Counting

and some examples around us

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Outline

Acknowledgement and disclaimer

All mistakes (if any) are mine.
I have used several other sources which I have referred to in the appropriate places.

Section 1

India: Its states and UTs

India: Its states and UTs

There are 28 states and 8 UTs in India. The Election Commission of India holds the elections for the state legislative assemblies and central parliament every 5 years.

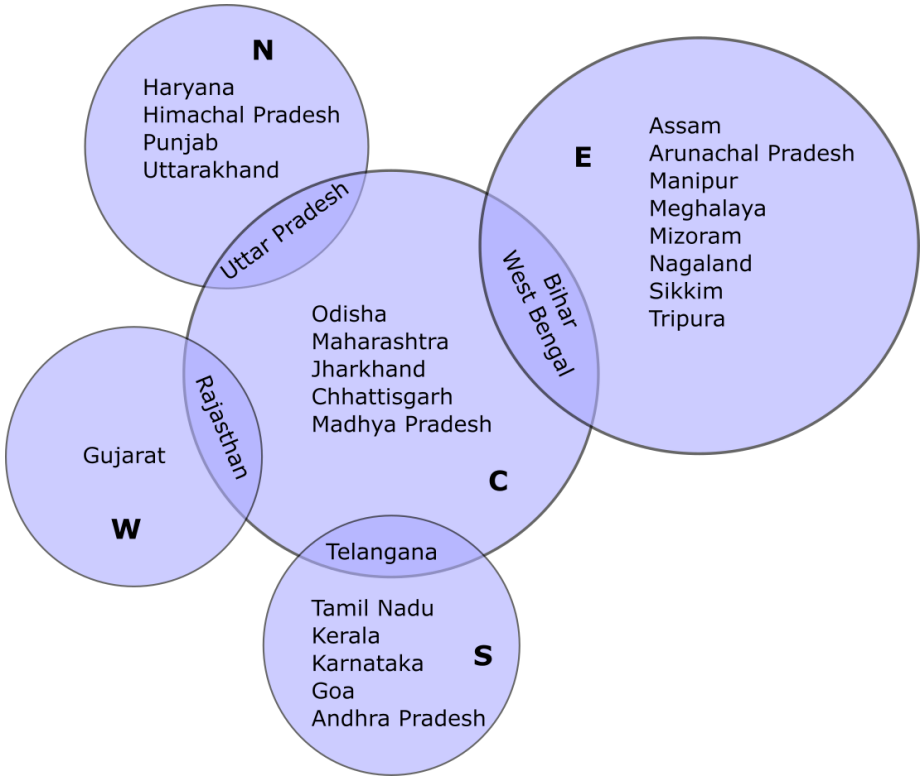
Not an expert at Geography, but we can (roughly) divide the states into the following geographical regions:

- North: Haryana, Himachal Pradesh, Punjab, Uttarakhand, Uttar Pradesh
- South: Tamil Nadu, Kerala, Karnataka, Goa, Andhra Pradesh, *Telangana*
- West: Gujarat, *Rajasthan*
- East: *Bihar*, *West Bengal*, Assam, Arunachal Pradesh, Manipur, Meghalaya, Mizoram, Nagaland, Sikkim, Tripura
- Central: Odisha, Maharashtra, Jharkhand, Chhattisgarh, *Telangana*, Madhya Pradesh, *Rajasthan*, *Bihar*, *West Bengal*, *Uttar Pradesh*

However, some neighboring states can be put in two categories.

Set form

Looks something like this?



Section 2

Principles for counting

The additive principle (or)

Definition

The additive principle states that if event A can occur in m ways, and event B can occur in n disjoint ways, then the event "A or B" can occur in $m + n$ ways.

