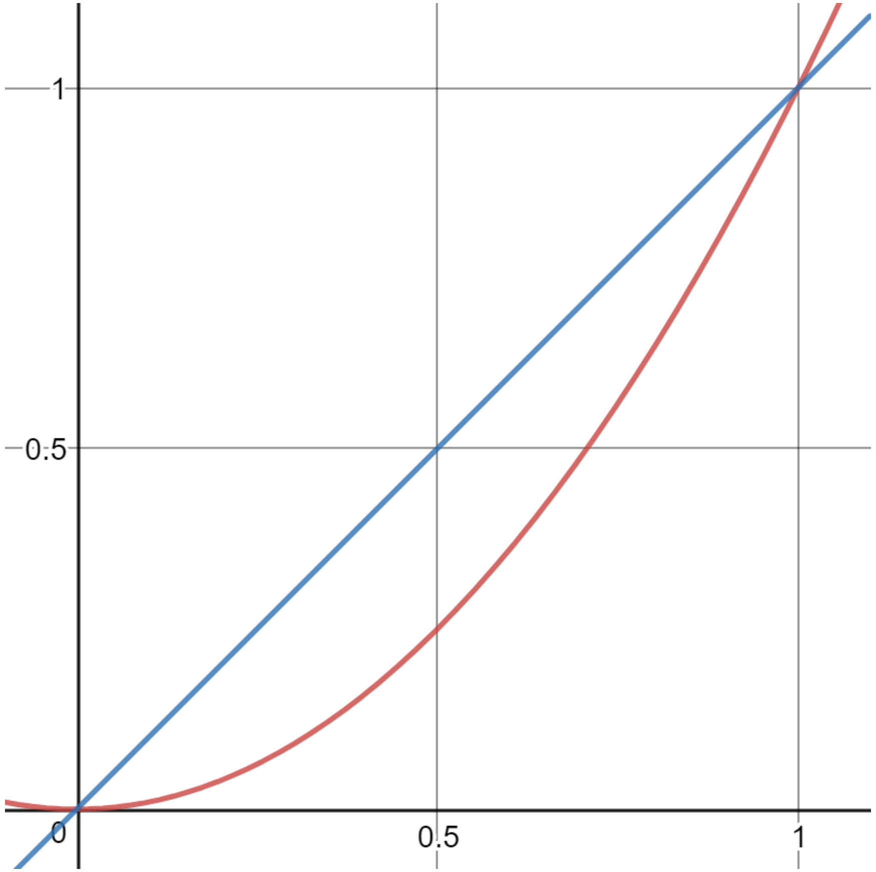
DOUBLE INTEGRALS (Cont.)

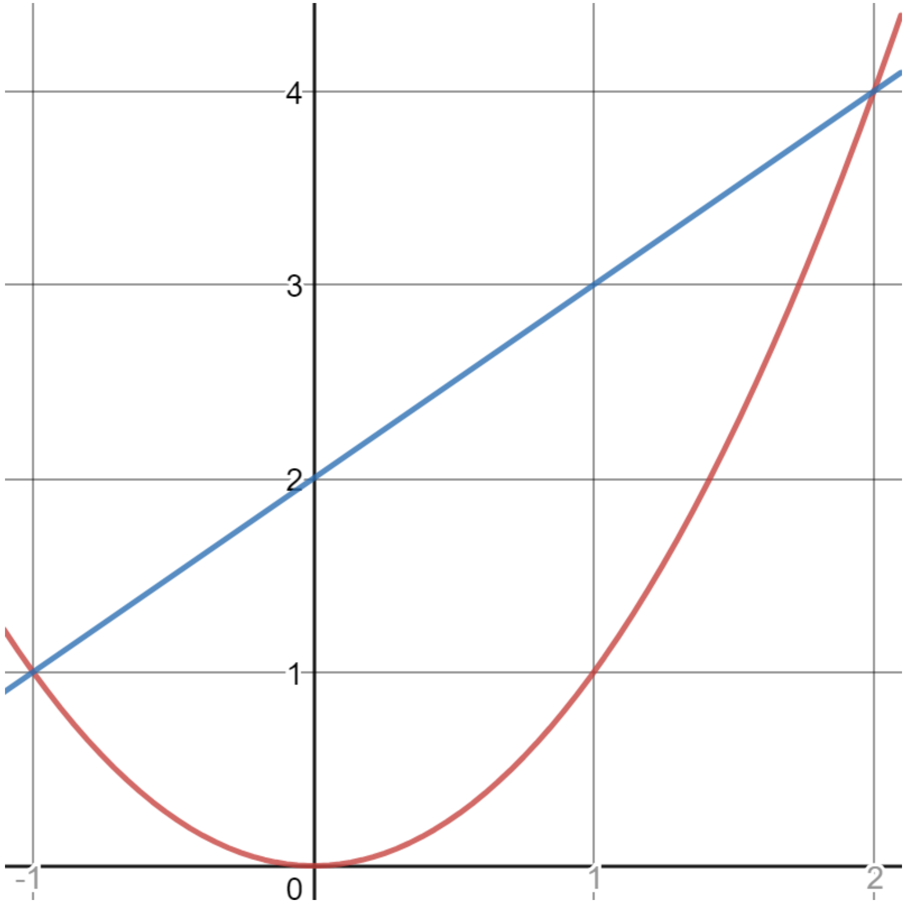
Applications

Problem - 1 Using a double integral find the area of the region enclosed by the parabola $y = x^2$ and the line $y = x$.



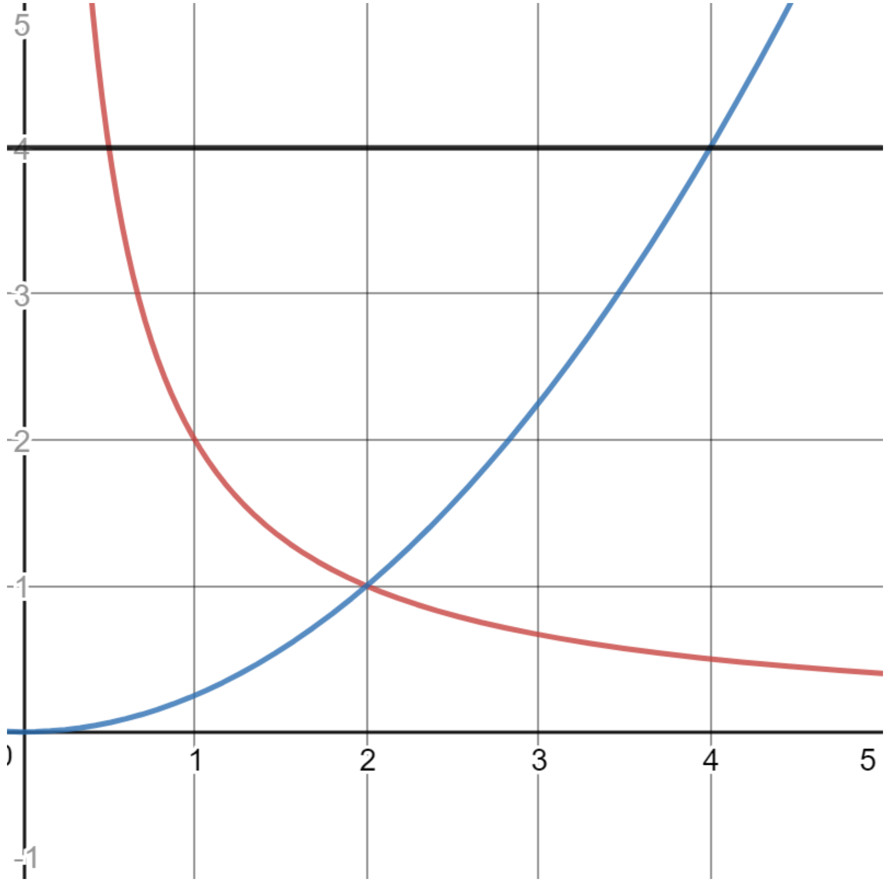
$$\int_0^1 \int_{x^2}^{x} \mathrm{d}y \, \mathrm{d}x = \frac{1}{6}$$

