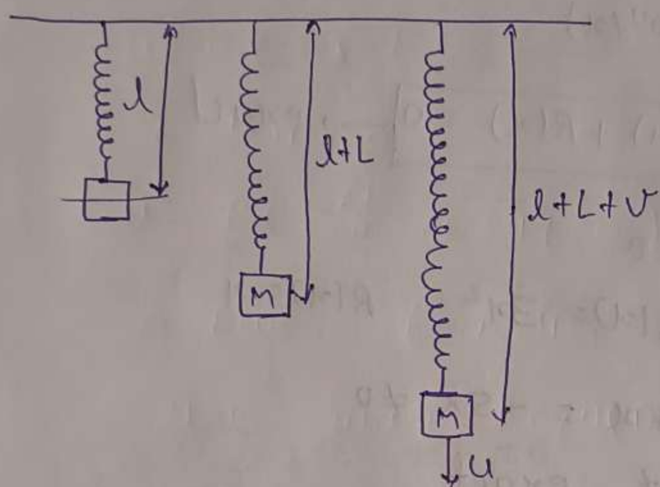
$$P(x)y'' + Q(x)y' + R(x)y = f(x)$$

$$ay'' + by' + cy = g(t)$$

## ☆ Spring mass system:-



## ☆ Hook's law :-

$$F_s = -KL$$

spring const

$$Mg = KL$$

(i) $\omega = mg$ act downward

(ii) $F_s = -k(L+u)$

(iii) $F_d = -\nu u'(t)$

(iv) $F = ma$

$$mu(t) = mg + F_s + F_d + F(t)$$

$$= mg - KL - Ku - \nu u'(t) + F(t)$$

$$mu''(t) + \nu u'(t) + Ku(t) = F(t)$$

No Damping $\gamma = 0$

Free-vibration

$$mu''(t) + Ku(t) = 0$$

$$u = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\omega_0^2 = \frac{K}{M}$$

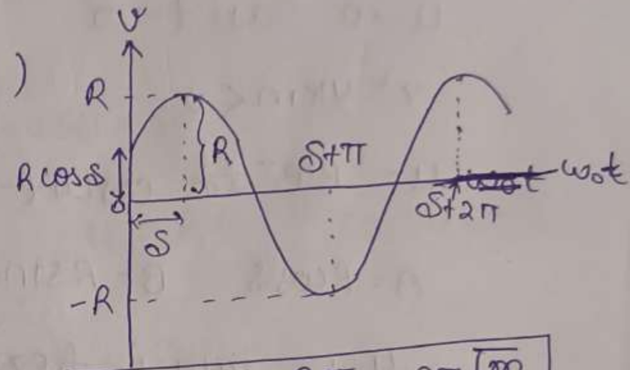
$$u = R\cos(\omega_0 t - \delta)$$

$$A = R\cos\delta$$

$$B = R\sin\delta$$

$$R^2 = A^2 + B^2$$

$$\tan\delta = \frac{B}{A}$$




$$\boxed{\text{Period} = T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}}$$

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

Damped vibration and no extra force

$$mu''(t) - \gamma u'(t) + Ku(t) = 0$$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4Km}}{2m} = \cancel{\frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{}}$$

$$= \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4Km}{\gamma^2}}\right)$$

