- Conservative Field
- Independence of Path

Conservative Vector Field

A vector field $\vec{V}$ is said to be conservative if the vector function can be written as the gradient of a scalar function f, i.e., $\vec{V} = \nabla f$.

The function f is called a potential function or a potential of $\vec{V}$.

Example : Show that the vector field $\vec{F} = (2x + y, x, 2z)$ is conservative.

$\vec{F}$ is conservative if it can be written as $\vec{F} = \nabla f$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x + y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial f}{\partial x} = 2x + y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x + y \ \Rightarrow f = x^2 + xy + h(y,z) \ \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial h}{\partial y}$$

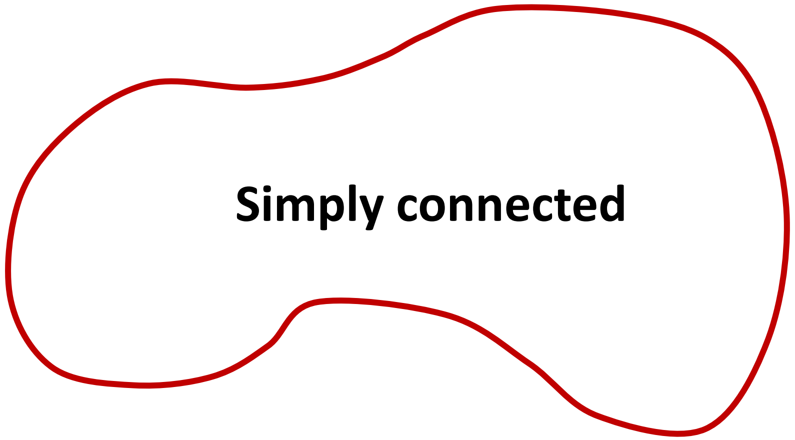
$\Rightarrow x = x + \dfrac{\partial h}{\partial y} \ \Rightarrow \dfrac{\partial h}{\partial y} = 0 \ \Rightarrow h$ is independent of y, i.e., $h = h(z)$

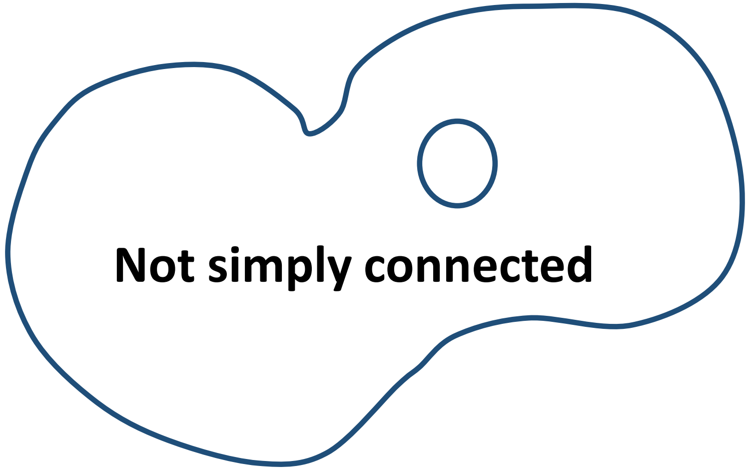
Using the last equation $2z = 0 + \dfrac{dh}{dz} \ \Rightarrow h = z^2 + c$

$$\Rightarrow f = x^2 + xy + z^2 + c$$

Simply Connected domain

A domain D (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is simply connected if it consists of a single connected piece and if every simple, closed curve C in D can be continuously shrunk to a point while remaining in D throughout the deformation.





Test for Conservative Field

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a vector field whose components have continuous first order partial derivatives in a simply connected domain D.

$\vec{F}$ is conservative if and only if $\nabla \times \vec{F} = 0$ at all points of D

Equivalently, $\vec{F}$ is conservative if and only if

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \quad \& \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad \& \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Proof : (conservative $\Longrightarrow \nabla \times \vec{F} = 0$)

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$$

$$= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$ (assuming that $\vec{F}$ is conservative)

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$ (partial derivatives are continuous)

$$= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial F_2}{\partial z}$$

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

Similarly other relations can be proved.

Problem Show that $\vec{F} = (e^x \cos y + yz)\,\hat{\imath} + (xz - e^x \sin y)\,\hat{\jmath} + (xy + z)\,\hat{k}$ is conservative.

