$$y^{(4)} - 5y^{(3)} + 6y'' + 4y' - 8y = 0$$

$$m^4 - 5m^3 + 6m^2 + 4m - 8 = 0$$

$$m_1 = m_2 = m_3 = 2 \;\&\; m_4 = -1$$

GS $\quad y(x) := (\alpha_1 + \alpha_2 x + \alpha_3 x^2) e^{2x} + \alpha_4 e^{-x}$

Reduction of orders

This method is used to find a linearly independent solution corresponding to a given solution

Process $\quad y'' + p(t) y' + q(t) y = 0 \quad \text{--- (1)}$

Suppose $y_1(t)$ is one soln of (1).

To find a second solution, let us consider

$$y(t) = v(t) y_1(t)$$

then $y'(t) = v'(t) y_1(t) + v(t) y_1'(t)$

$$y''(t) = v''(t) y_1(t) + 2v'(t) y_1'(t) + v(t) y_1''(t)$$

Now, we substitute the value of $y(t)$, $y'(t)$ & $y''(t)$ in (1) we have

$$[v''(t) y_1(t) + 2v'(t) y_1'(t) + v(t) y_1''(t)]$$
$$+ p(t) [v'(t) y_1(t) + v(t) y_1'(t)] + q(t) v(t) y_1(t) = 0 \quad \text{(2)}$$

As y_1 is a solution of (1)

$$y_1''(t) + p(t) y_1'(t) + q(t) y_1(t) = 0 \quad \text{--- (3)}$$

$(2) \Rightarrow$

$$v''(t) y_1(t) + 2v'(t) y_1'(t) + p(t) v'(t) y_1(t) = 0.$$

$\cancel{+ p(t) v'(t) y_1(t)}$

$$\Rightarrow y_1(t) v''(t) + (2y_1'(t) + p(t) y_1(t)) v'(t) = 0.$$

Consider $v'(t) = w(t)$.

$$y_1(t) w'(t) + (2y_1'(t) + p(t) y_1(t)) w(t) = 0$$

$\downarrow$

A First order ODE

We solve in terms of $W(t)$

Once we obtain $W(t)$, using $v'(t) = w(t)$,

we get the solution.

- Method is called reduction of order, as an important step of ~~solving~~ solution is solving a first order ODE.

Q Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0,$$

find a fundamental set of solutions

Ans

$$y(t) = v(t) t^{-1}$$

$$y'(t) = v'(t) t^{-1} - v(t) t^{-2}$$

$$y''(t) = v''(t) t^{-1} - 2v'(t) t^{-2} + 2v(t) t^{-3}$$

Putting y, y' & y'' into (1),

$$2t^2 (v''(t) t^{-1} - 2v'(t) t^{-2} + 2v(t) t^{-3})$$
$$+ 3t(v'(t) t^{-1} - v(t) t^{-2}) - vt^{-1} = 0$$

$$\Rightarrow 2t v''(t) + v'(-4+3) + v\underbrace{(4t^{-1} - 3t^{-1} - t^{-1})}_{0} = 0$$

$$\Rightarrow 2t v''(t) - v' = 0$$

$$\frac{v''}{v'} = \frac{dt}{2t} \Rightarrow \log(v') = \log(t^{1/2} c)$$

Q Prove that if y_1 & y_2 have maxima at the same point on I, then they can't be a fundamental set of solutions on that interval

where y_1 & y_2 are solution of $y'' + p(t)y' + q(t)y = 0$

& $p(t)$ & $q(t)$ are continuous on $\underline{I}$.

Ans As y_1 & y_2 are solutions, they are differentiable

Suppose $t_0 \in \underline{I}$ is the pt. where y_1 & y_2 have either a maxima or a minima.

$$\Rightarrow y_1'(t_0) = 0 = y_2'(t_0)$$

$$W(y_1, y_2)(t_0) = 0$$

Hence $W(y_1, y_2) = 0 \quad \forall t \in \underline{I}$

& hence y_1 & y_2 can't form a fundamental set of solutions

✓. $y'' + p(t)y' + q(t)y = 0$

where p, q are continuous functions on an open interval $\underline{I}$. Then TFAE:-

1) y_1 & y_2 form a fundamental set of solutions on I.

