MA102 - Linear Algebra and Integral Transforms and Special Functions

Second Semester of Academic Year 2023-24

Tutorial sheet -4

1. Let V be a vector space defined as $V = M_{2\times 2}(\mathbb{R})$ over the field $\mathbb{R}$. Let $W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V; a, b, c \in \mathbb{R} \right\}$ and $W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V; a, b \in \mathbb{R} \right\}$. Prove that W_1 and W_2 are subspaces of V and find the dimension of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

2. Let W_1 and W_2 be subspaces of a vector space $V(\mathbb{R})$ having dimensions m and n, respectively, where $m \le n$.
   (a) Prove that $\dim(W_1 \cap W_2) \le m$.
   (b) Prove that $\dim(W_1 + W_2) \le m + n$.

3. Let $M_4(\mathbb{R})$ be the space of all (4×4) matrices over the field $\mathbb{R}$. Let
$$W = \left\{ (a_{ij}) \in M_4(\mathbb{R}) \middle| \sum_{i+j=k} a_{ij} = 0,\ k = 2, 3, 4, 5, 6, 7, 8 \right\},$$
then find the $\dim(W)$.

4. Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9(\mathbb{R})$. Prove that $U \cap W \ne \{0\}$.

5. Suppose that U and W are both four-dimensional subspaces of $\mathbb{C}^6(\mathbb{C})$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is scalar multiple of other.

6. Suppose V is a finite-dimensional vector space over the field $\mathbb{R}$ with $\dim(V) = n \ge 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots U_n$ of V such that
$$V = U_1 \oplus \cdots \oplus U_n.$$

7. Determine which of the following mappings are linear transformation.
   (a) $T : \mathbb{R}^3(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R}),\ T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ yz \end{bmatrix}$.
   (b) $T : \mathbb{R}^3(\mathbb{R}) \to \mathbb{R}^3(\mathbb{R}),\ T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+1 \\ y \\ z \end{bmatrix}$.
   (c) Transpose mapping $T : \mathbb{R}^{m\times n}(\mathbb{R}) \to \mathbb{R}^{n\times m}(\mathbb{R}),\ T(A) = A^T$.
   (d) Trace mapping $tr : \mathbb{R}^{n\times n}(\mathbb{R}) \to \mathbb{R}(\mathbb{R}),\ \mathrm{tr}(A) = \mathrm{trace}(A)$.
   (e) The evaluation mapping $\varepsilon_u : \mathbb{R}[x](\mathbb{R}) \to \mathbb{R}(\mathbb{R}),\ u \in \mathbb{R}$, defined by $\varepsilon_u(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1u + \cdots + a_nu^n$, where $\mathbb{R}[x]$ is the space of all polynomials over $\mathbb{R}$.

8. Find out whether the following statements are true or false.
   (a) The differential mapping $\mathcal{D} : \mathcal{C}^1(\mathcal{I}) \to \mathcal{C}^0(\mathcal{I})$, defined by $\mathcal{D}(f(x)) = f'(x)$ is injective, where $\mathcal{I}$ is an open interval in $\mathbb{R}$, $\mathcal{C}^1(\mathcal{I})$ is the space of all continuously differentiable functions on $\mathcal{I}$ and $\mathcal{C}^0(\mathcal{I})$ is the space of all continuous functions on $\mathcal{I}$.

(b) The evaluation mapping is surjective.

(c) The trace mapping is injective but not surjective.

9. Find Null space and Range space of the following mappings.

(a) S: $\mathbb{R}^n(\mathbb{R}) \to \mathbb{R}^n(\mathbb{R})$ such that $S \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \end{bmatrix}$

(b) $T : \mathbb{R}^n(\mathbb{R}) \to \mathbb{R}^n(\mathbb{R})$ such that $T \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ . \\ . \\ . \\ x_n - x_{n-1} \end{bmatrix}$

10. Let T be a linear operator on a vector space $V(\mathbb{R})$. Let $v \in V$ and let m be a positive integer such that $T^m v = 0$ and $T^{m-1}v \neq 0$. Then show that $v, Tv, \ldots, T^{m-1}v$ are linearly independent.

11. Let $T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ is a map, first reflects this map through x-axis and then reflects through the line $y = x$. Find the mapping T.

12. Find a 2×2 singular matrix B that maps $(1, 1)^T$ to $(1, 3)^T$.

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