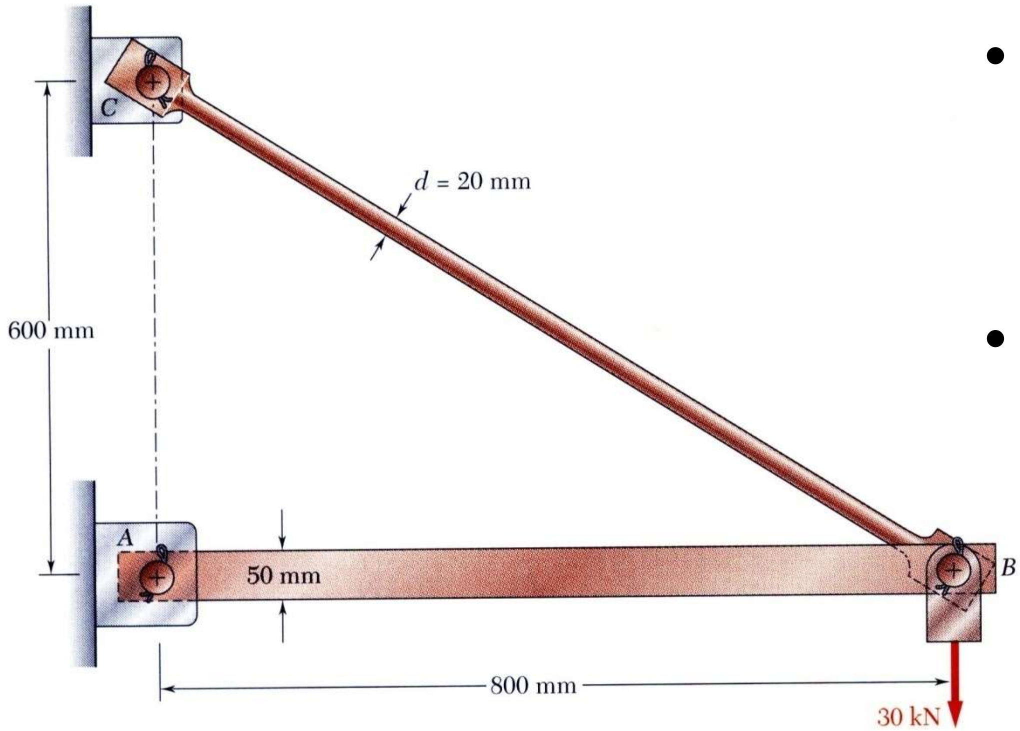
Review of Statics

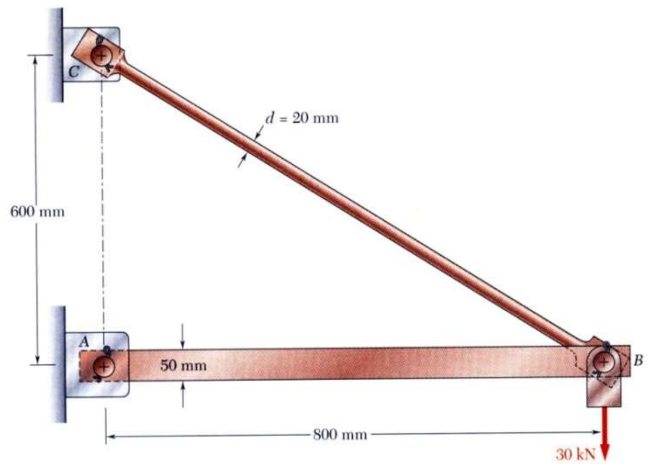


- The structure is designed to support a 30 kN load

- Determine the internal force in each structural member and the reaction forces at the supports

Review of Statics

- Can the structure safely support the 30 kN load?



- From a statics analysis

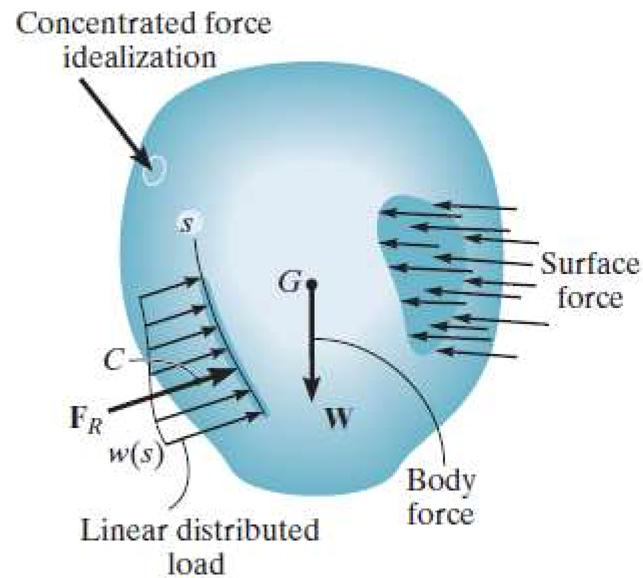
  F_{AB} = 40 kN (compression)

  F_{BC} = 50 kN (tension)

- Can we really answer this question by static analysis?
- Why not??
- We have used equilibrium conditions alone and assumed rigid bodies
- In practice, all the bodies are deformable

