Discrete Math, Sets and Finite State Automation

Briefly

Sukrit Gupta and Nitin Auluck

January 10, 2024

Outline

Acknowledgement and disclaimer

All mistakes (if any) are mine.
I have used several other sources which I have referred to in the appropriate places.

Section 1

Discrete math as the language of computation

What is (discrete) math? [1]

- Defining discrete mathematics is hard because defining mathematics is hard. What is mathematics?
- The study of numbers? In part, but you also study functions, lines, triangles and vectors, et cetera.
- Or perhaps mathematics is a collection of tools that allow you to solve problems.
- Some of the math fundamentally deals with stuff that is individually separate and distinct.
- In an algebra or calculus class, you might have found a particular set of numbers represented as an interval, for example, the set of numbers in the range of a function.
- For example: $[0, \infty)$ is the range of $f(x) = x^2$. This set of numbers is NOT discrete.

What is (discrete) math? [2]

- $[0, \infty)$ is the range of $f(x) = x^2$
- The numbers in the set are not separated by much at all. In fact, take any two numbers in the set and there are infinitely many more between them which are also in the set.
- Consider a function which gives the number of students attending this class.
- What is the range?
- Something like $\{0, 1, 2, 3, \cdots\}$.
- But certainly there is nobody reading this that has 180.32419 children. This set is discrete because the elements are *separate*.
- *Discrete mathematics is the study of mathematical structures that are countable or otherwise distinct and separable.*

Topics that we will study in discrete math

The following topics are deeply intertwined with computation (and programming):

- Set Theory
- Functions
- Symbolic Logic and Proofs
- Counting
- Sequences
- Graph Theory (studied at the end)

Section 2

Sets

Sets

- A set is an unordered collection of objects.
- The most fundamental objects we will use in our studies (and really in all of math) are sets.
- Denoted by an uppercase (NOT bold) letter.
- Two examples: We could consider the set, C, of all colors in the rainbow $C = \{'V', 'I', 'B', 'G', 'Y', 'O', 'R'\}$ and set of all natural numbers, S, below 5 given by $S = \{1, 2, 3, 4\}$.

Set notation

- $A = \{1, 2, 3\}$: Read as 'A is the set containing the elements 1, 2 and 3'.
- $\{, \}$: We use these braces to enclose the elements of a set. So $\{1, 2, 3\}$ is the set containing 1, 2, and 3.
- $\in$: $a \in \{a, b, c\}$ asserts that a is an element of the set $\{a, b, c\}$.
- Thus the above means that a is an element of the set containing the letters a, b, and c.
- $\notin$: $d \notin \{a, b, c\}$ because d is not an element of the set $\{a, b, c\}$.

Problem

$A = \{1, b, \{x, y, z\}, \emptyset\}$. Is x in A?

This set contains four elements: the number 1, the letter b, the set $\{x, y, z\}$, and the empty set $\emptyset = \{\}$, the set containing no elements.

The answer is no. None of the four elements in A are the letter x, so we must conclude that $x \notin A$.

Some useful sets

- $\emptyset$: The empty set is the set which contains no elements.
- $\mathbb{U}$: The universe set is the set of all elements.
- $\mathbb{N}$: The set of natural numbers. That is, $\mathbb{N} = \{1, 2, 3, ...\}$.
- $\mathbb{Z}$: The set of integers. That is, $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, 3, ...\}$.
- $\mathbb{Q}$: The set of rational numbers.
- $\mathbb{R}$: The set of real numbers.

Writing (infinite) Sets

- We have described the sets above by listing their elements.
- Sometimes this is hard to do, especially when there are a lot of elements in the set (perhaps infinitely many).
- For instance, if we want A to be the set of all even natural numbers, would could write, $A = \{2, 4, 6, \ldots\}$. However, this is imprecise.
- A better way would be:

$$A = \{x \in \mathbb{N} : \exists n \in \mathbb{N}(x = 2n)\}$$

- Here:
  - $x \in \mathbb{N}$ means x is in the set $\mathbb{N}$
  - ':' is read 'such that'. $|$ or $\ni$ can also be used for the 'such that' symbol.
  - '$\exists n \in \mathbb{N}(x = 2n)$' is read 'there exists an n in the natural numbers for which x is two times n' (in other words, x is even)