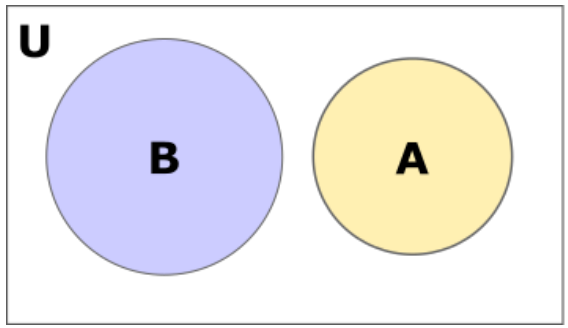
It is important that the events be disjoint: i.e., that there is no way for A and B to both happen at the same time.

For example, a standard deck of 52 cards contains 26 red cards and 12 face cards. What is the number of ways to select a card which is either red or a face card?

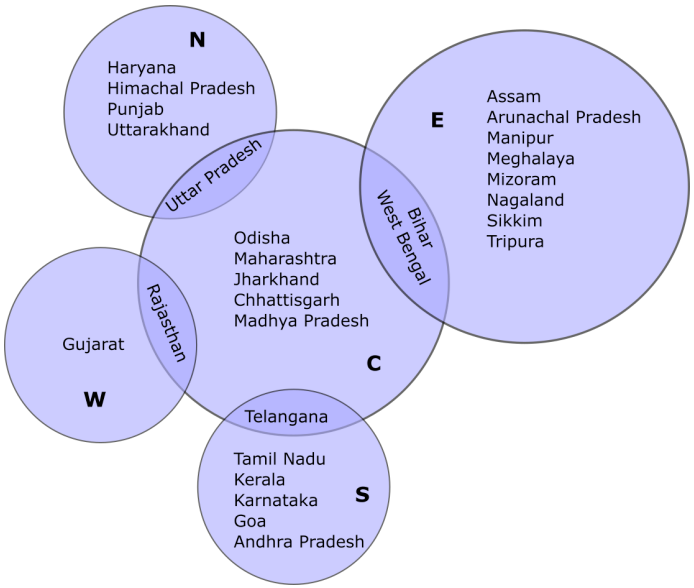
It is not $26 + 12 = 38$. 6 cards are both red and face cards.

The additive principle: from sets

Set definition: Given two sets A and B, if $A \cap B = \emptyset$ (that is, if there is no element in common to both A and B), then $|A \cup B| = |A| + |B|$.



Q: The additive principle



Number of ways to select a state which is in the North or the South?

- North: Haryana, Himachal Pradesh, Punjab, Uttarakhand, Uttar Pradesh
- South: Tamil Nadu, Kerala, Karnataka, Goa, Andhra Pradesh, Telangana
- Number of ways $= |N| + |S| = 5 + 6 = 11$

Subtraction Rule/Inclusion-exclusion Principle (or)

If your events are not *mutually exclusive*, find the total number of possibilities use the subtraction rule.

Definition

Suppose event A can occur m ways, event B can occur n ways, and there are k ways that A and B both occur. Then there are $m + n - k$ ways A or B can occur.

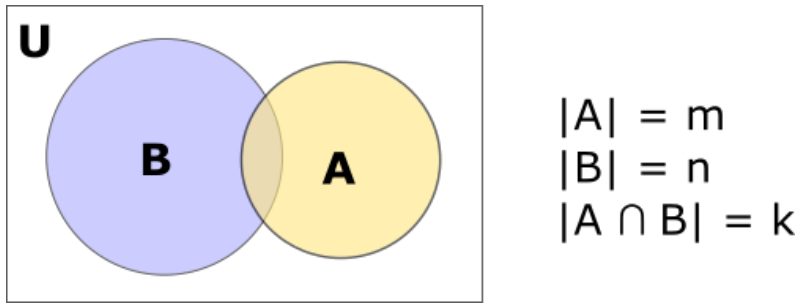
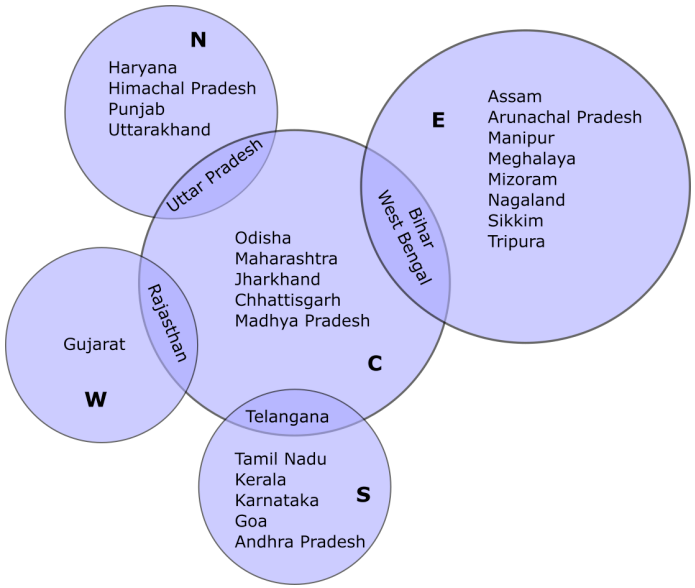