$$u = Ae^{r_1 t} + Be^{r_2 t} \qquad \gamma^2 - 4km > 0$$

$$= (A + Bt)e^{-\frac{\gamma t}{2m}} \qquad \gamma^2 - 4km = 0$$

$$= e^{-\frac{\gamma t}{m}}(A\cos\mu t + B\sin\mu t) \qquad \gamma^2 - 4km < 0$$

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m} > 0$$

$m, \gamma, k$ are +ve

$\gamma^2 - 4km < \gamma^2$

If $\gamma^2 - 4km \geq 0$ $r_1$ and $r_2$ are negative

$u \to 0$ as $t \to \infty$

$\gamma^2 - 4km < 0$

$u = Re^{-\frac{\gamma t}{2m}} \cos(\mu t - \delta)$

$A = R\cos\delta$ $B = R\sin\delta$

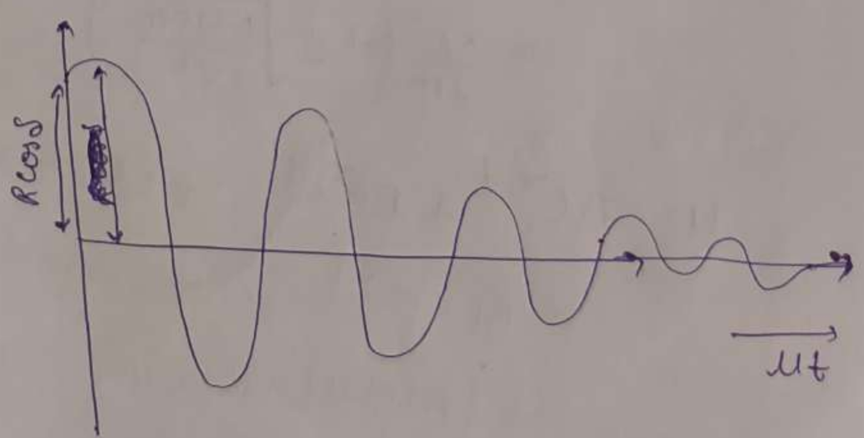
$\mu$ is quasi-frequency

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m} > 0$$

$$\frac{\mu}{\omega_0} = \frac{\frac{\sqrt{4km - \gamma^2}}{2m}}{\sqrt{\frac{k}{m}}} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \simeq 1 - \frac{\gamma^2}{8km}$$

when $\frac{\gamma^2}{4km}$ is small

quasi-period $= T_d = \frac{2\pi}{\mu}$



$$u = u_c(t) + u_p(t)$$ $$\boxed{f(t) = F_0 \cos(\omega t)}$$

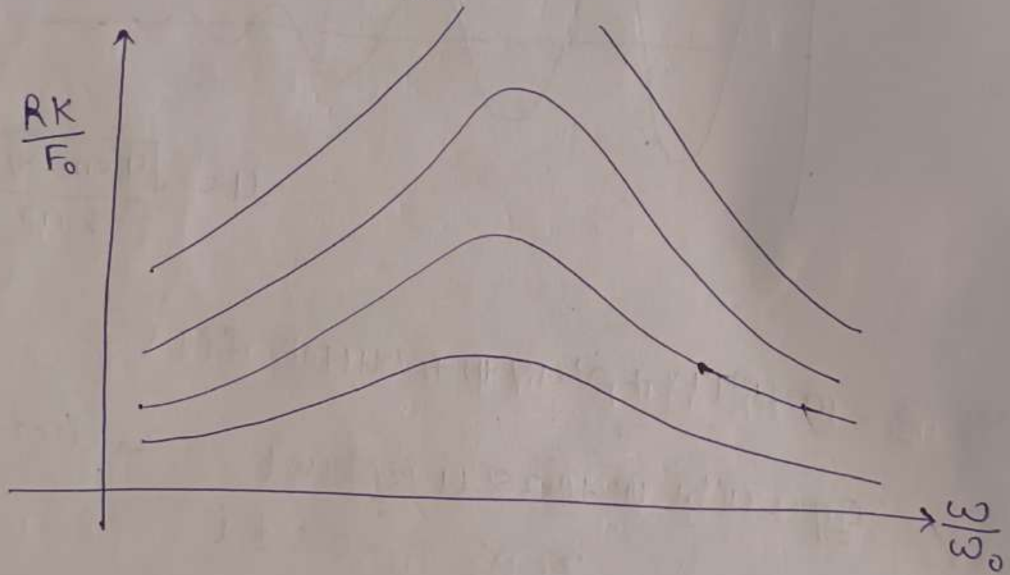
$$u_p(t) = R\cos(\omega t - \delta)$$

$$R = \frac{F_0}{\Delta} \qquad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta} \qquad \sin\delta = \frac{\gamma\omega}{\Delta}$$

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \quad , \omega_0^2 = \frac{k}{m}$$

$$\frac{R}{F_0/k} = \frac{Rk}{F_0} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}}$$

$$\Gamma = \frac{\gamma^2}{mk}$$




$$R_{max} = \frac{dR}{d\omega} = 0$$

$$\omega = \omega_{max} = \omega_0^2 - \frac{\gamma^2}{2m}$$ $R_{max}$ occurs

e.g.

$$y' - 2xy = 0$$

$$y(x) = ce^{x^2}$$

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

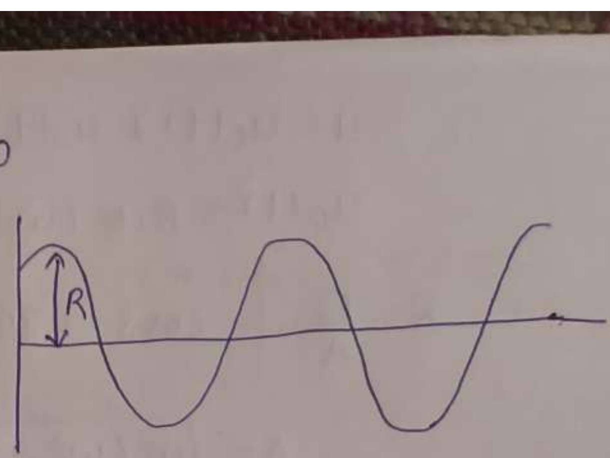
$$y = \sum_{n=0}^{\infty} a_n x^n$$

$a_0, a_1, a_2, \dots a_n \rightarrow$ sequence

$\Sigma a_n \rightarrow$ series.

