Indian Institute of Technology Ropar
Department of Mathematics
MA102 - Linear Algebra and Integral Transforms and Special Functions
Second Semester of Academic Year 2023-24

Tutorial sheet - 3

1. Let $\{v_1, v_2, v_3\}$ be a basis of vector space $V(\mathbb{R})$. Show that the set $\{v_1, v_1+v_2, v_1+v_2+v_3\}$ is also a basis of $V(\mathbb{R})$.

2. Show that $\{\ (1,4),\ (0,1)\ \}$ form a basis of $\mathbb{R}^2$ over $\mathbb{R}$.

3. Find a basis and hence give the dimension of each of the following subspaces of $V = M_n(\mathbb{R})$
(a) $W_1 = \{A \in V \mid A_{ij} = 0, \forall\ i \neq j\}$
(b) $W_2 = \left\{A \in V \mid A = -A^T\right\}$
(c) $W_3 = \{A \in V \mid \mathrm{Trace}(A) = 0\}$
(d) $W_4 = \{A \in V \mid A_{ij} = 0, \forall\ i < j\}$.

4. Find a basis and dimension of following subspace S of vector space of polynomials $P_n(\mathbb{R})$ over $\mathbb{R}$, where:
(a) $S = \{p(x) \in P_n(\mathbb{R}) \mid p(0) = 0\}$.
(b) $S = \{p(x) \in P_n(\mathbb{R}) \mid p(x)$ is an odd function$\}$.
(c) $S = \{p(x) \in P_n(\mathbb{R}) \mid p(0) = p''(0) = 0\}$.

5. Check whether the vector space $V = P(\mathbb{R})$, set of all real polynomials over $\mathbb{R}$ is finite dimensional or not.

6. Find a basis and dimension for the subspaces W_1 and W_2 of $\mathbb{R}^5(\mathbb{R})$, where:
(a) $W_1 = \left\{(a_1,\ a_2,\ a_3,\ a_4,\ a_5) \in \mathbb{R}^5 \mid a_1 - a_3 - a_4 = 0\right\}$.
(b) $W_2 = \left\{(a_1,\ a_2,\ a_3,\ a_4,\ a_5) \in \mathbb{R}^5 \mid a_2 = a_3 = a_4,\ a_1 + a_5 = 0\right\}$.

7. Prove that if W_1 and W_2 are finite dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite dimensional and $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. Hint: Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots v_m\}$ for W_1 and to a basis $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots w_p\}$ for W_2.

8. Let V and W be following subspaces of $\mathbb{R}^4(\mathbb{R})$:
$V = \{(a,b,c,d) \mid b - 2c + d = 0\}, W = \{(a,b,c,d) \mid a = d,\ b = 2c\}$.
Find bases and the dimensions of V, W and $V \cap W$. Hence prove that $\mathbb{R}^4(\mathbb{R}) = V + W$.

9. Suppose that U and V are subspaces of $\mathbb{R}^8(\mathbb{R})$ such that $\dim U = 3$ and $\dim W = 5$ and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus V$.

10. Suppose V is finite dimensional vector space over the field $\mathbb{F}$ and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

11. Suppose $U_1, U_2 \ldots, U_m$ are finite-dimensional subspaces of V over the field $\mathbb{F}$. Prove that $U_1 + U_2 + \ldots + U_m$ is finite-dimensional and

$$\dim(U_1 + U_2 + \ldots + U_m) \leq \dim(U_1) + \dim(U_2) + \ldots + \dim(U_m)$$

12. Find an example of subspaces W_1 and W_2 of $\mathbb{R}^3(\mathbb{R})$ with dimensions m and n, where $m \geq n$, such that both $\dim(\mathrm{W}_1 \cap \mathrm{W}_2) < n$ and $\dim(W_1 + W_2) < m + n$.

13. Let $\mathrm{M}_{m\times n}(\mathbb{F})$ is the collection of all m$\times$n matrix over the field $\mathbb{F}$.
Define $\mathrm{W}_1 = \{A \in \mathrm{M}_{m\times n}(\mathbb{F}) : A_{ij} = 0$ whenever $i > j\}$ and $\mathrm{W}_2 = \{A \in \mathrm{M}_{m\times n}(\mathbb{F}) : A_{ij} = 0$ whenever $i \leq j\}$. Show that $\mathrm{M}_{m\times n}(\mathbb{F}) = \mathrm{W}_1 \oplus \mathrm{W}_2$.

14. Give two different bases for $\mathbb{R}^2(\mathbb{R})$ and for $\mathrm{M}_{2\times 2}(\mathbb{R})$.

15. Let V be a vector space having dimension n, and let S be a subset of V that generates V then Prove that S contains at least n vectors.

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