Problem - 2 Using a double integral find the area of the region enclosed by parabola $y = x^2$ and the line $y = x + 2$.



$$\int_{-1}^{2}\int_{x^2}^{x+2} \mathrm{d}y\,\mathrm{d}x = \frac{9}{2}$$

Problem - 3 Using a double integral, determine the area bounded by the curves $xy = 2$, $y = \frac{x^2}{4}$ and $y = 4$.

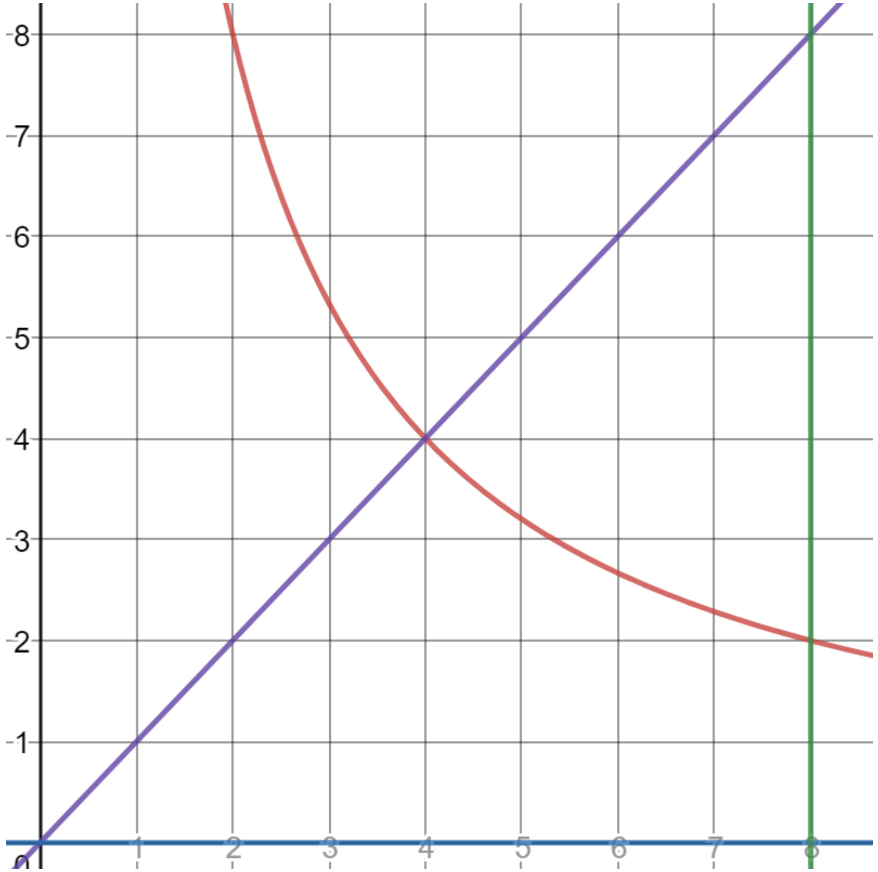


$$\int_{y=1}^{4} \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\, dy = \frac{28}{3} - 2 \ln 4$$

Problem - 4 Using double integrals find the volume of the solid below the $z = xy$ over the region enclosed by $y = 4 - x^2, x = 1, x = 2$ and the x-axis.

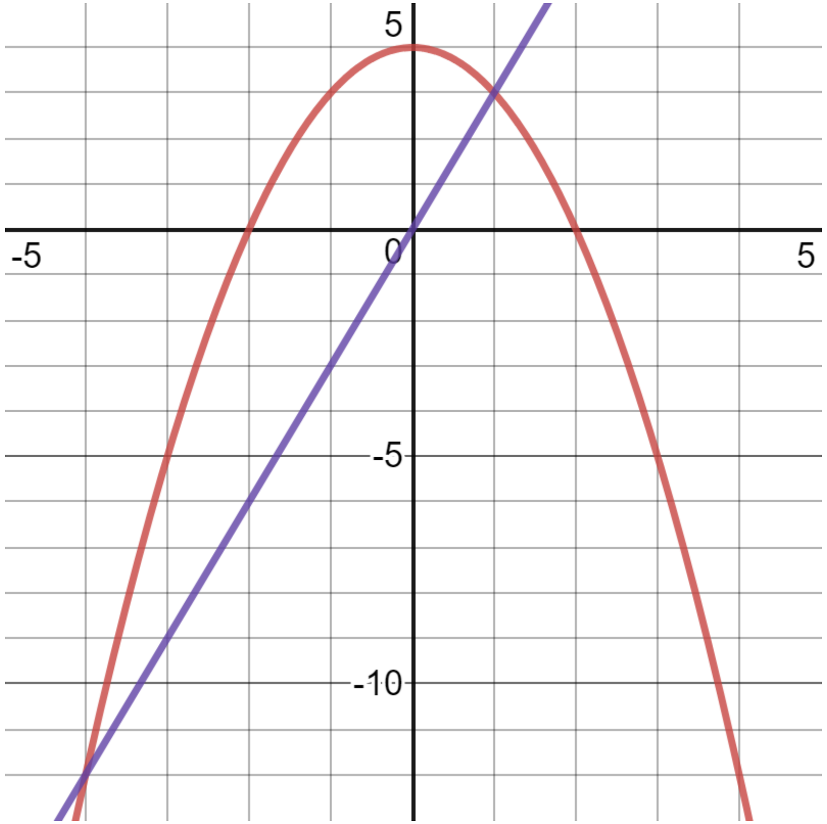
$$V = \int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} xy \;\, \mathrm{d}x \, \mathrm{d}y = \frac{9}{4}$$

Problem - 5 Calculate the volume of a solid whose base is in a $xy-$plane and is bounded by the curve $xy = 16$ and the line $y = x, y = 0, x = 8$ while the top of the solid is in the plane $z = x$.



$$\int_0^4 \int_0^x x\,dy\,dx + \int_4^8 \int_0^{16/x} x\,dy\,dx = \frac{256}{3}$$

