2+5A)\sin t+3A\cos t=0$ — ⊛

We want ⊛ to hold $\forall t$.

Thus it must hold for $t=0$ & $t=\pi/2$

⊛ $\Rightarrow 3A=0 \Rightarrow A=0$

$2+5A=0$

Hence, our assumption about $y(t)$ is not proper.

Modify

Let $y(t)=A\sin t+B\cos t$, where A & B are constants to be determined

$y'(t)=A\cos t-B\sin t$

$y''(t)=-A\sin t-B\cos t$

$(-A\sin t-B\cos t)-3(A\cos t-B\sin t)-4(A\sin t+B\cos t)$

$=2\sin t$

$\Rightarrow (-A+3B-4A)\sin t+(-B-3A-4B)\cos t=2\sin t$

Equating coeffs. of sin & cos

$$\left.\begin{aligned}-5A+3B&=2\\-5B-3A&=0\end{aligned}\right\}\quad \begin{aligned}A&=-5/17\\B&=3/17\end{aligned}$$

PI is $y(t)=\frac{-5}{17}\sin t+\frac{3}{17}\cos t$

The method illustrated can be used when Right side of eqn is polynomial

NOTE :- When the RHS of ODE is a polynomial

$$y'' - 3y' - 4y = 4t^2 - 1$$

we assume PI $y(t) = At^2 + Bt + C$

same degree poly as the non homogeneous term.

eqn

-coeff

SUMMARY

Consider ODE $ay'' + by' + cy = g(t)$.

(1) If $g(t) = e^{\alpha t}$ then $y(t) = Ae^{\alpha t}$.

(2) If $g(t) = \sin \alpha t$ or $\cos \alpha t$, then

$$y(t) = A \sin \alpha t + B \cos \alpha t$$

(3) If $g(t)$ is a polynomial, then $y(t)$ is a polynomial of same degree.

- The same principle extends to the case when $g(t)$ is a product of any two, or all three, of these types of functions

Q Find a p.s. of $y'' - 3y' - 4y = -8e^t \cos 2t$

PI $y(t) = A e^t \cos 2t + B e^t \sin 2t$

($y(t)$ is the product of e^t & a linear combination of $\cos 2t$ & $\sin 2t$)

$Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$

$\Rightarrow \quad 0 = 2e^{-t} \quad \rightarrow \leftarrow$

Why such a situation arise?

Corresponding homogeneous eqn

$$y'' - 3y' - 4y = 0$$

CE $\quad m^2 - 3m - 4 = 0$

$\Rightarrow (m+1)(m-4) = 0$

$\Rightarrow m_1 = -1 \quad m_2 = 4$

$\therefore y_1(t) = e^{-t} \qquad y_2(t) = e^{4t}$ they form a fundamen set of solutions of $L[y] = 0$

- Thus our assumed PI is a sol of corresponding $L[y]=0$

To find a particular sol of ① $y(t) = Ate^{-t}$

$y'(t) = -Ate^{-t} + Ae^{-t}$

$y''(t) = Ate^{-t} - Ae^{-t} - Ae^{-t}$

$= -2Ae^{-t} + Ate^{-t}$

$-5A = 2 \Rightarrow A = -2/5$

Hence PI is $y(t) = -\frac{2}{5}te^{-t}$

<u>NOTE</u> If the assumed form of PI duplicates a soln of corresponding eqn, then modify the assumed PI by multiplying t. If this modification is insuffi then multiply by t second time.

Summary

$$ay'' + by' + cy = g(t)$$

a, b, c are constants

1) Find g.s of corresponding homogeneous eqn

2. $g(t)$ — exp, sines, cosin, poly, sum or pdt

3. $g(t) = g_1(t) + \cdots g_n(t)$

form n subproblem

4. $y_i'(t)$

any duplication multiply t, or by t^2

5) $Y_1(t) + \cdots + Y_n(t)$

6) Form sum CF + PI

7) Use initial cond

Summary

$$ay'' + by' + cy = g(t)$$

$$g(t) = \sum_{k=1}^{n} g_k(t)$$

$$ay'' + by' + cy = g_1(t)$$

$$ay'' + by' + cy = g_2(t)$$

$$\vdots$$

$$ay'' + by' + cy = g_n(t)$$

The particular soln of

$ay'' + by' + cy = g(t)$ is of the below form

$g(t)$	PI $y(t)$
$P_n(t) = a_0 t^n + \cdots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots A_n)$
$P_n(t) e^{\alpha t}$	$t^s(A_0 t^n + \cdots + A_n) e^{\alpha t}$
$P_n(t) e^{\alpha t} \begin{cases} \sin\beta t \\ \cos\beta t \end{cases}$	$t^s [A_0 t^n + \cdots + A_n] e^{\alpha t} \cos\beta t + [B_0 t^n + B_1 t^{n-1} + \cdots + B_n] e^{\alpha t} \sin$

Here s is the smallest non neg integer (0, 1, 2) that will ensure that no term in $y_i(t)$ is a solution of corres homogeneous eqn

Q Solve the ODE

$$y'' + 2y' + 5y = \begin{cases} 1 & 0 \le t \le \pi/2 \\ 0 & t > \pi/2 \end{cases}$$

with initial conditions $y(0) = 0$ & $y'(0) = 0$.