$$mu''(t) + \nu u'(t) + k u(t) = 0$$

Case-1. $mu''(t) + k u(t) = 0$

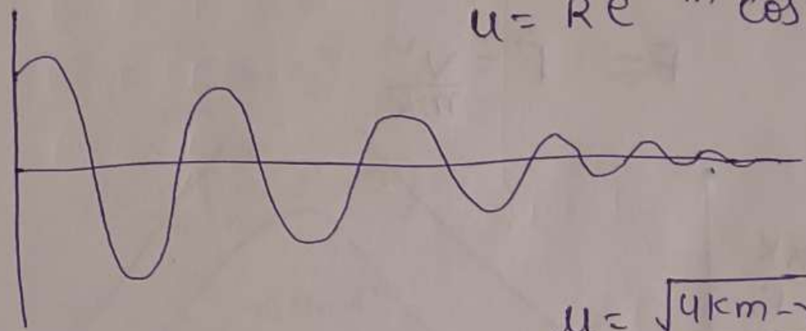



$$u = R\cos(\omega_0 t - \delta)$$

$$\omega_0^2 = \frac{k}{m}$$

Case-2. $mu''(t) + \nu u'(t) + k u(t) = 0$

$$u = Re^{-\frac{\nu t}{2m}} \cos(\mu t - \delta)$$



$$\mu = \frac{\sqrt{4km - \nu^2}}{2m}$$

Case. 3. $mu''(t) + \nu u'(t) + k u(t) = f(t)$

e.g. $u'' + u' + 1.25\, u = 3\cos t \quad \rightarrow u(0) = 2 \quad u'(0) = 3$

$$r_1, r_2 = -0.5 \pm i$$

$$u_c(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t$$

$$u_p(t) = A\cos t + B\sin t$$

$$= \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

$$u(t) = \underbrace{\frac{22}{17} e^{-t/2}\cos t}_{(i)} + \underbrace{\frac{14}{17} e^{-t/2}\sin t}_{(ii)} + \underbrace{\frac{12}{17}\cos t}_{(iii)} + \underbrace{\frac{48}{17}\sin t}_{(iv)}$$

(i), (ii) → Transient solutions → Due to decay term

(iii), (iv) → Steady solution

$$y' = a_1 + 2a_2x + 3a_3x^2 + \dots\dots$$

$$(a_1 + 2a_2x + 3a_3x^2 + \dots) = 2x(a_0 + a_1x + a_2x^2 + \dots)$$

$$a_1 + (2a_2 - 2a_0)x + (3a_3 - 2a_2)x^2 + \dots\dots = 0$$

$$\{1, x, x^2, \dots\} \rightarrow L.I.$$

$a_1 = 0$ $\quad 2a_2 - 2a_0 = 0 \quad 3a_3 - 2a_2 = 0$

$\quad a_2 = a_0 \quad a_3 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 2x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} 2a_{n-1}x^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} ((n+1)a_{n+1} - 2a_{n-1})x^n = 0$$

$$a_1 = 0$$

$$\sum_{n=1}^{\infty} ((n+1)a_{n+1} - 2a_{n-1})x^n = 0$$

$$(n+1)a_{n+1} = 2a_{n-1}$$

$$a_{n+1} = \frac{2}{n+1}a_{n-1} \longrightarrow \text{Recurrence Relation}$$

## Radius of Convergence :-

$\sum a_n (x-x_0)^n$

$\exists$ a $\rho \in [0,\infty)$ , $\rho \geq 0$

The series converges $|x-x_0| < \rho$
$\forall$ all $x$

diverges. $|x-x_0| > \rho$

## Root Test

$$\lim_{n\to\infty} (a_n)^{1/n} = L$$

only for $\sum a_n$
not for $\sum a_n x^n$

## Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \qquad 0 \leq L \leq \infty$$

if $L < 1 \rightarrow$ converges
$L > 1 \rightarrow$ diverges
$L = 1 \rightarrow$ no conclusion.

