Allowable Stress

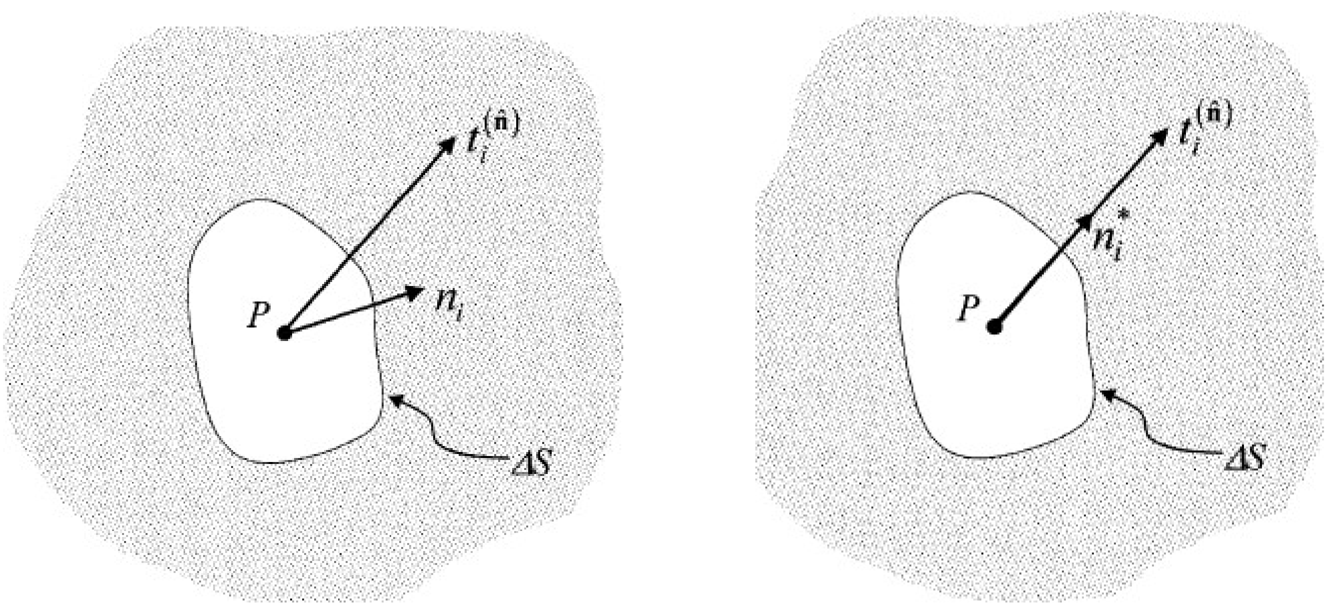
- How to get the value of allowable stress??
- Easier to get these numbers for an uni-axial or simple loading
- How about generalized loading??
- Combined effect of all loads has to be taken into account
- What is the way out??

Principal stresses and planes

- In general, the stress vector does not act in the direction of unit normal n_i
- Normal and shear stress components can be determined on any arbitrary plane with normal vector n_i
- However, we may ask
  - Are there any planes on which only normal stresses act i.e. the resultant stress vector is along the area vector
  - On which plane normal stress is maximum and what is its magnitude
  - What is the plane on which shear stress is maximum
  - All this knowledge will help for the safe design of mechanical element

Principal stresses and planes

- For some special directions, stress vector does act in the direction of n_i and hence can be expressed as a scalar multiple of that normal