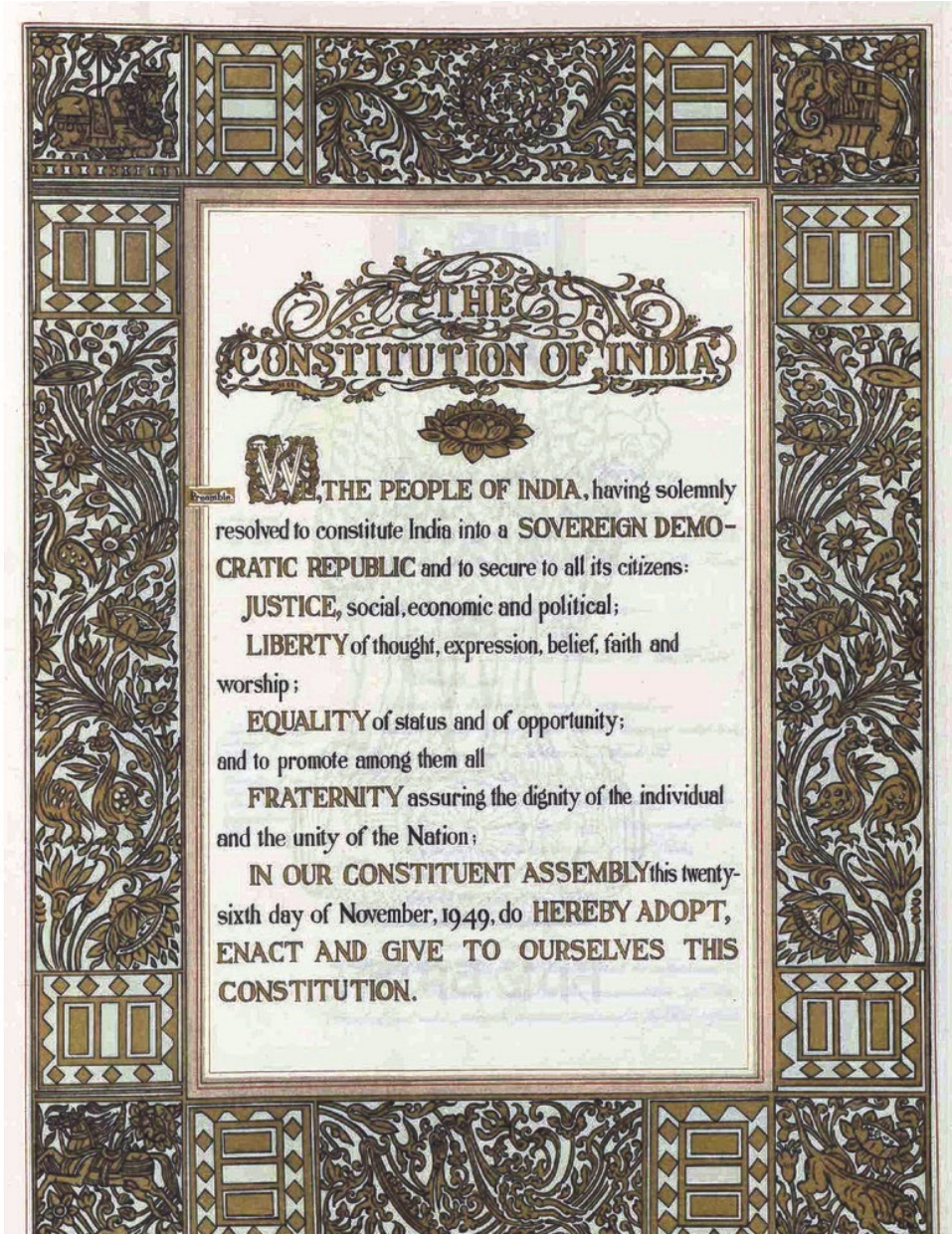
Equal Sets

Two sets are equal exactly if they contain the exact same elements, i.e. same elements and the same number of elements.

Order does not matter.

For example: The set, P, containing all the vowels in the Preamble to the Constitution of India is precisely the same set as the set of vowels in the word 'questionably' Q, i.e. $P = Q$.

Preamble to the Constitution of India



THE CONSTITUTION OF INDIA

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a SOVEREIGN DEMOCRATIC REPUBLIC and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity;

and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

Subsets

- What about the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$? Clearly $A \neq B$, but notice that every element of A is also an element of B.
- Because of this we say that A is a subset of B, or in symbols $A \subset B$ or $A \subseteq B$. Both symbols are read 'is a subset of'.
- $A \subset B$: Proper subset.
- $A \subseteq B$: A is either equal to or is a subset of B.
- This is analogous to the difference between $<$ and $\leq$.

Cardinality

- The size of a set is called the set's cardinality.
- For sets that have a finite number of elements, the cardinality of the set is simply the number of elements in the set.
- If set A has 6 elements (cardinality of 6), we would denote this as $|A| = 6$.