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

Solution $F_1 = (e^x \cos y + yz)$ $\quad F_2 = (xz - e^x \sin y)$ $\quad F_3 = (xy + z)$

$$\frac{\partial F_3}{\partial y} = x = \frac{\partial F_2}{\partial z} \qquad \frac{\partial F_2}{\partial x} = z - e^x \sin y = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = y = \frac{\partial F_3}{\partial x}$$

$\Rightarrow \vec{F}$ is conservative that is $\vec{F} = \nabla f$

Path Independence

Let $\vec{F}$ be a vector field defined on a simply connected domain D and suppose that for any two points A and B in D the integral

$$\int_A^B \vec{F} \cdot d\vec{r}$$

is same over all paths from A to B in the domain D.

Then the integral $\int_A^B \vec{F} \cdot d\vec{r}$ is called path independent in D.

Independence of Path and Conservative Vector Fields

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a vector field whose components are continuous throughout a simply connected region D in space. Then there exists a differentiable function f such that

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \qquad (\vec{F} \text{ is conservative in } D)$$

if and only if the integral $\int_A^B \vec{F}.d\vec{r}$ is independent of the path in D.

Proof $\vec{F} = \nabla f \Rightarrow$ Path Independence

Let the curve C be given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$\int \vec{F}.d\vec{r} = \int_a^b \vec{F}\big(r(t)\big) \cdot \frac{d\vec{r}}{dt}dt$$

$$= \int_a^b \frac{df}{dt} \cdot dt = f(b) - f(a)$$

$$\Longrightarrow \boxed{\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)}$$

KEY TAKEAWAY

A vector field $\vec{V}$ is said to be conservative $\vec{V} = \nabla f$.

Equivalent Conditions:

1. The field $\vec{F}$ is conservative.

2. $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D

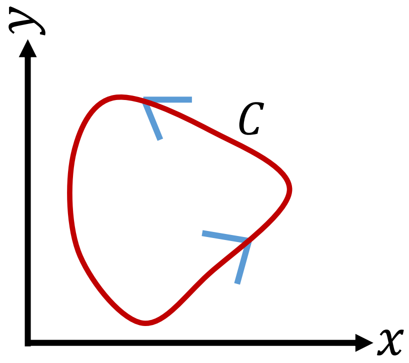
3. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve in D

C be a piecewise smooth curve in a simply connected domain D.

Green’s Theorem

- Transformation between double integrals and line integral

Green's theorem:



Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C which is piecewise smooth (oriented counter clockwise – when traversed on C the region R always lies left).

Let $\vec{F} = F_1(x,y)\,\hat{\imath} + F_2(x,y)\hat{\jmath}$ be smooth vector field (F_1 & F_2 are C^1 functions) on both R and C, then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Note that the above can also be written as

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\mathrm{curl}\,\vec{F}\right)\cdot\hat{k}\;\,dA$$

Problem -1 Verify Green's theorem for the vector field $\vec{F}(x,y) = (x-y)\hat{\imath} + x\,\hat{\jmath}$

The region R is bounded by the circle $C: \vec{r}(t) = \cos t\ \hat{\imath} + \sin t\ \hat{\jmath} \qquad 0 \le t \le 2\pi$

Solution: $F_1 = x - y \implies \dfrac{\partial F_1}{\partial y} = -1 \qquad F_2 = x \implies \dfrac{\partial F_2}{\partial x} = 1$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = 2 \iint_R dxdy = 2\pi$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left((\cos t - \sin t)\hat{\imath} + \cos t\,\hat{\jmath} \right) \cdot (-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath})\,dt$$

$$= \int_0^{2\pi} (-\cos t\ \sin t\ + \sin^2 t + \cos^2 t)\,dt = 2\pi - \frac{1}{2}\int_0^{2\pi} \sin 2t\,dt = 2\pi$$

Problem -2 Evaluate the integral $\oint_C xy\,dy - y^2\,dx$ using Green's theorem.

Here C is the square cut from the first quadrant by the lines $x = 1$ & $y = 1$.

Solution:

$$\oint_C \underbrace{xy}_{F_2}\ dy - \underbrace{y^2}_{F_1} dx = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$= \int_0^1 \int_0^1 (y + 2y)\,dx\,dy$$

$$= \frac{3}{2}$$

Problem-3 Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

Solution: Green's theorem: $\frac{1}{2}\oint_C \underbrace{x}_{F_2}\,dy - \underbrace{y}_{F_1}\,dx = \frac{1}{2}\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$

$$= \frac{1}{2}\iint_R [1-(-1)]dxdy$$

$$= \iint_R dx\,dy$$

$=$ Area of R

Problem - 4 Using Green's theorem, find the area of the ellipse $x = a\cos\theta\ ,\ y = b\sin\theta$