Radius of convergence $|x-x_0| < \rho$
where $\rho = \frac{1}{L}$

**Example:-** $\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} (x-1)^n$

**Soln:-** $a_n = \frac{2^{-n}}{n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)}}{n+2} \times \frac{n+1}{2^{-n}} = \frac{(n+1)}{2(n+2)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$$

$l = \frac{1}{2} < 1 \rightarrow$ converges

$x_0 = 1 \qquad |x - x_0| < \rho \qquad \rho = \frac{1}{l} = 2$

$|x-1| < 2$

$-1 < x < 3$

★ Properties:-

(1) $\sum a_n x^n \longrightarrow \rho_1 \qquad \sum b_n x^n \longrightarrow \rho_2$

$$\sum (a_n + b_n) x^n \longrightarrow \rho_1 + \rho_2$$

(2) $\frac{d}{dx}\left(\sum a_n x^n\right) = \sum a_n \cdot n x^{n-1}$

$$\int \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot x^{n+1}$$

Term by term differentiation and integration

(3) $\underbrace{\sum a_n (x-x_0)^n}_{\rho_1} \times \underbrace{\sum b_n (x-x_0)^n}_{\rho_2} \Rightarrow \rho_3$

$\rho_3 = \min(\rho_1, \rho_2)$

(4) $\dfrac{\sum a_n (x-x_0)^n}{\sum b_n (x-x_0)^n} \longrightarrow \rho_3 = \min(\rho_1, \rho_2)$

---

$y'' + y = 0 \qquad -\infty < x < \infty$

$y = c_1 \cos x + c_2 \sin x$

$y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$a_{2k+1} = 0 \quad \forall k$

$a_{2k} = \frac{a_0}{k!}$

$\downarrow$

$a_{2k} = \frac{2}{2k} a_{2k-2} = \frac{1}{k} a_{2k-2}$

$a_3 = \frac{2a_1}{3} = 0$

$a_2 = \frac{2}{2} a_0 = a_0$

$a_4 = \frac{a_0}{2}$

$a_5 = 0$

$a_6 = \frac{a_0}{6}$

$\downarrow$

$a_{2k+1} = 0$

$a_{2k} = \frac{a_0}{k!}$

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!} a_0 x^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{a_0}{k!} (x^2)^k = a_0 e^{x^2}$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots \frac{x^n}{n!}$$

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \dots + \frac{(x^2)^n}{n!}$$

## Power Series

$$\underbrace{\sum_{n=0}^{\infty} a_n (x - x_0)^n}_{\text{converges if}} = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0)^n$ → exists and finite → converge

doesn't exist → diverge

Ex:- $\sum_{n=0}^{\infty} 2^n (x-3)^n$

Soln:- $\lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(x-3)^{n+1}}{(x-3)} = \lim_{n\to\infty} |2\cdot(x-3)|$

For convergence $|2(x-3)| < 1$

$|(x-3)| < \frac{1}{2}$ $\quad R = \frac{1}{2}$

$\frac{5}{2} < x < \frac{7}{2}$

Ex:- $y' = 2y$

Soln:- $y = \sum a_n x^n$

Find recurrence relation

$(n+1) a_{n+1} = 2a_n$

$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}$

$\lim_{n\to\infty} \frac{2x^{n+1}}{(n+1)!} \times \frac{n!}{2x} = \lim_{n\to\infty} \frac{2x}{n+1} = 0$

$\rho = \frac{1}{0} = \infty$

Airy's Equation

$y'' - xy = 0$ $\qquad y = \sum a_n x^n$

$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$

$y_1(x)$

$y = a_0 \left[ 1 + \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} + \cdots \frac{x^{3n}}{2.3.\ldots(3n-1)3n} + \cdots \right] +$

$a_1 \left[ x + \frac{x^4}{3.4} + \frac{x^7}{3.4.6.7} + \cdots + \frac{x^{3n+1}}{3.4.\ldots(3n)(3n+1)} + \cdots \right]$

$y_2(x)$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(m+2)(m+1) a_{m+2} x^m.$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) x^n a_{n+2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} + a_n\right) x^n = 0$$

$$(n+2)(n+1) a_{n+2} + a_n = 0$$

$$a_{n+2} = \frac{-a_n}{(n+2)(\cancel{m}n+1)}$$

$$\left.\begin{aligned} n=0 &\rightarrow a_2 = -\frac{a_0}{2} \\ n=1 &\rightarrow a_3 = \frac{-a_1}{6} \\ n=2 &\rightarrow a_4 = \frac{a_0}{24} \\ n=3 &\rightarrow a_5 = \frac{a_1}{5\times4\times3\times2\times1} \end{aligned}\right\}$$

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}$$

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!} \cdot a_1$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0\left(1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n}{(2n)!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right)$$

$$\cos x \;\; \cancel{\sin x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \sin x \;\; \cancel{\cos x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

at $x_0 = 0$

$$W = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$y_1(0)=1 \quad y_1'(0)=0$

$y_2(0)=0 \quad y_2'(0)=1$

$$\omega(y_1, y_2) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\omega(y_1, y_2) \neq 0 \rightarrow$ Fundamental set of solution.

