Calculus: MA101

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https://sites.google.com/iitrpr.ac.in/ma101/home

Text Books/Reference Books

- G. B. Thomas and R. L. Finney, Calculus and Analytic Geometry, 9thed., Addison-Wesley/Narosa, 1998.
- James Stewart, Calculus, Brooks Cole , 2015.
- Gilbert Strang, Calculus, Wellesley-Cambridge Press; 2nd edition,2010.
- Tom M. Apostol, Calculus, Vol 1 and Vol II, Wiely, 2007.
- Thomas, Weir, Hass, Calculus, Thirteenth Edition, Pearson.

Grading Scheme:

Apart from Mid Semester and End Semester examinations, there will be 2 quizzes (1 quiz before Mid Semester Examination, and 1 quiz after Mid Semester Examination). The minimum pass mark for this course will be 30%.

Mark Distribution:

- Mid Semester: 25 Marks,
- End Semester: 45 Marks,
- Quizzes: 20 Marks,
- Quizzes during Tutorials: 10 Marks (TA's will inform the details)

Information/Instructions:

- Tentative dates of Quizzes: Quiz - 1: 09 September, 2023
  Quiz - 2: 04 November, 2023
- If anyone fails to write any exam, then NO make-up exam will be conducted except on a medical grounds (in this case, there MUST be a prior information to the course coordinator/instructor).
- Institute rules will be followed if a student fails to have minimum 75 percent attendance both in lectures and tutorials.
- Students can approach corresponding instructor for any doubts / clarifications during the office hour given in the course plan.
- All details of this course including lecture notes (PPT) will be uploaded in the ACADLY: https://app.acadly.com/
- Attendance will be taken through ACADLY App.

Limit, Continuity, Differentiability

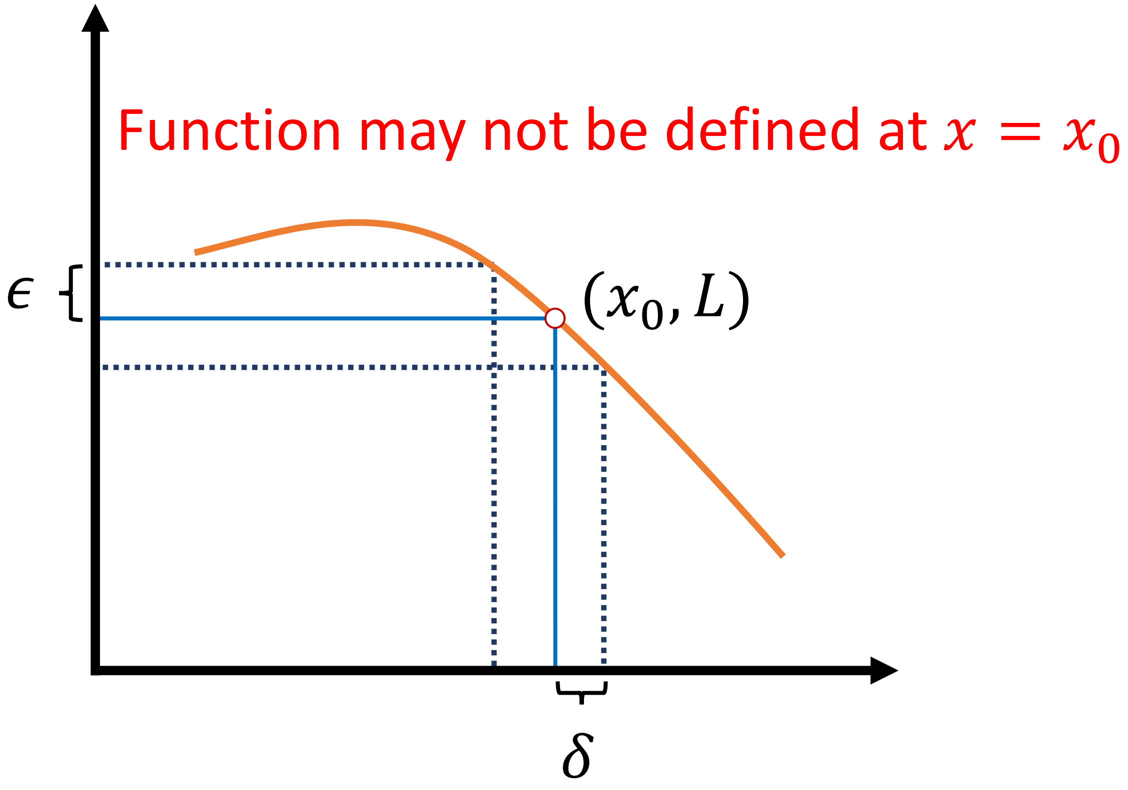
- **Limit & Continuity:** $\epsilon - \delta$ Definition
- **Differentiability:** Differentials, Geometrical Interpretation