Solution: Using Green's theorem

$$\text{Area of ellipse} = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)d\theta - (b\sin\theta)(-a\sin\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)d\theta$$

$$= \pi ab$$

Problem - 5 Evaluate $\oint_C (x^2 + y^2)\, dx + 2xy\, dy$, C is the boundary of the region

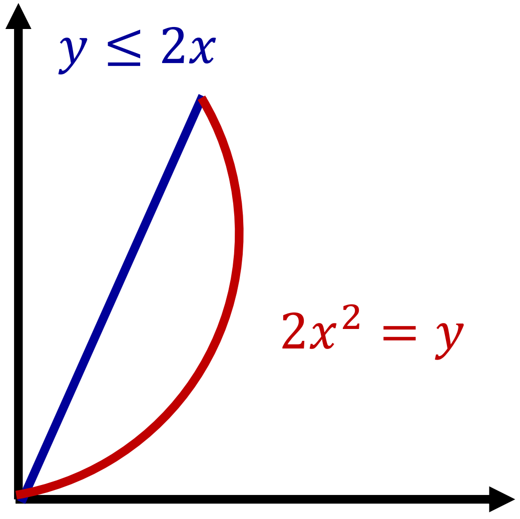
$$R = \{(x, y): 0 \le x \le 1, 2x^2 \le y \le 2x\}$$

Solution: Using Green's theorem

$$\oint_C (x^2 + y^2)\, dx + 2xy\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$= \frac{1}{2}\iint_R (2y - 2y)\, dxdy$$

$$= 0$$



Note: $(x^2 + y^2)\,\hat{\imath} + 2xy\,\hat{\jmath} = \nabla\left(\frac{1}{3}x^3 + xy^2 + c\right)$ conservative vector field

NOTE: Consider $\vec{F}(x,y) = -\dfrac{y}{x^2+y^2}\hat{\imath} + \dfrac{x}{x^2+y^2}\hat{\jmath}$ $\quad R = \{(x,y) \colon 0 < x^2+y^2 \le 1\}$

$C\colon\ x = \cos\theta\,, y = \sin\theta$ $\quad \vec{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ $\quad \Rightarrow \dfrac{d\vec{r}}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

$$\oint_C \vec{F}.d\,\vec{r} = \int_{\theta=0}^{2\pi} (-\sin\theta)(-\sin\theta) + \cos\theta\cos\theta\ d\theta = 2\pi$$

Whereas: $\displaystyle\iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right) dxdy$

$$= \iint_R \left(\frac{(x^2+y^2) - x\,2x}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y\,2y}{(x^2+y^2)^2}\right) dxdy = 0$$

Does it contradict Green’s theorem?

KEY TAKEAWAY

- GREEN's THEOREM

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_1\, dx + F_2\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\text{curl } \vec{F} \right) \cdot \hat{k}\; dxdy$$

- Smooth Surfaces
- Evaluation of Surface Area
- Surface Integral of a Scalar Function

Smooth Surface

Recall that a curve is called smooth if it has a continuous tangent.

Similarly, a surface is smooth if it has a continuous normal vector.

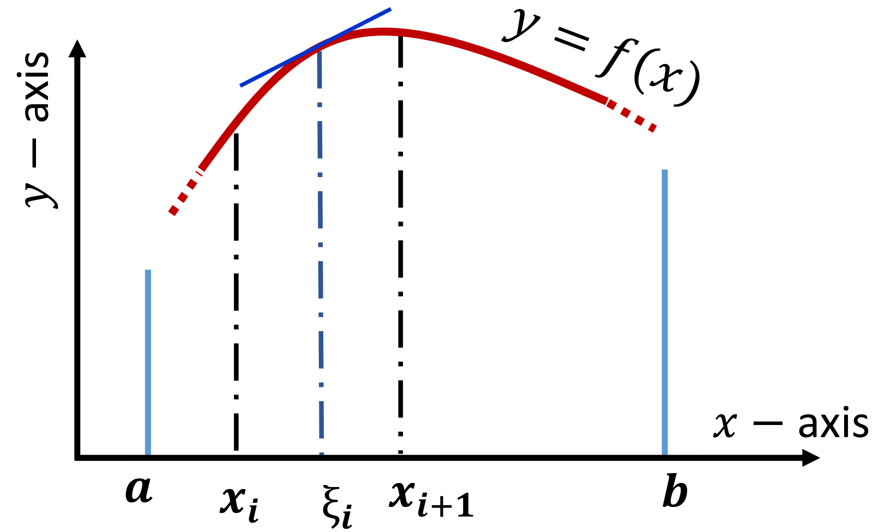
A surface is called piecewise smooth if it consists of finite number of smooth surfaces.

