CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Introduction to Partial Derivatives
- Continuity and Partial Derivatives
- Worked Problems

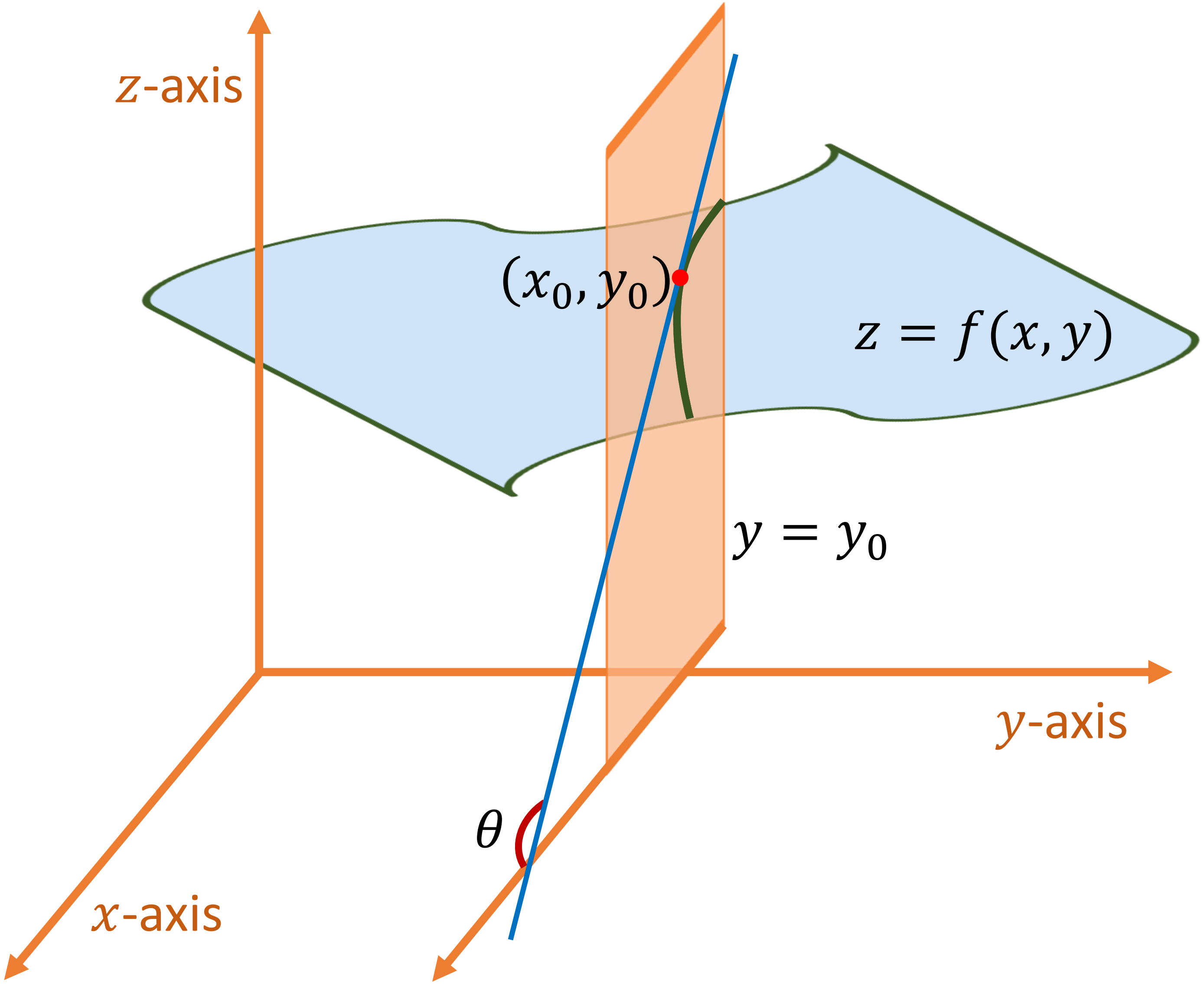
Partial Derivatives

The usual derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables as constant is called the partial derivatives of the function with respect to that variable.

Let $z = f(x, y); \quad (x, y) \in \mathbb{R}^2, \quad z \in \mathbb{R}$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \frac{d}{dx} f(x, y_0)\Big|_{x = x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = \frac{d}{dy} f(x_0, y)\Big|_{y = y_0}$$



Geometrical Interpretation of Partial Derivatives $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

$$\tan\theta = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

Problem – 1: Find the value of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (x, y) of the function $f(x, y) = ye^{-x}$ from the first principal.