Problem - 6 Calculate the volume of a solid whose base is in a $xy-$plane and is bounded by the parabola $y = 4 - x^2$ and the straight line $y = 3x$ while the top of the solid is in the plane $z = x + 4$.



$$V = \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)\,\mathrm{d}y\,\mathrm{d}x = \frac{625}{12}$$

Conclusion:

Some Applications of Double Integrals

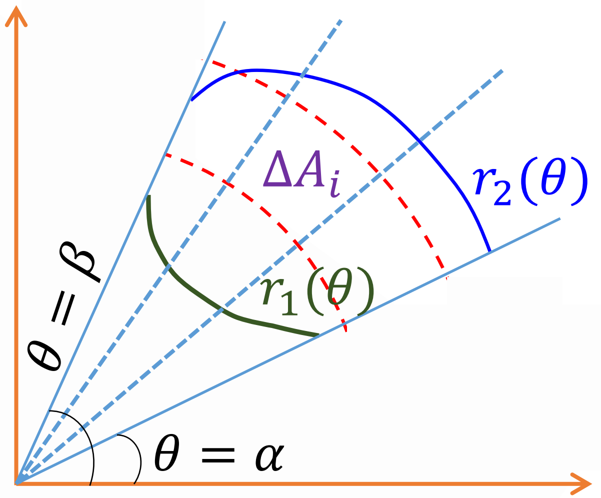
- Computation of Area
- Computation of Volume

DOUBLE INTEGRALS (Cont.)

Double Integrals in Polar Form

Double Integrals: Change of Variables

Double Integrals in Polar Forms



$$\Delta A_i = (r_i + \Delta r_i)^2 \frac{\Delta\theta_i}{2} - r_i^2 \frac{\Delta\theta_i}{2}$$

$$= \left(2r_i\Delta r_i + \Delta r_i{}^2\right)\frac{\Delta\theta_i}{2}$$

$$= \left(r_i + \frac{\Delta r_i}{2}\right)\Delta r_i\,\Delta\theta_i$$

$$= r_i^*\Delta r_i\Delta\theta_i$$

$$I = \lim_{n\to\infty}\sum_{j=1}^{n} f(r_j^*, \theta_j^*)\,\Delta A_j = \int_{\theta=\alpha}^{\beta}\int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Changing Cartesian integral to polar integrals

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

- Substitute $x = r\cos\theta$, $y = r\sin\theta$
- Replace $dx\, dy$ by $r\, dr\, d\theta$
- G is same as R but described in polar corrdinates

Example: Compute area of first quadrant of a circle of radius a.

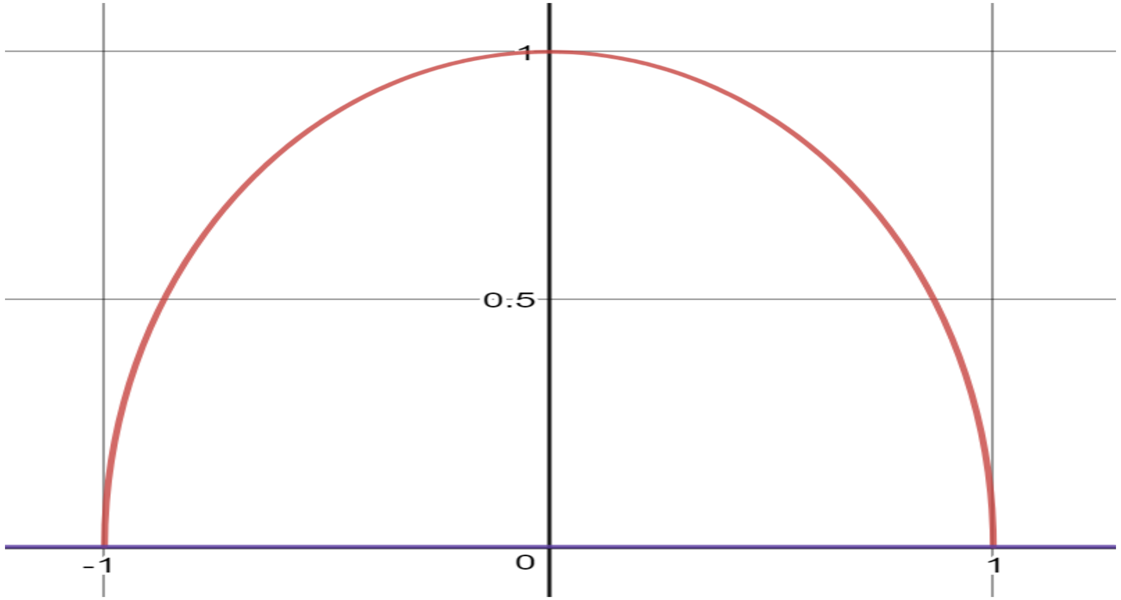
$$A = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r dr d\theta$$

$$= \frac{a^2}{2} \frac{\pi}{2}$$

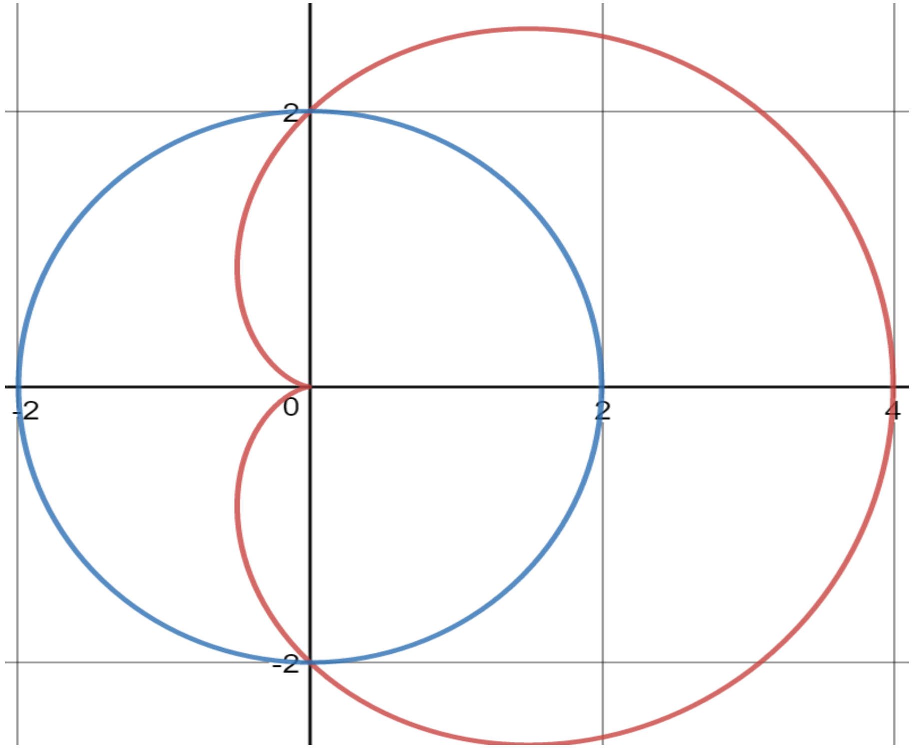
$$= \frac{\pi a^2}{4}$$

Problem -1: Evaluate $\iint_R e^{x^2+y^2}\ dy\, dx$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$



$$\int_0^{\pi}\int_0^{1} e^{r^2}\, r\, dr\, d\theta = \frac{1}{2}\int_0^{\pi} e^{r^2}\Big|_0^1\ d\theta \quad = \frac{1}{2}\int_0^{\pi}(e-1)\ d\theta \quad = \frac{\pi}{2}(e-1)$$



