MA 102: Linear Algebra, Integral Transforms and Special Functions
Tutorial Sheet - 2
Second Semester of the Academic Year 2023-2024

1. Let W_1, W_2 be two subspaces of a vector space $V(\mathbb{F})$ and define $W_1+W_2 = \{w_1+w_2 \mid w_1 \in W_1, w_2 \in W_2\}$. Prove that $W_1 + W_2$ is a subspace of $V(\mathbb{F})$. Also establish that it is the smallest subspace of $V(\mathbb{F})$ containing both W_1 and W_2, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

2. A vector space $V(\mathbb{F})$ is called the direct sum of two subspaces W_1 and W_2 and denoted by $V = W_1 \bigoplus W_2$ if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Show that the direct sum of any two subspaces is again a subspace and every vector has a unique representation as sum of vectors from W_1 and W_2.

3. Let W_1 be the subspace of $M_{n\times n}(\mathbb{R})$ consisting of all the skew symmetric $n \times n$ matrices and W_2 be the subspace of $M_{n\times n}(\mathbb{R})$ consisting of all the symmetric $n \times n$ matrices. Prove that $M_{n\times n}(\mathbb{R}) = W_1 \bigoplus W_2$.

4. Determine whether the following sets are subspaces of $\mathbb{R}^3(\mathbb{R})$ under the component wise addition and scalar multiplication. Find $W_1 \cap W_2$, where
$W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$,
$W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$.

5. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \textit{and} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ generate } M_{2\times 2}(\mathbb{R}) \text{ over } \mathbb{R}.$$

6. Write the vector $v = (a, b, c) \in \mathbb{R}^3$ over $\mathbb{R}$ as linear combination of the vectors $u_1 = (1, 2, 0)$, $u_2 = (-1, 1, 2)$ and $u_3 = (3, 0, -4)$.

7. Examine in each case which of the following sets are linearly independent over $\mathbb{R}$:

(a) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$.

(b) $\left\{x^3 + 2x^2,\ -x^2 + 3x + 1,\ x^3 - x^2 + 2x - 1\right\}$ in $P_3(\mathbb{R})$.

(c) $\{(1, -1, 2),\ (2, 0, 1),\ (-1, 2, -1)\}$ in $\mathbb{R}^3$.

7. Check the linearly independence of the sets $\{f,\ g\}$ over $\mathbb{R}$, where:

(a) $f(x) = x,\ g(x) = |x|$.
(b) $f(x) = \cos(x),\ g(x) = \sin(x)$.
(c) $f(x) = e^{rx},\ g(x) = e^{sx}$, for $r \neq s$ and $r, s \in \mathbb{R}$.

8. Consider the vector space $M_{3\times 3}(\mathbb{R})$ over $\mathbb{R}$. Let $A \in M_{3\times 3}(\mathbb{R})$ and $v = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $w = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. Suppose that $Av = -v$ and $Aw = 2w$. Then find the vector $A^5 \begin{pmatrix} -1 \\ 8 \\ -9 \end{pmatrix}$.

9. Let $S = \{(1+i,\ 2\ i,\ 2),\ (1,\ 1+i,\ 1-i)\} \subset \mathbb{C}^3$. Check the linear independence of S over $\mathbb{R}$.

10. Is the set S considered in problem 9 linearly independent over $\mathbb{C}$?

11. Let $v_1 = (a,\ b,\ c),\ v_2 = (d,\ e,\ f),\ v_3 = (g,\ h,\ i)$ be any three vectors in $\mathbb{R}^3(\mathbb{R})$. Show that the set $\{v_1,\ v_2,\ v_3\}$ is linearly dependent over $\mathbb{R}$ if and only if there exists a non zero vector $x = (x_1,\ x_2,\ x_3)$ such that $Ax^T = O$, where A is the matrix $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$.

12. Prove that any set S in a vector space $V(\mathbb{F})$ containing the $\mathbf{0}$ vector is linearly dependent, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

13. Find the value of h for which the following set of vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} h \\ 1 \\ -h \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 2h \\ 3h+1 \end{pmatrix}$ are linearly independent over $\mathbb{R}$.

14. Let V be a vector space over a field $\mathbb{F}$ and $S_1 \subseteq S_2 \subseteq V$, then prove that:

    (a) If S_2 is linearly independent set over $\mathbb{F}$ then so is S_1.

    (b) If S_1 is linearly dependent set over $\mathbb{F}$ then so is S_2, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

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