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x}$$

$$= ye^{-x} \lim_{\Delta x \to 0} \frac{e^{-\Delta x} - 1}{\Delta x} = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(y + \Delta y)e^{-x} - ye^{-x}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} e^{-x} = e^{-x}$$

Relationship: Partial Derivatives & Continuity

A function can have partial derivatives with respect to both x and y at a point without being continuous there. On the other hand a continuous function may not have partial derivatives.

Problem – 2: Show that the function

$$f(x,y) = \begin{cases} (x+y)\sin\left(\dfrac{1}{x+y}\right), & (x+y) \neq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives do not exist at $(0,0)$

Continuity at $(0,0)$ $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Partial Derivatives at $(0,0)$

$$f(x,y) = (x+y)\sin\left(\frac{1}{x+y}\right)$$

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\Delta x\sin\left(\frac{1}{\Delta x}\right) = \lim_{\Delta x\to 0}\sin\left(\frac{1}{\Delta x}\right)$$

$\Longrightarrow$ The partial derivative w.r.t. x does not exist.

Similarly, the partial derivative w.r.t. y does not exist.

Problem – 3: Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives exist at $(0,0)$

Choosing the path $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2} = \frac{m}{(1+2m^2)}$$ Limit depends on the path

The limit does not exist. Hence the function is not continuous at $(0,0)$.

Partial Derivatives at $(\mathbf{0}, \mathbf{0})$

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{elsewhere} \end{cases}$$

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = \lim_{\Delta x \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$\Longrightarrow$ The partial derivatives w.r.t. x & y exist at $(0, 0)$.

Problem – 4: Let $f(x,y) = \begin{cases} \dfrac{2x^3+3y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$

Compute $f_x(0,0)$ & $f_y(0,0)$.

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 2$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 3$$

The function $f(x,y)$ is continuous and also partial derivatives exist.

Sufficient Condition for Continuity of f at (x_0, y_0) – Two Variants

If the first order partial derivatives of f exist and are bounded in the neighborhood of (x_0, y_0), then the function f is continuous at (x_0, y_0).

If one of the first order partial derivatives of f exists and is bounded in the neighborhood of (x_0, y_0) and the other exists at (x_0, y_0), then the function f is continuous at (x_0, y_0).

Sufficient Condition for Continuity of f at (x_0, y_0)

$\sqrt{h^2+k^2} < \delta$

Let $f_x(x,y) \leq M$ and $f_y(x,y) \leq M$ for all $(x,y) \in N_\delta(x_0,y_0)$. Consider

$$f(x_0+h, y_0+k) - f(x_0,y_0) = f(x_0+h, y_0+k) - f(x_0+h, y_0) + f(x_0+h, y_0) - f(x_0,y_0)$$

$$= k f_y(x_0+h, \xi_1) + h f_x(\xi_2, y_0)$$ Using mean value theorem

$$|f(x_0+h, y_0+k) - f(x_0,y_0)| = |k f_y(x_0+h, \xi_1) + h f_x(\xi_2, y_0)|$$

$$\leq M(|k|+|h|) \leq 2M\sqrt{(h^2+k^2)} \leq 2M\delta \leq \epsilon$$

For $\epsilon > 0$, choose $\delta \leq \frac{\epsilon}{2M}$. Hence the function is continuous.

Problem – 5: Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$

Compute $\dfrac{\partial f}{\partial x}(x,y)$ & $\dfrac{\partial f}{\partial y}(x,y)$ and discuss the continuity of these partial derivatives

$$f_x(x,y) = \begin{cases} \dfrac{y^3}{(x^2+y^2)^{3/2}}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

$$f_y(x,y) = \begin{cases} \dfrac{x^3}{(x^2+y^2)^{3/2}}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

Continuity of partial derivatives

$$f_x(x,y) = \begin{cases} \dfrac{y^3}{(x^2+y^2)^{3/2}}, (x,y) \neq (0,0) \\ 0, \qquad\qquad\qquad \text{elsewhere} \end{cases}$$

$$f_y(x,y) = \begin{cases} \dfrac{x^3}{(x^2+y^2)^{3/2}}, (x,y) \neq (0,0) \\ 0, \qquad\qquad\qquad \text{elsewhere} \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} = \lim_{r\to 0} \sin^3\theta = \sin^3\theta$$