Ans Characteristic eqn is

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$

$\therefore$ CF is $c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

Now, by inspection, PI is

$$y_p(t) = \begin{cases} 1/5 & 0 \le t \le \pi/2 \\ 0 & t > \pi/2 \end{cases}$$

For $0 \le t \le \pi/2$, the general soln is

$$y(t) = y_c(t) + y_p(t)$$

$$= e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{5}$$

ICs is $y(0) = 0 = y'(0)$

$$c_1 = \frac{-1}{5}, \quad c_2 = \frac{-1}{10}$$

$$\therefore y(t) = \frac{1}{5} - \frac{1}{10}(2e^{-t}\cos 2t + e^{-t}\sin 2t)$$

on $0 \le t \le \pi/2$

$$y'' + p(t)y' + q(t)y = g(t)$$

Assume C.F of (1)

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Soln corresponding to

$$y'' + p(t)y' + q(t)y = 0$$

KEY $\boxed{y = u_1(t) y_1(t) + u_2(t) y_2(t)}$

$$y' = u_1'(t) y_1(t) + u_2'(t) y_2(t) + u_1(t) y_1'(t) + u_2(t) y_2'(t)$$

SET $u_1'(t) y_1(t)\ \ u_2'(t) y_2(t) = 0$

Then,

$$y' = u_1(t) y_1'(t) + u_2(t) y_2'(t)$$

$$y'' = u_1'(t) y_1'(t) + u_1(t) y_1''(t) + u_2'(t) y_2'(t) + u_2(t) y_2''(t)$$

Now put y, y' & y'' into (1)

$$\cancel{u_1'(t) y_1''(t) + u_1(t) y_1''(t) + u_2'(t) y_2'(t) + u_2(t) y_2''(t)}$$

$$u_1'(t) y_1'(t) + u_2'(t) y_2'(t) = g(t)$$

Continue after pg 3

Now $|A| \neq 0$ as y_1 & y_2 are fundamental set of solutions

$|A| = W(y_1, y_2)(t)$

Solving (6) & (10), we get

$$\left.\begin{aligned} u_1'(t) &= \frac{-y_2(t) g(t)}{W(y_1, y_2)(t)} \\ u_2'(t) &= \frac{y_1(t) g(t)}{W(y_1, y_2)(t)} \end{aligned}\right\} \quad (11)$$

where $W(y_1, y_2)$ is non zero, as y_1 & y_2 are fundamental set of solutions.

By integrating (11) we get :-

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W(y_1, y_2)(t)} dt + c_1$$

$$\& \; u_2(t) = \int \frac{y_1(t) g(t)}{W(y_1, y_2)(t)} dt + c_2$$

— (12)

Putting $u_1(t)$ & $u_2(t)$ in (4) we get the general soln of (1) as.

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{CF} \underbrace{\left[-y_1 \int \frac{y_2(t) g(t)}{W} dt + y_2 \int \frac{y_1(t) g(t)}{W} dt \right]}_{PI}$$

Q. Apply the method of variation of parameter to solve the following ODE :-

$$y'' + a^2 y = \sec(ax) \quad —(1)$$

Ans Characteristic eqn is $m^2 + a^2 = 0$

$m = \pm ai$

CF of (1) is

$$y_c(x) = A\cos(ax) + B\sin(ax)$$

where A & B are constant.

Assume

$$Y = A(x)\cos(ax) + B(x)\sin(ax) \quad —(2)$$

is a GS of (1)

$$\frac{dY}{dx} = -aA(x)\sin(ax) + aB(x)\cos(ax) + \cos(ax)A'(x) + \sin(ax)B'(x)$$

We choose A & B s.t.

$$\cos(ax)\frac{dA}{dx} + \sin(ax)\frac{dB}{dx} = 0 \quad —(3)$$

Hence $\frac{dY}{dx} = -aA(x)\sin(ax) + aB(x)\cos(ax) \quad —(4)$

$$\Rightarrow \frac{d^2y}{dx^2} = -a^2A(x)\cos(ax) - a^2B(x)\sin(ax) - aA'(x)\sin(ax) + aB'(x)\cos(ax)$$

Putting $y, y' \& y''$ from (2), (4) & (5) into (1) we get

$-a^2 A\cancel{(x)}\cos(ax) - a^2\cancel{B(x)}\sin(ax)$

$-aA'(x)\sin(ax) + aB'(x)\cos(ax)$

$+a^2\cancel{A(x)}\cos(ax) + a^2\cancel{B(x)}\sin(ax) = \sec(ax)$

$\Rightarrow -aA'(x)\sin(ax) + aB'(x)\cos(ax) = \sec(ax)$ (6)

