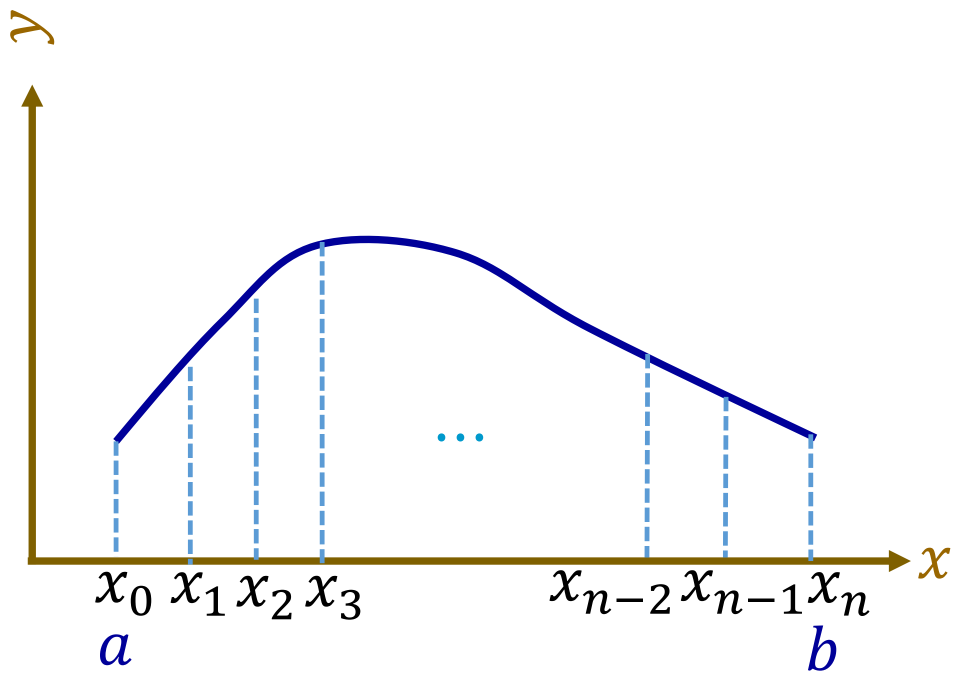
DOUBLE INTEGRALS

- ❑ **Double Integrals**
- ❑ **Evaluation**

Integrals of Functions of Single Variable

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$



Double Integrals

Let $f(x,y)$ be defined in a closed region D of the xy plane.

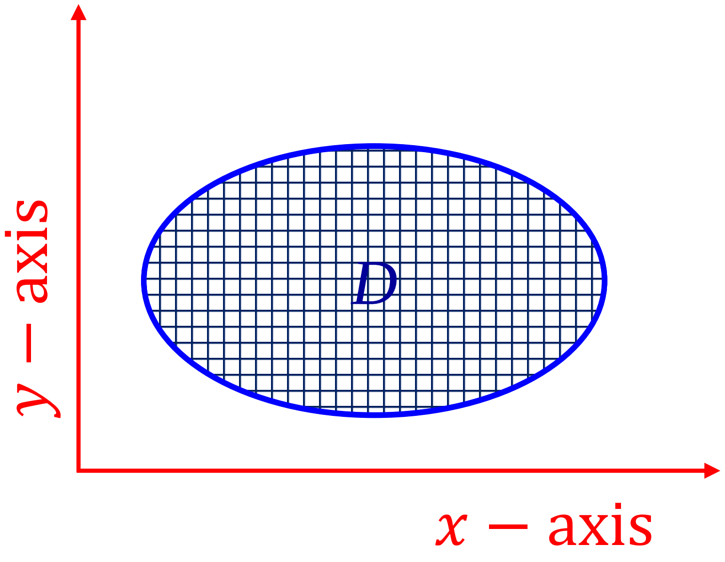
Divide D into n sub-regions of area ΔA_j, $\ j = 1,2,\dots,n$.

Let (x_j, y_j) be some point of ΔA_j.

Then consider $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n} f(x_j, y_j)\,\Delta A_j$

If this limit exists, then it is denoted by

$$\iint_D f(x,y)\,dA \quad \text{OR} \quad \iint_D f(x,y)\,dx\,dy \quad \text{OR} \quad \iint_D f(x,y)\,dy\,dx$$



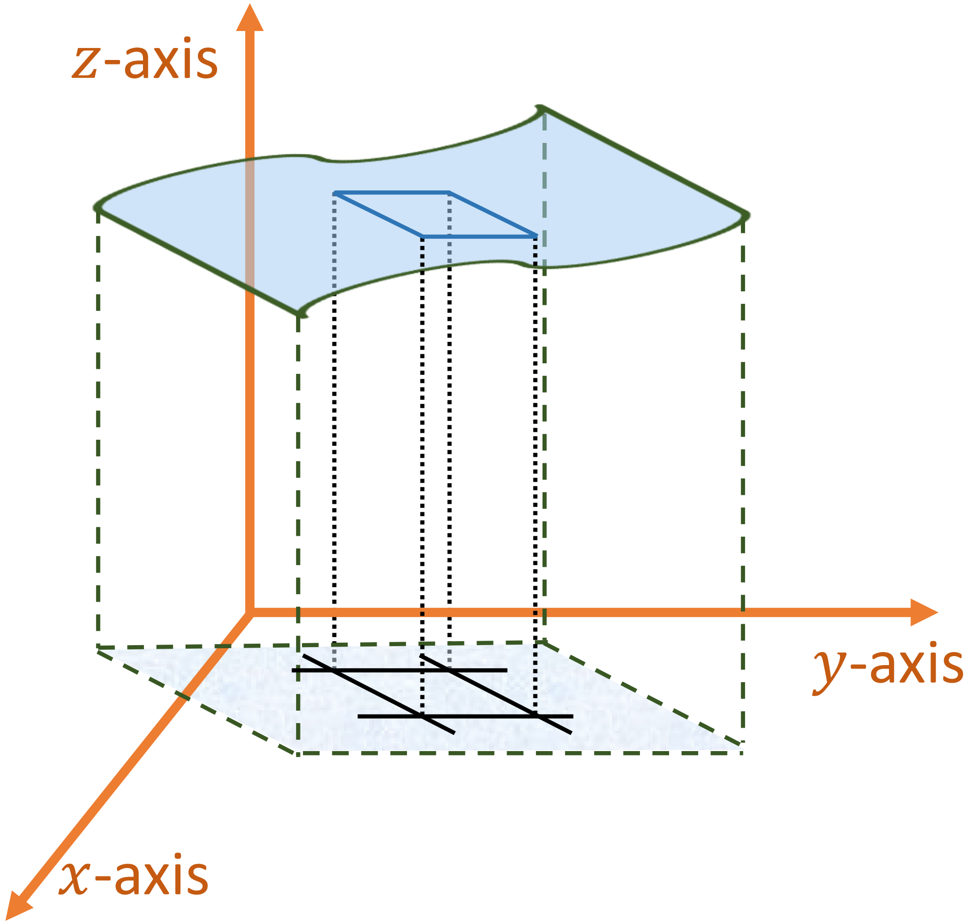
Note: It can be proved that the above limit exists if $z = f(x, y)$ is continuous or piecewise continuous in D.