Example: Surface of a Sphere - a smooth surface

Surface of a cube - a piecewise smooth surface

Does not have a normal vector along any of its edges

Evaluation of Arc Length (Recall from Integral Calculus)

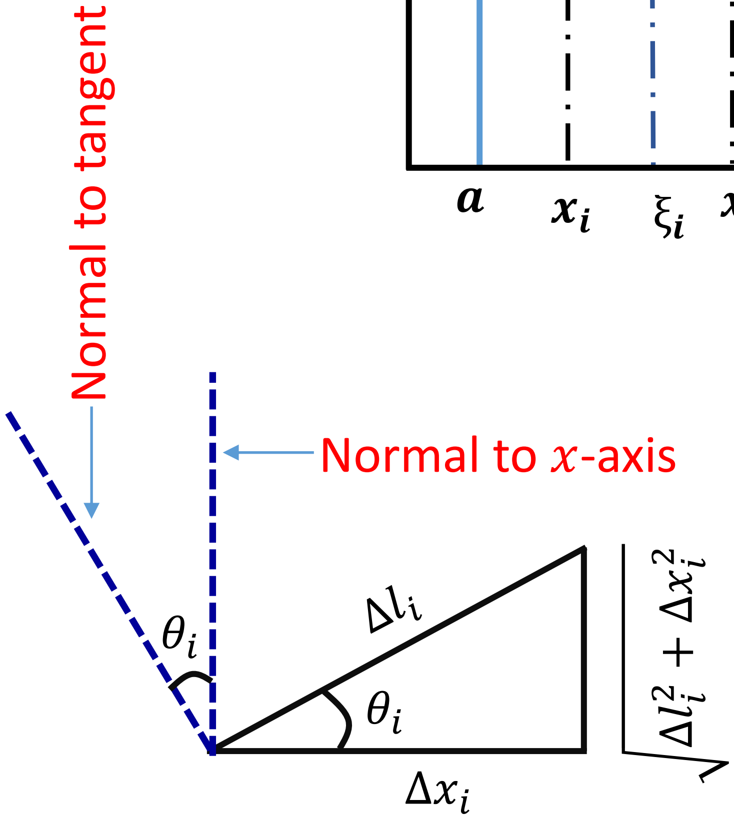


Let θ be the angle of the tangent at ξ_i with the positive x axis

$$\frac{\Delta x_i}{\Delta l_i} = |\cos\theta_i| \Rightarrow \Delta l_i = \frac{1}{|\cos\theta_i|}\Delta x_i$$

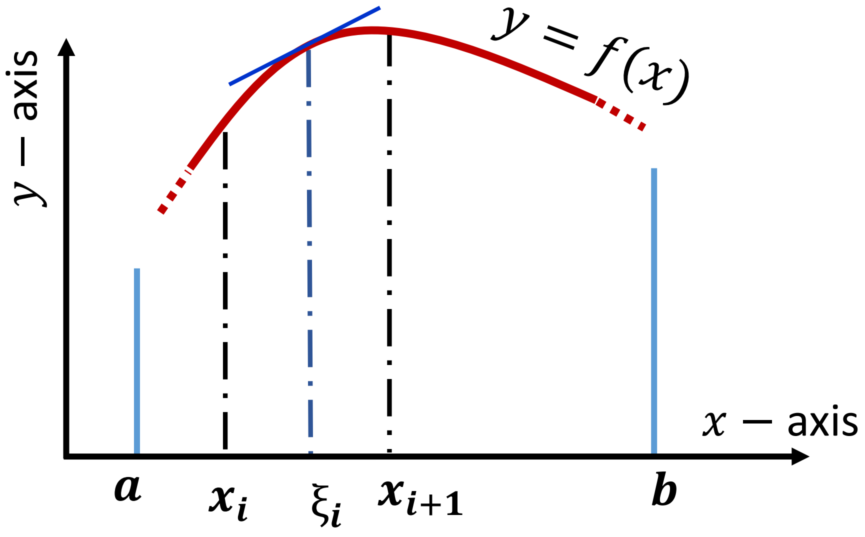
Alternatively $\quad f'(\xi_i) = \dfrac{\sqrt{\Delta l_i^2 + \Delta x_i^2}}{\Delta x_i}$

