CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Limit Test for Differentiability
- Worked Problems

Differentiability of Functions of Two Variables (Previous Lecture)

The function $z = f(x, y)$ is said to be differentiable at the point (x, y), if at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

Necessary conditions

- Continuity of f
- Existence of partial derivatives f_x & f_y

Sufficient conditions

- Continuity of one/both partial derivatives

Testing Differentiability

$$\text{Differentiability} \iff \lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = 0, \qquad \Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

Let f be differentiable

$$\Delta z = \underbrace{a\,\Delta x + b\,\Delta y}_{dz} + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y \quad \Longrightarrow \quad \frac{\Delta z - dz}{\Delta\rho} = \epsilon_1 \frac{\Delta x}{\Delta\rho} + \epsilon_2 \frac{\Delta y}{\Delta\rho}$$

$$\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho\to 0} \epsilon_1 \frac{\Delta x}{\Delta\rho} + \lim_{\Delta\rho\to 0} \epsilon_2 \frac{\Delta y}{\Delta\rho} \quad = 0$$

Note that $\frac{\Delta x}{\Delta\rho} \leq 1$ & $\frac{\Delta y}{\Delta\rho} \leq 1$ and ϵ_1, ϵ_2 tend to zero as $\Delta\rho \to 0$

Testing Differentiability (cont.)

$$\text{Differentiability} \iff \lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = 0, \qquad \Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\text{Let } \lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = 0 \implies \frac{\Delta z - dz}{\Delta\rho} = \epsilon \qquad \epsilon \to 0 \text{ as } \Delta\rho \to 0$$

$$\implies \Delta z - dz = \epsilon\ \Delta\rho = \epsilon\sqrt{\Delta x^2 + \Delta y^2} = \epsilon\frac{\Delta x^2 + \Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \left(\frac{\epsilon\,\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}\right)\Delta x + \left(\frac{\epsilon\,\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}\right)\Delta y$$

$$\implies \Delta z = dz + \epsilon_1\ \Delta x + \epsilon_2\ \Delta y$$ $\implies$ **Differentiability of *f***

Problem – 1 (Continuous, partial derivatives exist but not differentiable)

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Existence of Partial Derivatives

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0 \qquad f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$$

Continuity Changing to polar coordinates ($x=r\cos\theta$, $y=r\sin\theta$)

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{r\to 0} r\cos\theta\sin\theta=0=f(0,0)$$

Differentiability $\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = ?$

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0) = \frac{\Delta x \, \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta \mathrm{y} = 0$$

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = \lim_{\Delta\rho \to 0} \left(\frac{\Delta x \, \Delta y}{\Delta x^2 + \Delta y^2} \right)$$

Along the path $\Delta y = m \, \Delta x$ $\quad \lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = \frac{m}{1+m^2}$

The given function is NOT differentiable.

Problem – 2 (Continuous, partial derivatives exist but not differentiable)

$$f(x,y) = \begin{cases} \dfrac{x^3 + 2y^3}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Existence of Partial Derivatives

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 1 \qquad f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 2$$

Continuity Changing to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$)

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 + 2y^3}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 \cos^3\theta + 2r^3 \sin^3\theta}{r^2} = 0 = f(0,0)$$

Differentiability $\lim_{\Delta\rho\to 0}\frac{\Delta z - dz}{\Delta\rho} = ?$

$$f(x,y)=\begin{cases}\dfrac{x^3+2y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0) = \frac{\Delta x^3 + 2\Delta y^3}{\Delta x^2+\Delta y^2}$$

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta \mathrm{y} = \Delta x + 2\,\Delta \mathrm{y}$$

$$\lim_{\Delta\rho\to 0}\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\left(\frac{\Delta x^3+2\Delta y^3}{\Delta x^2+\Delta y^2} - (\Delta x + 2\,\Delta \mathrm{y})\right) = \lim_{\Delta\rho\to 0}\frac{-\Delta x\,\Delta y^2 - 2\Delta x^2\Delta y}{(\Delta x^2+\Delta y^2)^{3/2}}$$

Along the path $\Delta y = m\,\Delta x$ $$= \frac{-m^2-2m}{(1+m^2)^{3/2}}$$

The given function is NOT differentiable.