Set algebra: Union

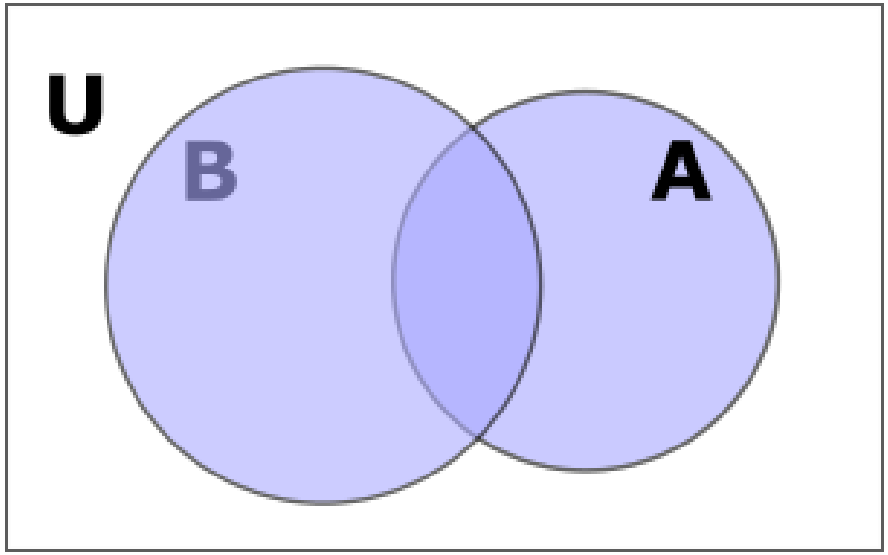


Figure: Venn Diagram for A ∪ B

Set algebra: Intersection

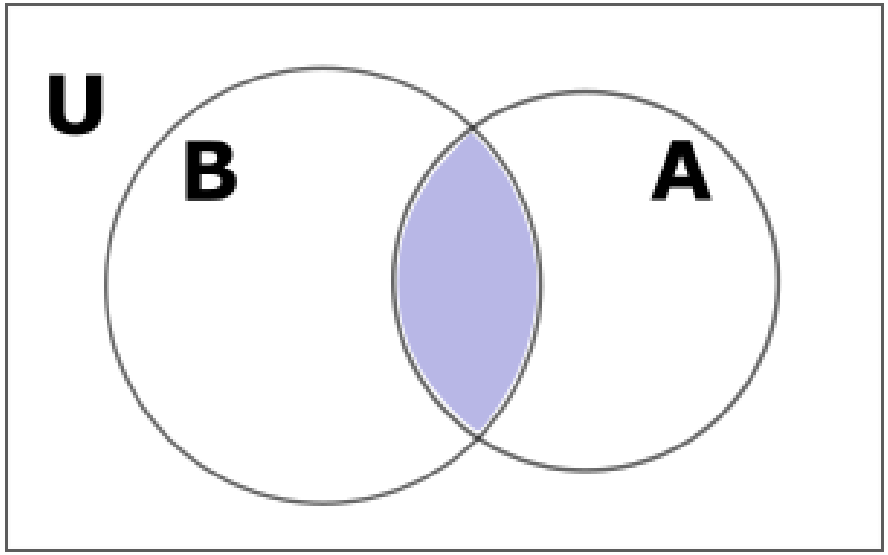


Figure: Venn Diagram for A ∩ B

Set algebra: Complement

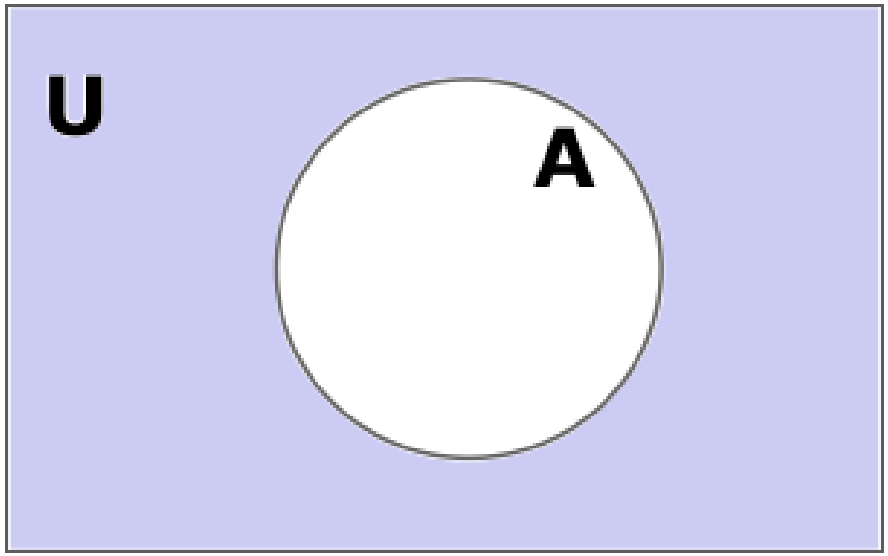


Figure: Venn Diagram for $\overline{A}$

Set algebra: Set difference

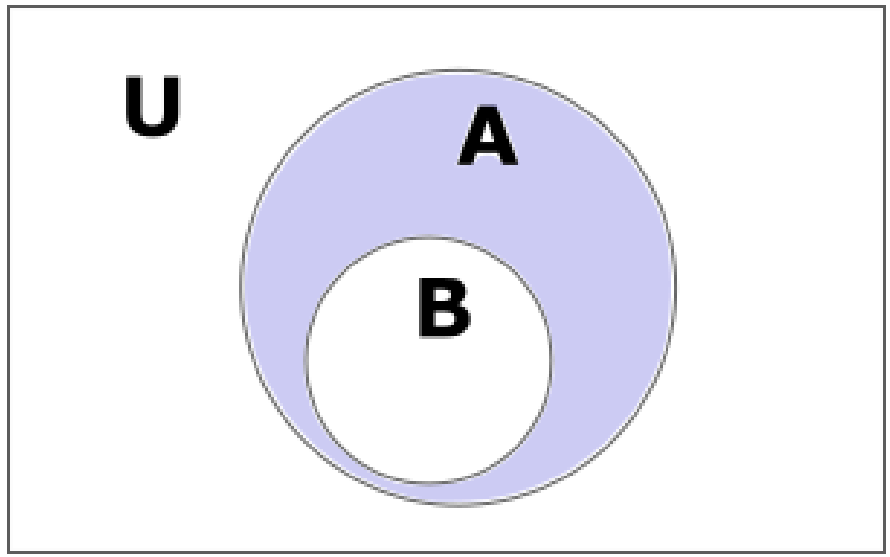


Figure: Venn Diagram for $A \setminus B$

Start with A and remove all of the elements which were in B. Another way to write this is the set difference $= A \setminus B$

Section 3

Computation with Finite State Machines

What do computer systems do mainly?

- At a very very high level?
- Systems are functions that transform signals.
- The domain and the range of these functions are both signal spaces, which significantly complicates specification of the functions.