Limit of a Functions of One Variable

We say $\lim_{x \to x_0} f(x) = L$, if for every $\epsilon > 0$,

there exists $\delta > 0$, such that $\forall\, x$,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$



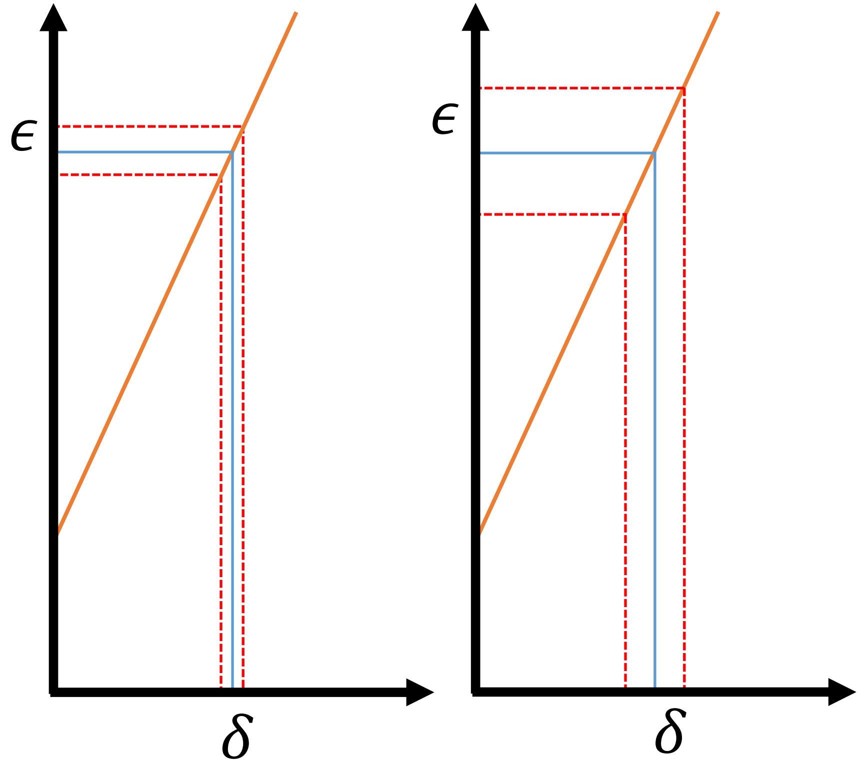
In other words,

If we can make the difference $|f(x) - L|$ as small as we like by considering a small enough neighborhood around x_0, then we say that $\lim_{x \to x_0} f(x) = L$

Example: $\lim_{x \to 1}(3x+1) = 4$

Show that for a given $\epsilon > 0$, there exist a δ so that

$$0 < |x-1| < \delta \quad \Longrightarrow |(3x+1)-4| < \epsilon$$



We start with the difference

$$|(3x+1)-4| = |3x-3| = 3|x-1| < 3\delta \le \epsilon$$

If we choose $\delta \le \dfrac{\epsilon}{3}$ Then for any given ϵ, we have

$$|(3x+1)-4| < \epsilon \quad \text{whenever} \quad 0 < |x-1| < \delta$$

Example: $\lim_{x \to 1}(3x+1) = 4$ Alternative Approach!