External loads

- Surface Forces
  - Point load
  - Distributed load
- Body forces
  - Weight
  - Gravitational force
  - Electromagnetic force



External loads

- Support reactions

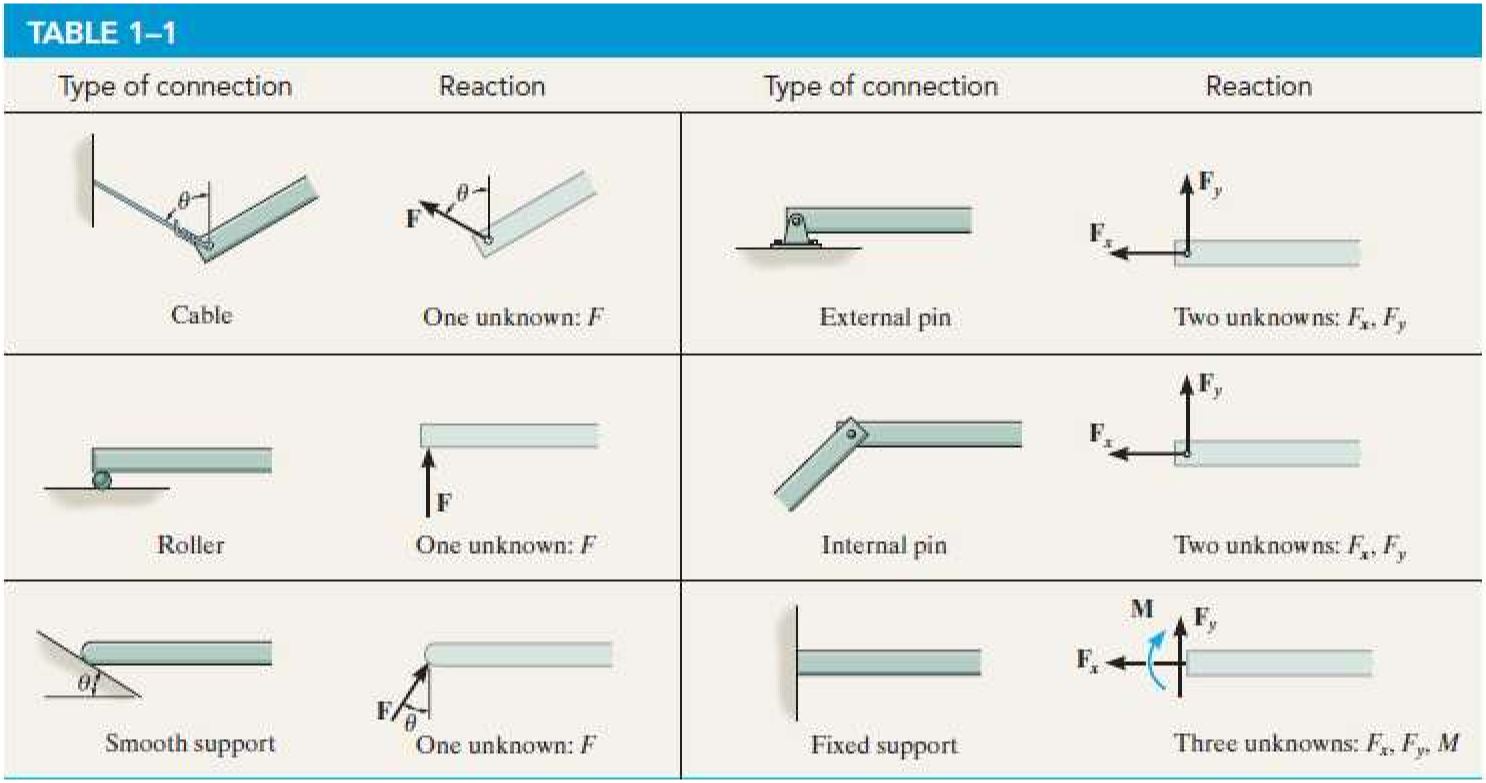


TABLE 1–1

Type of connection	Reaction	Type of connection	Reaction
Cable	One unknown: F	External pin	Two unknowns: F_x, F_y
Roller	One unknown: F	Internal pin	Two unknowns: F_x, F_y
Smooth support	One unknown: F	Fixed support	Three unknowns: F_x, F_y, M

Equilibrium of a Deformable Body

- Requires
  - Balance of forces – to prevent translation or acceleration
  - Balance of moments – to prevent rotation

$$\Sigma \mathbf{F} = \mathbf{0}$$
$$\Sigma \mathbf{M}_O = \mathbf{0}$$

  - In 3D

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0$$
$$\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0$$

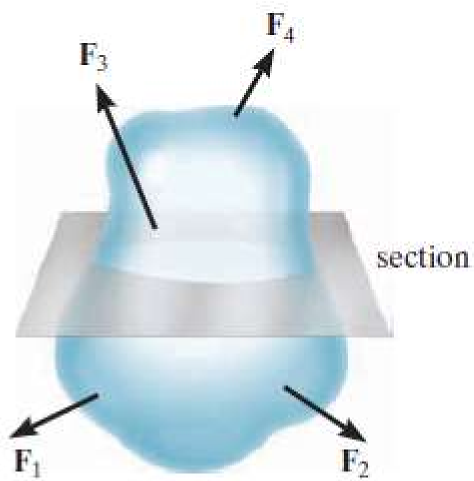
  - In 2D

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_O = 0$$

  - In 1D - ??