- A broad class of systems can be characterized using the concept of state and the idea that a system evolves through a sequence of changes in state, or state transitions.

An example to build confidence

A turnstile is a form of gate which allows one person to pass at a time.
A turnstile can be configured to enforce one-way human traffic.

Figure: Image Courtesy: Wikipedia.

An example

Current State	Input	Next State	Output
Locked	coin	Unlocked	Unlocks the turnstile for the customer
Locked	push	Locked	None
Unlocked	coin	Unlocked	None
Unlocked	push	Locked	Locks the turnstile once the customer has pushed through.

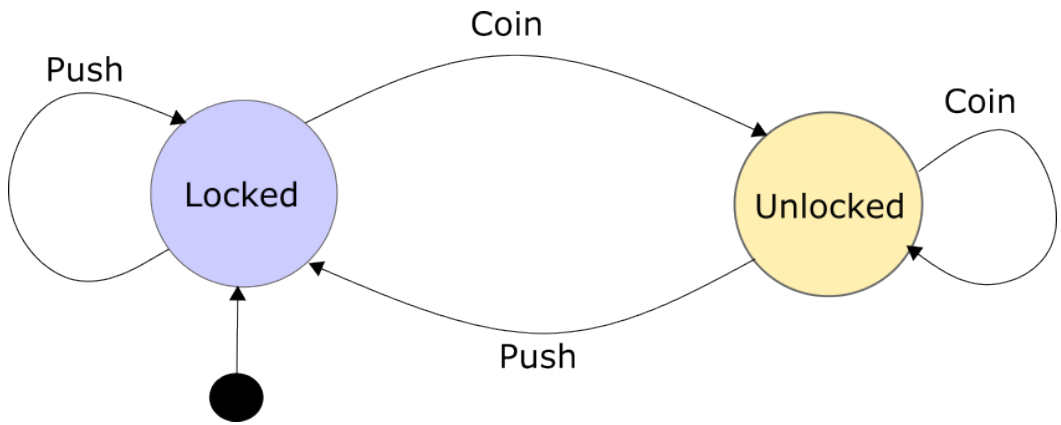


Figure: A State Space Model representing a turnstile.