Problem -2:

Calculate the area which is inside the cardioid $r = 2(1 + \cos\theta\ )$ and outside the circle $r = 2$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2}^{2(1+\cos\theta\ )} r\,dr\,d\theta = \pi + 8$$

Problem - 3: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1\, d\theta = \frac{\pi}{4}$$

Note: $I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

$$= \frac{\pi}{4}$$

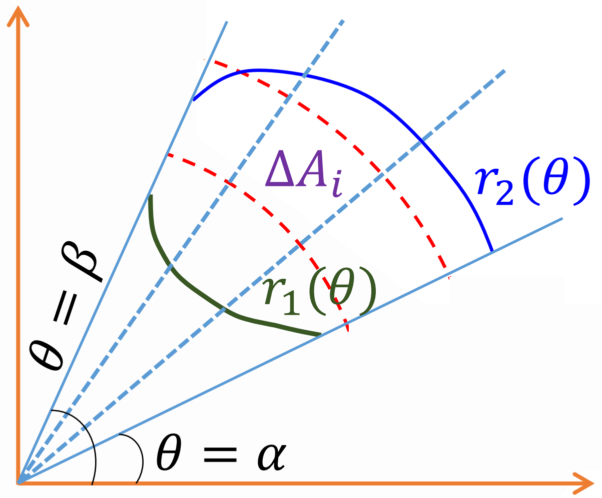
$$\Rightarrow \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Conclusion:

Double Integrals in Polar form

- Some integrals become easier by changing to polar coordinate due to
  - Integrands
  - Domain

Double Integrals in Polar Forms (Previous Lecture)



$$\iint_R f(x,y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\; r\, dr\, d\theta$$

Double Integrals – Change of Variable

$$\int_a^b f(x)\,dx = \int_c^d f\big(g(t)\big)\,g'(t)\,dt$$

Substitution: $x = g(t)$.

where $a = g(c)$ and $b = g(d)$

Double Integrals – Change of Variables

$$\iint_R f(x,y)\,dx\,dy$$

Substitution $x = \Phi(u,v), y = \psi(u,v)$

$$\iint_{R'} f\big(\Phi(u,v),\psi(u,v)\big)\,|J|\,dudv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

R' is the region in uv plane which corresponds to the region R in the xy-plane.

Double Integrals – Change of Variables (Special Case)

$$\iint_R f(x, y)\, dx\, dy$$

Cartesian to polar co-ordinates:

$$x = r\cos\theta\,, \quad y = r\sin\theta\,;$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & \mathrm{r}\cos\theta \end{vmatrix} = r$$

$$\Rightarrow \quad \iint_R f(x, y)\, dx\, dy = \iint_{R'} f(r\cos\theta\,, r\sin\theta)\; r\, dr\, d\theta$$

Problem -1 Find the volume in one octant of a sphere of radius a.

$$V = \iint_S \sqrt{a^2 - x^2 - y^2}\, dx\, dy$$

S is the first quadrant of the circular disc $x^2 + y^2 \leq a^2$

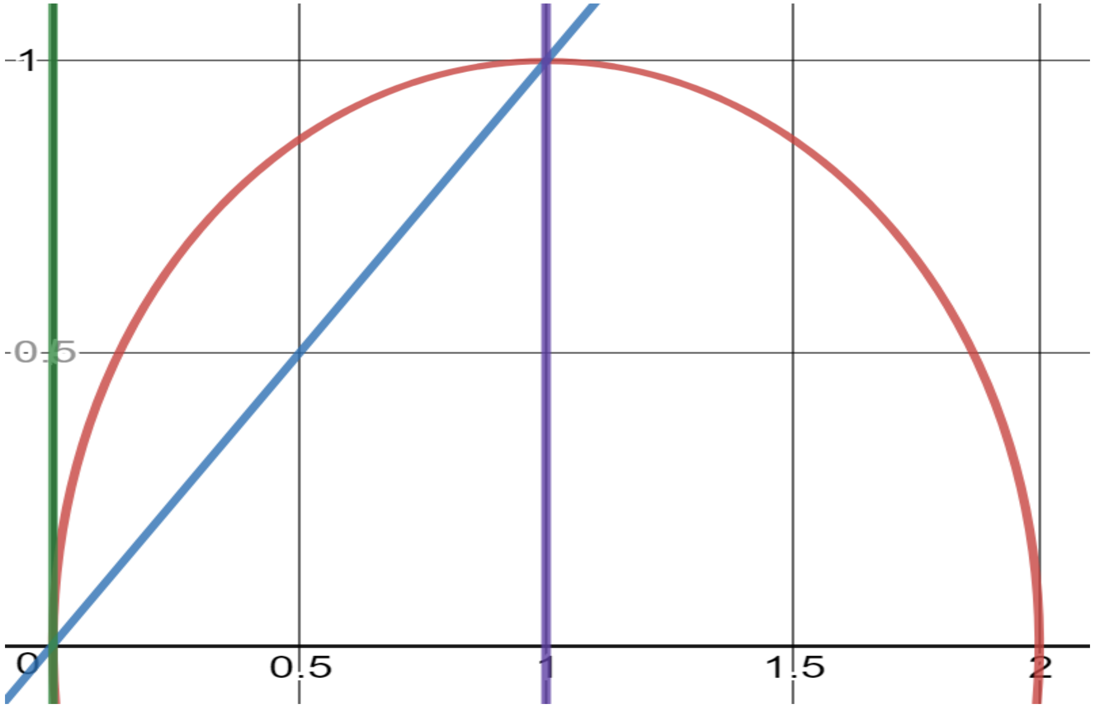
Change of variables $x = r\cos\theta\,,\quad y = r\sin\theta\,,\quad |J| = r$

$$\int\int_S \sqrt{a^2 - x^2 - y^2}\, dx\, dy = \int\int_R \sqrt{a^2 - r^2}\; r\, dr\, d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} \sqrt{a^2 - r^2}\; r\, dr\, d\theta$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(a^2 - r^2)^{\frac{3}{2}}\Big|_0^a \quad = \frac{\pi}{6}a^3$$

Problem -2: Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ by changing to polar coordinates.