---

$P(x)y'' + Q(x)y' + R(x)y = 0$

at $x = x_0$ if $P(x_0) \neq 0$

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0$$

we say $x_0$ is an ordinary point

$$y'' + p(x)y' + q(x)y = 0$$

$$p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

$\underline{y'' - xy = 0}$

near $x=0$

$y = \Sigma a_n x^n \Rightarrow x=0$ is an ordinary point

otherwise $x = x_0$ is a singular point.

near $x=1$

$$y = \Sigma a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} a_n x (x-1)^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^{n} = \sum_{n=0}^{\infty} a_n x (x-1)^n$$

$$x = 1 + (x-1)$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1}$$

$$\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n + 2a_2 = \sum_{n=1}^{\infty}a_n(x-1)^n + a_0 + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n$$

$$2a_2 = a_0$$

$$\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_n - a_{n-1}\right](x-1)^n = 0$$

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1} \quad \text{for } n \geq 1$$

we didn't find any relation like $a_{2k+1}$ or $a_{2k}$ so, we can't find nth term. we can't apply ratio test.

Now, Assume $y = \phi(x)$ is a solution

$$= \sum a_n(x-x_0)^n$$

$$\phi'(x) = \sum n a_n (x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + \cdots$$

$$\phi'(x_0) = a_1$$

$$\phi''(x) = 2a_2 + 3\cdot 2a_3(x-x_0) + \cdots$$

$$\phi''(x_0) = 2a_2$$

$$\vdots$$

$$\phi^m(x_0) = m!\,a_m$$

To get $a_n$ for $\sum a_n(x-x_0)^n$ we can determine $\phi^n(x_0)$, $n = 0, 1, 2, \ldots$

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0$$

$$\phi''(x) = -p(x)\,\phi'(x) - q(x)\,\phi(x)$$

$$p(x) = \frac{Q(x)}{P(x)} \qquad q(x) = \frac{R(x)}{P(x)}$$

$$\phi''(x_0) = -p(x_0)\,\phi'(x_0) - q(x_0)\,\phi(x_0)$$

$$2!\,a_2 = -p(x_0)\,a_1 - q(x_0)\,a_0$$

$$\phi'''(x_0) = -p'(x)\,\phi'(x) - p(x)\,\phi''(x) - q'(x)\,\phi(x) - q(x)\,\phi'(x)$$

$$\phi'''(x_0) = 3!\,a_3 = -2\,p(x_0)\left(-p(x_0)\,a_1 - q(x_0)\,a_0\right) - \left(p'(x_0) + q(x_0)\right)a_1 - q'(x_0)\,a_0$$

$p(x)$, $q(x)$ are analytic function at $x = x_0$

$\downarrow$

Have a Taylor series Expansion.

$$f(x)\Big|_{x=x_0} = f(x_0) + (x-x_0)f'(x) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$$

~~f(x)~~

$\uparrow$

Taylor Series.

<u>Theorem</u>:- If $x_0$ is an ordinary point of the

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (*)$$

The general solution of $(*)$ is.

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where $a_0$, $a_1$ are arbitary and $y_1$ and $y_2$ are are the power series solution that are analytic at $x = x_0$ and $y_1$ and $y_2$ form a fundamental solution.

The radius of convergence of each $\text{sol}^n$ $y_1$ and $y_2$ is at least as large as the minimum of radius of the convergence of the series $p(x)$ and $q(x)$

$$p(x) = \frac{Q(x)}{P(x)} \quad , \quad q(x) = \frac{R(x)}{P(x)}$$

**Note**:- Since $P, Q, R$ are polynomials it can be seen that the ratio $\frac{Q}{P}$ has a convergent Power series expansion at $x_0$ if $P(x_0) \neq 0$

**Thus**, The radius of the convergence of $\frac{Q}{P}$ is precisely the distance from $x_0$ to the nearest zero of $P(x)$

So $P(x) = 0$ real or complex.

$$p(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$$

$$\frac{Q(x)}{P(x)} = \frac{1}{1+x^2} \qquad \rho = 1$$

$$P(x) = 1 + x^2 = 0 \quad x = \pm i$$

**Ex**:- $y'' + \sin x \, y' + (1+x^2) y = 0$

$Q/P$ :- $1/(1+x^2)$ What is the radius of convergence of solution about $x_0 = 0$.

Soln:- $p(x) = \frac{Q(x)}{P(x)} = \sin x \rightsquigarrow \rho_1 = \infty$

$q(x) = \frac{R(x)}{P(x)} = 1 + x^2 \rightsquigarrow \rho_2 = \infty$

So, $\rho = \min(\rho_1, \rho_2)$

$\rho = \infty$

Ex:- $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

Soln:- $x = 0 \to$ ordinary point $\because P(0) \neq 0$

$x = 1 \to$ not ordinary point $\because P(1) = 0$

$x = \pm 1 \to$ singular point.