Figure: Overlapping sets

Q: The Inclusion-exclusion Principle



Number of ways to select a state which is in the East or the Centre?

- East: *Bihar*, *West Bengal*, Assam, Arunachal Pradesh, Manipur, Meghalaya, Mizoram, Nagaland, Sikkim, Tripura
- Central: Odisha, Maharashtra, Jharkhand, Chhattisgarh, *Telangana*, Madhya Pradesh, *Rajasthan*, *Bihar*, *West Bengal*, *Uttar Pradesh*

- Number of ways $= |E| + |C| - |E \cap C|$ $= 10 + 10 - 2 = 18$

The multiplicative principle (and)

Definition

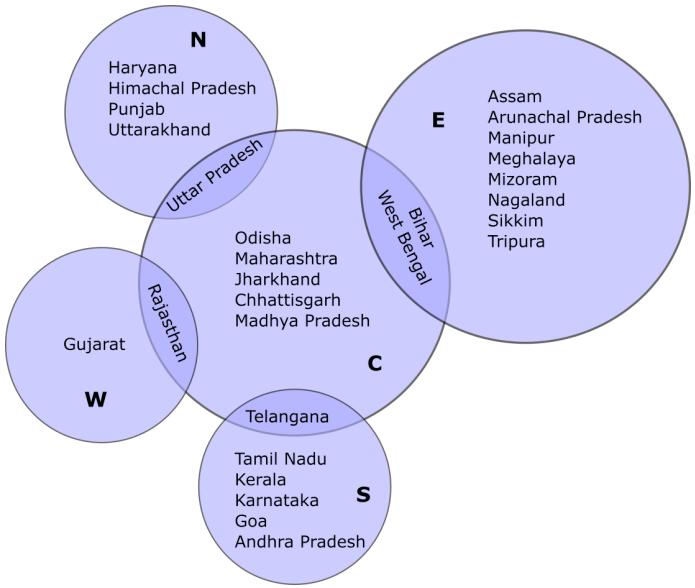
The multiplicative principle states that if event A can occur in m ways, and each possibility for A allows for exactly n ways for event B, then the event "A and B" can occur in $m \times n$ ways.

The multiplicative principle: From Sets

Given sets A and B, we can form the set
$A \times B = \{(x, y) : x \in A \wedge y \in B\}$ to be the set of all ordered pairs (x, y) where x is an element of A and y is an element of B. We call $A \times B$ the Cartesian product of A and B.

Q: The Election Commission Gets Involved

The EC of India decides that it wants to conduct elections in a state from North India in 2021 and subsequently in a state from South India in the year 2022. How many ways can this be done?



- North: Haryana, Himachal Pradesh, Punjab, Uttarakhand, Uttar Pradesh
- South: Tamil Nadu, Kerala, Karnataka, Goa, Andhra Pradesh, *Telangana*
- Number of ways $= |N| \times |S| = 5*6 = 30$

Section 3

Permutations

Permutation

Definition

A permutation is a (possible) rearrangement of objects.