Geometrical Interpretation of Double Integral

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j)\, \Delta x\, \Delta y$$

$$= \iint_D f(x, y)\, dx\, dy \quad \text{represents volume}$$

OR area of D if $f(x, y) = 1$



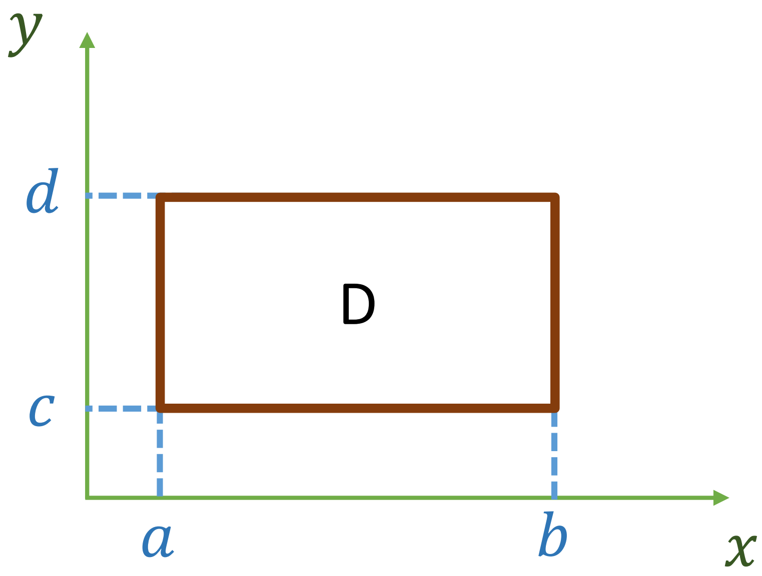
Properties

- $$\iint_D k\, f(x,y)\, dA = k \iint_D f(x,y)\, dA$$
- $$\iint_D [f(x,y) \pm g(x,y)]\, dA = \iint_D f(x,y)\, dA \pm \iint_D g(x,y)\, dA$$
- $$\iint_D f(x,y)\, dA \geq 0 \text{ if } f(x,y) \geq 0 \text{ on } D$$
- $$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA \text{ if } f(x,y) \geq g(x,y) \text{ on } D$$
- $$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA \text{ if } D = D_1 \cup D_2$$

Evaluation of Double Integral

- If $f(x, y)$ is continuous (or defined and bounded) on rectangular region

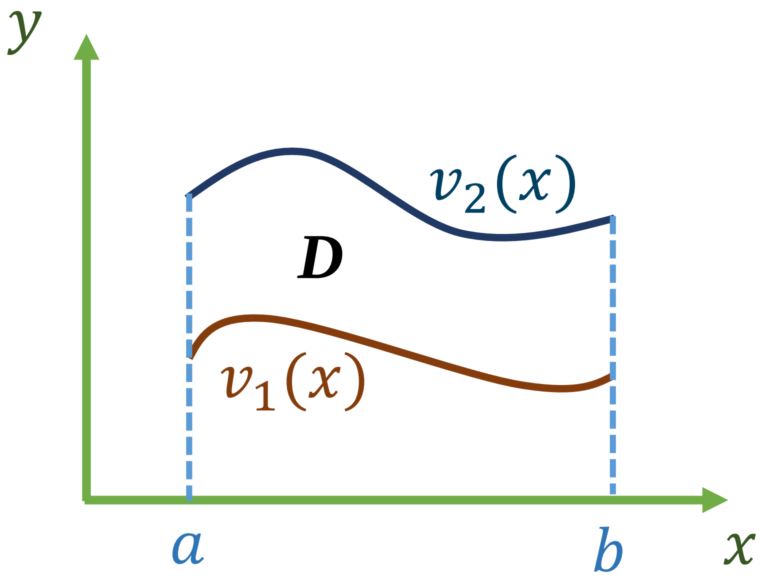
D: $a \leq x \leq b, c \leq y \leq d,$



$$\iint\limits_{D} f(x, y)\, dA = \qquad \underbrace{\int_a^b f(x, y)\, dx}_{\Psi(y)} \qquad = \qquad \underbrace{\int_c^d f(x, y)\, dy}_{\Phi(x)}$$