The region of integration is bounded by $y = x$, $y = \sqrt{2x - x^2}$, $x = 0$ and $x = 1$

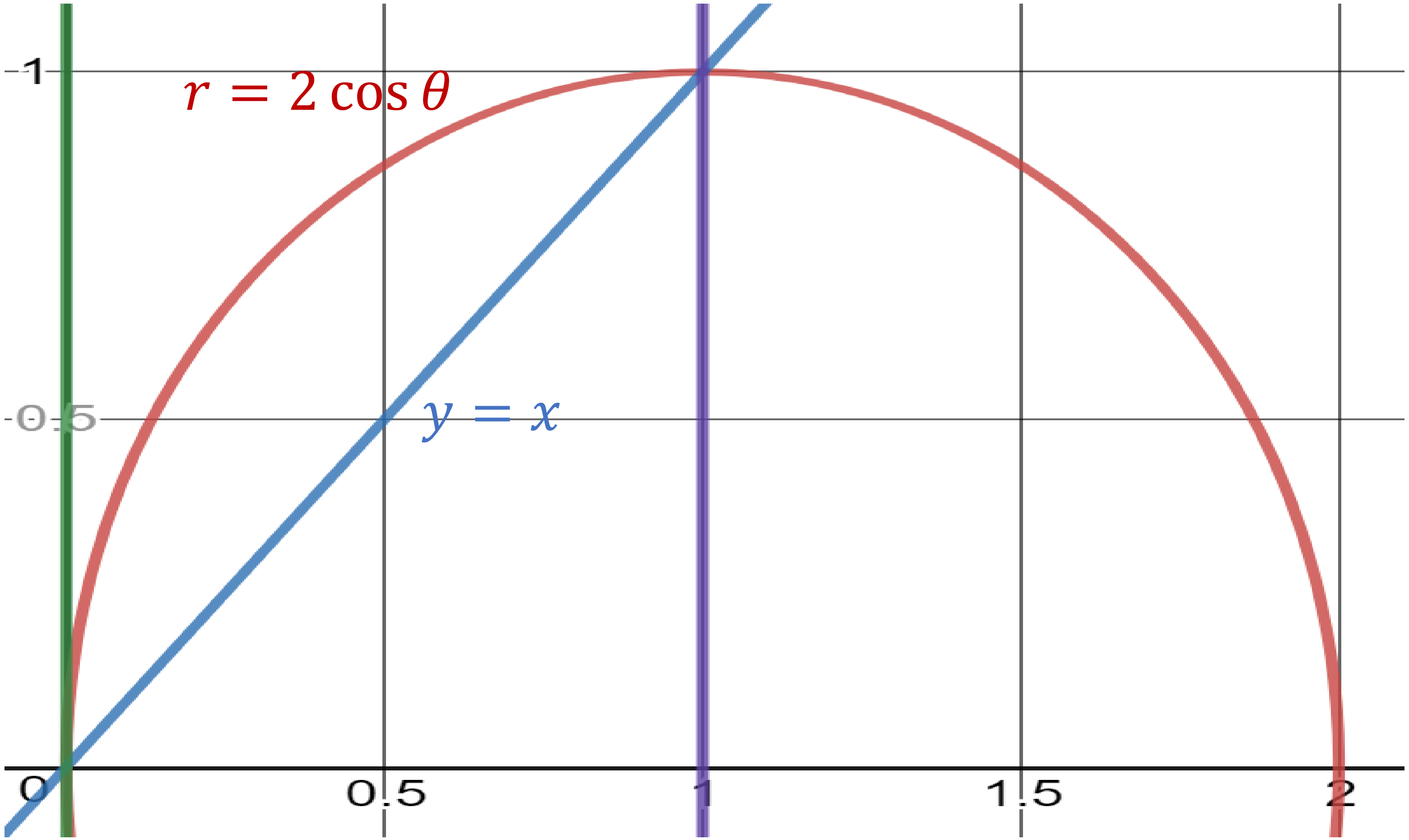


Polar equation of the circle

$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1,$

$r^2 - 2r\cos\theta = 0,$

$r = 2\cos\theta$



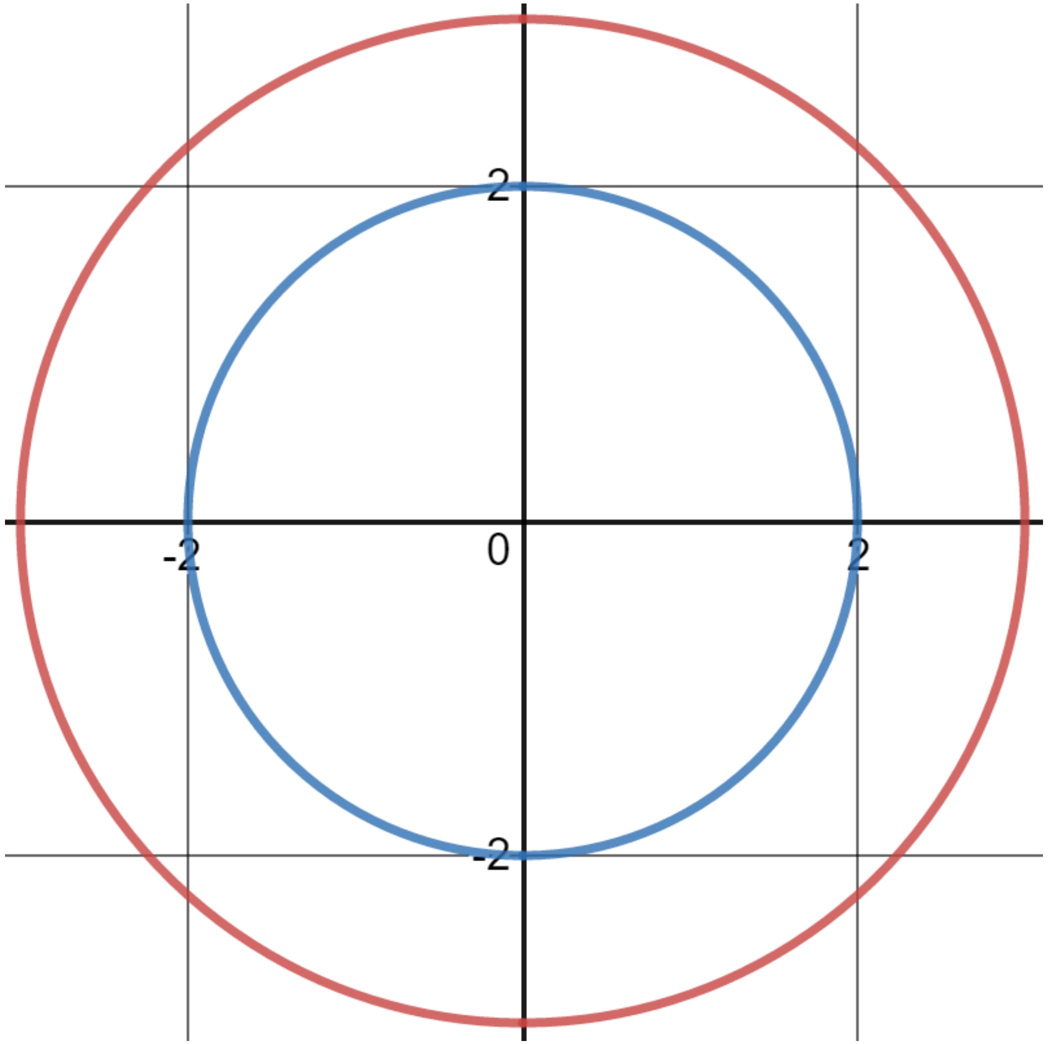
$$\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$$\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^4\theta\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos^2\theta)^2 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2\theta)^2\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos^2 2\theta + 2\cos 2\theta) d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1+\frac{1}{2}(1+\cos 4\theta)+2\cos 2\theta\right) d\theta = \frac{1}{8}(3\pi - 8)$$