$\frac{Q}{P} = \frac{-2x}{1-x^2}$ $\qquad P(x) = 0 \to x = \pm 1$

$\rho_1 =$ distance $x_0$ to $P(x) = 0$

$x_0 \downarrow 0$, $\quad P(x)=0 \downarrow \pm 1$

$\underline{\underline{\rho_1 = 1}}$

## Legendre's Equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

$$\alpha > -1$$

$x = 1$ and $-1$ - singular point

$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2}$

$$(1-x^2)\sum_{n=2}^{\infty}(n)(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$y_1 = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}x^4$$

$$\sum_{m=3}^{\infty} \frac{(-1)^m \alpha(\alpha-2) \cdots (\alpha-2m+2)(\alpha+1) \cdots (\alpha+2m-1)x^{2m}}{(2m)!}$$

$$y_2 = x - \frac{(\alpha-1)(\alpha-3)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 +$$

$$\cdots \sum_{m=3}^{\infty} \frac{(-1)^m (\alpha-1) \cdots (\alpha-2m+1)(\alpha+2)(\alpha+2m)}{(2m+1)!} x^{2m+1}$$

not for singular point

$|x| < 1$ radius of convergence

$w(y_1, y_2)\big|_{x=0}$ | if $\neq 0 \Rightarrow$ L.I. / if $0 \Rightarrow$ L.D.

$y = a_0 y_1 + a_1 y_2$

if $\alpha = 0$ $\quad y_1 = 1$, $\quad \alpha = 2n \quad a_{2m} = 0 \quad \forall\, m \geq n+1$

$$y_1(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m 2^m n(n-1)(n-m+1)(2n+1)(2n+3) \cdots (2n+2m-1)x^{2m}}{(2m)!}$$

$\rightarrow P_\alpha(x) = y_1(x)$

$\alpha = 0 \quad P_0(x) = 1$

$\alpha = 2 \quad P_2(x) = 1 - 3x^2$

$\alpha = 4 \quad P_4(x) = 1 - 10x^2 + \frac{35}{3}x^4$

$$(1-x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} - 2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} +$$

$$\alpha(\alpha+1)\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}x^n - \sum_{n=2}^{\infty}(n)(n-1)\,a_n x^n - 2\sum_{n=1}^{\infty}(n)a_n x^n +$$

$$\alpha(\alpha+1)\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty}\Big((n+2)(n+1)a_{n+2} - (n)(n-1)\,a_n - 2(n)\,a_n + \alpha(\alpha+1)a_n\Big) +$$

$$2a_2 + 6a_3x - 2a_1x + \alpha(\alpha+1)a_0 + \alpha(\alpha+1)a_1x = 0$$

$$\boxed{(n+2)(n+1)a_{n+2} = (n^2-n+2n-\alpha^2-\alpha)\,a_n}$$ Recurrence relation.

$2a_2 + \alpha(\alpha+1)a_0 = 0$

$6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0$

$2^{nd}$ degree $\rightarrow a_0, a_1 \rightarrow$ arbitrary constant

$a_0, a_2, a_4, \ldots$ in term of $a_0$

$a_1, a_3, \ldots$ in term of $a_1$

$a_{2n} =$

Case-1 $\rightarrow a_0 = 1$, $a_1 = 0$

$\rightarrow a_3, a_5, a_7 \cdots = 0$

$a_{2n+1} =$

Case-2 $\rightarrow a_0 = 0$, $a_1 = 1$

$a_2, a_4, \ldots a_{2n} = 0$

$L\{y\}: x^2y'' + \alpha x y' + \beta y = 0$

$y = x^r$

$$L\{x^r\} = x^2 \frac{d^2(x^r)}{dx^2} + \alpha x \frac{d(x^r)}{dx} + \beta x^r = 0$$

$$L(x^r) = x^r(r(r-1) + \alpha r + \beta) = 0$$

$$L(x^r) = x^r F(r) \qquad F(r) = r(r-1) + \alpha r + \beta$$

$$= r^2 + (\alpha-1)r + \beta$$

$$x^r F(r) = 0$$

$$F(r) = 0$$

$$r_1, r_2 = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 \cancel{4} - 4\beta}}{2}$$

(i) $r_1 \neq r_2$

$$y_1(x) = x^{r_1}, \quad y_2(x) = x^{r_2}$$

(#) $\boxed{y = C_1 x^{r_1} + C_2 x^{r_2}}$

$x^r = e^{r \ln x}$ , $r$ is not a rational no.

$$W\{y_1, y_2\} - (r_2 - r_1) x^{r_1 + r_2 - 1} \neq 0$$

(ii) $r_1 = r_2$

$$F(r) = (r - r_1)(r - r_2) = (r - r_1)^2$$

$$F(r_1) = 0 \qquad F'(r_1) = 0$$

$$L\{x^r\} = r^2 F(r)$$

$$\frac{\partial}{\partial r} L\{x^r\} = \frac{\partial}{\partial r}\{x^r F(r)\} = x^r F'(r) + \frac{\partial}{\partial r} x^r \cdot F(r)$$

$$\frac{\partial x^r}{\partial r} = x^r \ln x$$

Defination:-

$\lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)}$ is finite , $\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)}$ is finite

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0$$

Then $x = x_0$ is regular singular point

if not then it is irregular singular point

---

Ex:- $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$

singular point $\to x = 0, 2$.