Problem – 3 (Differentiable but f_x & f_y are not continuous)

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Existence of Partial Derivatives

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0 \qquad f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

Continuity Changing to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$)

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0 = f(0,0)$$

Differentiability

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$$

$$= (\Delta x^2 + \Delta y^2)\cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right)$$

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = 0$$

$$\lim_{\Delta\rho\to 0}\frac{\Delta z - dz}{\Delta \rho} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{(\Delta x^2+\Delta y^2)}{\sqrt{\Delta x^2+\Delta y^2}}\cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right)$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sqrt{\Delta x^2+\Delta y^2}\cos\left(\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}\right) = 0$$

Hence, the given function is differentiable

Continuity of f_x & f_y

$$f(x,y)=\begin{cases}(x^2+y^2)\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

At $(x,y)\neq(0,0)$

$$f_x(x,y)=-(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\left(-\frac{1}{2}\frac{2x}{(x^2+y^2)^{\frac{3}{2}}}\right)+2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$=\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\left(\frac{x}{\sqrt{(x^2+y^2)}}\right)+2x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Along x-axis

$$\lim_{x\to 0} f_x(x,y)=\lim_{x\to 0}\left(\frac{x}{|x|}\sin\left(\frac{1}{|x|}\right)+2x\cos\left(\frac{1}{|x|}\right)\right)\neq 0$$

$\Longrightarrow$ f_x is not continuous at $(0,0)$. Similarly, f_y is not continuous at $(0,0)$

Remark: The above example shows that continuity of partial derivatives is not a necessary condition for differentiability. A function can be differentiable without having continuous first order partial derivatives.

Example (Differentiable but f_x & f_y are not continuous) – Homework Problem

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \cos\left(\frac{1}{y}\right), & x \neq 0, y \neq 0 \\ 0 & \text{elsewhere} \end{cases}$$

CONCLUSIONS

Differentiability $\iff \lim\limits_{\Delta\rho \to 0} \dfrac{\Delta z - dz}{\Delta\rho} = 0$

- The function may not be differentiable at a point $P(x, y)$ even if the partial derivatives f_x and f_y exists at P.

(Existence of partial derivatives is a necessary condition)

- A function may be differentiable even if f_x and f_y are not continuous.

(Continuity of the f_x and/or f_y is a sufficient condition)

Differentiability of Functions of Two Variables (Previous Lecture)

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y \quad \Longleftrightarrow \quad \lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = 0, \qquad \Delta\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

Necessary Conditions

- Continuity of f
- Existence of partial derivatives f_x & f_y

Sufficient Conditions

- Existence of one partial derivative and continuity of the other

Problem – 1

Discuss the differentiability at origin of the function $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Necessary Conditions

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \qquad f_x(0,0) = 0 \qquad f_y(0,0) = 0$$

Sufficient Conditions

$$f_x(x,y) = \begin{cases} \dfrac{-x^2y^3 + y^5}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$$

Hence f_x is continuous.

Therefore the function f is differentiable at $(0,0)$.

Problem – 2

Discuss the differentiability at origin of the function $f(x, y) = \begin{cases} y^3 \sin\left(\dfrac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Continuity of f and existence of partial derivatives at $(0,0)$ can easily be shown.

$$f_x(x, y) = \begin{cases} -\dfrac{2y^3}{x^3} \cos\left(\dfrac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0, y = 0 \\ \text{Does not exist}, & x = 0, y \neq 0 \end{cases}$$

Since f_x does not exist in the neighborhood of $(0, 0)$, f_x is NOT continuous.

Differentiability at origin of the function $f(x,y) = \begin{cases} y^3 \sin\left(\dfrac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$f_y(x,y) = \begin{cases} 3y^2 \cos\left(\dfrac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{(x,y)\to(0,0)} f_y(x,y) = 0$ Hence f_y is continuous.

