CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- ❑ Continuity
- ❑ Worked Problems

Continuity

A function $z = f(x, y)$ is said to be continuous at a point (x_0, y_0) if

I. $f(x, y)$ is defined at (x_0, y_0)

II. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists

III. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

If a function $f(x, y)$ is continuous at every point in a domain D, then it is said to be continuous in D.

Continuity ($\epsilon - \delta$ Definition)

A function $z = f(x, y)$ is said to be continuous at a point (x_0, y_0), if for a given $\epsilon > 0$, there exist a real number $\delta > 0$ such that

$$|f(x, y) - f(x_0, y_0)| < \epsilon \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Removable Discontinuity

I. $f(x, y)$ is defined at (x_0, y_0)

II. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists

III. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) \neq f(x_0, y_0)$

Problem – 1:

Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{xy}, & xy \neq 0 \\ 0, & \text{elsewhere} \end{cases}$ at origin.

Choosing the path $y = mx, m \neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = \frac{1+m^2}{m}$$

Limit depends on the path

The limit does not exist. Hence the function is not continuous at $(0,0)$

Problem – 2:

Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ at origin.

Choosing the path $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = \frac{(1-m)^2}{(1+m^2)}$$ Limit depends on the path

The limit does not exist. Hence the function is not continuous at $(0,0)$

Problem – 3:

Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^4y^4}{(x^2+y^4)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ at origin.

Choosing the path $y^2 = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3} = \frac{m^2}{(1+m^2)^3}$$

Limit depends on the path

The limit does not exist. Hence the function is not continuous at $(0,0)$

Problem – 4:

Discuss the continuity of $f(x, y) = \begin{cases} \dfrac{2x^4 + 3y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0,0) \end{cases}$ at origin.

We need to check if $$\lim_{(x,y)\to(0,0)} \frac{2x^4 + 3y^4}{x^2 + y^2} = 0$$

Changing to polar coordinate:

$$\lim_{(x,y)\to(0,0)} \frac{2x^4 + 3y^4}{x^2 + y^2} = \lim_{r\to 0} \frac{2r^4 \cos^4\theta + 3r^4 \sin^4\theta}{r^2} = 0$$

Hence the function is continuous

Problem – 5:

Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ at origin.

Changing to polar coordinate:

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{\sin\sqrt{r^2}}{\sqrt{r^2}} = \lim_{r\to 0} \frac{\sin r}{r} = 1$$

The limit exists. But the function is not continuous at $(0,0)$ as

$\lim_{(x,y)\to(0,0)} f(x,y) \neq 0$ (Removable Discontinuity)

Problem – 6:

Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^3+y^3}, & (x^3+y^3) \neq 0 \\ 0, & \text{elsewhere} \end{cases}$ at origin.

Changing to polar coordinate ($x = r\cos\theta\,,\ \ y = r\sin\theta$):

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3} = \lim_{r\to 0} r \underbrace{\left(\frac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}\right)}_{\text{Unbounded}}$$

Note that If we fix θ the limit is ZERO

Thus we cannot conclude that the limit is ZERO

Continuity of $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^3+y^3}, & (x^3+y^3) \neq 0 \\ 0, & \text{elsewhere} \end{cases}$

Along the path $y = mx$ $$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3} = 0$$

Along the path $y = -xe^x$ $$\lim_{x\to 0} \frac{x^2x^2e^{2x}}{x^3 - x^3e^{3x}} = \lim_{x\to 0} \frac{xe^{2x}}{1-e^{3x}} = -\frac{1}{3}$$

The limit does not exist. Hence the function is not continuous at $(0,0)$

Problem – 7:

Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$ at origin.

Changing to polar coordinate ($x = r\cos\theta$, $y = r\sin\theta$):

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = \lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{r^4(\cos^4\theta + \sin^4\theta)} = \lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{r^4(1 - 2\cos^2\theta\sin^2\theta)}$$

$$= \lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{r^4\left(1 - \frac{1}{2}\sin^2 2\theta\right)}$$

$$\lim_{r\to 0}\frac{e^{-\frac{1}{r^2}}}{r^4\left(1-\frac{1}{2}\sin^2 2\theta\right)} = 0$$

Noting: $$\frac{1}{2} \le \left(1-\frac{1}{2}\sin^2 2\theta\right) \le 1$$

For $r \neq 0$

$$0 < \frac{e^{-\frac{1}{r^2}}}{r^4\left(1-\frac{1}{2}\sin^2 2\theta\right)} < \frac{2e^{-\frac{1}{r^2}}}{r^4}$$

$$\lim_{r\to 0}\frac{2e^{-\frac{1}{r^2}}}{r^4} = \lim_{t\to\infty} 2e^{-t^2}t^4$$

$$= \lim_{t\to\infty}\frac{2t^4}{e^{t^2}} = \lim_{t\to\infty}\frac{4t^2}{e^{t^2}}$$

$$= \lim_{t\to\infty}\frac{4}{e^{t^2}} = 0$$

Using Sandwich (Squeeze) Theorem, we get the limit.

Thus the given function is continuous.

CONCLUSIONS

CONTINUITY

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$