(3) $\Rightarrow$

$\cos(ax)\, A'(x) + \sin(ax)\, B'(x) = 0$

Solving it, we get

$$A'(x) = \frac{-\sin(ax)\sec(ax)}{W(y_1, y_2)(x)} = -\frac{\tan(ax)}{a}$$

$$B'(x) = \frac{\cos(ax)\sec(ax)}{W(y_1, y_2)(x)} = \frac{1}{a}$$

Now, $W(y_1, y_2) = a$

As $B'(x) = \frac{1}{a} \Rightarrow B(x) = \frac{x}{a} + c_2$

$A'(x) = -\frac{\tan(ax)}{a} \Rightarrow A(x) = \frac{1}{a^2}\log(\cos(ax)) + c_1$

GS of (1) is

$y = A(x)\cos(ax) + B(x)\sin(ax)$

Remark Suppose $g(t)$ is sum of two terms.

$$g(t) = g_1(t) + g_2(t)$$

Suppose Y_1 & Y_2 are solutions of eqns

$$ay'' + by' + cy = g_1(t)$$

&

$$ay'' + by' + cy = g_2(t)$$

Then $Y_1 + Y_2$ is a solution of

$$ay'' + by' + cy = g(t)$$

Check $\hookrightarrow$ $g(t)$ any finite no. of terms

Q Find a PI of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos 2t$$

$\downarrow$

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^t\cos 2t$$

Q Find a PI of

$$y'' - 3y' - 4y = 2e^{-t} \quad \text{— (I)}$$

so Assume $Y(t) = Ae^{-t}$

$\therefore Y'(t) = -Ae^{-t}$, $Y''(t) = Ae^{-t}$

Putting Y, Y' & Y'' into (I), we get

From eqn(1)

$$e^{2t}\left[e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right] + a e^{t} e^{-t}\frac{dy}{dt} + by = 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0} \quad \text{—(2)}$$

which has constant coefficients

If $y_1(t)$ & $y_2(t)$ form a fundamental set of solutions of eqn(2), then they are also fundamental set of solutions of eqn(1)

Hence, $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$\Rightarrow y(x) = c_1 y_1(\ln x) + c_2 y_2(\ln x)$ — $\square$

Eg $x^2y'' + xy' - 4y = 0$, $x > 0$

Solution :- Here $a = 1$, $b = -4$

Let $x = e^t$ i.e. $t = \ln x$.

Here the given eqn reduces to

$y'' + (a-1)y' + by = 0$

$\Rightarrow y'' + 0.y' \cancel{*} - 4y = 0$

$\Rightarrow y'' - 4y = 0$

$m^2 = 4 \quad m = \pm 2$

GS $y(t) = c_1 e^{2t} + c_2 e^{-2t}$

$y(x) = c_1 e^{2\ln x} + c_2 e^{-2\ln x}$

$\boxed{y(x) = c_1 x^2 + c_2 x^{-2}}$

mark

If the eqn is of this form

$$\boxed{x^2y'' + axy' + by = f(x)},\ x>0$$

then you write

$$y'' + a\frac{1}{x}y' + \frac{b}{x^2}y = \frac{f(x)}{x^2} = g(x)$$

Then you solve for y_c & y_p ie solve

1. $y'' + a\frac{1}{x}y' + \frac{b}{x^2}y = 0 \Rightarrow x^2y'' + axy' + by' = 0$ to get y_c.

2. $y'' + a\frac{1}{x}y' + \frac{b}{x^2}y = g(x)$ to get y_p by method of variation of parameters

Then GS $\boxed{y(x) = y_c + y_p}$

Linear ODE with variable coefficients

- The method of undetermined coefficients to find a particular integral is valid only for a restricted class of constant coefficients & linear ODE.
- Even among with constant coefficients ODE it's not applicable to all. For instance, it can't be applied for the ODE $y'' + y = \tan x$.
- We now introduce the method of variation of parameters for variable coefficients linear ODE

Lecture

Method 3:- Changing independent variable: Euler type eqn can be transferred into 2nd order D E with constant coefficients (1)

An eqn of the form

$$x^2 y'' + axy' + by = 0, \; x>0 \quad —(1)$$

is called an Euler eqn, where a, b, c are real constants.

We can transform this equation to an equation with constant coefficients by change of independent variable

Let $t = \ln x$ (i.e. $x = e^t$)

$$\frac{dt}{dx} = \frac{1}{x} = e^{-t}, \; \frac{d^2t}{dx^2} = -\frac{1}{x^2} = -e^{-2t}$$

$$\left(\frac{dt}{dx}\right)^2 = (e^{-t})^2 = e^{-2t}.$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{1}{x}\frac{dy}{dt} = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right)$$

Product rule:- $\frac{d}{dx}(e^{-t})\frac{dy}{dt} + e^{-t}\frac{d}{dx}\left(\frac{dy}{dt}\right)$

$$= -e^{-t}\frac{dt}{dx}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$= -e^{-2t}\frac{dy}{dt} + e^{-2t}\frac{d^2y}{dt^2}$$