$\Rightarrow f_x$ exist at (0,0) and f_y is continuous at (0,0)

$\Rightarrow f$ is differentiable at (0,0)

LIMIT TEST

Differentiability at origin of the function $f(x,y) = \begin{cases} y^3 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta \mathrm{y} = 0 \qquad \Delta z = f(\Delta x, \Delta y) - f(0,0) = \Delta y^3 \sin\left(\frac{1}{\Delta x^2}\right)$$

$$\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho\to 0} \frac{\Delta y^3 \sin\left(\frac{1}{\Delta x^2}\right)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

Polar Coordinates:

$\Delta x = r\cos\theta, \Delta x = r\sin\theta$

$$= \lim_{r\to 0} r^2 \sin\left(\frac{1}{r^2\cos^2\theta}\right) = 0$$

$\Longrightarrow f$ is differentiable at (0,0)

Problem – 3 Let $f(x,y) = \begin{cases} \sqrt{xy}, & xy \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$

Determine whether the function is differentiable at the origin.

Continuity: $\lim_{(x,y)\to(0,0)} \sqrt{xy} = 0$

Existence of Partial Derivatives:

$$f_x = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0 \qquad f_y = \lim_{\Delta y\to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

Differentiability $f(x, y) = \begin{cases} \sqrt{xy}, & xy \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = ?$$

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = 0$$

$$\Delta z = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = \sqrt{\Delta x\, \Delta y}$$

(assuming $\Delta x \Delta y \geq 0$)

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \to 0} \frac{\sqrt{\Delta x\, \Delta y}}{\sqrt{\Delta x^2 + \Delta y^2}}$$

Along the path $\Delta y = \Delta x$

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = \frac{1}{\sqrt{2}} \neq 0$$

The given function is NOT differentiable

Problem – 4 Discuss the differentiability at the origin of the function

$$f(x,y)=\begin{cases} x^{\frac{5}{2}}\sin\left(\frac{1}{\sqrt{x}}\right)+y^{\frac{5}{2}}\cos\left(\frac{1}{\sqrt{y}}\right), & x\neq 0, y\neq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Recall the definition of differentiability $\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$

$$f(\Delta x,\Delta y)-f(0,0)=\Delta x^{\frac{5}{2}}\sin\left(\frac{1}{\sqrt{\Delta x}}\right)+\Delta y^{\frac{5}{2}}\cos\left(\frac{1}{\sqrt{\Delta y}}\right)$$

$$=0\cdot\Delta x+0\cdot\Delta y+\Delta x\underbrace{\left(\Delta x^{\frac{3}{2}}\sin\left(\frac{1}{\sqrt{\Delta x}}\right)\right)}_{\epsilon_1}+\Delta y\underbrace{\left(\Delta y^{\frac{3}{2}}\cos\left(\frac{1}{\sqrt{\Delta y}}\right)\right)}_{\epsilon_2}$$

Problem – 5 Let $f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

Check the existence of f_x & f_y at origin. Is f differentiable at origin?

$$f_x = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0 \qquad f_y = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

Continuity Check of f

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ along } (x = y) = 0 \neq f(0,0)$$

$\Longrightarrow$ the function f is not continuous at $(0,0)$

$\Longrightarrow$ the function f is NOT differentiable at $(0,0)$

Problem – 6 Let $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ Is f differentiable at origin?

Continuity Check

$$|f(x,y) - 0| = \frac{x^2+y^2}{|x|+|y|} < \frac{(|x|+|y|)^2}{|x|+|y|} = |x|+|y| < \sqrt{2}\,\sqrt{x^2+y^2} < \sqrt{2}\,\delta < \epsilon$$

Choose $\delta < \dfrac{\epsilon}{\sqrt{2}}$, then $|f(x,y) - f(0,0)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

This implies the function $f(x,y)$ is continuous.

Differentiablity of $\quad f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Existence of Partial Derivatives

$$\lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^2}{|\Delta x|\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{|\Delta x|}$$

limit does not exist

$\Longrightarrow$ $f_x(0,0)$ does not exist.

Similarly, $f_y(0,0)$ does not exist.

$\Longrightarrow$ The function f is NOT differentiable at $(0,0)$

CONCLUSIONS

Necessary Conditions:

Continuity & existence of partial derivatives

Sufficient Conditions:

Continuity of one of the partial derivatives

Final Check (Limit Test):

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = 0 \text{ or } \Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$