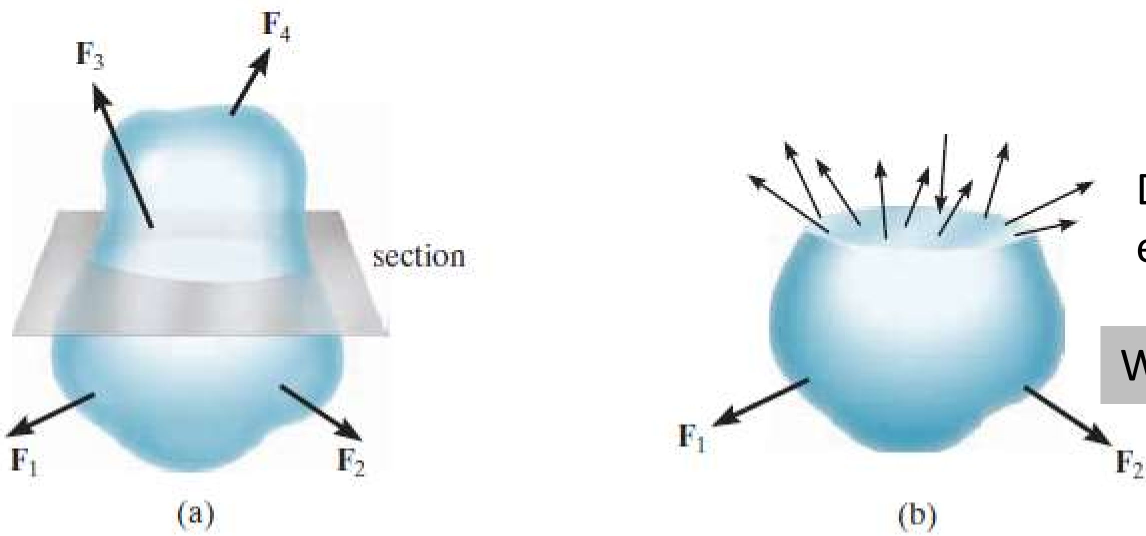
Equilibrium of a Deformable Body

- Complete specification of all the known and unknown forces is needed to apply equations of equilibrium
- Which can be achieved by drawing free body diagrams
- External applied loading



Equilibrium of a Deformable Body

- Internal resultant loading
  - it is necessary to pass an imaginary section or "cut" through the region where the internal loadings are to be determined



Distribution of forces on the exposed area of section.

What these forces represent??

  - Exact measure of this internal loading is unknown

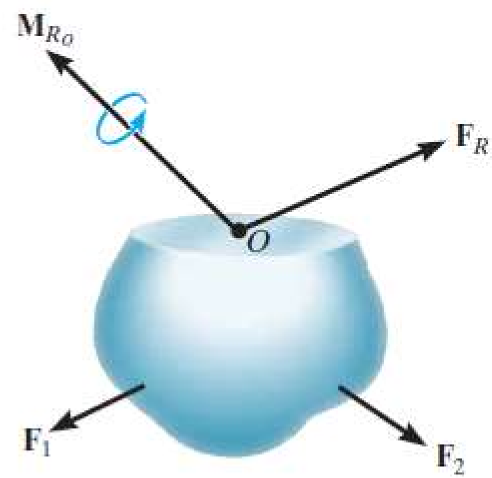
Equilibrium of a Deformable Body

- Equations of equilibrium can be used to relate the external forces to the resultant force and moments of distributed internal forces at any point *O*.

Most of the time, centroid of the area is chosen as *O*.

For long and slender members such as beam, column, the section perpendicular to longitudinal axis is considered.

Can you see why???



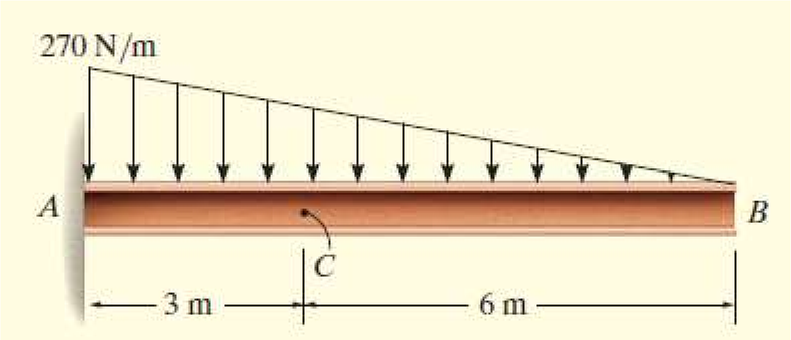
(c)