Show that for a given $\epsilon > 0$, there exist a δ so that

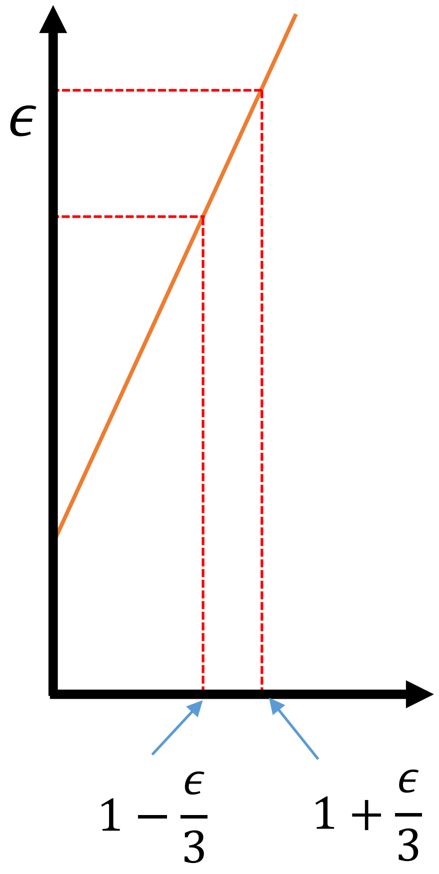
$$0 < |x-1| < \delta \quad \Longrightarrow |(3x+1) - 4| < \epsilon$$

Solve the inequality:

$$|(3x+1) - 4| < \epsilon \;\Rightarrow |3x-3| < \epsilon \Rightarrow 1 - \frac{\epsilon}{3} < x < 1 + \frac{\epsilon}{3}$$

If we choose $\delta \leq \frac{\epsilon}{3}$ Then for any given ϵ, we have

$$|(3x+1) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x-1| < \delta$$



Note that the interval $\left(1 - \frac{\epsilon}{3}, 1 + \frac{\epsilon}{3}\right)$ contains the point $x_0 = 1$

Example: Suppose we test $\lim_{x \to 1}(3x+1) = 7$

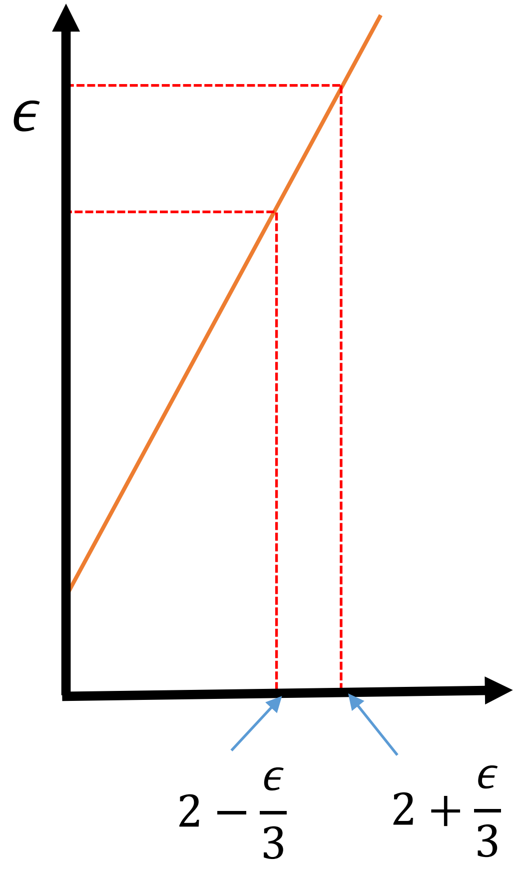
Trying to show that for a given $\epsilon > 0$, there exist a δ so that

$$|x-1| < \delta \quad \Longrightarrow \quad |(3x+1)-4| < \epsilon$$

Solve the inequality:

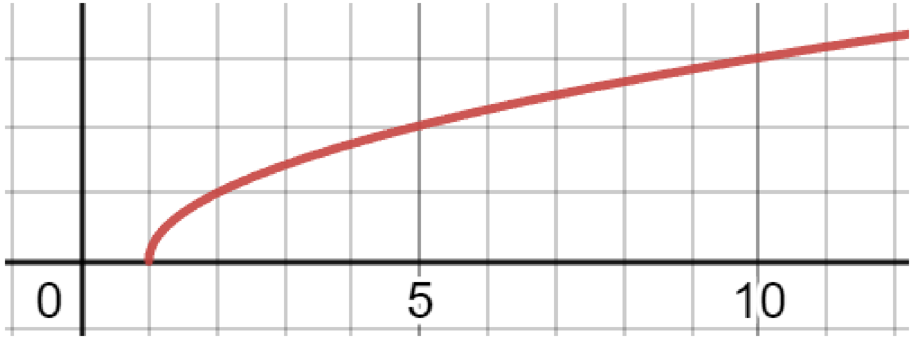
$$|(3x+1)-7| < \epsilon \;\Rightarrow |3x-6| < \epsilon \Rightarrow 2-\frac{\epsilon}{3} < x < 2+\frac{\epsilon}{3}$$

Note that the interval $\left(2-\frac{\epsilon}{3}, 2+\frac{\epsilon}{3}\right)$ does not contains the point $x_0 = 1$ for any values of ϵ.



For an arbitrary given ϵ, δ does not exist and hence the limit can not be 7

Example: Show that $\lim_{x \to 5} \sqrt{x-1} = 2$



Show that for a given $\epsilon > 0$, there exist a δ so that

$$|x-5| < \delta \implies \left|\sqrt{x-1} - 2\right| < \epsilon$$

Solve the inequality:

$$\left|\sqrt{x-1} - 2\right| < \epsilon \quad \Rightarrow -\epsilon < \sqrt{x-1} - 2 < \epsilon \quad \Rightarrow (2-\epsilon)^2 + 1 < x < (2+\epsilon)^2 + 1$$

Note that there exists a δ such that the interval $(5-\delta, 5+\delta)$ lies inside the interval $((2-\epsilon)^2 + 1, (2+\epsilon)^2 + 1)$

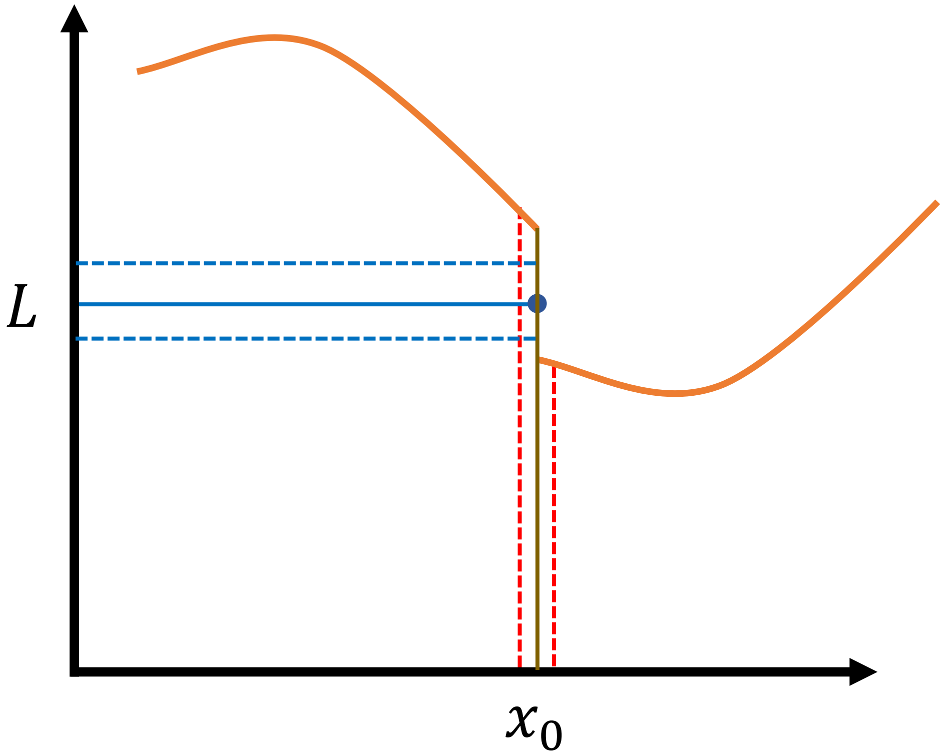
$$\delta \leq \min(5 - (2-\epsilon)^2 - 1, (2+\epsilon)^2 + 1 - 5)$$