$$t_i^{(\hat{\mathbf{n}})} = \sigma n_i$$

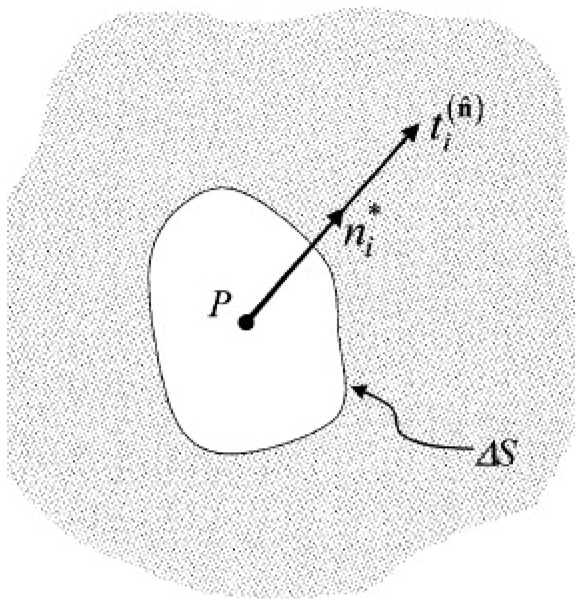
Scalar multiple

Principal stresses and planes

- How to get the direction of principal plane and magnitude of principal stress
  - Let us assume that there is a plane *n* on which the stress is wholly normal

$$\overset{n}{T} = \sigma n$$ and components

$$\overset{n}{T}_x = \sigma n_x, \qquad \overset{n}{T}_y = \sigma n_y, \qquad \overset{n}{T}_z = \sigma n_z$$



In general we have

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$

$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

Principal stresses and planes

- For *n* to be a principal direction, we get

$$(\sigma_x - \sigma)\, n_x + \tau_{xy}\, n_y + \tau_{xz}\, n_z = 0$$

$$\tau_{xy}\, n_x + (\sigma_y - \sigma)\, n_y + \tau_{yz}\, n_z = 0$$

$$\tau_{xz}\, n_x + \tau_{yz}\, n_y + (\sigma_z - \sigma)\, n_z = 0$$

- Three simultaneous homogenous equation with trivial solution

$$n_x = n_y = n_z = 0$$

- For existence of non-trivial solution

Principal stresses and planes

- Non-trivial solution exist if

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0$$

On expanding the determinant we get the following cubic equation:

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\,\sigma -$$

$$(\sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2) = 0$$

Three roots of equation:

σ_1, σ_2 and σ_3.

Principal stresses and planes

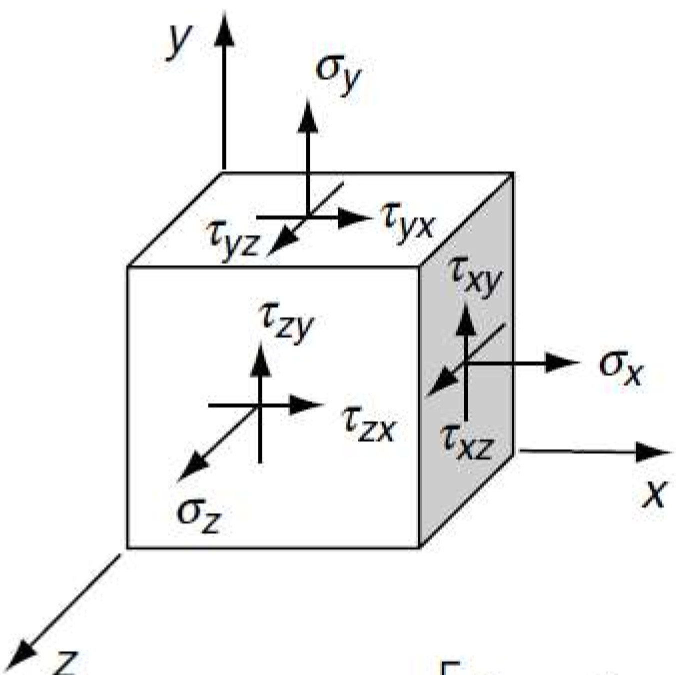
- For each root we can calculate the directions cosines of associated plane
- These roots are called principal stresses and corresponding direction represents the normal of principal plane
- To solve for direction cosines we need to satisfy