Limit does not exist

The same observation for f_y

Hence, both f_x & f_y are not continuous

CONCLUSIONS

PARTIAL DERIVATIVES

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0}$$

CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Partial Derivatives of Higher Order
- Worked Problems

First Order Partial Derivatives of f (Previous Lecture)

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Second Order Partial Derivatives of f

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xx} \qquad f_{yx} \qquad f_{yy} \qquad f_{xy}$$

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial^2 f}{\partial y^2} \qquad \frac{\partial^2 f}{\partial x \partial y}$$

The derivatives f_{xy} and f_{yx} are called mixed derivatives.

Problem – 1:

Compute $\dfrac{\partial^2 f}{\partial x \partial y}$ at the origin of $f(x,y) = \begin{cases} \dfrac{x^2 y \sin(2x-3y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f_y}{\partial x} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(2\Delta x) - 0}{\Delta x} = 2$$

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y)}{\Delta y} = \sin(2\Delta x)$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

Problem – 2:

Compute $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ at the origin of $f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f_y}{\partial x} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y} = \Delta x$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} \text{ at the origin of } f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f_x}{\partial y} = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-\Delta y - 0}{\Delta y} = -1$$

$$f_x(0, \Delta y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, \Delta y) - f(0, \Delta y)}{\Delta x} = -\Delta y$$

Note that $\dfrac{\partial^2 f}{\partial x \partial y} = 1$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

The equality of mixed partial derivatives

If (i) f_x, f_y, f_{yx} all exist in the neighborhood of the point (x_0, y_0)

& (ii) f_{yx} is continuous at (x_0, y_0), then

a) f_{xy} also exists at (x_0, y_0), and

b) $f_{xy} = f_{yx}$

OR

If the mixed derivatives f_{yx} & f_{xy} are continuous in an open domain D, then at any point $(x, y) \in D$

$$f_{xy} = f_{yx}$$

Problem – 3:

Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function $f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & x \neq -y^2 \\ 0, & \text{elsewhere} \end{cases}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f_y}{\partial x} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

$f_{yx}(0,0)$ for the function $f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & x \neq -y^2 \\ 0, & \text{elsewhere} \end{cases}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f_x}{\partial y} = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y} = 1$$

$$f_x(0,\Delta y) = \lim_{\Delta x \to 0} \frac{f(\Delta x,\Delta y) - f(0,\Delta y)}{\Delta x} = \Delta y \qquad f_x(0,0) = 0$$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$, f_{xy} and f_{yx} are not continuous at $(0,0)$.

Continuity Check of f_{xy} & f_{yx}

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & x \neq -y^2 \\ 0, & \text{elsewhere} \end{cases}$$

For $x \neq -y^2$

$$f_x(x,y) = \frac{y^5}{(x+y^2)^2}$$

$$f_{yx}(x,y) = \frac{y^6 + 5xy^4}{(x+y^2)^3} = f_{xy}\,(x,y)$$

Along the path $x = my^2$ the limit $\lim_{(x,y)\to(0,0)} f_{yx}(x,y) = \dfrac{1+5m}{(m+1)^3}$

Limit depends on the path

The limit does not exist. Hence f_{yx} and f_{xy} are not continuous at $(0,0)$

Problem 4: Showing existence of second order partial derivative though the function is not continuous

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & \text{elsewhere} \end{cases}$$

Take a path $y = x\cos x$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3 + x^3\cos^3 x}{x - x\cos x} = 4$$

The function is not continuous at $(0,0)$.

Evaluation of $f_{xx}(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{xx}(0,0) = \lim_{\Delta x \to 0} \frac{f_x(\Delta x, 0) - f_x(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x - 0}{\Delta x} = 2$$

$$f_x(x,0) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, 0) - f(x,0)}{\Delta x} = 2x$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

Evaluation of $f_{yy}(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{yy}(0,0) = \lim_{\Delta y \to 0} \frac{f_y(0,\Delta y) - f_y(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-2\Delta y - 0}{\Delta y} = -2$$

$$f_y(0,\mathrm{y}) = \quad -2y$$

$$f_y(0,0) = \ 0$$

CONCLUSIONS

Partial Derivatives of Higher Order

Continuity of f_{yx} & $f_{xy} \Longrightarrow f_{yx} = f_{xy}$