Resultant loading

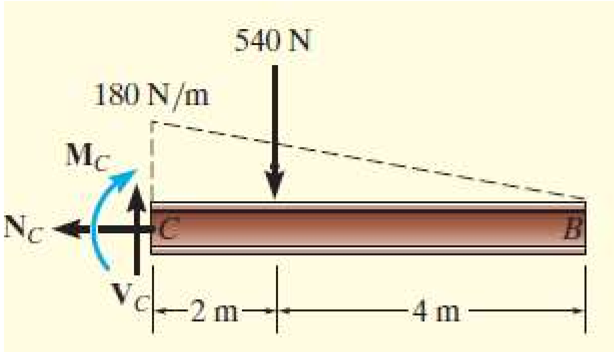
- Types of resultant loading can be defined as
  - Normal force N
  - Shear force V
  - Torsional moment or torque T
  - Bending moment M

Example - 1

- Determine the internal loading at *C*



- FBD of part *CB*

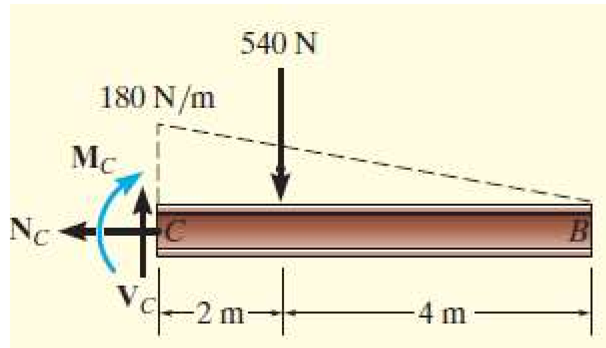


Example - 1

- Applying equations of equilibrium

$$V_C = 540 \text{ N}$$

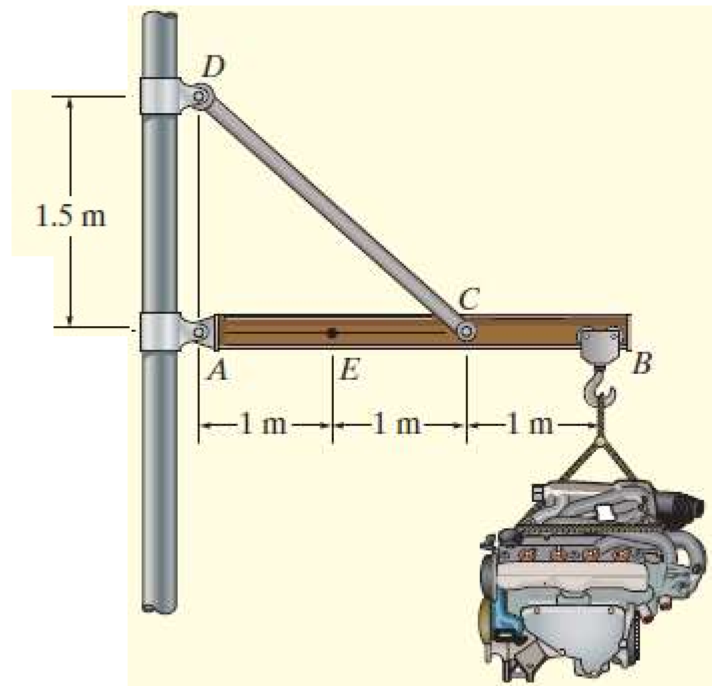
$$M_C = -1080 \text{ N} \cdot \text{m}$$



- Solve the same problem with segment *AC*

Example - 2

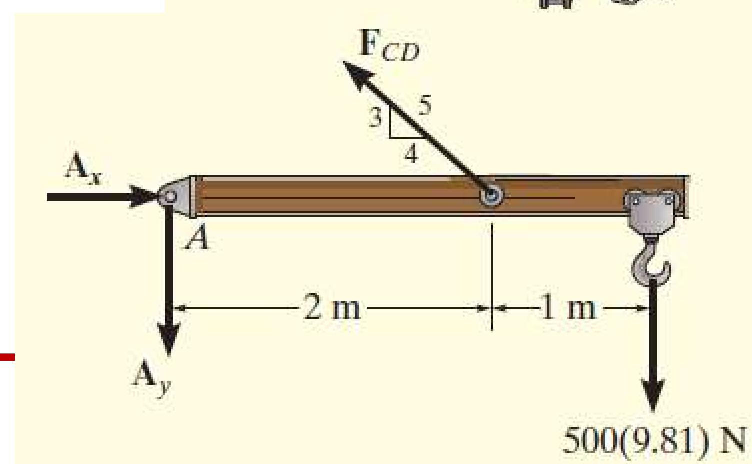
- Determine the internal loading at E
  - Reactions at A