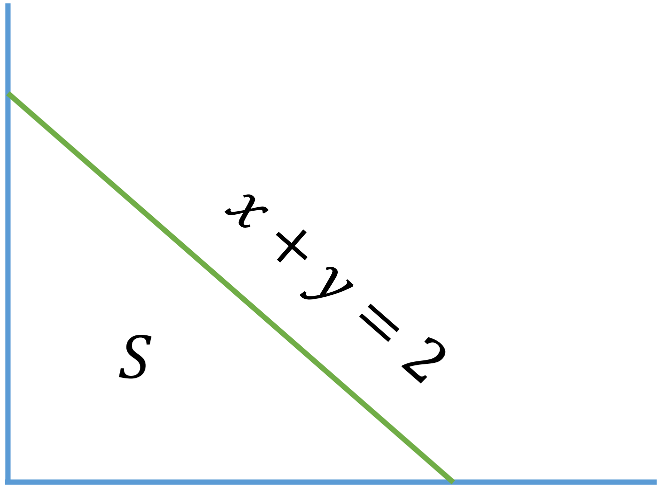
Problem - 3: Evaluate $\iint_R \sqrt{x^2+y^2}\, dx\, dy$ by changing to polar coordinates, where R is the region in the xy plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$

$x = r\cos\theta\,,\quad y = r\sin\theta\,,\quad |J| = r$

$$I = \int_0^{2\pi}\int_2^3 r\, r\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^3}{3}\right]_2^3 d\theta$$

$$= \left(\frac{27-8}{3}\right)2\pi = \frac{38}{3}\pi$$

Problem-4 $\displaystyle\iint_S e^{\frac{y-x}{y+x}}\,dx\,dy$

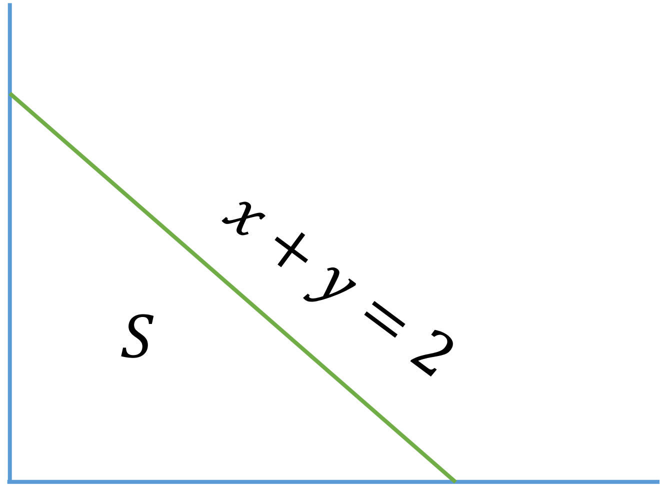


Change of variables $y-x=u,\quad y+x=v$ implies

$$x=\frac{v-u}{2},\qquad y=\frac{v+u}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{vmatrix}=-\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}$$

$$\iint_S e^{\frac{y-x}{y+x}}\, dx\, dy$$

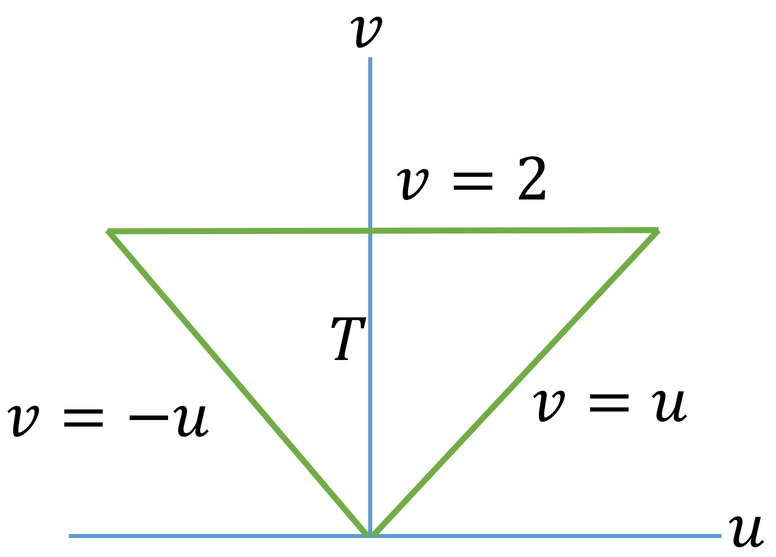


Change of variables

$$y-x=u, \quad y+x=v$$

$$x=\frac{v-u}{2}, \quad y=\frac{v+u}{2}$$

Domain in the uv-plane.

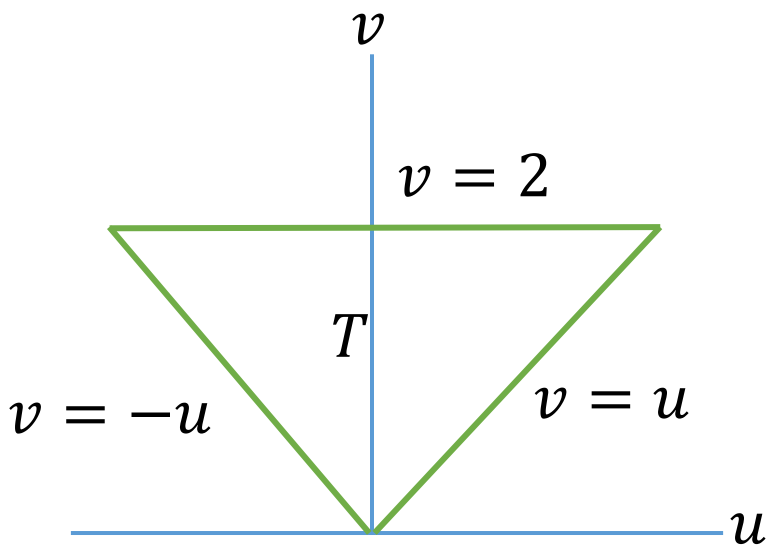


Line $x=0$ maps to

Line $y=0$ maps to

Line $x+y=2$ maps to

$$\iint_S e^{\frac{y-x}{y+x}}\, dx\, dy \qquad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$



$$\iint_S e^{\frac{y-x}{y+x}}\, dx\, dy = \iint_T e^{\frac{u}{v}} \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2}\int_{v=0}^{2}\int_{u=-v}^{v} e^{\frac{u}{v}}\, du\, dv$$

$$= \frac{1}{2}\int_0^2 v\left(e - \frac{1}{e}\right) dv = e - \frac{1}{e}$$

Problem-5 $\iint_R (x^2y - x^3)\, e^{(y-x)^2} dA$ where $R: 0 \le x \le 1 \;\&\; x \le y \le x+1$

Substitute $x = u \;\&\; y - x = v \Rightarrow x = u \;\&\; y = u + v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

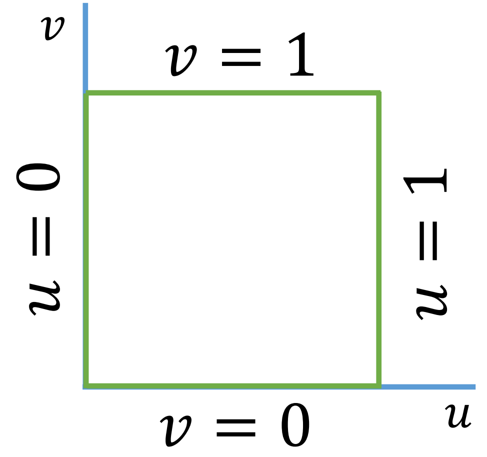
Domain in the uv-plane.

