Probability and Statistics
MA-202
Supplementary Notes

Limit Superior: Let $\{A_n\}$ be a sequence of sets. The set of all points $\omega \in \Omega$ that belong to A_n for infinitely many values of n is known as the *limit superior* of the sequence and is denoted by

$$\limsup_{n\to\infty} A_n \quad \text{or} \quad \overline{\lim}_{n\to\infty} A_n.$$

Limit Inferior: The set of all points that belong to A_n for all but a finite number of values of n is known as the *limit inferior* of the sequence $\{A_n\}$ and is denoted by

$$\liminf_{n\to\infty} A_n \quad \text{or} \quad \underline{\lim}_{n\to\infty} A_n.$$

Limit: If

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n,$$

we say that the limit exists and write $\lim_{n\to\infty} A_n$ for the common set and call it the *limit set*.

We have

$$\underline{\lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \overline{\lim}_{n\to\infty} A_n.$$

If the sequence $\{A_n\}$ is such that $A_n \subseteq A_{n+1}$, for $n = 1, 2, \ldots$, it is called *nondecreasing*; if $A_n \supseteq A_{n+1}$, $n = 1, 2, \ldots$, it is called *nonincreasing*. In the both the cases, the limit exists and we have

$$\lim_n A_n = \bigcup_{n=1}^{\infty} A_n \quad \text{if } A_n \text{is non-decreasing}$$

and

$$\lim_n A_n = \bigcap_{n=1}^{\infty} A_n \quad \text{if } A_n \text{is non-increasing.}$$

Theorem Let $\{A_n\}$ be a non-decreasing sequence of events in $\mathcal{F}$, then

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n\right). \tag{1}$$

Proof. Let

$$A = \bigcup_{j=1}^{\infty} A_j.$$

Then

$$A = A_n + \bigcup_{j=n}^{\infty}(A_{j+1} - A_j).$$

By countable additivity, we have

$$P(A) = P(A_n) + \sum_{j=n}^{\infty} P(A_{j+1} - A_j).$$

Letting $n \to \infty$, we see that

$$P(A) = \lim_{n\to\infty} P(A_n).$$

Note that $\lim_{n\to\infty} \sum_{j=n}^{\infty} P(A_{j+1} - A_j)$ tends to zero as $n \to \infty$ since the sum $\sum_{j=1}^{\infty} P(A_{j+1} - A_j) \leq 1$ and each summand is non-negative.

Corollary. Let $\{A_n\}$ be a non-increasing sequence of events in $\mathcal{F}$. Then

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} A_n\right). \tag{2}$$

(Hint: Consider the non-decreasing sequence of events $\{A_n^c\}$, and apply the result from the above theorem.)

Distribution Function

Theorem Given a random variable X, the function $F(x) = P(X \leq x)$ is a distribution function.

Proof.

a) **Non-decreasing:** Let $x_1 < x_2$. Then $(-\infty, x_1] \subset (-\infty, x_2]$, and we have

$$F(x_1) = P\{X \leq x_1\} \leq P\{X \leq x_2\} = F(x_2).$$

b) **Right continuous:** Since F is non-decreasing, it is sufficient to show that for any sequence of numbers $x_n \downarrow x$, $x_1 > x_2 > \cdots > x_n \to x$, $F(x_n) \to F(x)$.

Let $A_k = \{\omega : x < X(\omega) \leq x_k\}$. Then $A_k \in S$ is non-increasing, and we have

$$\lim_{k\to\infty} A_k = \bigcap_{k=1}^{\infty} A_k = \emptyset,$$

From Equation (2), we get $\lim_{k\to\infty} P(A_k) = 0$.

But

$$P(A_k) = P\{X \leq x_k\} - P\{X \leq x\} = F(x_k) - F(x),$$

so that

$$\lim_{k\to\infty} F(x_k) = F(x),$$

and F is right continuous.

c) **Proving $F(-\infty) = 0$ and $F(\infty) = 1$:**

Finally, let $\{x_n\}$ be a sequence of numbers decreasing to $-\infty$. Then

$$\{X \leq x_n\} \supseteq \{X \leq x_{n+1}\} \quad \text{for each } n,$$

and

$$\lim_{n\to\infty} \{X \leq x_n\} = \bigcap_{n=1}^{\infty} \{X \leq x_n\} = \emptyset.$$

Therefore,

$$F(-\infty) = \lim_{n\to\infty} P\{X \leq x_n\} = P\left[\lim_{n\to\infty} \{X \leq x_n\}\right] = 0.$$

Similarly, we can show

$$F(+\infty) = \lim_{x_n\to\infty} P\{X \leq x_n\} = 1,$$

and the proof is complete.