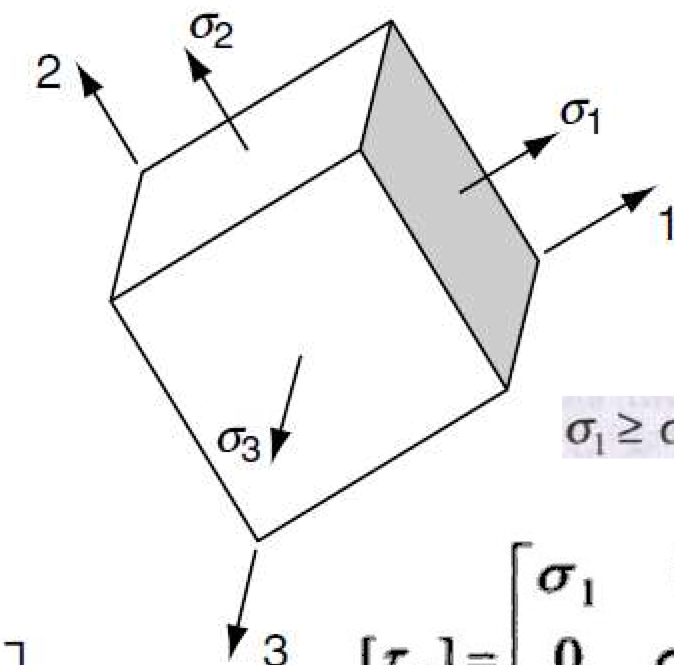
$$n_x^2 + n_y^2 + n_z^2 = 1$$

- Since the resultant stress is along the normal, there is no shear stress component on principal planes

Principal stresses and planes

- The stress can be represented as





$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$\boldsymbol{\sigma} = [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$$[\tau_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Principal stresses and Principal directions

- Some remarks
  - If stress tensor has real components, the three Invariants will be real and the roots of characteristic equation will also be real.
  - If principal values are distinct then the principal directions associated with these will be mutually perpendicular.
  - If two principal values are equal say $\sigma_1 = \sigma_2$, then principal direction n_3 associated with σ_3 will still be unique and any direction in the plane perpendicular to n_3 may serve as principal direction.
  - If all three principal stresses are equal then every direction is a principal direction and state of stress will represent hydrostatic or isotropic state of stress
  - Knowledge of principal stresses is very important from the point of view of the strength and failure of materials and hence in designing the structural or mechanical member. Can you see how??

Stress invariants

- We have the cubic equation for σ

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\,\sigma_y + \sigma_y\,\sigma_z + \sigma_z\,\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\,\sigma -$$

$$(\sigma_x\,\sigma_y\,\sigma_z + 2\tau_{xy}\,\tau_{yz}\,\tau_{zx} - \sigma_x\,\tau_{yz}^2 - \sigma_y\,\tau_{xz}^2 - \sigma_z\,\tau_{xy}^2) = 0$$

- It can be written as

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$= \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix}$$

Stress invariants

- And

$$I_3 = \sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{zx}^2 - \sigma_z \tau_{xy}^2$$

$$= \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix}$$

- I_1, I_2 and I_3 are known as first, second and third invariants
- Invariant
  - Whose value does not change with the change in frame of reference

Invariant

- Suppose we have another frame of reference x’, y’ and z’ for the same point P
- The components of stress in this new frame will be

$$\sigma_{x'}, \sigma_{y'}, \sigma_{z'}, \tau_{x'y'}, \tau_{y'z'}\ \tau_{z'x'},$$

- But the values of I_1, I_2 and I_3 will remain unchanged or

$$\sigma_x + \sigma_y + \sigma_z = \sigma_x' + \sigma_y' + \sigma_z'$$

$$I_1 = I_1'$$

$$I_2 = I_2' \quad \text{and} \quad I_3 = I_3'$$

Stress invariants

- In terms of principal stresses, we can write invariants as

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

Orthogonality of principal planes

- Let n and n' are principal planes and σ_1 and σ_2 are corresponding principal stresses
- Using projection theorem we can write

$$\sigma_1 \boldsymbol{n}' \cdot \boldsymbol{n} = \sigma_2 \boldsymbol{n} \cdot \boldsymbol{n}'$$