Another series of examples

- You're sleeping peacefully. You hear the sound of the alarm clock and push the button to shut it down.
- You use the key to turn the lock in your apartment. The right key makes the lock go from unlocked to the locked state.
- Your car alarm beeps if you don't wear the seat belts.
- The traffic lights change based on the time allocated.
- You go to a friend's place and you play a game of snakes and ladders.

Finite State Machine [1]

- Finite State Machine is the simplest machine that is used for recognizing patterns. Informally, you can think of it to be like a computer with extremely limited memory.
- It is a mathematical model of computation. Essentially, it is pretty much an abstract model of a very simple digital computer.
- It possesses a set of states and rules that are used for moving from one state to another, but it is dependent on the applied input symbol.
- States represent the status of a system.
  - Automatic sliding doors in supermarkets have two states: either close or open.
  - Traffic lights contain three states: red, yellow, green.
  - More sophisticated examples can contain quite a lot of states, but never an infinite number.

Finite State Machine [2]

Definition

$M = (S, I, O, f_S, f_O)$ is a Finite State Machine if S is a finite set of states, I is a finite set of input symbols (the input alphabet), O is a finite set of output symbols (the output alphabet), and f_S and f_S are functions where $f_S : S \times I \rightarrow S$ and $f_O : S \rightarrow O$. The machine is fixed to begin in a fixed starting state s_0.

- Function f_S is the **next state function** which maps the $(state, input)$ pair to the $state$. Thus:

$$state_{t+1} = f_S(state_t, input_t)$$

- The function f_O is an output function, which gives the output of the state at time t. Thus:

$$output_t = f_O(state_t)$$

- The effect of application of f_O is visible instantly, but the effect of applying f_S is not visible until the next clock pulse.

Finite State Machine [3]

- The FSM describes a system procedurally, giving a sequence of step-by-step operations for the evolution of a system.
- It shows how the input signal drives changes in state, and how the output signal is produced.
- Implementing a system described by a FSM in software or hardware is straightforward. The hardware or software simply needs to sequentially carry out the steps given by the model.
- Conversely, given a piece of software or hardware, it is often useful to describe it using a state-space model, which yields to better analysis than more informal descriptions.

Exercise: Going through a FSM

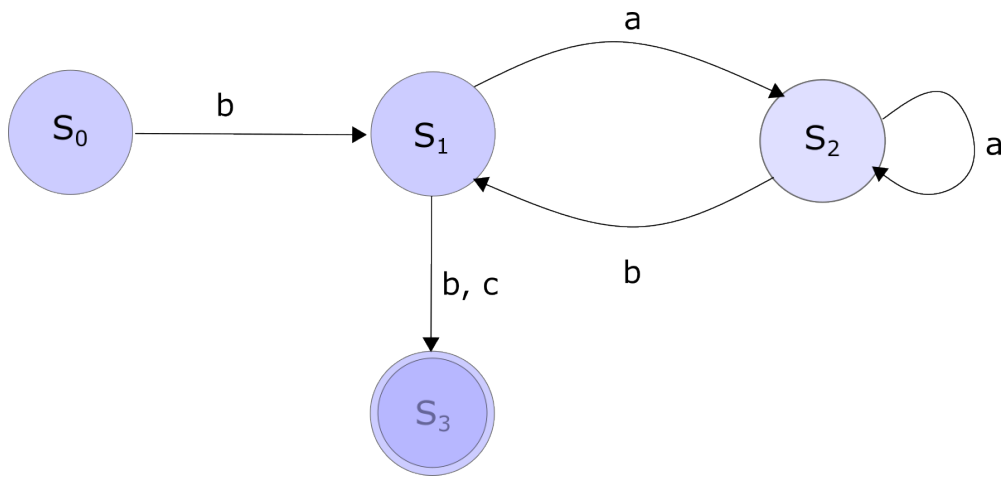


Figure: State Space Model for a FSM.

What will be the final state for the inputs? Try with the given inputs and look at the transitions with different inputs.

- 'baabb'
- 'cababb'
- 'bacacbc'
- 'bacacba'

Exercise: Homework

When a call arrives, the phone rings.
If the phone is not picked up, then on the third ring, the machine answers.
It plays a pre-recorded greeting requesting that the caller leave a message ("Hello, sorry I can't answer your call right now. Please leave a message after the beep"), then records the caller's message, and then automatically hangs up.
If the phone is answered before the third ring, the machine does nothing.

What did we learn today?

1. Discrete math as the language of computation
2. Sets
3. Computation with Finite State Machines

Thank you!