2) y_1 & y_2 are linearly independent on I

3) $W(y_1, y_2)(t_0) \neq 0$ for a $t_0 \in \underline{I}$.

4) $W(y_1, y_2)(t) \neq 0$ for all $t \in \underline{I}$.

Def If y_1 & y_2 are 2 solutions of the ODE

$$y''+p(t)y'+q(t)y=0$$

& the $W(y_1,y_2)$ does not vanish everywhere.

$$\left[W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \right]$$

then $y=c_1y_1+c_2y_2$ with arbitrary constants c_1 & c_2 is known as "general solution" of the ODE.

- The solutions y_1 & y_2 are said to form a fundamental set of solutions if and only if their Wronskian is non zero.

Q. Show $y_1(t)=e^{r_1t}$ $y_2(t)=e^{r_2t}$ are two solutions of eqn

$$L[y]=y''+p(t)y'+q(t)y=0$$

Show they form a fundamental set of solns.

Q $y_1(t)=\sqrt{t}$ & $y_2(t)=t^{-1}$ form a fundamental set of solns of

$$2t^2y''+3ty'-y=0, t>0$$

Question Whether for an ODE :-

$$y''+p(t)y'+q(t)y=0$$

always has a fundamental set of solns?

W

<u>Theorem</u> If y_1, y_2 are solution of 2nd order homogeneous ODE $Ly = 0$ in I

then $W(y_1, y_2)$ is either identically zero or is never zero on I.

<u>Proof</u> Suppose $W(y_1, y_2)(x_0) = 0$.

then y_1 & y_2 are linearly dependent

but then by previous theorem

Wronskian vanishes everywhere.

<u>Eg</u> Wronskian of $y_1(x) := x$ & $y_2(x) := \sin x$ is

$x \cos x - \sin x$.

This Wronskian is non zero, for eg, at $x = \pi$, then functions y_1 & y_2 are linearly independent

However, Wronskian is zero at $x = 0$

<u>Note</u> x & $\sin x$ can't span solutions of a second order ODE.

$\boxed{\text{Why?}}$

because by Abel theorem & Theorem 3.2.4, ~~that~~ if ~~y~~ they were sol, then

$W(y_1, y_2)$ must never be zero

but $W(y_1, y_2) = 0$ at $x = 0$

$y'' + py' + qy = 0$

$p + qx = 0$

$-\sin x + p\cos x + q\sin x = 0$

$q^2 x$

$(q-1)\sin x - \cancel{pq}\cos x$

$= 0$

x

To find general soln of the ODE

$$y'' + p(t)y' + q(t)y = 0 \quad t \in I.$$

- Find 2 fns y_1 & y_2 that satisfy the ODE in I.
- There is a pt $t_0 \in I$ s.t.

$$W(y_1, y_2)(t_0) \neq 0$$

- Then y_1 & y_2 form a fundamental set of solns

- General solution is

$$y = c_1 y_1(t) + c_2 y_2(t)$$

where c_1 & c_2 are arbitrary const.

3.3. Complex roots of the characteristic equation

$$ay'' + by' + cy = 0 \quad \text{—— (1)}$$

where a, b, c are given real no.s

If we seek solutions of form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0 \quad \text{—— (2)}$$

If roots r_1 & r_2 are real & distinct, ie when

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ we have $b^2 - 4ac > 0$.

then the general solution of (1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

If $b^2 - 4ac < 0$, then (2) has ~~complex conjugate roots~~ conjugate pair of complex roots

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu.$$

where λ, μ are real

Then $y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos\mu t + i \sin\mu t)$

& $y_2 = e^{(\lambda - i\mu)t} = e^{\lambda t}(\cos\mu t - i\sin\mu t)$

are complex solutions of the ODE.