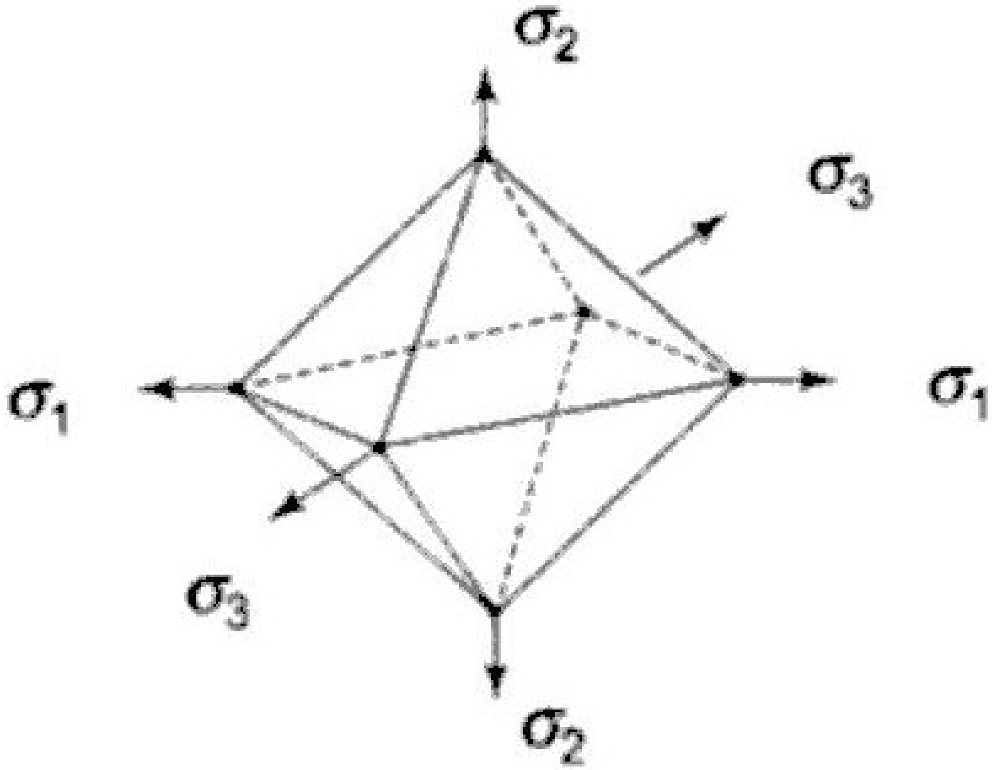
$$\sigma_1 \left(n_x n'_x + n_y n'_y + n_z n'_z \right) = \sigma_2 \left(n_x n'_x + n_y n'_y + n_z n'_z \right)$$

σ_2 and σ_2 are not equal so we have,

$$n_x n'_x + n_y n'_y + n_z n'_z = 0$$

$\Rightarrow$ Principal planes are orthogonal.

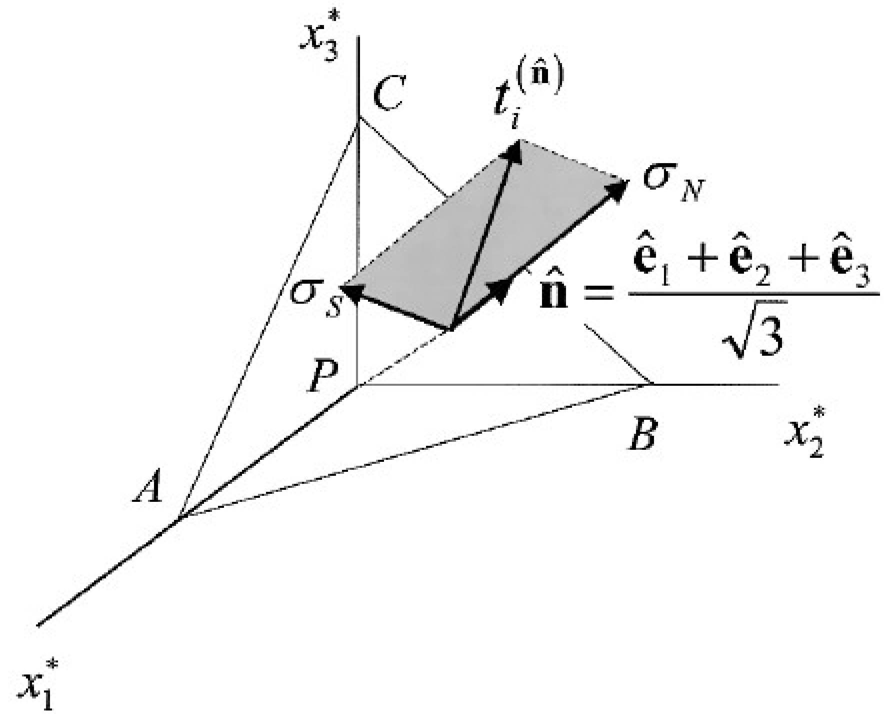
Octahedral stresses



Octahedral stresses

- Consider a plane at point *P* which makes equal angles with all three principal directions

$$n_x = n_y = n_z = \pm 1/\sqrt{3}$$



Normal stress:

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2$$

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$= \frac{1}{3} I_1$$

Mean stress!!