Evaluation of Double Integral

Non-rectangular Region

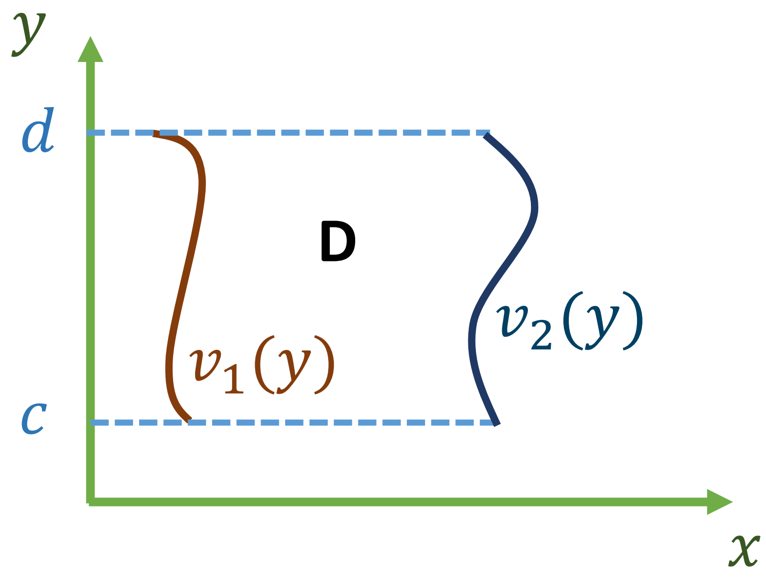


- If $f(x, y)$ is defined and bounded in D
- v_1 and v_2 are continuous in (a, b)

$$\iint_D f(x,y)\,dA = \int_{v_1(x)}^{v_2(x)} f(x,y)\,dy$$

Evaluation of Double Integral

Non-rectangular Region



- If $f(x,y)$ is defined and bounded in D
- v_1 and v_2 are continuous in (a,b)

$$\iint_D f(x,y)\,dA = \int_{v_1(y)}^{v_2(y)} f(x,y)\,dx$$

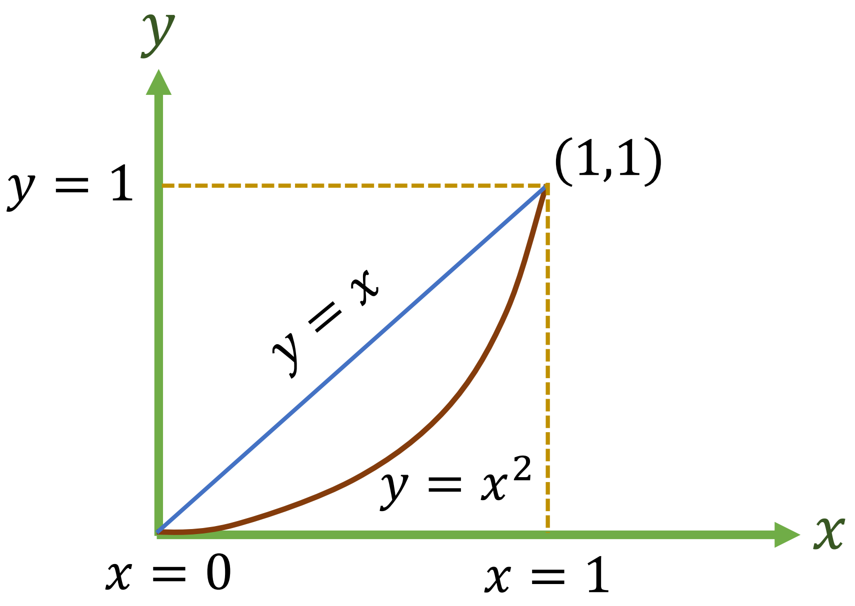
Example - 1 $$\iint_R xy(x+y)\,dA =$$

where R is the region bounded by the line $y = x$ and the curve $y = x^2$.

$$\int_{y=x^2}^{x} xy\,(x+y)\,dy$$

OR

$$\int_{x=y}^{\sqrt{y}} xy(x+y)\,dx$$



Consider $\displaystyle\int_{x=0}^{1}\int_{x^2}^{x} xy(x+y)dy\,dx$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{56}$$

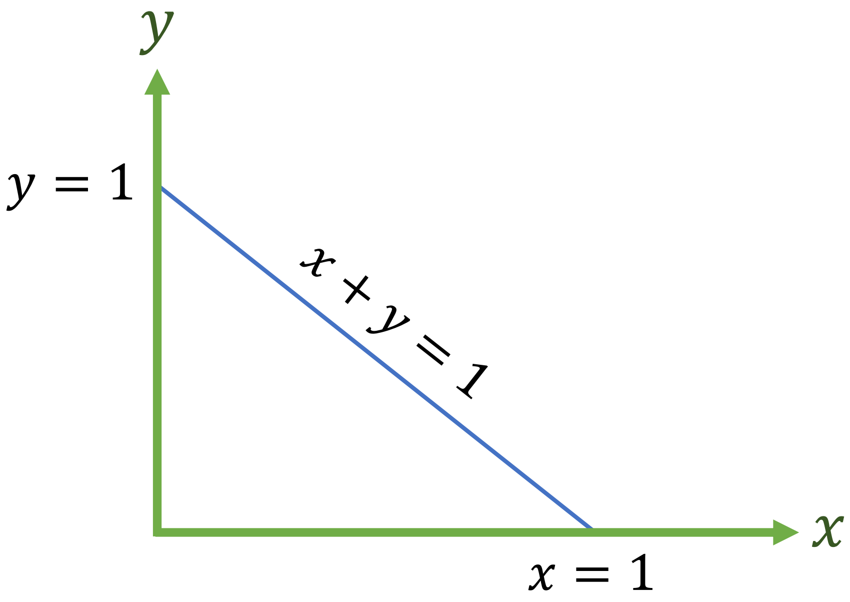
Example - 2 Evaluate $\int\int_R e^{2x+3y}\,dxdy$,

where R is the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

$$\int_{y=0}^{1-x} e^{2x+3y}\,dy$$

OR

$$\int_{x=0}^{1-y} e^{2x+3y}\,dx$$



Consider $\int_{x=0}^{1}\int_{0}^{1-x} e^{2x+3y}\, dy\, dx$

$$= \frac{1}{3}\int_{0}^{1} e^{2x}\left(e^{3-3x} - 1\right) dx$$

$$= \frac{1}{3}\left[-\frac{3e^{2}}{2} + e^{3} + \frac{1}{2}\right]$$

Conclusion:

$$\lim_{n\to\infty}\sum_{j=1}^{n} f\left(x_j, y_j\right)\Delta A_j = \iint_D f(x,y)\,dA$$