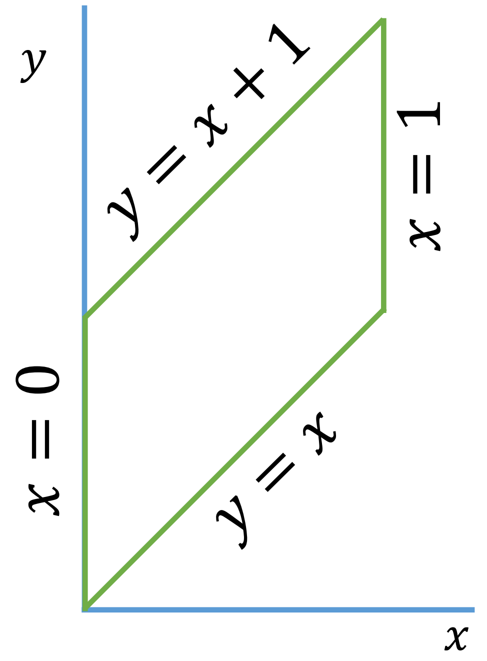
Line $x = 0$ maps to

Line $x = 1$ maps to

Line $y = x$ maps to

Line $y = x + 1$ maps to





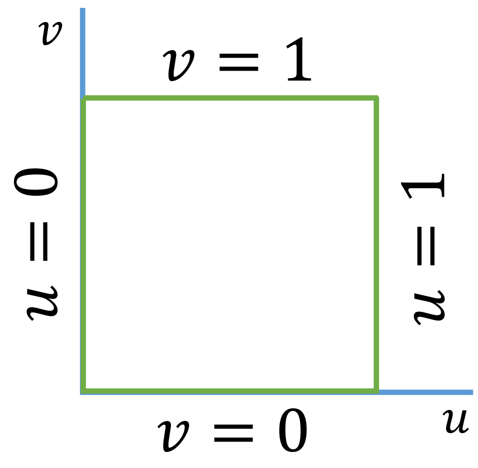
Substitute $x = u$ & $y - x = v$ $\Rightarrow x = u$ & $y = u + v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$\int\int_R x^2 (y-x)\, e^{(y-x)^2}\, dA = \int_0^1 \int_0^1 u^2\, v\, e^{v^2}\, du\, dv$$

$$= \frac{1}{2}\int_0^1 u^2\, e^{v^2}\Big|_0^1\, dv$$

$$= \frac{(e-1)}{6}$$



Conclusion:

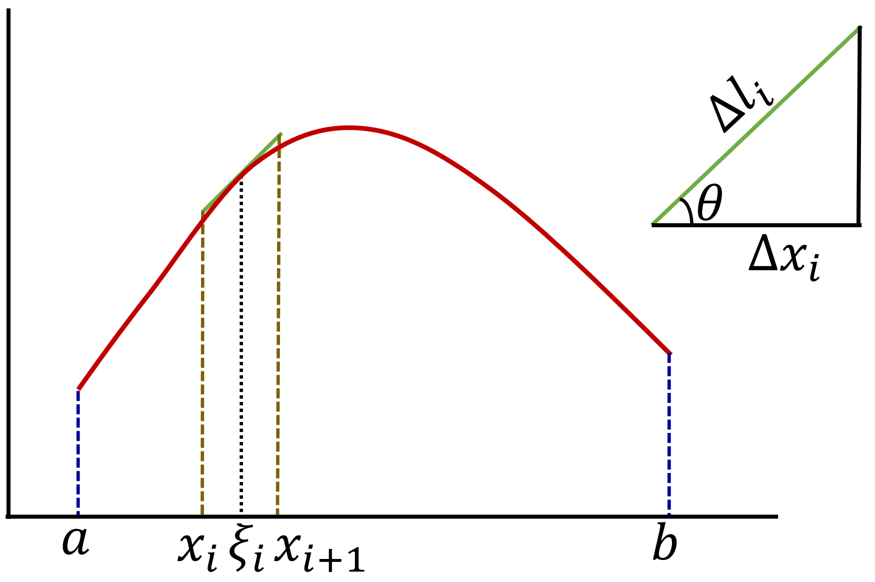
Double Integrals – Change of Variables

- Important for evaluation of integrals
- Changing to polar coordinate is a particular case

Topic

Integral Calculus – Double Integrals: Surface Area

Recall: Computation of curve length



Length of the curve $L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta l_i$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(\xi_i))^2}\, \Delta x_i$$

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

$$\frac{\Delta x_i}{\Delta l_i} = \cos\theta \quad \Longrightarrow \Delta l_i = \frac{1}{\cos\theta}\, \Delta x_i$$

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} \quad \Longrightarrow \frac{1}{\cos\theta} = \sqrt{1 + (f'(\xi_i))^2}$$

Computation of Surface Area $(z = f(x, y))$

Curve Length $L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dx\, dy$$

where D is the projection of the surface in the xy-plane.

Similarly, if the equation is given in the form: $x = \mu(y, z)$ or in the form $y = \psi(x, z)$ then

$$S = \iint_{\widehat{D}} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\; dy\, dz \quad OR \quad \iint_{\widehat{\widehat{D}}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\; dx\, dz$$

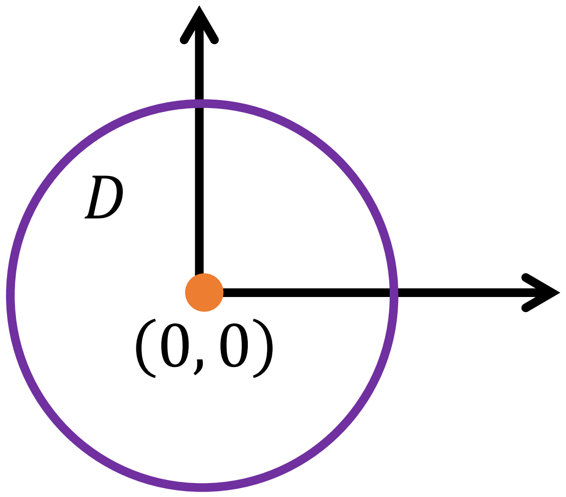
where $\widehat{D}$ and $\widehat{\widehat{D}}$ are the domains in the yz and xz planes in which the given surface is projected.