For example, there are 6 permutations of the letters a, b, c: a b c, a c b, b a c, b c a, c a b, c b a
What's the intuition behind computing the number of possible permutations?

3 choices for which letter we put first, then 2 choices for which letter comes next, which leaves only 1 choice for the last letter. The multiplicative principle says we multiply $3 \times 2 \times 1$

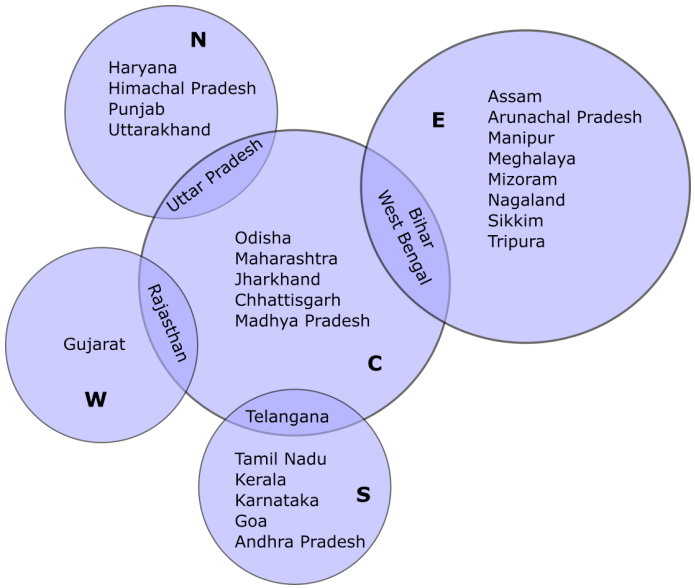
Factorial

A piece of notation is helpful here: $n!$ read "n factorial", is the product of all positive integers less than or equal to n (for *reasons of convenience*, we also define $0!$ to be 1).

Let's write a program to compute factorial of a number n.

Q: The Election Commission's Conundrum

The ECI wants to conduct elections for all the northern states in 2022. How many ways can it do it?



- North: Haryana, Himachal Pradesh, Punjab, Uttarakhand, Uttar Pradesh
- Number of ways $= 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$
- Phew! That's a lot of choices for the ECI.

k-permutations of n elements

We can ask how many permutations exist of k objects choosing those objects from a larger collection of n objects.

$P(n, k)$ is the number of k-permutations of n elements, the number of ways to arrange k objects chosen from n distinct objects.

Intuitively, how do we compute $P(n, k)$?

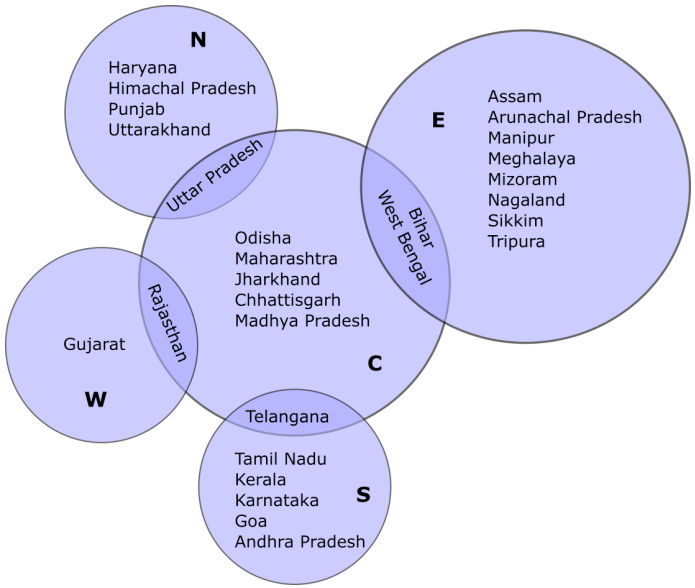
$P(n, k)$ is given by:

$$P(n, k) = n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!}$$

Note that this includes the permutations of the k objects.