- We seek real linearly independent solutions

$\Rightarrow v'(t) = t^{-\frac{1}{2}} c$

$v(t) = c\int t^{-1/2} dt + K$

$= \frac{2}{3} c t^{3/2} + K$

$\Rightarrow v(t) = \frac{2}{3} c_1 t^{3/2} + c_2$

Hence $y(t) = v(t) y_1(t)$

$= \frac{2}{3} c_1 t^{\frac{1}{2}} + c_2 t^{-1}$

$y_1(t) = t^{-1}$, $y_2(t) = \frac{2}{3} t^{1/2}$

$W(y_1, y_2) = \frac{3}{2} t^{-3/2}, t > 0$

Hence, y_1 & y_2 form a fundamental set of soluti

For second order linear hom eqn with constant coeffs:-

$$ay'' + by' + cy = 0$$

Let r_1 & r_2 be roots of corresponding characteristic eqn

$$ar^2 + br + c = 0$$

- r_1 & r_2 are real but not equal,

GS $\quad y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

- r_1 & r_2 are complex conjugates $\lambda \pm i\mu$,

$$y = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$$

- $r_1 = r_2$ $\quad y = c_1 e^{r_1 t} + c_2 t e^{r_2 t}$

3.5 Non homogeneous equations, method of undetermined coefficients

Recall

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \quad (1), \quad g(t) \neq 0$$

p, q & g are given (continuous) functions on the open interval I

$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad -(2)$$

is called the homogeneous eqn corresponding to (1)

Theorem

Y_1, Y_2 solution of (1)

Then $Y_1 - Y_2$ solution of (2)

If in addition, y_1 & y_2 are a fundamental set of solutions, then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem GS of $L[y] = g(t)$ is

$$y = \phi(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{C.F.}} + \underbrace{Y(t)}_{\text{PI}}$$

Eg Non homogenous ODE

$$y'' + y = x$$

C.F. $y'' + y = 0$

$$y_c = \alpha_1 \sin x + \alpha_2 \cos x$$

PI $y'' + y = x \quad y_p(x) = x$

Thm Consider ODE

$$y'' + p(t)y' + q(t)y = 0 \quad (1)$$

whose coeffs p & q are continuous on some open interval I. Choose $t_0 \in I$.

Let y_1 be a soln of (1) that also satisfies

$$y(t_0) = 1,\ y'(t_0) = 0$$

Let y_2 be a soln of (1) that satisfies

$$y(t_0) = 0,\ y'(t_0) = 1$$

Then y_1 & y_2 form a fundamental set of soln.

Q Find fundamental set of solutions specified by above theorem for ODE

$$y'' - y = 0 \quad (1)$$

using initial pt. $t_0 = 0$.

Ans

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$y_1(t) = e^t \qquad y_2 = e^{-t}$$

$$W(y_1, y_2) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2 \neq 0$$

So y_1 & y_2 form a fundamental set of solutions

NOTE An ODE has more than one fundamental set of solutions - infinitely many fundamental set of solutions

ABEL'S THEOREM

If y_1 & y_2 are solns of the ODE

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p & q are continous on an open interval I, then $W(y_1, y_2)(t) = c\exp\left[-\int p(t)\,dt\right]$

where c is a certain constant that depends on y_1 & y_2, but not on t.

Furthermore,

$W(y_1, y_2)(t)$ is either zero $\forall\, t \in I$ (if $c = 0$) or else is never zero in I (if $c \neq 0$)

(Since exponential function is never zero)

Eg $y_1(t) = \sqrt{t}$ & $y_2(t) = t^{-1}$ are solns of

$$2t^2y'' + 3ty' - y = 0$$

$$\Rightarrow y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$$

$$p(t) = \frac{3}{2t}$$

$$W(y_1, y_2)(t) = c\exp\left[-\int\frac{3}{2t}\,dt\right]$$

$$= c\exp\left(-\frac{3}{2}\ln t\right) = ct^{-3/2}$$

They are not fundamental solns indicated by them
$y_1(0)=1, y_1'(0)=1, y_2(0)=1, y_2'(0)=-1.$