$\Sigma M_A = 0;$ $F_{CD} = 12\,262.5\text{ N}$

$\Sigma F_x = 0;$ $A_x = 9810\text{ N}$

$\Sigma F_y = 0;$ $A_y = 2452.5\text{ N}$



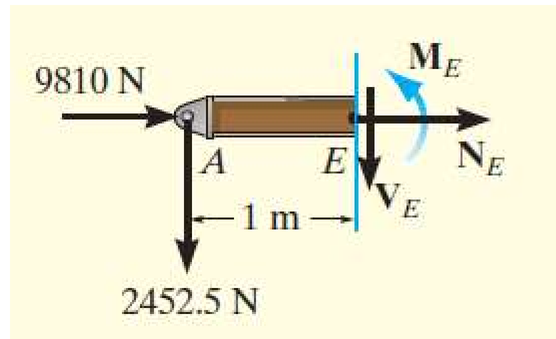
Example - 2

- Consider the FBD of segment AE

$\Sigma F_x = 0;$ $N_E = -9810\ \text{N} = -9.81\ \text{kN}$

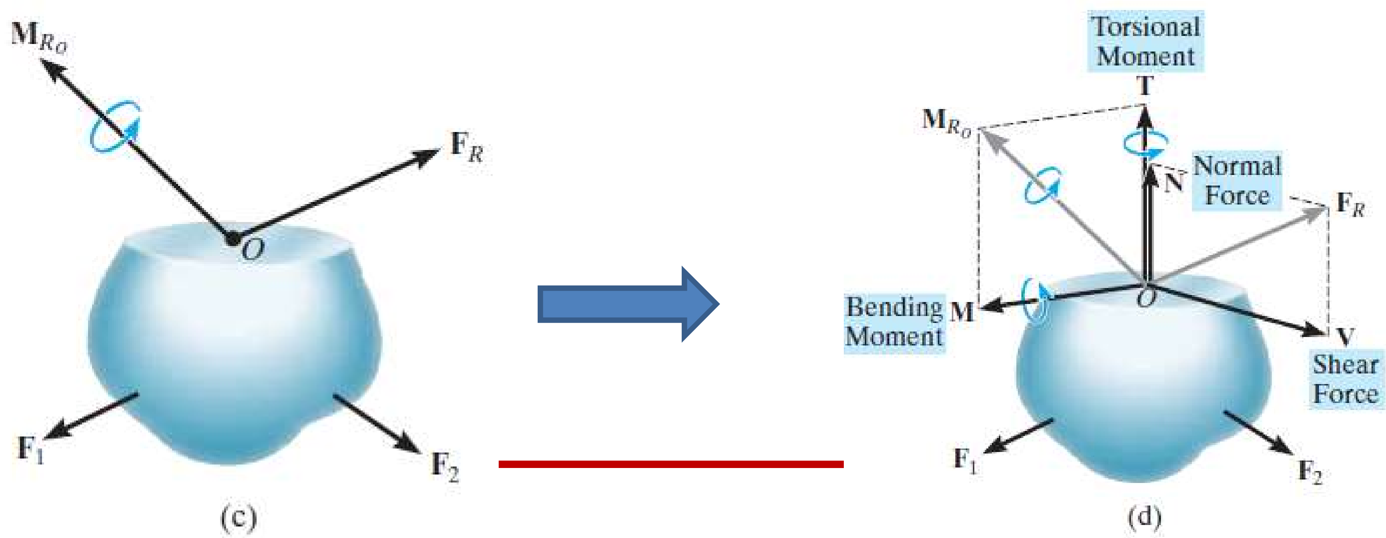
$\Sigma F_y = 0;$ $V_E = -2452.5\ \text{N}$

$\curvearrowleft + \Sigma M_E = 0;$ $M_E = -2452.5\ \text{N} \cdot \text{m}$



Equilibrium of a Deformable Body

- The resultant loading (force and moments) will be used to develop equations that can be used for analysis and design.
- To do this, the components of resultant force and moments acting perpendicular and normal to the sectioned area needs to be determined



Concept of stress

- Formulating the concept of stress took 500 years of struggle
- Why?
  - It is not a single idea, it is a package of ideas that may be repackaged in many ways.
  - We will see this as we move along the course
- Lenardo da Vinci – 1452-1519
  - Testing the strength of iron wires of various lengths
- Galileo Galilei (1564–1642)
  - derived theories on the bearing capacities of rods and beams
- Jakob Bernoulli (1655–1705) and Leonhard Euler (1707–1783)
  - deformation of beams
- Augustin Louis Cauchy (1789–1857)
  - Basis for theory of elasticity, state of stress and strain
  - Used in almost all design problems

Concept of stress

- What is stress??
  - Stress is the internal resistance, or counterforce, of a material to the distorting effects of an external force or load. These counterforces tend to return the atoms to their normal positions. The total resistance developed is equal to the external load. This resistance per unit area is known as *stress*.
  - Although it is impossible to measure the intensity of this stress, the external load and the area to which it is applied can be measured.
- Solid mechanics
  - is a branch of mechanics that studies the internal effects of stress and strain in a solid deformable body that is subjected to an external loading.

Concept of stress

- Unit of stress
  - force per unit area N/mm2
  - Same as pressure
  - What is relation between stress and pressure?
  - Is it a scalar or vector?
  - What information it stores?