It represents volume (or area if $f(x,y) = 1$)

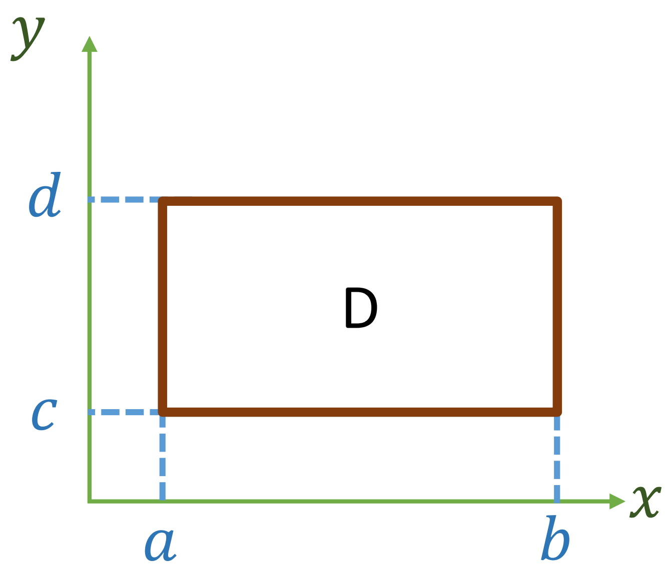
- Hardest part in evaluating multiple integral is finding the limit of integration

- Sketch of region of integration is important

DOUBLE INTEGRALS (Cont.)

- Double Integrals - Change of Order

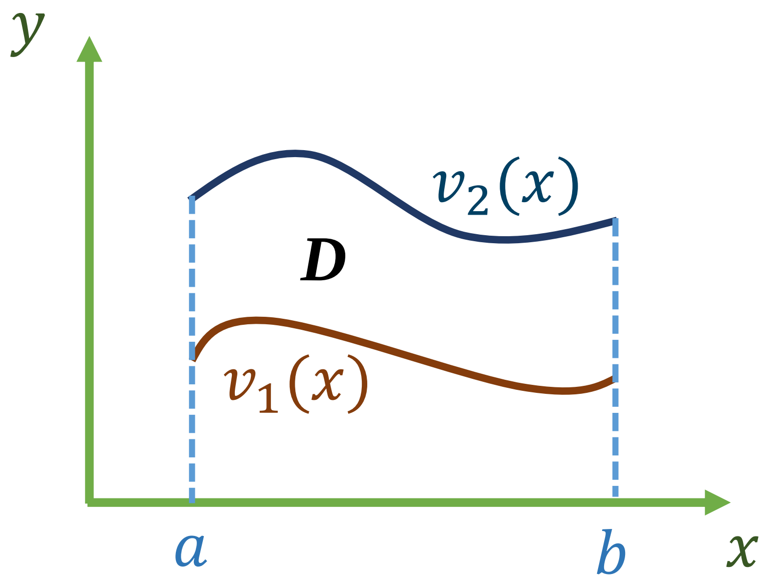
Evaluation of Double Integral (Recall)

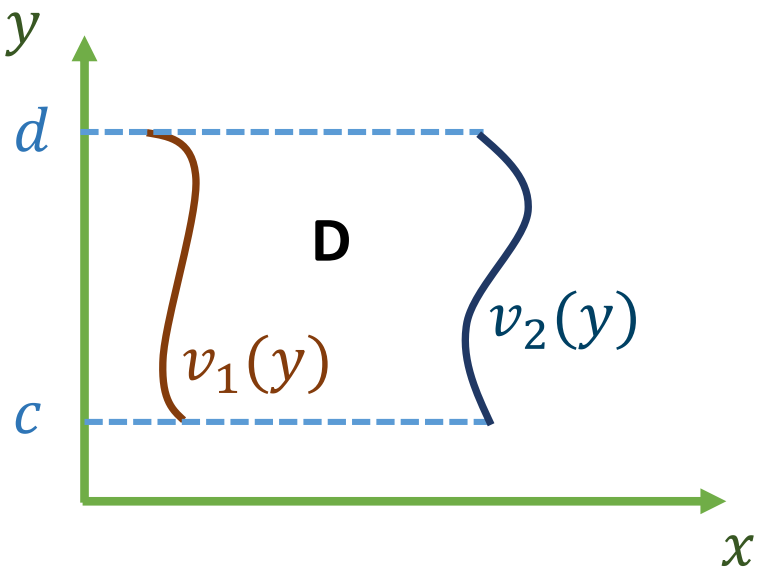


$$\iint_{D} f(x,y)\, dA = \int_c^d \left\{ \int_a^b f(x,y)\, dx \right\} dy = \int_a^b \left\{ \int_c^d f(x,y)\, dy \right\} dx$$

Evaluation of Double Integral (Recall)

$$\iint_D f(x,y)\,dA = \int_a^b \left\{ \int_{v_1(x)}^{v_2(x)} f(x,y)\,dy \right\} dx$$





$$\iint_D f(x,y)\,dA = \int_c^d \left\{ \int_{v_1(y)}^{v_2(y)} f(x,y)\,dx \right\} dy$$