What if first invariant is zero??

Octahedral stresses

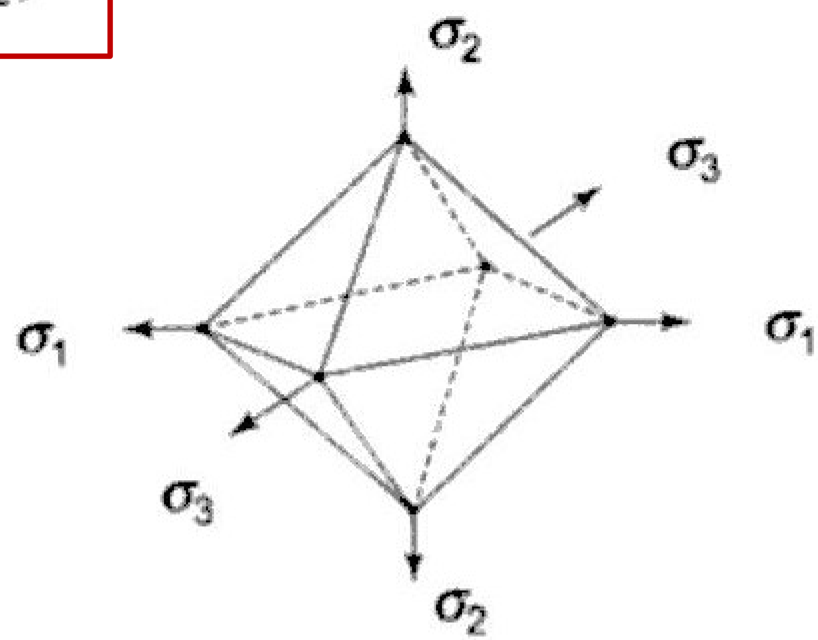
- Shear stress

$$\left|\overset{n}{T}\right|^2 = \sigma^2 + \tau^2 = \sigma_1^2 n_x^2 + \sigma_2^2 n_y^2 + \sigma_3^2 n_z^2$$

$$\tau_{oct}^2 = \frac{1}{9}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$\boxed{= \frac{\sqrt{2}}{3}(I_1^2 - 3I_2)^{1/2}}$$

Stresses acting on such plane are called octahedral stresses because there can be 8 such planes which will make equal angles with principal directions.



Octahedral stresses

- Defined with respect to principal axes
- Can also be expressed in terms of stress components

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \ldots$

State of pure shear

- If, at least for one choice of coordinate system, we find that $\sigma_x = \sigma_y = \sigma_z = 0,$
- For such a stress, the stress matrix will be

$$\left[\tau_{ij}\right] = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix}$$

- For this system

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 0.$$

Stress invariant so must be true for any choice of coordinate system at P.

State of pure shear exist if first invariant is zero and octahedral planes are free from normal stress.

Decomposition of state of stress

- Let the given state of stress at point *P* is

$$[\tau_{ij}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

- Let the mean stress *p* is given as

$$p = 1/3(\sigma_x + \sigma_y + \sigma_z) = 1/3 I_1$$

- The given state can be resolved as

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_x - p & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - p & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - p \end{bmatrix}$$

Decomposition of state of stress

- First part is called hydrostatic or spherical stress. Why??
- Second part is referred as state of pure shear or deviatoric part. Why??