Scalar and Vector

- Example of house numbers, distance and position
- Magnitude of a vector can be changed by a multiplying it with a scalar
- What to do if want to change the magnitude and direction both?
- Cross product will change the direction by right angle
- What we will do for general case

Another look ahead →

Scalar and Vector

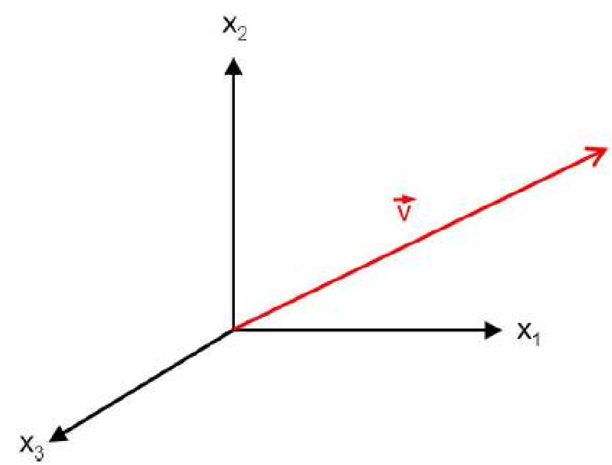
- Scalar
  – physical quantity which is represented by a dimensional number at a particular point in space and time *e.g.* temperature, dist etc.
- Vector
  – A vector is a tool to keep track of two pieces of information (typically magnitude and direction) for a physical quantity e.g. position, force velocity etc. It has 3 components in 3D space

Velocity vector

$$\vec{v} = v_1\vec{x}_1 + v_2\vec{x}_2 + v_3\vec{x}_3 = \sum_i v_i\vec{x}_i = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Magnitude of velocity vector

$$v \equiv |\vec{v}|$$

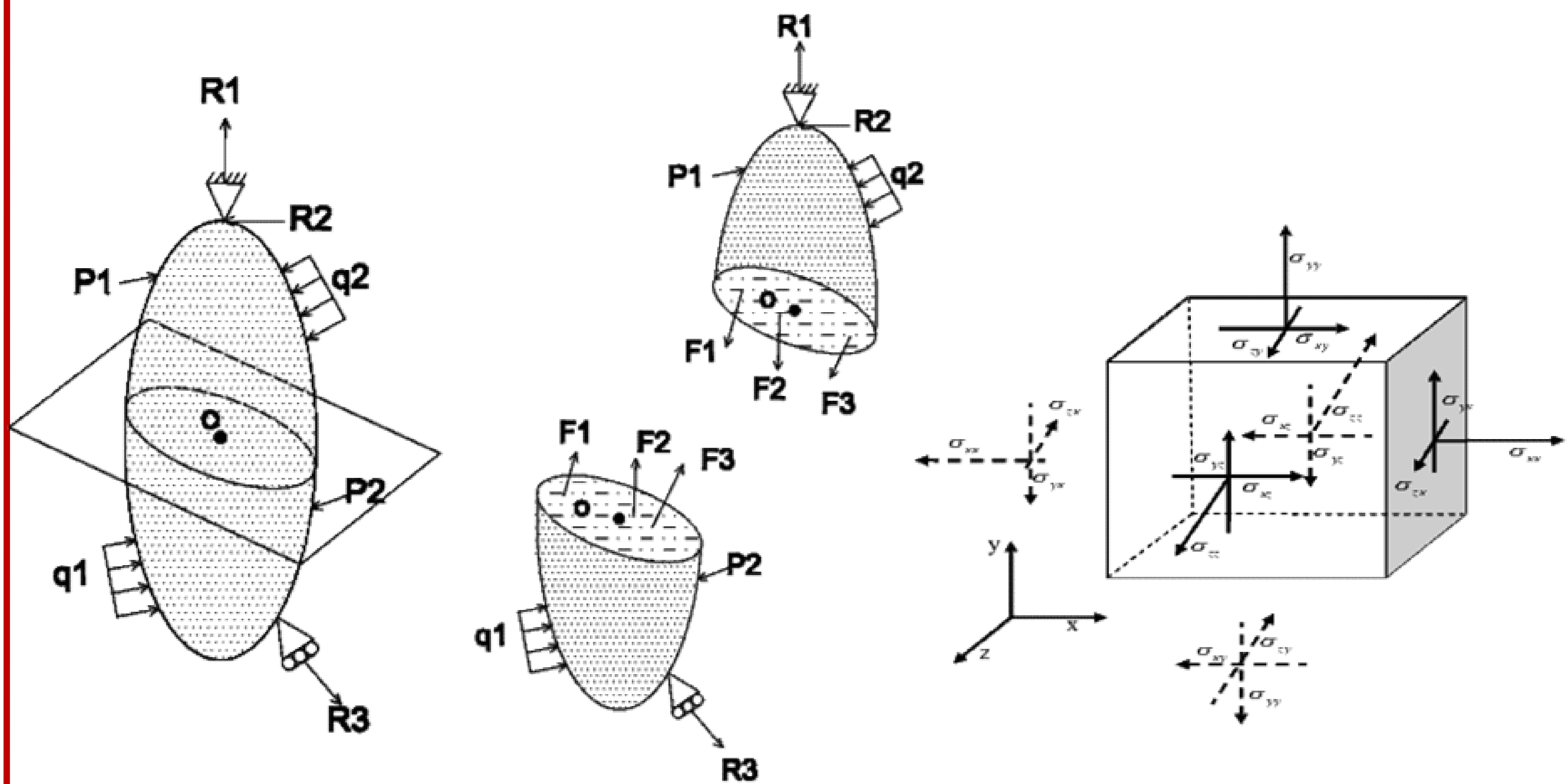


Tensor

- What if we need to keep three pieces of information?
- Tensors – Mapping between two vectors
  - Stress
  - We keep track of magnitude, direction and the plane the components act on
  - The rank (or order) of a tensor is defined by the number of directions (and hence the dimensionality of the array) required to describe it