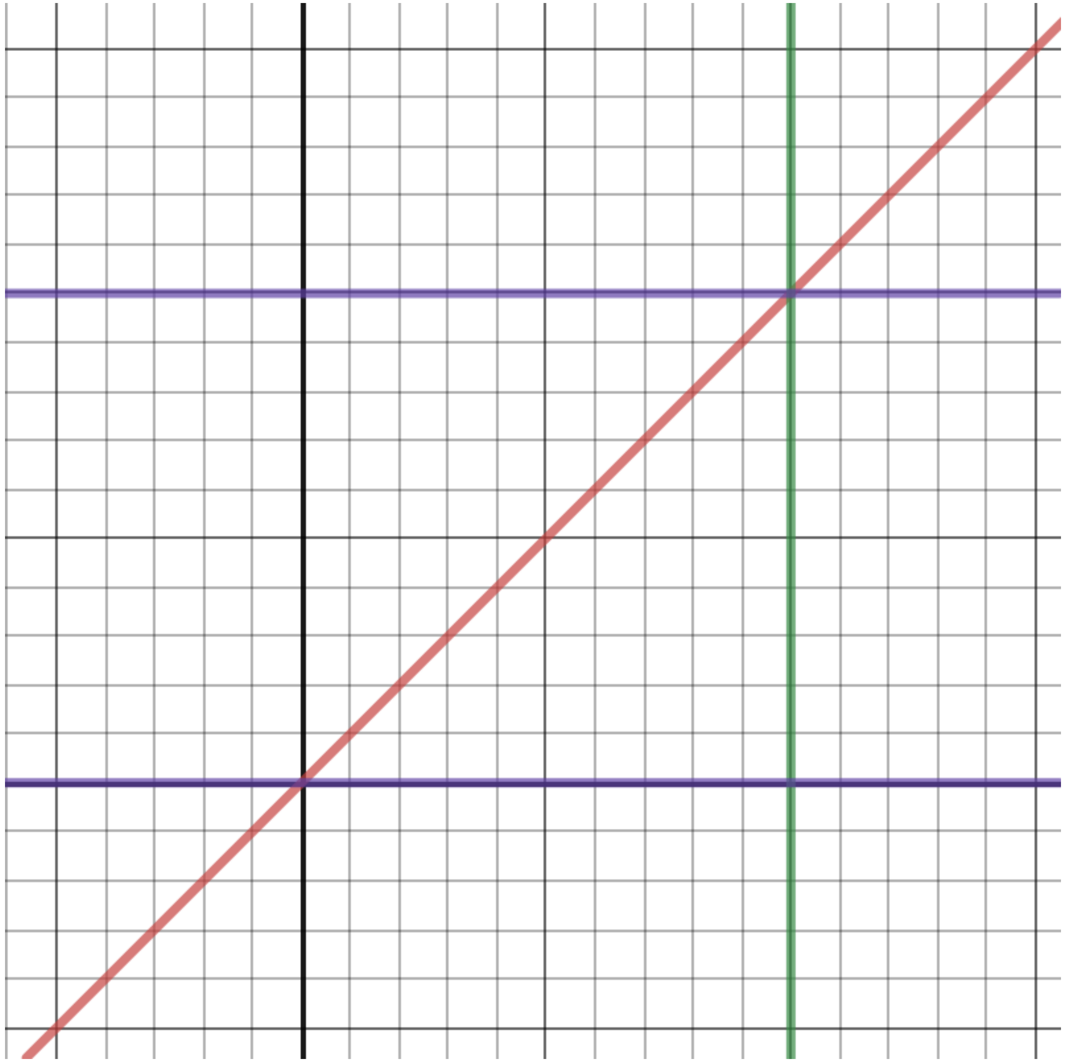
Evaluation of Double Integral - Change of Order of Integration

Why do we change order ?

Example : Evaluate $\int_{y=0}^{1}\int_{x=y}^{1}\frac{x}{x^2+y^2}\,dx\,dy$

Changing the order of Integration

$$\int_{x=0}^{1}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dy\,dx = ?$$