$$\Longrightarrow \Delta l_i = \sqrt{1 + \left(f'(\xi_i)\right)^2}\,\Delta x_i$$



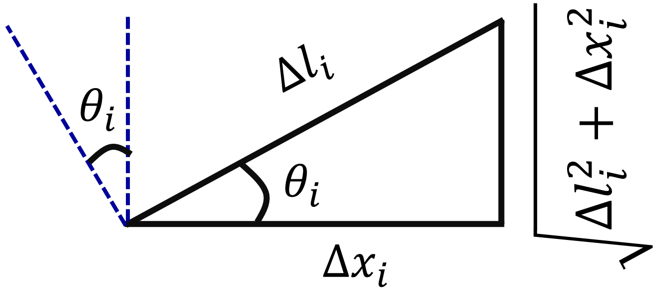
Evaluation of Arc Length (Recall from Integral Calculus)

$$\Delta l_i = \frac{1}{|\cos\theta_i|}\Delta x_i \qquad \Delta l_i = \sqrt{1+\left(f'(\xi_i)\right)^2}\,\Delta x_i$$



Arc length $L = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta l_i = \int_C dl = \int_a^b \frac{1}{|\cos\theta|}dx$

$$= \int_a^b \sqrt{1+\left((f'(x)\right)^2}\;dx$$



Arc length differential $\quad dl = \frac{1}{|\cos\theta|}dx = \sqrt{1+f'^2}\;dx$

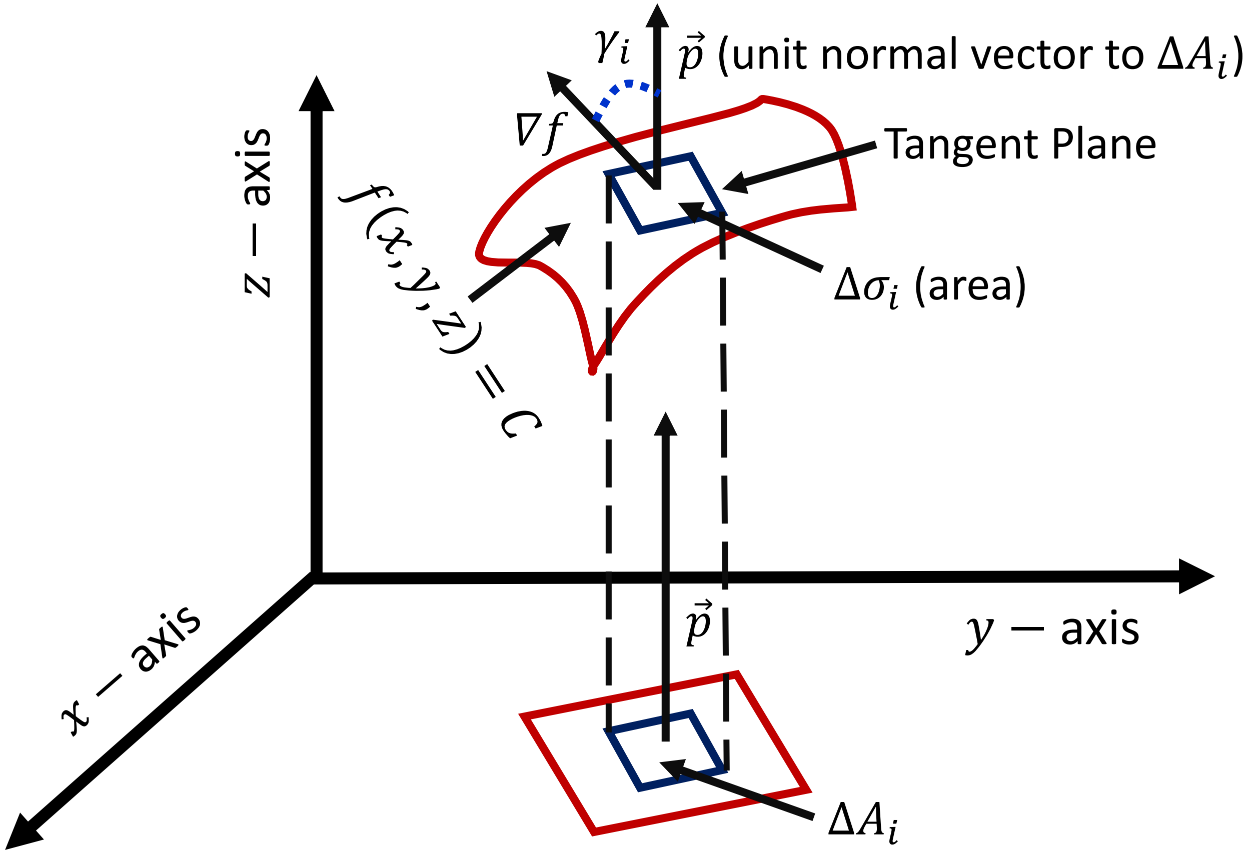
Evaluation of Surface Area

$$\frac{\Delta A_i}{\Delta \sigma_i} = |\cos\gamma_i| \Rightarrow \Delta\sigma_i = \frac{1}{|\cos\gamma_i|}\Delta A_i$$