Soln:- $x=0$

$$\lim_{x \to 0} x \cdot \frac{3x}{2x(x-2)^2} = 0$$

$$\lim_{x \to 0} x^2 \cdot \frac{(x-2)}{2x(x-2)^2} = 0 \longrightarrow \text{regular.}$$

$x=2$.

$$\lim_{x \to 2} \frac{(x-2) \cdot 3x}{2x(x-2)^2} \to \infty \to \text{irregular.}$$

$$\lim_{x \to 2} (x-2)^2 \frac{(x-2)}{2x \cdot (x-2)^2} = 0$$

Euler's Equation:-

$$x^2 y'' + xy' + y = 0$$

$$y = x^r$$

$$y_2(x) = x + \sum_{m=1}^{\infty} \frac{(-1)^m (2^m)\, n(n-1) \cdots}{(2m+1)!}$$

$$P_\alpha(x) = y_2(x)$$

$$P_1(x) = x, \quad P_3(x) = x - \frac{5}{3}x^3$$

$$P_5(x) = x - \frac{14}{3}x^3 + \cdots$$

if $n \neq m$

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \longrightarrow \text{both solution are orthogonal solution}$$

$$y'' + p(x)y' + q(x)y = 0$$

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$y = \Sigma a_n x^n$ → we were assuming this.

but if $x_0$ is singular point then assume

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}$$

→ Find $y'$ & $y''$, put in eqn

↓

$$\sum_{n=0}^{\infty} (\quad) x^{r+n} = 0$$

↓

$$a_0 \underbrace{\{r(r-1) + rp_0 + q_0\}}_{F(r)} x^r + \sum_{n=1}^{\infty} (\quad) x^{r+n} = 0$$

$F(r) = 0$

↓

$$a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] \right\} x^{r+n} = 0$$

$F(r) = 0 \rightarrow r_1, r_2 \Rightarrow$ exponents at the singularity

---

Ex:- $2x^2 y'' + 3xy' - (x^2+1) y = 0$. Find indicial equation

Soln:- $F(r) = r(r-1) + p_0 r + q_0 = 0$

$$p_0 = \lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \times \frac{3x}{2x^2} = \frac{3}{2}$$

$$L\left\{\frac{\partial x^r}{\partial x}\right\} = x^r F'(r) + x^r \ln x \, F(r)$$

$$L\{x^{r_1}\} = 0$$
$$L\left\{\frac{\partial x^r}{\partial r}\right\} = 0 \longrightarrow x^{r_1}, \; x^{r_1}\ln x$$

$$y_1 = x^{r_1} \qquad y_2 = x^{r_1}\ln x.$$

$$\boxed{y = c_1 x^{r_1} + c_2 x^{r_1}\ln x}$$

$$W\{y_1, y_2\} = x^{2r_1 - 1} \neq 0 \;, \; x > 0$$

(iii) $r_1 = \lambda + i\mu \qquad r_2 = \lambda - i\mu$

$$y = c_1 x^{\lambda + i\beta} + c_2 x^{\lambda - i\mu}$$

$$\boxed{y = c_1 x^{\lambda}\{\cos(\mu \ln x)\} + c_2 x^{\lambda}\{\sin(\mu \ln x)\}}$$

---

Ex:- $y'' + p(x)y' + q(x)y = 0 \qquad p(x) = \frac{Q(x)}{P(x)} \quad q(x) = \frac{R(x)}{P(x)}$

$x_0 = 0$ is a singular point

$\left.\begin{matrix} x p(x) \\ x^2 q(x) \end{matrix}\right\}$ are analytic

$$x p(x) = \sum_{n=0}^{\infty} p_n x^n$$
$$x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

$$x^2 y'' + x(x p(x)) y' + x^2 q(x) y = 0$$
$$x^2 y'' + x\{p_0 + p_1 x + p_2 x^2 + \dots\} + \{q_0 + q_1 x + q_2 x^2 + \dots\} y = 0$$

$$x^2 y'' + x p_0 y' + q_0 y = 0$$

$$\lim_{x \to 0} x p(x) = p_0 \qquad \lim_{x \to 0} x^2 q(x) = q_0$$

$$q_0 = \lim_{x\to 0} x^2 q_x = \lim_{x\to 0} x^2 \times \frac{(-(x^2+1))}{2x^2} = \frac{-1}{2}$$