Let us denote

$y_3(t)$ the soln of (1) s.t.

$$y_3(0)=1 \ \& \ y_3'(0)=0.$$

The general soln of (1) is $y=c_1e^t+c_2e^{-t}$.

and the ICs are.

$$y_3(0)=1 \Rightarrow c_1+c_2=1$$
$$y_3'(0)=0 \Rightarrow c_1-c_2=0$$

$$\Rightarrow c_1=c_2=\frac{1}{2}$$

$$\boxed{y_3(t)=\frac{e^t+e^{-t}}{2}=\cosh t}$$

$y_4(t)$ is a soln of (1) s.t.

$$y_4(0)=0, \ y_4'(0)=1$$

$$\Rightarrow y_4(t)=\frac{e^t-e^{-t}}{2}=\sinh t.$$

Now $W(y_3,y_4)(t)=\cosh^2 t-\sinh^2 t$
$$=1\neq 0$$

$\therefore$ y_3 & y_4 form a fundamental set of solutions

- $y'' - 6y' + 25y = 0$
- $16y'' - 8y' + 145y = 0$

$y(0) = -2,\ y'(0) = 1.$

$\hookrightarrow$ CE is $16r^2 - 8r + 145 = 0$

$r = \frac{1}{4} \pm 3i$

GS is $y = e^{\frac{1}{4}t}(c_1 \cos 3t + c_2 \sin 3t)$.

$y' = \frac{1}{4} e^{\frac{1}{4}t}(c_1 \cos 3t + c_2 \sin 3t)$

$\quad + e^{\frac{1}{4}t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$.

$y(0) = -2 \Rightarrow \boxed{c_1 = -2}$

$y'(0) = 1 \Rightarrow 1 = \frac{1}{4}c_1 + 3c_2$

$\Rightarrow \frac{1}{4}(-2) + 3c_2 = 1.$

$\Rightarrow 3c_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$\Rightarrow \boxed{c_2 = \frac{1}{2}}$

$\therefore y = e^{\frac{1}{4}t}(-2\cos 3t + \frac{1}{2}\sin 3t)$ soln of IVP.

Repeated roots, reduction of order

$$ay'' + by' + cy = 0 \quad -(1)$$

Characteristic eqn is $ar^2 + br + c = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

When $b^2-4ac=0$

then $r_1=r_2=\frac{-b}{2a}$

Both the roots are same & results in soln $y_1(t)=e^{\frac{-b}{2a}t}$

$\underline{\underline{\text{TF}}}$ a second solution

Assume

$$y(t)=v(t)\,y_1(t)$$
$$=v(t)\,e^{-\frac{b}{2a}t} \quad \text{——— (I)}$$

[Since $y_1(t)$ is a solution $cy_1(t)$ is also a solution. Basic idea - generalize this observation by replacing c by a function $v(t)$ so that $v(t)y_1(t)$ is also a solution]

$$y'(t)=v'(t)\,\cancel{y}\,e^{-\frac{b}{2a}t}+\left(\frac{-b}{2a}\right)e^{\frac{-b}{2a}t}v(t)$$

$$y''(t)=v''(t)\,e^{-\frac{b}{2a}t}+\left(\frac{-b}{2a}\right)v'(t)\,e^{-\frac{b}{2a}t}$$
$$+\left(-\frac{b}{2a}\right)v'(t)\,e^{-\frac{b}{2a}t}+\frac{b^2}{4a^2}v(t)\,e^{-\frac{b}{2a}t}$$

$$=v''(t)\,-\frac{b}{a}v'(t)e^{\frac{-b}{2a}t}+\frac{b^2}{4a^2}v(t)\,e^{-\frac{b}{2a}t}.$$