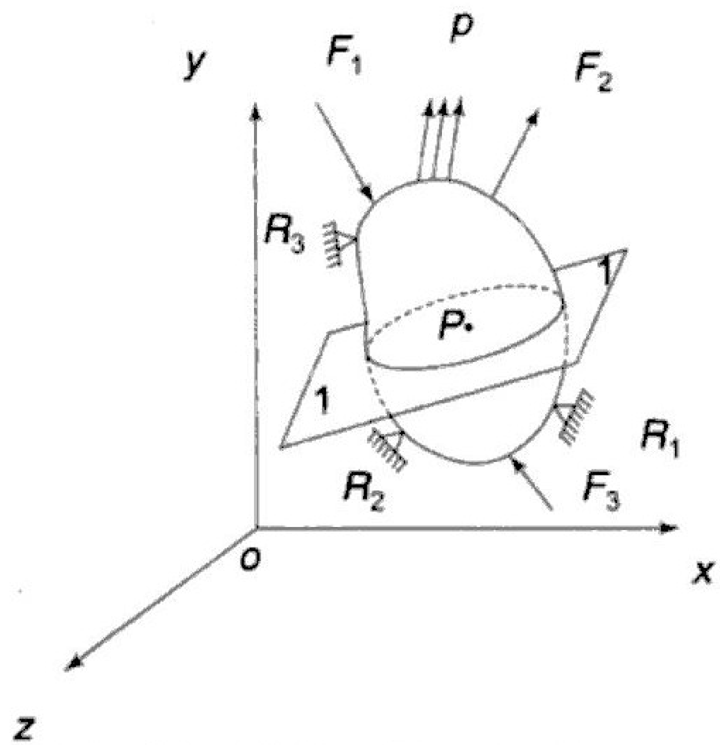
- Scalar: Tensor of rank 0 (only magnitude - 1 comp)
- Vector: Tensor of rank 1 (magnitude with one direction – 3 comp – why?)
- Why stop at rank 1, we can go on 2, 3, 4 etc.
- Dyad: Tensor of rank 2
- Triad: Tensor of rank 3

Stress vector or surface traction



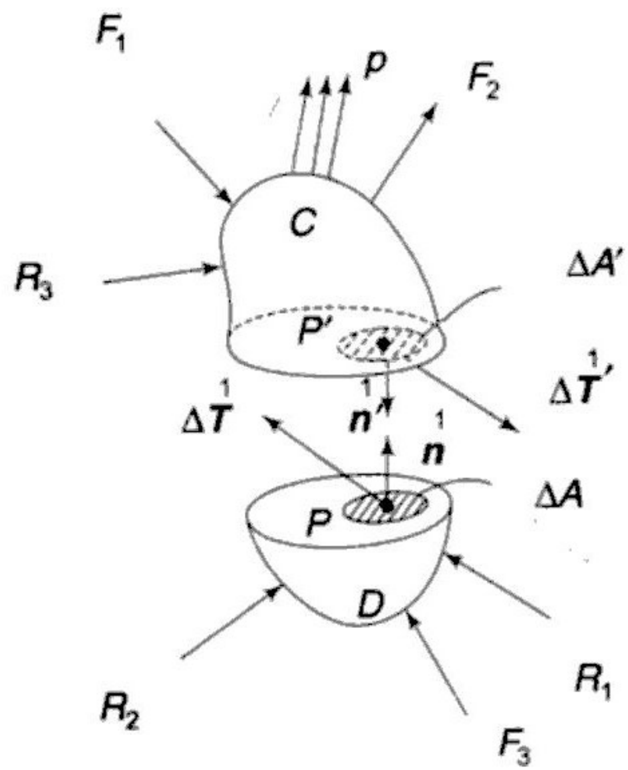
Concept of stress

- Stress (Traction) vector



Concept of stress vector

- Let us have an imaginary cut across the body



$$\lim_{\Delta A' \to 0} \frac{\Delta \overset{1}{\boldsymbol{T}}'}{\Delta A'} = \frac{d\overset{1}{\boldsymbol{T}}'}{dA'} = \overset{1}{\boldsymbol{T}}'$$

$$\lim_{\Delta A \to 0} \frac{\Delta \overset{1}{\boldsymbol{T}}}{\Delta A} = \frac{d\overset{1}{\boldsymbol{T}}}{dA} = \overset{1}{\boldsymbol{T}}$$