- First invariant of deviatoric part

$$
\begin{aligned}
I'_1 &= (\sigma_x - p) + (\sigma_y - p) + (\sigma_x - p) \\
&= \sigma_x + \sigma_y + \sigma_z - 3p
\end{aligned}
$$

Decomposition of state of stress

- If the given state of stress is referred to principal axes

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_1 - p & 0 & 0 \\ 0 & \sigma_2 - p & 0 \\ 0 & 0 & \sigma_3 - p \end{bmatrix}$$

$$p = 1/3(\sigma_1 + \sigma_2 + \sigma_3) = 1/3 I_1$$

Example

- Find the principal stress and planes for given state of stress

$$[\tau_{ij}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\sigma^3 - 3\sigma^2 - 3\sigma + 1 = 0$$

$$(\sigma + 1)(\sigma^2 - 4\sigma + 1) = 0$$

$$\Rightarrow \left\{ \begin{aligned} &\sigma_1 = -1, \\ &\sigma_2 = 2 + \sqrt{3}, \\ &\sigma_3 = 2 - \sqrt{3} \end{aligned} \right.$$

$$[\tau_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

Check for stress invariants!!

Example

- Principal directions

$$\boxed{\sigma_1 = -1,}$$

$$(1+1)n_x + 2n_y + n_z = 0$$
$$2n_x + (1+1)n_y + n_z = 0$$

$$n_z = 0.$$

$$\boxed{n_x + n_y} + (1+1)n_z = 0$$

$$\boxed{n_x^2 + n_y^2} + n_z^2 = 1$$

$$n_x = \pm\left(1/\sqrt{2}\right),\ n_y = \pm\left(1/\sqrt{2}\right)$$

$$\boxed{\sigma_2 = 2 + \sqrt{3}}$$

$$(-1-\sqrt{3})n_x + 2n_y + n_z = 0$$
$$2n_x + (-1-\sqrt{3})n_y + n_z = 0$$
$$n_x + n_y\,(-1-\sqrt{3})n_z = 0$$
$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$n_x = n_y = \left(1 + \frac{1}{\sqrt{3}}\right)^{1/2}$$

$$n_z = \frac{1}{\left(3+\sqrt{3}\right)^{1/2}}$$

$$\boxed{\boldsymbol{n}_3 = \boldsymbol{n}_1 \times \boldsymbol{n}_2}$$

Home work

- Determine the principal stress and direction for the given state of stress

$$[\sigma_{ij}] = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix}$$

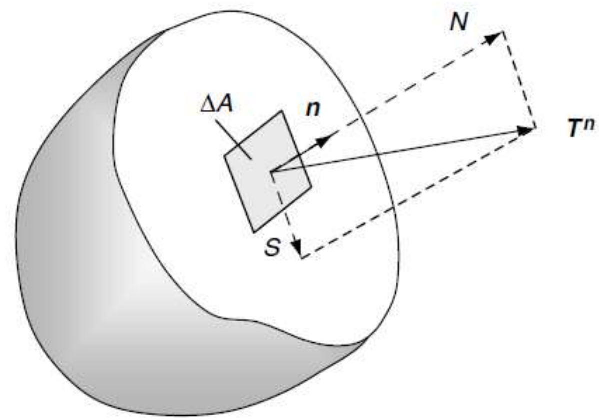
- Determine the principal stress and direction for given state of stress

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = \rho$$

- Prob no 1.4, 1.5, 1.7 from L. S. Srinath

Mohr's circle

- Choose the frame of reference such that the axes are along the principal axes
- Normal and shear component



$$\left|\overset{n}{T}\right|^2 = \sigma^2 + \tau^2 = \sigma_1^2 n_x^2 + \sigma_2^2 n_y^2 + \sigma_3^2 n_z^2$$

$$\sigma = \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2$$

$$1 = n_x^2 + n_y^2 + n_z^2$$

Above set of equations can be solved for direction cosines n_x, n_y and n_z.

$$n_x^2 = \frac{(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

$$n_y^2 = \frac{(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}$$

$$n_z^2 = \frac{(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$

Mohr's circle

- We get

$$n_x^2 = \frac{(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

$$n_y^2 = \frac{(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}$$

$$n_z^2 = \frac{(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

- Case 1: $\sigma_1 > \sigma_2 > \sigma_3$

$$\left.\begin{aligned}(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2 \geq 0\\ (\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2 \leq 0\\ (\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2 \geq 0\end{aligned}\right\}$$

Equation of a circle.

First derived by Otto Mohr.
Generally called Mohr's circle.