Surface Area: $S = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta\sigma_i$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{|\cos\gamma_i|}\Delta A_i$$

$$= \iint_R \frac{1}{|\cos\gamma|}\,dA$$

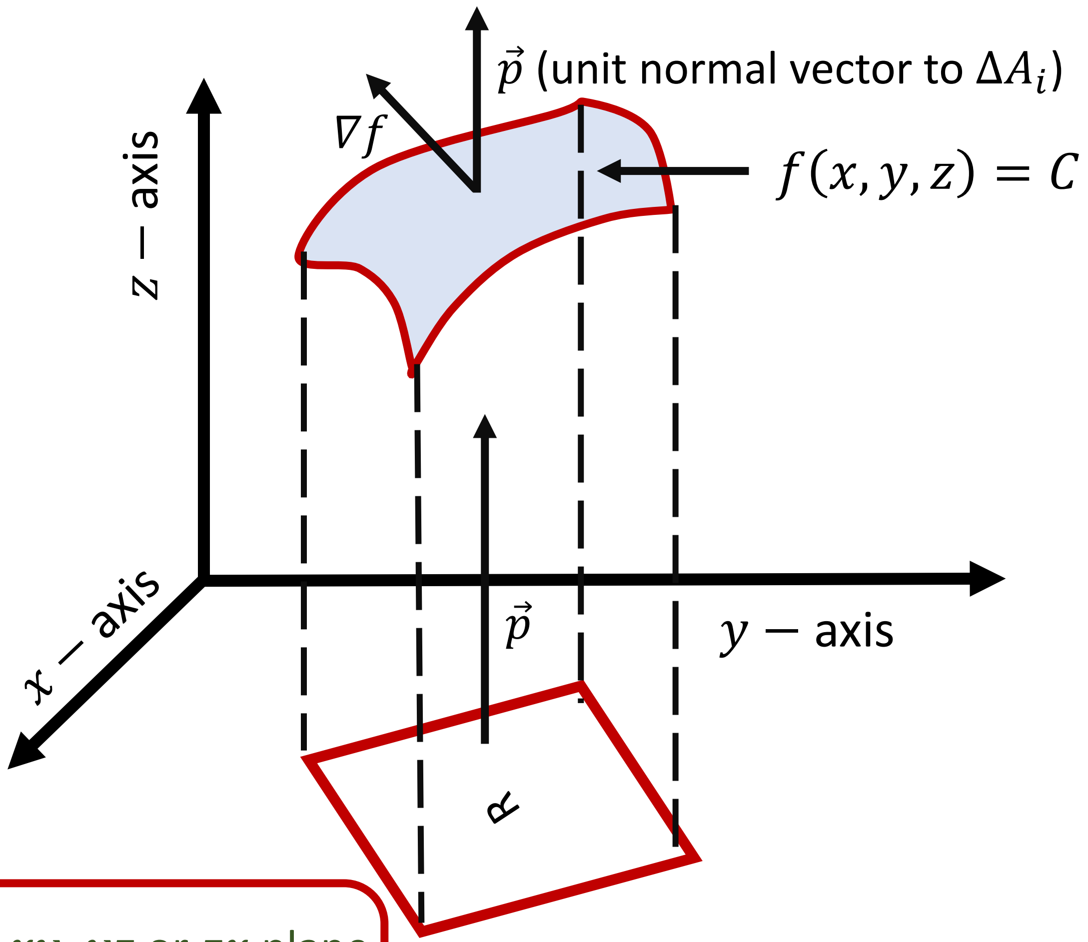


R is the projection of the surface on the xy, yz or zx plane.

$$S = \iint_R \frac{1}{|\cos\gamma|}\, dA$$

Note that : $|\nabla f.\vec{p}| = |\nabla f||\vec{p}||\cos\gamma|$

$$\Rightarrow \frac{1}{|\cos\gamma|} = \frac{|\nabla f|}{|\nabla f.\vec{p}|}$$



The **area of the surface** $f(x,y,z) = C$ **over a closed and bounded plane** $\boldsymbol{R}$:

$$S = \iint_S d\sigma = \iint_R \frac{|\nabla f|}{|\nabla f.\vec{p}|}\, dA$$

R is the projection of S on on the xy, yz or zx plane

$\vec{p}$ is the unit normal to R and $\nabla f.\vec{p} \neq 0$

REMARK: **Recall from Integral Calculus:**

Let $z = g(x, y)$ be the equation of a surface.