Evaluate $\displaystyle\int_{y=0}^{1}\int_{x=y}^{1}\frac{x}{x^2+y^2}\,dx\,dy$

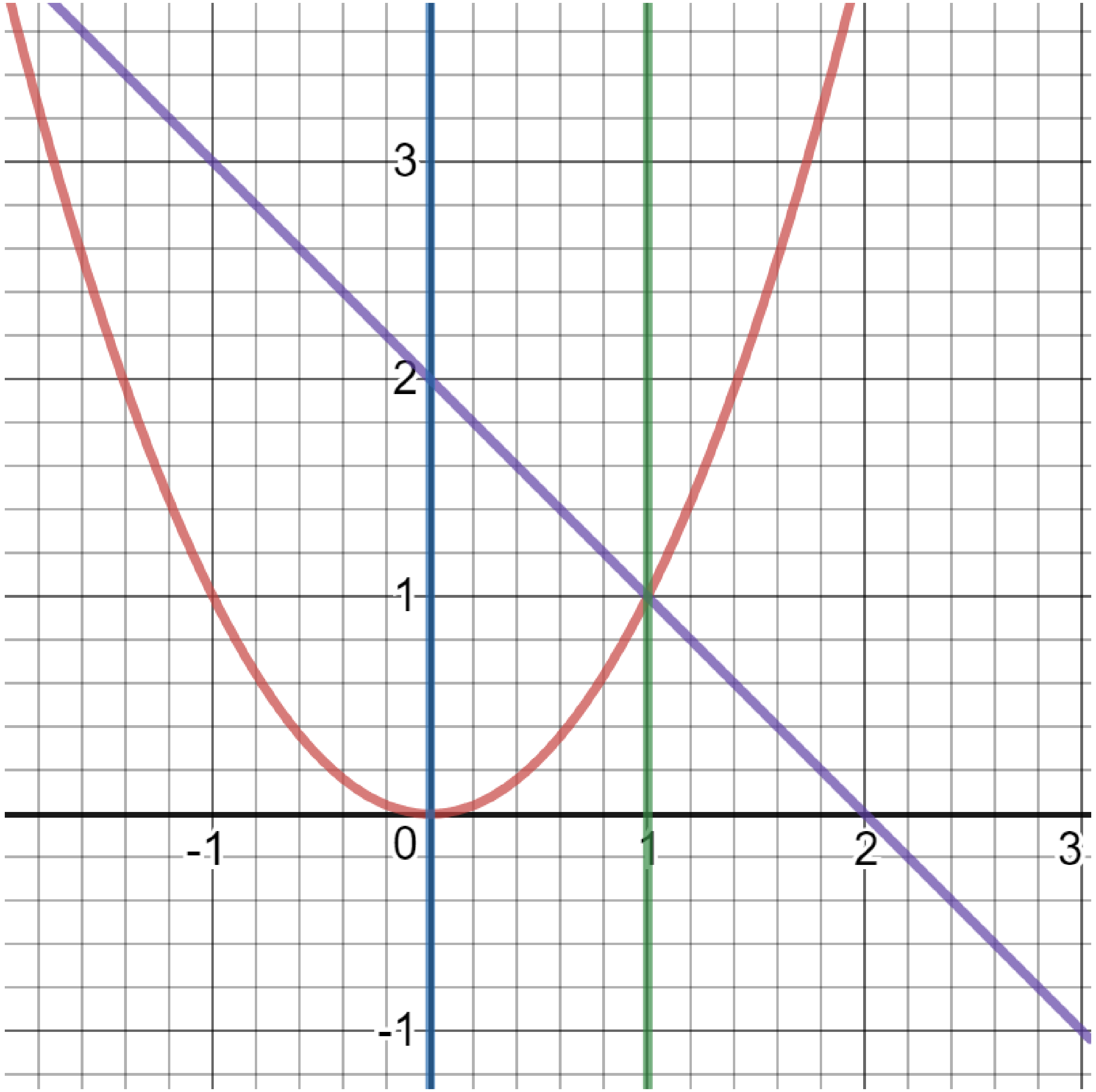
Changing the order of Integration

$$\int_{x=0}^{1}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dy\,dx = \int_{x=0}^{1}\tan^{-1}\left(\frac{y}{x}\right)\Big|_{0}^{x}\,dx$$

$$=\frac{\pi}{4}$$

Problem - 1 Consider $\int_0^1 \int_{y=x^2}^{2-x} xy \; dy\, dx.$

Change the order of integration and evaluate.

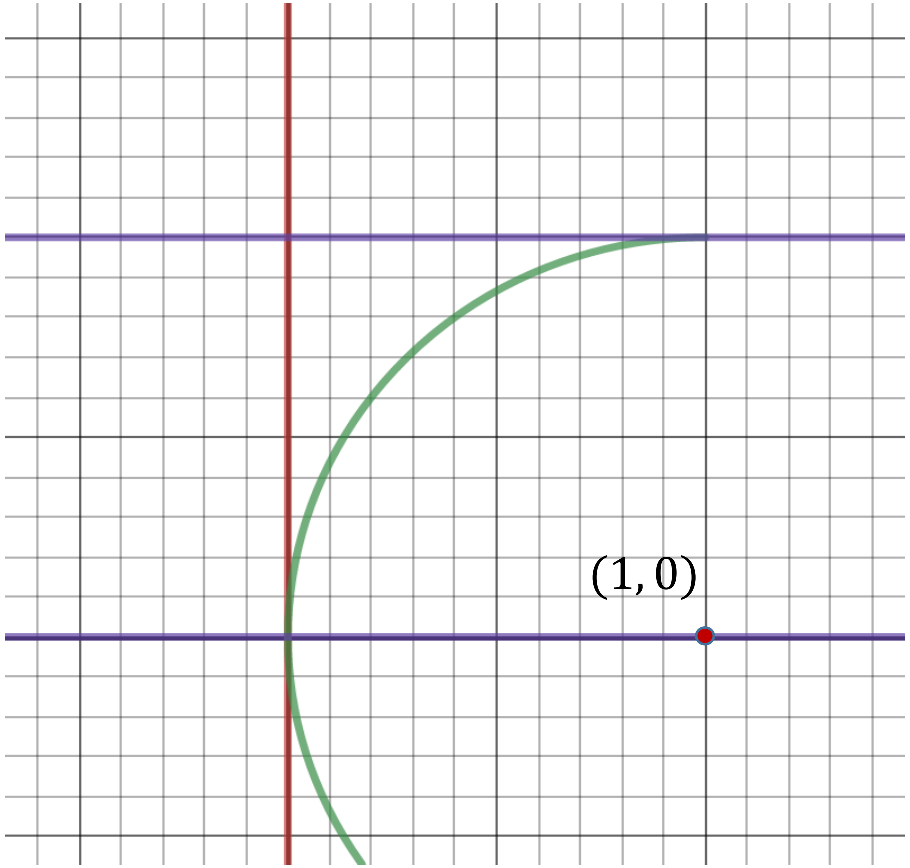


$$\int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\, dx\, dy$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

Problem - 2 $$\int_{y=0}^{1}\int_{x=0}^{1-\sqrt{(1-y^2)}} \frac{xy\ln(x+1)}{(x-1)^2}\,dx\,dy$$

Change the order of integration and evaluate.



$$I = \int_{x=0}^{1}\int_{y=\sqrt{1-(x-1)^2}}^{1} \frac{xy\ln(x+1)}{(x-1)^2}\,dy\,dx$$

$$I = \frac{1}{2}\int_0^1 x\ln(x+1)\,dx$$

$$I = \frac{1}{2}\int_0^1 x\ln(x+1)\,dx$$

$$= \frac{1}{2}\left[\left\{\frac{1}{2}\ln(2)\right\} - \frac{1}{2}\int_0^1 \left[(x-1) + \frac{1}{x+1}\right] dx\right]$$