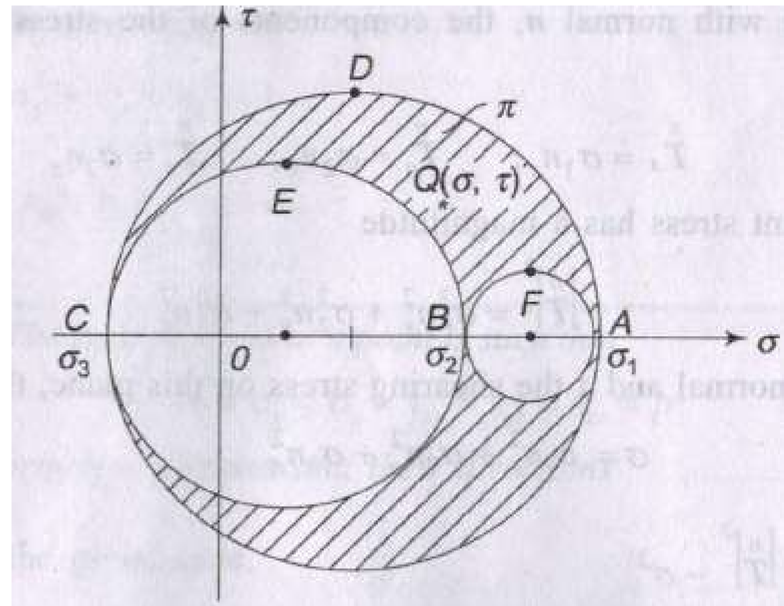
Mohr's circle

- Previous 3 equations can also be written as

$$\tau^2 + \left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 \geq \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

$$\tau^2 + \left(\sigma - \frac{\sigma_3 + \sigma_1}{2}\right)^2 \leq \left(\frac{\sigma_3 - \sigma_1}{2}\right)^2$$

$$\tau^2 + \left(\sigma - \frac{\sigma_1 + \sigma_2}{2}\right)^2 \geq \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2$$



What these equations tell us??

First relation:

The point with coordinate (σ,τ) must lie on or outside of the circle with,

Radius:- $\frac{1}{2}(\sigma_2 - \sigma_3)$

Centre:- $\frac{1}{2}(\sigma_2 + \sigma_3)$

Circle with *BC* as diameter.

Mohr's circle

- Second relation

The point with coordinate (σ,τ) must lie inside or on the circle with,

Radius:- $\frac{1}{2}(\sigma_1 - \sigma_3)$

Centre:- $\frac{1}{2}(\sigma_1 + \sigma_3)$

→ Circle *ADC*.

- Third relation

The point with coordinate (σ,τ) must lie on or outside of the circle with,

Radius:- $\frac{1}{2}(\sigma_1 - \sigma_2)$

Centre:- $\frac{1}{2}(\sigma_1 + \sigma_2)$

→ Circle *AFB*.

- Conclusion

The point with coordinate (σ,τ) must lie in the shaded region.

Mohr's circle

- Case 2: $\sigma_1 = \sigma_2 > \sigma_3$ We can express the previous relations as,

$$\tau^2 + \left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

$$\tau^2 + \left(\sigma - \frac{\sigma_3 + \sigma_1}{2}\right)^2 = \left(\frac{\sigma_3 - \sigma_1}{2}\right)^2$$

The point with coordinate (σ,τ) must lie on the circle with,

Radius:- $\frac{1}{2}(\sigma_1 - \sigma_3)$

Centre:- $\frac{1}{2}(\sigma_1 + \sigma_3)$

$$\boxed{\tau^2 + \left(\sigma - \frac{\sigma_1 + \sigma_2}{2}\right)^2 \geq \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2}$$

