CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Derivative of composite functions
  - Derivative of functions defined implicitly

Composite Functions

Consider $\left. z = f(x, y) \right\}$ (1)

Let $\left. \begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned} \right\}$ (2) or $\left. \begin{aligned} x &= \phi(u, v) \\ y &= \psi(u, v) \end{aligned} \right\}$ $(2')$

The equations $(1 \;\&\; 2)$ or $(1 \;\&\; 2')$ are said to define z as composite function of t or $u \;\&\; v$.

Differentiation of Composite Functions

Let $z = f(x, y)$ posses continuous partial derivatives (differentiable) and let $x = \phi(t)$, $\mathrm{y} = \psi(t)$ posses continuous derivatives (differentiable). Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Proof: Let $z = f(x, y)$, $x = \phi(t)$, $y = \psi(t)$ be a composite function of t.

Assuming z, ϕ, ψ to be differentiable

$z = f(x, y), x = \phi(t),\ y = \psi(t)$

$$\Delta z = z_x \Delta x + z_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Dividing by Δt

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

Taking limit $\Delta t \to 0$ $(\Delta x \to 0,\ \ \Delta y \to 0)$

$$\frac{\mathrm{dz}}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Differentiation of Composite Functions

For the case $z = f(x,y), x = \phi(t),\ y = \psi(t)$

$$\frac{\mathrm{dz}}{\mathrm{dt}} = \frac{\partial z}{\partial x}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\partial z}{\partial y}\frac{\mathrm{dy}}{\mathrm{dt}}$$

For the case $z = f(x,y)$

$$x = \phi(u,v), y = \psi(u,v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

For the case $z = f(x)$

$$x = \phi(u,v)$$

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v}$$

For the case $z = f(x)$

$$x = \phi(u,v,w)$$

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v}$$

$$\frac{\partial z}{\partial w} = \frac{dz}{dx}\frac{\partial x}{\partial w}$$

Problem - 1 Given $z = xy; \quad x = \cos t, \quad y = \sin t$. Find $\dfrac{\mathrm{dz}}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= y(-\sin t) + x\cos t$$

$$= -\sin^2 t + \cos^2 t$$

$$= \cos 2t$$

Problem - 2 Let z be a function of x & y. Further, it is given that

$$x = e^{u} + e^{-v} \qquad\qquad y = e^{-u} + e^{v}$$

Then show that $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^{u} + \frac{\partial z}{\partial y}(-e^{-u})$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x}e^{-v} + \frac{\partial z}{\partial y}e^{v}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u}+e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} + e^{v}) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Problem-3: Find $\partial z/\partial u$ and $\partial z/\partial v$ if $z = \tan^{-1} x$ and $x = e^u + \ln v$

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \frac{1}{1+x^2}e^u = \frac{1}{1+(e^u+\ln v)^2}e^u$$

$$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v} = \frac{1}{1+x^2}\frac{1}{v} = \frac{1}{1+(e^u+\ln v)^2}\frac{1}{v}$$

Derivative of a function defined implicitly

Case – I : Functions of single variable

Let the function y of x be defined as $F(x, y) = 0$

Let $z = F(x, y) = 0$

$$\frac{dz}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad \Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y}; \qquad \frac{\partial F}{\partial y} \neq 0$$

Case – II : Functions of two Variables

Let the function z of x & y be defined as $F(x, y, z) = 0$ Let $w = F(x, y, z) = 0$

Differentiating w with respect to x

$$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{OR} \quad \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial z}; \quad \frac{\partial F}{\partial z} \neq 0$$

Differentiating w with respect to y

$$\Rightarrow \frac{\partial w}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \quad \text{OR} \quad \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\bigg/\frac{\partial F}{\partial z}; \quad \frac{\partial F}{\partial z} \neq 0$$

Problem - 4: Let z be the function of x & y defined as $x^2 + y^2 + z^2 - c = 0$.

Find $\partial z/\partial x$ and $\partial z/\partial y$.

Differentiating with respect to x

$$2x + 2z\frac{\partial z}{\partial x} = 0 \quad \Longrightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$$

Differentiating with respect to y

$$2y + 2z\frac{\partial z}{\partial y} = 0 \quad \Longrightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$$