Then the surface area (Integral Calculus): $S = \iint\limits_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\, dy$

where R is the projection of the surface in the xy plane

In the vector form the same can be calculated using $S = \iint\limits_R \frac{|\nabla f|}{|\nabla f . \vec{p}|} dA$

Let $f = z - g(x, y) = 0 \ \Rightarrow \nabla f = (-g_x, -g_y, 1)$

$\Rightarrow\ |\nabla f| = \sqrt{1 + z_x^2 + z_y^2}$ $|\nabla f . \vec{p}| = 1$ (considering $\vec{p}$ as the unit normal to xy plane)

Surface Integral: $\iint\limits_S g(x,y,z)\,d\sigma$

Integrating a function over surface using the idea just developed for calculating surface area.

Suppose, for example, we have electrical charge distribution over the surface $f(x,y,z) = C$

Let the function $g(x,y,z)$ gives the change per unit area (charge density) at each point on S

Total charge on S $= \iint\limits_S g(x,y,z)\,d\sigma = \iint\limits_R g(x,y,z)\dfrac{|\nabla f|}{|\nabla f.\vec{p}|}dA$ **Surface integral of g over S**

NOTE:

- if g gives the mass density of a thin shell of material modeled by S, the integral gives the mass of the shell.
- if $g = 1$ then the integral will simply gives the total area of the surface

Problem - 1 Find the area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2$, $z \geq 0$ by the cylinder $x^2 + y^2 = 1$

Solution: Projection of the surface $f(x, y, z) = c$, i.e., $x^2 + y^2 + z^2 = 2$ onto the xy plane : $x^2 + y^2 \leq 1$

Note that $f(x, y, z) = x^2 + y^2 + z^2$

$$\Longrightarrow \ \nabla f = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$\Longrightarrow |\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2}$$

The vector $\vec{p} = \hat{k}$ is normal to the xy plane $\Longrightarrow |\nabla f . \vec{p}| = |2z| = 2z \ \ (\because z \geq 0)$

Surface Area: $S = \iint_R \frac{|\nabla f|}{|\nabla f . \vec{p}|} dA = \iint_R \frac{2\sqrt{2}}{2z} dA = \sqrt{2} \iint_R \frac{dA}{z}$

$x^2 + y^2 + z^2 = 2 \, , z \geq 0$

$|\nabla f| = 2\sqrt{2}$

$|\nabla f . \vec{p}| = 2z$

$$= \sqrt{2} \iint_R \frac{dA}{\sqrt{2 - (x^2 + y^2)}}$$

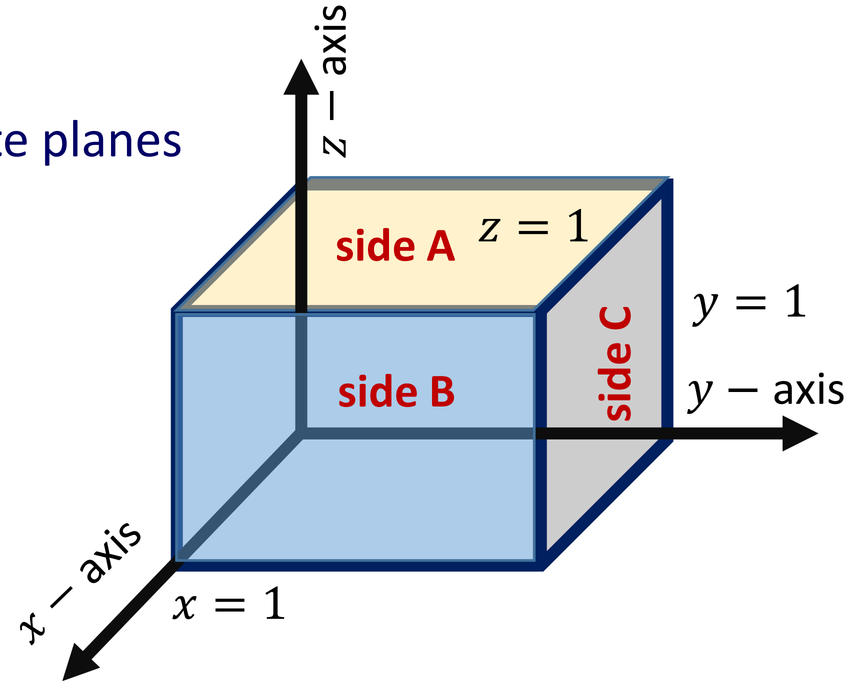
$$= \sqrt{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{r \, dr d\theta}{\sqrt{2 - r^2}}$$

$$= \sqrt{2} \int_0^{2\pi} \left[-\sqrt{(2 - r^2)} \right]_{r=0}^{1} d\theta$$

$$= \sqrt{2} \int_0^{2\pi} (\sqrt{2} - 1) d\theta = 2\pi(2 - \sqrt{2})$$

Problem-2 Integrate $g(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1, y = 1$ and $z = 1$