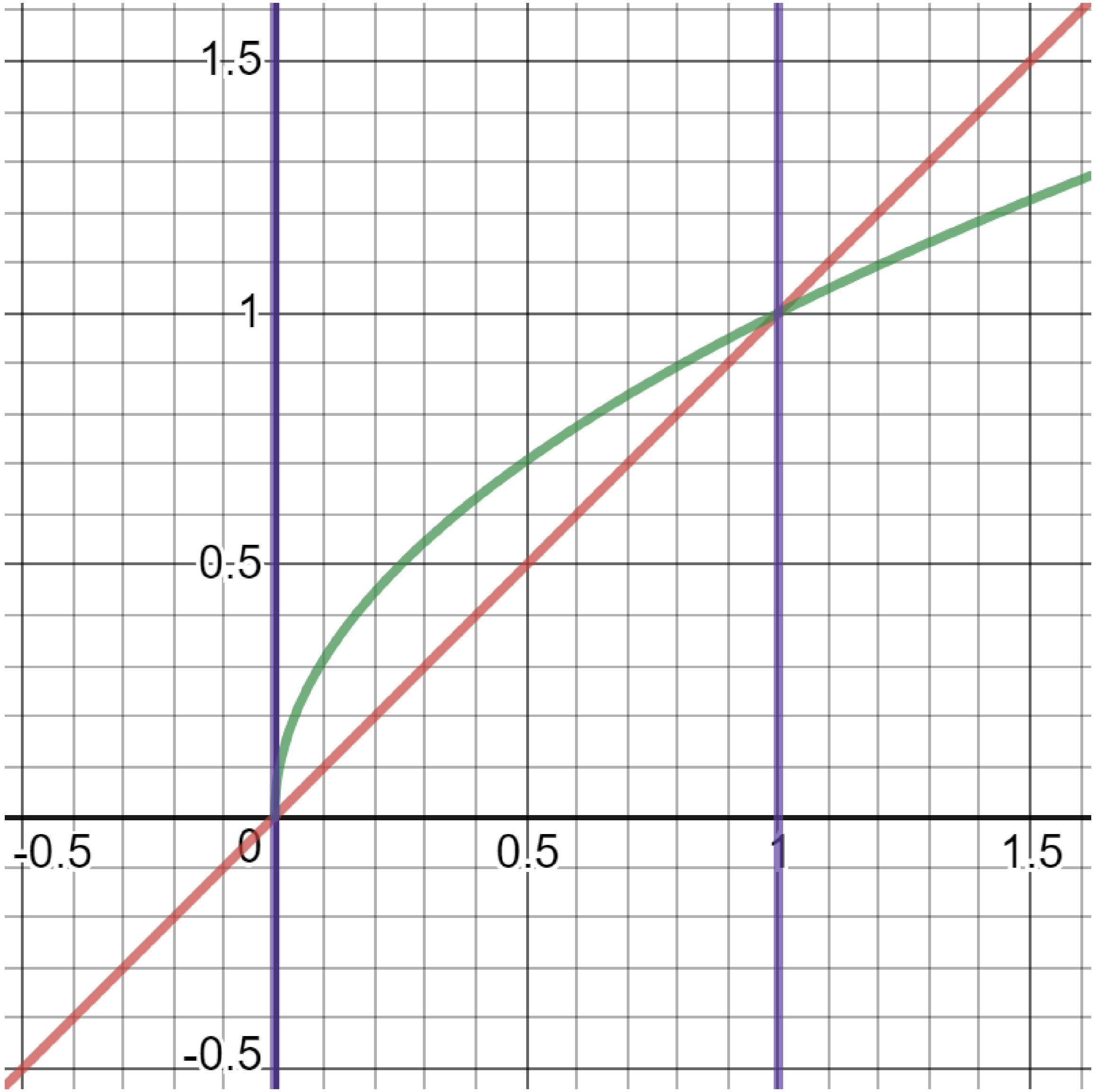
$$= \frac{1}{8}[1 + 2\ln 1] = \frac{1}{8}$$