$$F(r) = (r-\tfrac{1}{2})(r+1) = 0$$

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$$r = \tfrac{1}{2}, -1$$

$$r_1 = \tfrac{1}{2} \qquad r_2 = -1$$

$$y = \sum a_n x^{r+n}$$

$$2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{r+n} + 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r} -$$

$$\underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+2}}_{\displaystyle\sum_{n=2}^{\infty} a_{n-2} x^{n+r}} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$F(r) = 2\underbrace{(r^2 + \tfrac{1}{2} r - \tfrac{1}{2})}_{r_1 = \frac{1}{2},\ r_2 = -1} a_0 = 0$$

$$a_n = \frac{a_{n-2}}{2(n+r)^2 + (n+r) - 1} \quad n \geq 2$$

Case-1 $\quad r_1 = \frac{1}{2}$

$a_n(r_1)$

$$a_n = \frac{a_{n-2}}{2n^2 + 3n}$$

$$y_1(x) = a_0 x^{1/2}\left(1 + \frac{x^2}{14} + \frac{x^9}{616} + \cdots\right)$$

## Singularity at Infinity :-

$$\xi = \frac{1}{x_0} \quad \Rightarrow \xi_0 \to 0 \quad \because x_0 \to \infty$$

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} \cdot y = 0$$

$$x = \frac{1}{\xi}$$

$$\frac{dy}{dx} = \frac{dy}{d\xi} \cdot \frac{d\xi}{dx} = -\frac{1}{x^2} \cdot \frac{dy}{d\xi} = -\xi^2 \frac{dy}{d\xi}$$

$$\frac{d^2y}{dx^2} = \xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi}$$

$$P\left(\frac{1}{\xi}\right)\left(\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi}\right) + Q\left(\frac{1}{\xi}\right)(-\xi^2)\frac{dy}{d\xi} + R\left(\frac{1}{\xi}\right) y = 0$$

$$\underbrace{P\left(\frac{1}{\xi}\right)\xi^4}_{P_1(\xi)} \frac{d^2y}{d\xi^2} + \underbrace{\left(2P\left(\frac{1}{\xi}\right)\xi^3 - \xi^2 Q\left(\frac{1}{\xi}\right)\right)}_{Q_1(\xi)} \frac{dy}{d\xi} + \underbrace{R\left(\frac{1}{\xi}\right)}_{R_1(\xi)} y = 0$$

$$\lim_{\xi_0 \to \xi_0} (\xi - \xi_0) \frac{Q_1(\xi)}{P_1(\xi)} = \lim_{\xi \to 0} \xi \frac{Q_1(\xi)}{P_1(\xi)}$$

$$\lim_{\xi \to \xi_0} (\xi - \xi_0)^2 \frac{R_1(\xi)}{P_1(\xi)} = \lim_{\xi \to 0} \xi^2 \frac{R_1(\xi)}{P_1(\xi)}$$

} finite then $\xi \to 0$ is regular means $x \to \infty$ is regular

---

# Laplace Transform :-

$f(t)$

$$\boxed{F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt}$$

$f(t) = 1 \longrightarrow L\{f(t)\} = \frac{1}{s}$

$f(t) = t^n \longrightarrow L\{f(t)\} = \frac{n!}{s^{n+1}}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

$L\{e^{at}f(t)\} = F(s-a)$ $\quad L\{f(t)\} = F(s)$

$L\{t f(t)\} = -\frac{dF(s)}{ds}$

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$L\{u_c(t) f(t-c)\} = e^{-sc}F(s)$

Ex:- $y'' - y' - 2y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$

Soln:- m-1

$(m^2 - m - 2) = 0$

$m = 2, -1.$

$y = c_1 e^{-t} + c_2 e^{2t}$

m-2.

$L\{y'' - y' - 2y\} = 0$

$L\{y''\} - L\{y'\} - 2L\{y\} = 0$

$L\{\frac{d^2y}{dt^2}\} = s^2 L\{y\} - sy(0) - y'(0)$

$L\{y'\} = sL\{y\} - y(0)$

$s^2 L\{y\} - s - 0 - sL\{y\} + 1 - 2L\{y\} = 0$

$$y(s) = \frac{s-1}{s^2-s-2} = \frac{(s-1)}{(s-2)(s+1)}$$

$$y(s) = \frac{1}{3} \cdot \frac{1}{s-2} + \frac{2}{3} \cdot \frac{1}{s+1}$$

$$L^{-1}\{y(s)\} = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

Ex:- $y'' + y = \sin 2t \quad , y(0) = 2 , y'(0) = 1$

Soln:- $L\{y'' + y\} = L\{\sin 2t\}$

$$s^2 y(s) - sy(0) - y'(0) + y(s) = \frac{2}{s^2+4}$$

$$y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

$$L^{-1}\{y(s)\} = 2\cos t + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$$