Problem - 1 Compute the surface area of the sphere $x^2 + y^2 + z^2 = a^2$

Equation of the surface $z = \sqrt{a^2 - x^2 - y^2}$ (Upper half)

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

Domain of integration: $x^2 + y^2 \leq a^2$



$$S = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dydx$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

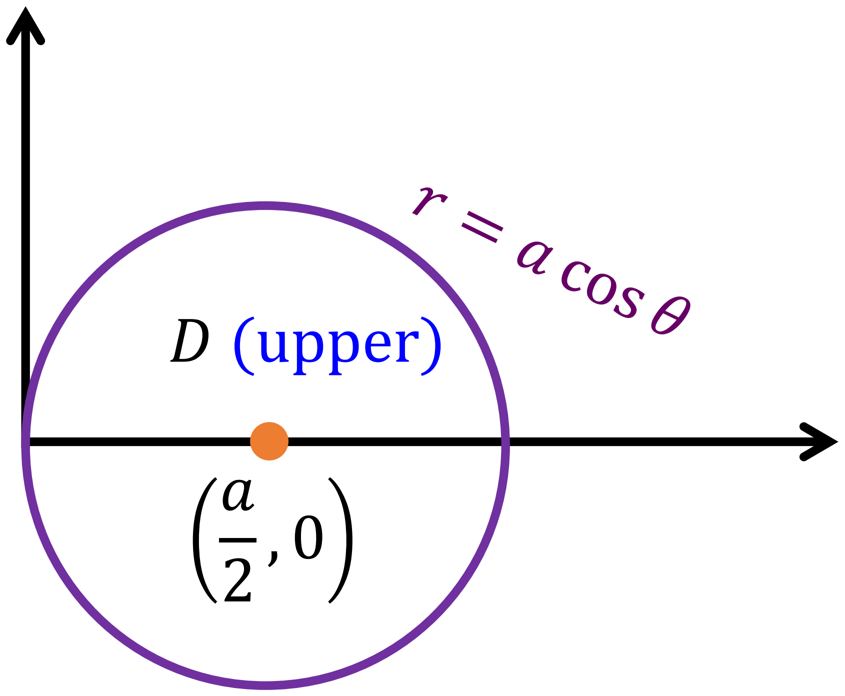
$$S = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dydx = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dydx$$

$$= 2\int_0^{2\pi}\int_0^{a} \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = -2a\, 2\pi \sqrt{a^2 - r^2}\,\Big|_0^a = 4\pi a^2$$

Problem - 2 Find the area of that part of the sphere $x^2 + y^2 + z^2 = a^2$ that is cut off by the cylinder $x^2 + y^2 = ax$.

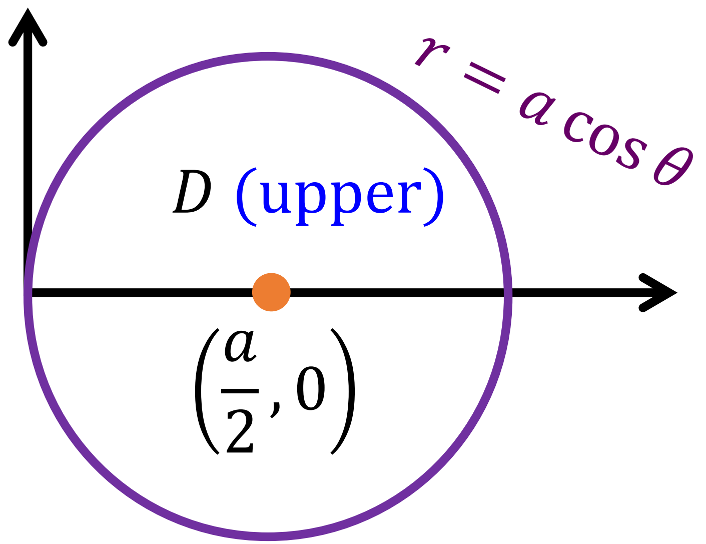
$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$



$$S = 2 \cdot 2 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = 4 \iint_D \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy$$

$$4\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy = 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

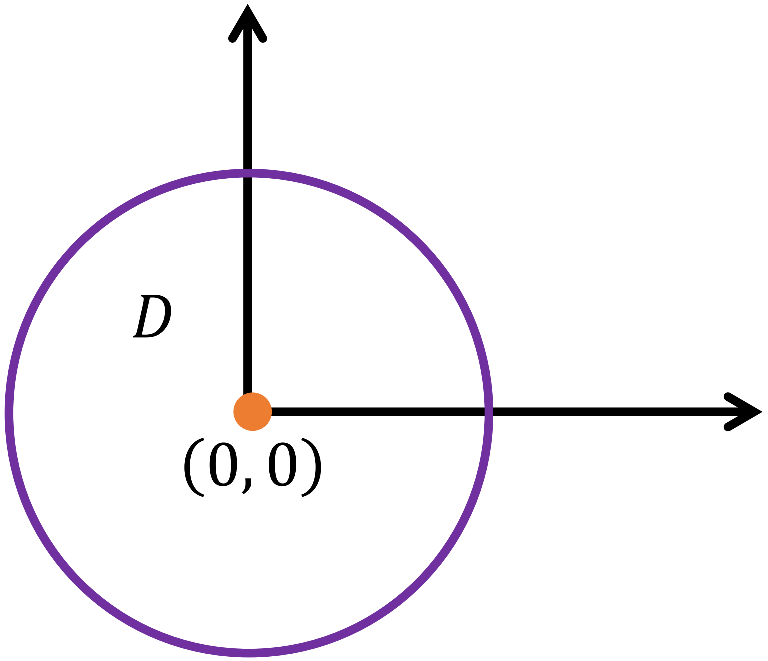


$$= 4\,a\int_0^{\pi/2}\left(-\sqrt{a^2-r^2}\right)_0^{a\cos\theta} d\theta$$

$$= 4a\int_0^{\pi/2}[-a\sin\theta + a]\,d\theta$$

$$= 4a\left[\{a\cos\theta\}_0^{\frac{\pi}{2}} + a\{\theta\}_0^{\frac{\pi}{2}}\right] = 4a\left[-a + a\frac{\pi}{2}\right] = 2a^2(\pi-2)$$

Problem - 3 Determine the surface area of the part of $z = xy$ that lies in the cylinder $x^2 + y^2 = 1$.



$$z = f(x, y) = xy \qquad z_x = y, \qquad z_y = x$$

$$S = \iint\limits_D \sqrt{1 + x^2 + y^2}\ dx\, dy$$

In polar coordinate $S = \int_0^{2\pi} \int_{r=0}^{1} \sqrt{1 + r^2}\ r\, dr d\theta$

$$= \int_0^{2\pi} \frac{1}{2} \frac{2}{3} \left[(1 + r^2)^{3/2}\right]_0^1 d\theta = \frac{2\pi}{3}\left(2^{3/2} - 1\right)$$

Conclusion:

Double Integrals – Application

- Surface area