Solution: Note that $xyz = 0$ on the sides that lie in the coordinate planes



The integral over the surface of the cube reduces to

$$\iint_{\substack{\text{cube}\\\text{surface}}} xyz\, d\sigma = \iint_{\text{side A}} xyz\, d\sigma + \iint_{\text{side B}} xyz\, d\sigma + \iint_{\text{side C}} xyz\, d\sigma$$

side A is the surface $f(x,y,z) = z - 1$ over the region

$\mathbb{R}_{xy}: 0 \le x \le 1, 0 \le y \le 1$ in the xy plane

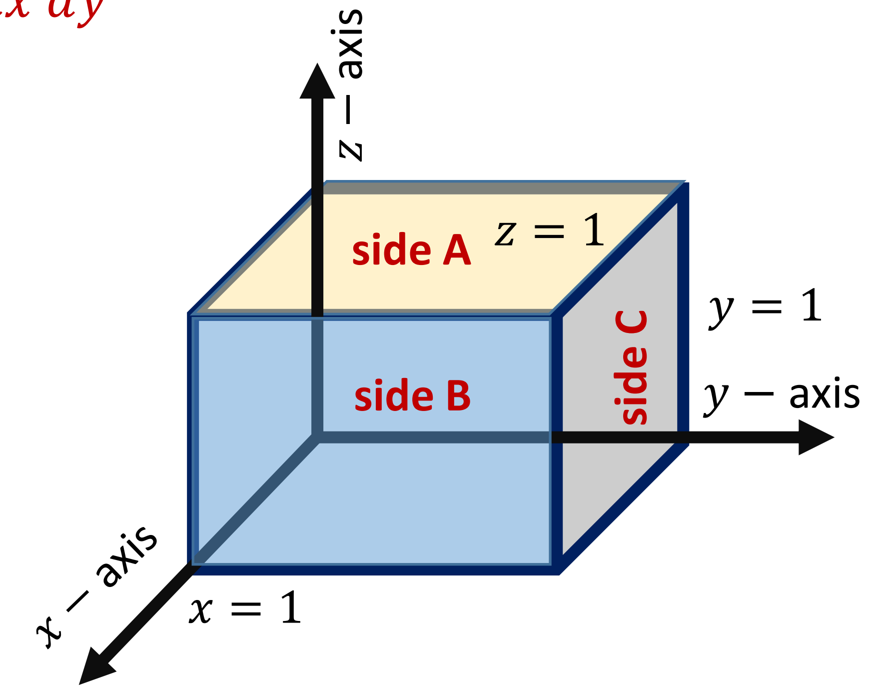
For this surface (side A) and region $\mathbb{R}_{xy}$:

$\vec{p} = \hat{k}, \nabla f = \hat{k} \Longrightarrow |\nabla f| = 1 \;\; \& \;\; |\nabla f.\vec{p}| = 1$

$$\Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f.\vec{p}|} dA = dxdy$$

$$\iint_{\text{side A}} xyz\, d\sigma = \int_0^1 \int_0^1 xy(1)\; dxdy = \frac{1}{4}$$

$$d\sigma = dx\, dy$$



Similarly, we obtain

$$\iint_{\text{side B}} xyz\, d\sigma = \frac{1}{4} \qquad \iint_{\text{side C}} xyz\, d\sigma = \frac{1}{4}$$

$$\iint_{\substack{\text{cube} \\ \text{surface}}} xyz\, d\sigma = 3 \times \frac{1}{4} = \frac{3}{4}$$

KEY TAKEAWAY

- Surface $z = g(x,y)$

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \; dx\, dy$$

- Surface $f = z - g(x,y) = 0$

$$S = \iint_R \frac{|\nabla f|}{|\nabla f . \vec{p}|} \, dA$$

$$\boxed{\iint_S g(x,y,z)\, d\sigma = \iint_R g(x,y,z) \frac{|\nabla f|}{|\nabla f . \vec{p}|} dA}$$