e.g., $\epsilon = 1$, the interval $(2, 10)$ contains $(5-\delta, 5+\delta)$ for $\delta \leq 3$.

Non-Existence of Limit

For a given ϵ, there does not exist any δ such that

$|f(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta$



Basic Properties:

If l, m, c and x_0 are real numbers, $n \epsilon \mathbb{N}$ and $\lim_{x \to x_0} f(x) = l$ and $\lim_{x \to x_0} g(x) = m$, then

(i) $\lim_{x \to x_0} (f(x) \pm g(x)) = l \pm m$

(ii) $\lim_{x \to x_0} \left(rf(x)\right) = r.l$

(iii) $\lim_{x \to x_0} \left(f(x).g(x)\right) = l.m$

(iv) If $m \neq 0,\ g(x) \neq 0\ \forall x$, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{l}{m}$

(v) If $\lim_{x \to x_0} f(x) \geq 0\ \forall x$ then $l \geq 0$ and $\lim_{x \to x_0} \sqrt[n]{f(x)} = \sqrt[n]{l}$

Consequences:

(i) If P is a Polynomial function i.e., $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then
$\lim_{x \to x_0} P(x) = P(x_0) = a_n x_0^n + a_{n-1}x_0^{n-1} + \cdots + a_0$.

(ii) If P and Q are Polynomial functions on $\mathbb{R}$ and if $Q(x) \neq 0$, then $\lim_{x \to x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)}$

(iii) (The sandwich theorem / The squeeze theorem)
Suppose that $g(x) \leq f(x) \leq h(x)\ \forall x$, in some open interval containing x_0, except possibly at $x = x_0$ itself.
Suppose also that $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = l$. Then

$$\lim_{x \to x_0} f(x) = l.$$

Example: Given that $1 - \frac{x^2}{4} \leq f(x) \leq 1 + \frac{x^2}{2}$; $\forall x \neq 0$. Find $\lim_{x \to 0} f(x)$.

(no matter how complicated $f(x)$ is)

Sol:- Since $\lim_{x \to 0} \left(1 - \frac{x^2}{4}\right) = 1$ and $\lim_{x \to 0} \left(1 + \frac{x^2}{2}\right) = 1$

By Sandwich theorem, $\lim_{x \to 0} f(x) = 1$.

One-sided Limit of a Functions of One Variable

We say $\lim_{x\to x_0+} f(x) = L$ (right hand limit), if for every $\epsilon > 0$, there exists $\delta > 0$, such that $\forall\, x$,

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \epsilon$$

We say $\lim_{x\to x_0-} f(x) = L$ (left hand limit), if for every $\epsilon > 0$, there exists $\delta > 0$, such that $\forall\, x$,

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \epsilon$$

Example: Show that $\lim\limits_{x \to 1+} \sqrt{x-1} = 0$

We need to show that for given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\left(\sqrt{x-1} - 0\right)\right| < \epsilon, \qquad \text{if} \quad 1 < x < 1 + \delta$$

OR $\sqrt{x-1} < \epsilon$, if $1 < x < 1 + \delta$

Solving the inequality $\sqrt{x-1} < \epsilon$, we get $1 < x < \epsilon^2 + 1$

So if we can choose $\delta \leq \epsilon^2$, then

$$1 < x < 1 + \delta \Rightarrow \left|\left(\sqrt{x-1} - 0\right)\right| < \epsilon$$