Problem - 3

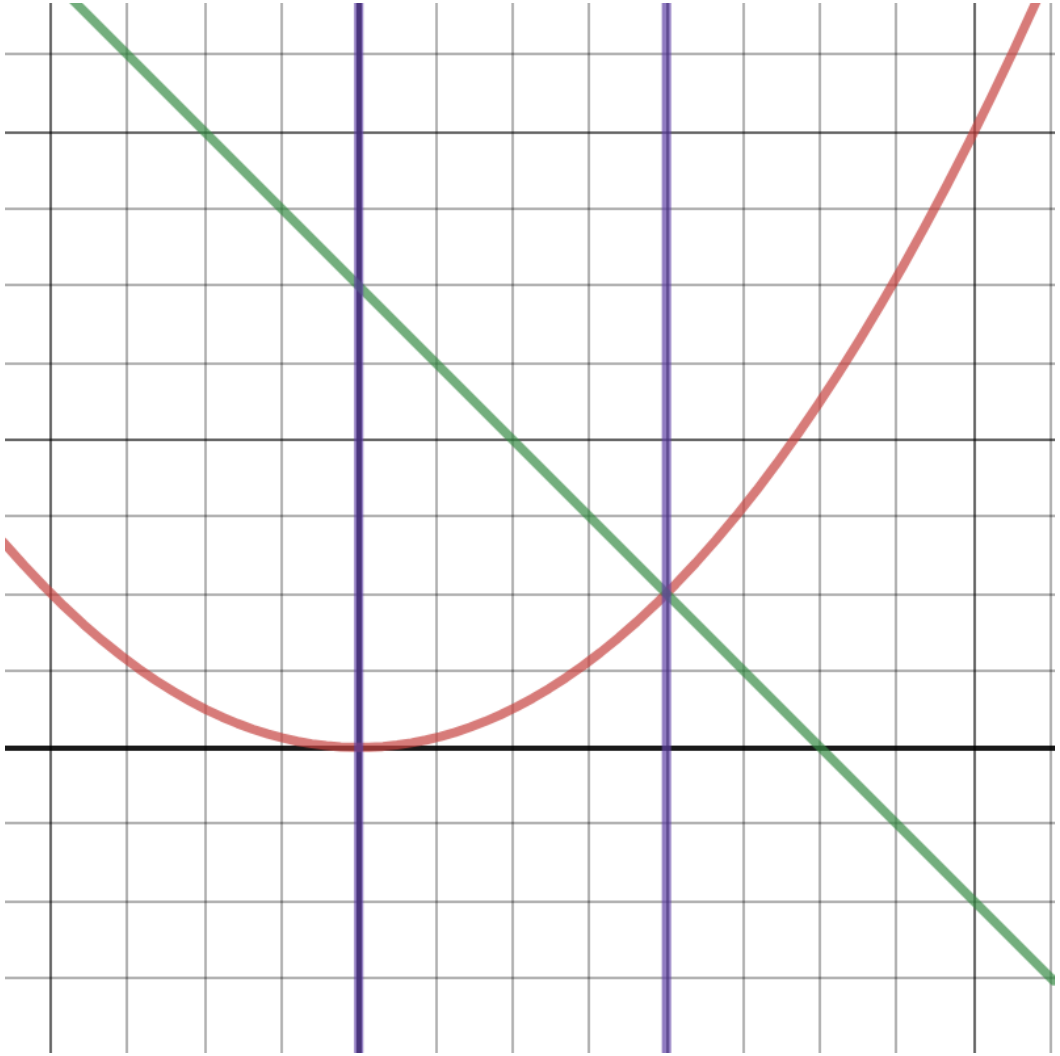
Change the order of integration

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)\, dy\, dx$$

ANS: $\int_0^1 \int_{y^2}^{y} f(x,y) dx dy$



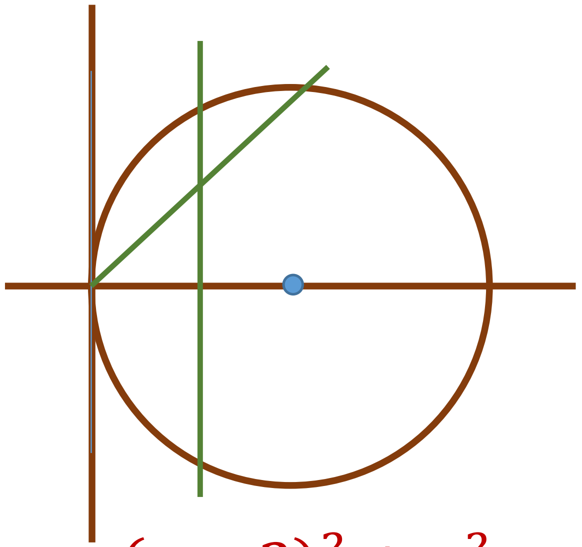
Problem - 4 Change the order of integration $\int_0^2 \int_{\frac{x^2}{4}}^{3-x} f(x,y)\,dy\,dx$



$$\int_0^1 \int_0^{2\sqrt{y}} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy$$

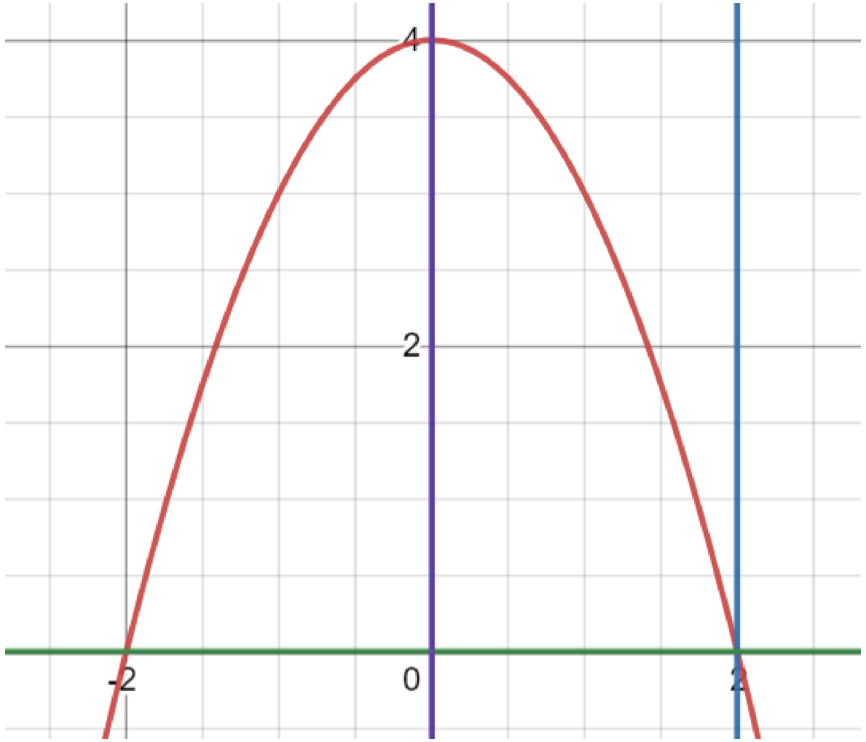
Problem 5: Change the order of integration of the integral

$$\int_0^1 \int_x^{\sqrt{4x-x^2}} f(x,y)\, dy\, dx$$



$(x-2)^2 + y^2 = 4$

$$\int_0^1 \int_{2-\sqrt{4-y^2}}^{y} f(x,y)\, dx\, dy + \int_1^{\sqrt{3}} \int_{2-\sqrt{4-y^2}}^{1} f(x,y)\, dx\, dy$$



Problem 6: Evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \frac{x\, e^{2y}}{4-y}\; dy\, dx$$

$$I = \int_0^4 \int_0^{\sqrt{4-y}} \frac{x\, e^{2y}}{4-y}\; dx\, dy = \frac{1}{2}\int_0^4 e^{2y}\; dx = \frac{1}{4}(e^8 - 1)$$

Conclusion:

- Sketching the region of integration
- Limit of integration