The point with coordinate (σ,τ) must lie on the circle with,

Radius:- 0

Centre:- σ_1

- Three circles will reduce to one circle *BC* and a point circle at *B*

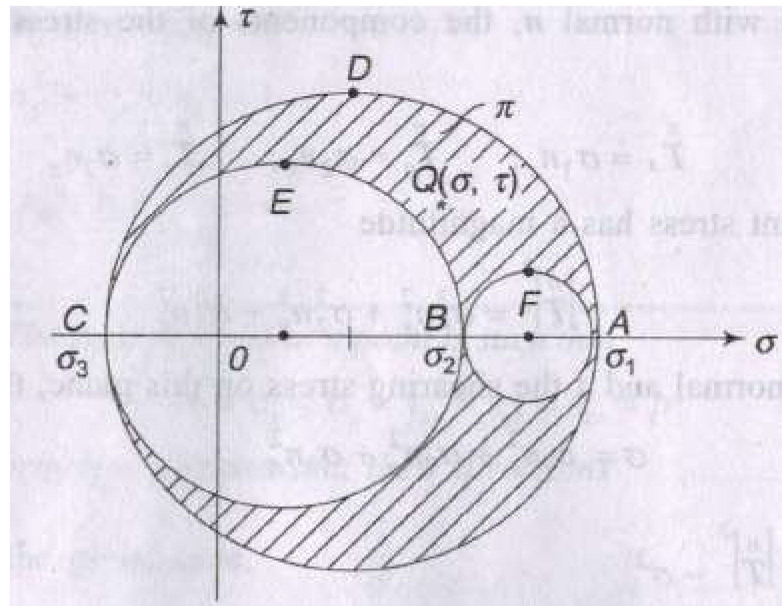
Mohr's circle

- Case 3: $\sigma_1 = \sigma_2 = \sigma_3$
  - Isotropic or hydrostatic state of stress
  - Mohr's circles will collapse to a point

$$n_x^2 = \frac{(\sigma - \sigma_2)(\sigma - \sigma_3) + \tau^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

$$n_y^2 = \frac{(\sigma - \sigma_3)(\sigma - \sigma_1) + \tau^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}$$

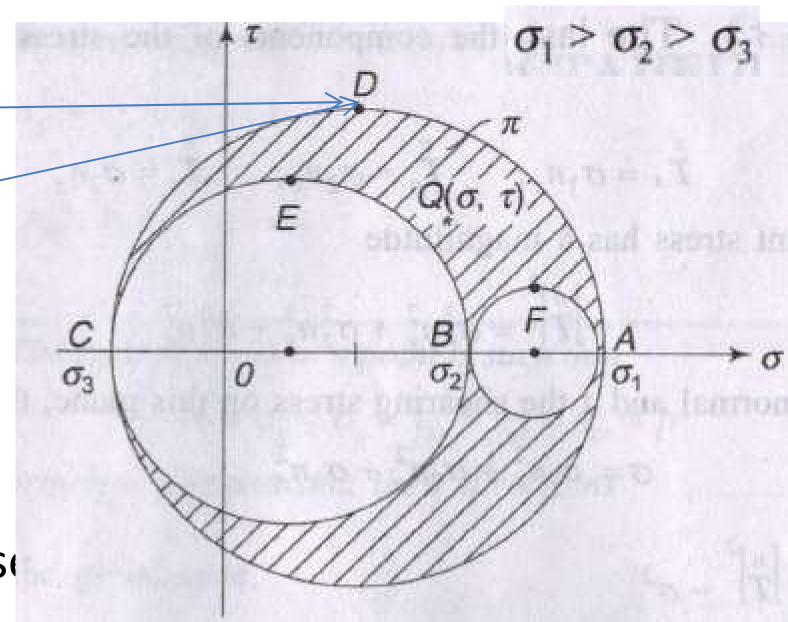
$$n_z^2 = \frac{(\sigma - \sigma_1)(\sigma - \sigma_2) + \tau^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$



Mohr's circle

- Some observations
  - Point A, B, C represents three principal stresses
  - Associated shear stresses are zero
  - Maximum shear stress $\frac{1}{2}(\sigma_1 - \sigma_3)$
  - Associated normal stress $\frac{1}{2}(\sigma_1 + \sigma_3)$
  - Three normal stresses
  - Three extremum values of τ

$$\frac{\sigma_1 - \sigma_3}{2}, \frac{\sigma_2 - \sigma_3}{2} \text{ and } \frac{\sigma_1 - \sigma_2}{2}$$



  - The planes on which these shear stresse planes
  - While principal normal stresses planes are free from shear stresses, principal shear stresses planes are not free from normal stresses

Mohr’s circle

- Application of Mohr’s circle
  - We need to know principal stresses in order to construct 3D Mohr’s circle then what is the use of it??