Substitute y, y' & y'' into (1),

$$a\left(v''-\frac{b}{a}v'+\frac{b^2}{4a^2}v\right)e^{-\frac{b}{2a}t}$$
$$+b\left(v'-\frac{b}{2a}v\right)e^{\frac{-b}{2a}t}+c\,v\,e^{-\frac{b}{2a}t}=0$$

$$\Rightarrow av''-\cancel{bv'}+\frac{b^2}{4a}v+\cancel{bv'}-\frac{b^2}{2a}v+cv=0$$

$$\Rightarrow av''-\frac{b^2}{4a}v+cv=0$$

Since $b^2 - 4ac = 0 \Rightarrow c = \frac{b^2}{2a}$

Hence, $av'' - cv + cv = 0$

$\Rightarrow v'' = 0$

$\Rightarrow v(t) = c_1 + c_2 t$

Hence $y(t) = v(t)\, e^{-\frac{b}{2a}t}$

$= (c_1 + c_2 t)\, e^{-\frac{b}{2a}t}$

$= \underline{c_1 e^{-\frac{b}{2a}t}} + \underline{c_2 t e^{-\frac{b}{2a}t}}$

$\therefore$ y is a linear combination of the two solution $e^{-\frac{b}{2a}t}$ & $te^{-\frac{b}{2a}t}$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-\frac{b}{2a}t} & te^{-\frac{b}{2a}t} \\ -\frac{b}{2a} e^{-\frac{b}{2a}t} & \left(1 - \frac{bt}{2a}\right) e^{-\frac{b}{2a}t} \end{vmatrix}$$

$$= \cancel{\left(1 - \frac{bt}{2a}\right)} e^{-\frac{bt}{a}} \neq 0$$

Hence, y_1 & y_2 form a fundamental set of solution

& general soln is

$$\boxed{y = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}}$$

- If m is a repeated root, then

  e^{mt} is one solution

- Now, using existing solution $u_1 = e^{mt}$, we find a linearly independent solution

  $e^{mt} v(t)$ for a suitable choice of v.

- $v(t) = t$ & te^{mt} is a l.i. solution

- For three repeated roots

  e^{mt}, te^{mt} & $t^2 e^{mt}$ are l.i. solutions

Q $y'' - 6y' + 9y = 0$

CE is $m^2 - 6m + 9 = 0$

Two repeated roots $m_1 = m_2 = 3$

Corresponding solution is e^{3x}

L.I solution - xe^{3x}.

general solution $y(x) := (\alpha_1 + \alpha_2 x) e^{3x}$

Q $y^{(3)} - 4y'' - 3y' + 18y = 0$

CE $m^3 - 4m^2 - 3m + 18 = 0$

$m_1 = m_2 = 3$ $\quad m_3 = -2$

e^{3x} $\quad e^{-2x}$

General sol $y(x) = (\alpha_1 + \alpha_2 x) e^{3x} + \alpha_3 e^{-2x}$

We can get it from linear combinations of y_1 & y_2.

- Sum the above two complex solutions & divide by 2,

$$y_1 + y_2 = 2e^{\lambda t}\cos\mu t.$$

to obtain $\boxed{u(t) = e^{\lambda t}\cos\mu t.}$

- Similarly, on subtraction & dividing by $2i$, we get

$$y_1 - y_2 = 2ie^{\lambda t}\sin\mu t.$$

$$\boxed{v(t) = e^{\lambda t}\sin\mu t}$$

$$W(u,v)(t) = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix}$$

$$= \boxed{\mu e^{2\lambda t}} \neq 0$$

Thus, as long as $\mu \neq 0$, the Wronskian W is not zero.

So $u(t)$ & $v(t)$ are real linearly independent solutions

& general solution is given by

$$y(t) = e^{\lambda t}(c_1\cos\mu t + c_2\sin\mu t),$$

where c_1 & c_2 are arbitrary constants

2 ODE $y'' + y = 0$

CE is $m^2 + 1 = 0$

$m_1 = i \quad m_2 = -i$

General solution is $y(t) := \alpha_1\sin t + \alpha_2\cos t$