Q: The Election Commission's Conundrum

The ECI wants to conduct elections in three of the northern states in 2022. The order in which it is conducted is important. How many permutations exist for this?



- North: Haryana, Himachal Pradesh, Punjab, Uttarakhand, Uttar Pradesh
- Number of ways $= \frac{5!}{(2)!} = 5 \times 4 \times 3 = 60$

Section 4

Combinations

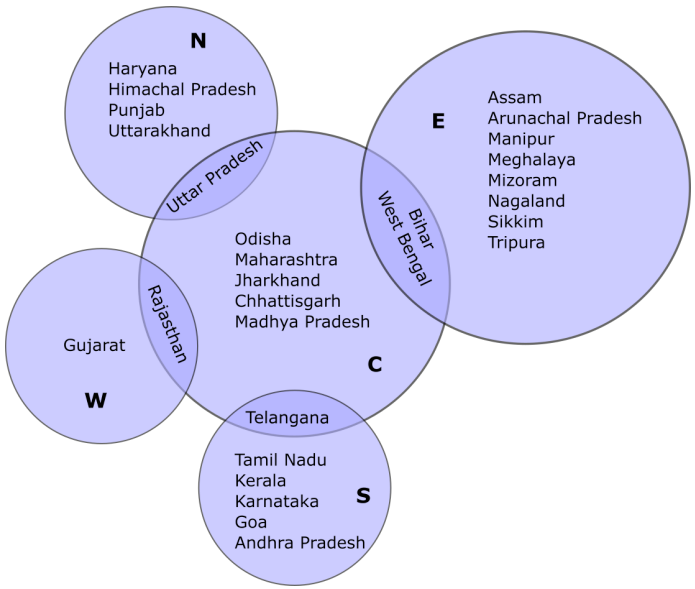
Combinations

What about if we want to discount the different orders or permutations of the k objects?

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Q: The Election Commission's Conundrum

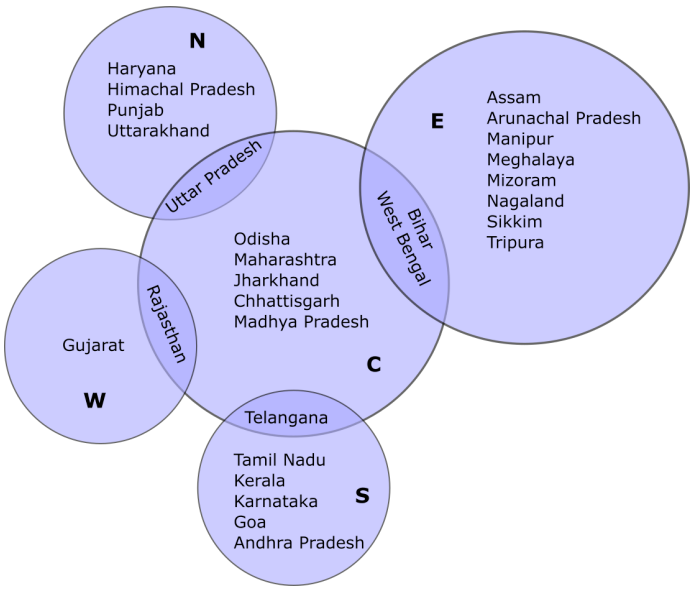
The ECI wants to conduct elections in three of the northern states in 2022. The order in which it is conducted is **not** important. How many combinations exist for this?



- North: Haryana, Himachal Pradesh, Punjab, Uttarakhand, Uttar Pradesh
- Number of ways $= \frac{5!}{(2)!3!} = \frac{120}{12} = 10$

Q: The Election Commission's Conundrum

The ECI wants to conduct elections in three of the northern states in 2021 and two of the southern states in 2022. *Order within the northern/southern states is not important.* How many ways can this be done?



- North: Haryana, Himachal Pradesh, Punjab, Uttarakhand, Uttar Pradesh
- South: Tamil Nadu, Kerala, Karnataka, Goa, Andhra Pradesh, *Telangana*
- Number of ways for North $= \frac{5!}{(2)!3!} = \frac{120}{12} = 10$
- Number of ways for South $= \frac{6!}{(2)!4!} = 15$
- These events are independent of each other. So, finally: $10 \times 15 = 150$

Section 5

Application: figuring out binomial coefficients

Binomial coefficients

Binomial coefficients are the coefficients in the expanded version of a binomial, such as $(x+y)^n$. Let us expand $(x+y)^n$ for various values of n:

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y2+4xy^3+y^4$$

$$(x+y)^5 = ?$$

Binomial coefficients

Essentially, we are simply multiplying out the binomial term. So $(x+y)^3$ can be seen as:

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

Imagine there are three pots, each containing both the variables x and y. You can only pick one variable from each pot to expand the equation.

Let's examine:

$$(x+y)^3 = 1 \cdot x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + 1 \cdot y^3$$

- How many ways can you pick three x from three pots? Pick 1 x from each pot. How many ways to do it? Just 1.
- How many ways can you pick 2 x and 1 y from three pots? Pick 2 x from two of the pots. How many ways to do it? $\binom{3}{2}$.

Binomial coefficients

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is the coefficient of $x^k \cdot y^{n-k}$ in the expansion of $(x+y)^n$.

Q: Expand $(x+y)^5$

Imagine there are five pots, each containing the variables x and y.
You can only pick one variable from each pot to expand the equation:

$$(x+y)^5 = ? \cdot x^5 + ? \cdot x^4y + ? \cdot x^3y^2 + ? \cdot x^2y^3 + ? \cdot xy^4 + ? \cdot y^5$$

Binomial coefficients

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is the coefficient of $x^k \cdot y^{n-k}$ in the expansion of $(x+y)^n$.

Section 6

Counting and Computer Science

Why did we study counting as a computer scientist?

- Some counting problems are so large and complex that we benefit from computation to solve. But we need to first estimate the steps that will be taken to solve a problem.
- Time estimation: While computers are fast, some problems require so much work that they would take an unreasonable amount of time to complete. Counting can help us estimate how much time a computer would take. How to estimate?
- Assuming a single processor machine, you have CPU speeds ranging from 3.5 - 4.0 GHz.

Example: Universe and computation

- A video to convey the scale of the universe.
- There are $\approx 10^{82}$ atoms in the universe [1]. Let that sink in.
- A chess board contains 8×8 blocks. What are the possible combinations of pieces on a chess board?
- It is estimated there are between 10^{111} and 10^{123} positions (including illegal moves) in chess. Legal moves: $\approx 10^{40}$
- There are even more possible variations of chess games than there are atoms in the observable universe.

[1] https://www.livescience.com/how-many-atoms-in-universe.html

Example: Pixels in a picture

- Let us say that you have a B&W picture.
- Each pixel in the picture can take values between 0 to 255 (2^8).
- Let us say that you have a 1 megapixel image. What are the combinations of values that you can have in that image?
- $(2^8)^{2^{10}*2^{10}*} = 2^{8192} \approx 10^{2467}$

Example: Parameters in a ML model

- I am sure most of you have heard about AI models around you. Can you give me some examples?
- Each of these models, needs to learn some parameters to be able to make predictions. The best set of parameters reduce the error of the model.
- For each parameter, you explore a range (possibly infinite!) of values. Let us assume that each parameter can take only 10^{10} values. Tall assumption.
- For models with 1 parameter, you have to search in 10^{10} values. For models with 2 parameters, you have to search in 10^{10*2} values.
- For the GPT-4 model, the number of parameters is 70 Billion $(70 * 10^9)$. The number of values that you have to search in?
- $10^{10*70*10^9} = 10^{7*10^{11}}$. Woah.

What did we learn today?

1. India: Its states and UTs
2. Principles for counting
3. Permutations
4. Combinations
5. Application: figuring out binomial coefficients
6. Counting and Computer Science

Thank you!