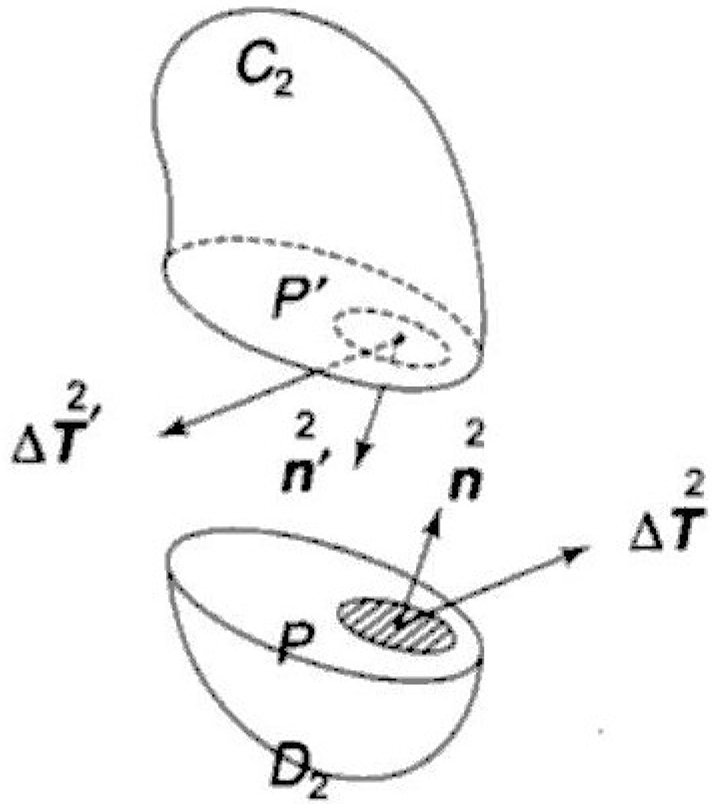
$$\boxed{\overset{1}{\boldsymbol{T}} = -\overset{1}{\boldsymbol{T}}'}$$

Concept of stress vector

- If body is cut by a different plane

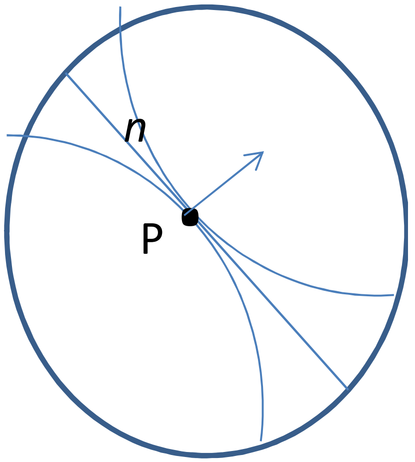
$$\overset{2}{\boldsymbol{T}} = \lim_{\Delta A \to 0} = \frac{\Delta \overset{2}{\boldsymbol{T}}'}{\Delta A}$$

- Too many (infinite) planes



Traction

- Traction vector for all three surfaces will be?
- There can be infinite planes associated with specific *n* which can pass through a point P
- There will be infinite associated traction vectors
- The totality of all these traction vector is defined as *state of stress* at P
- How to measure it???



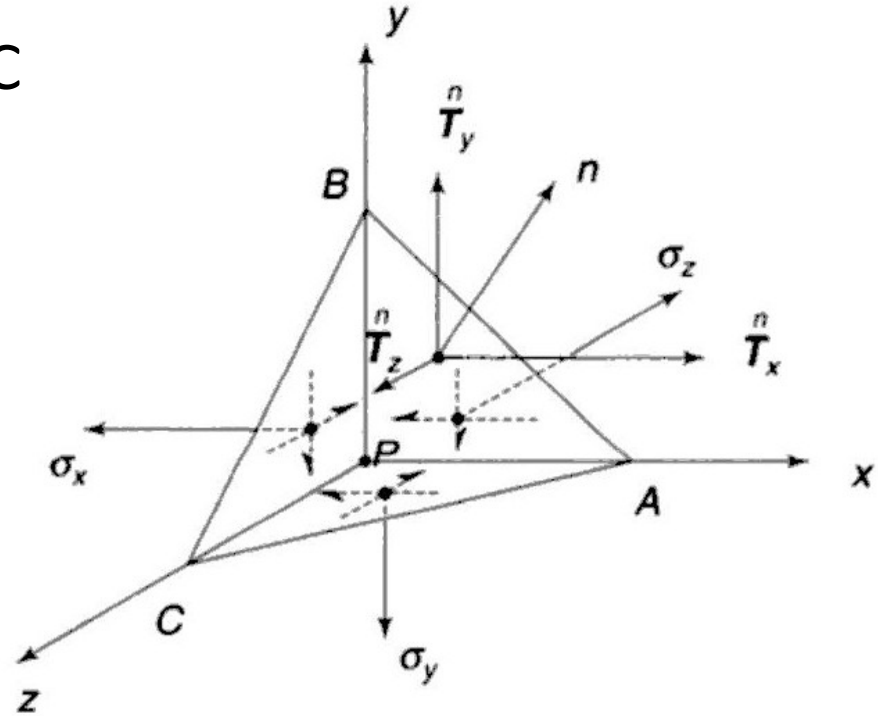
Stress on an arbitrary plane

- Traction vector on an oblique plane having arbitrary orientation
- Direction cosines of plane ABC

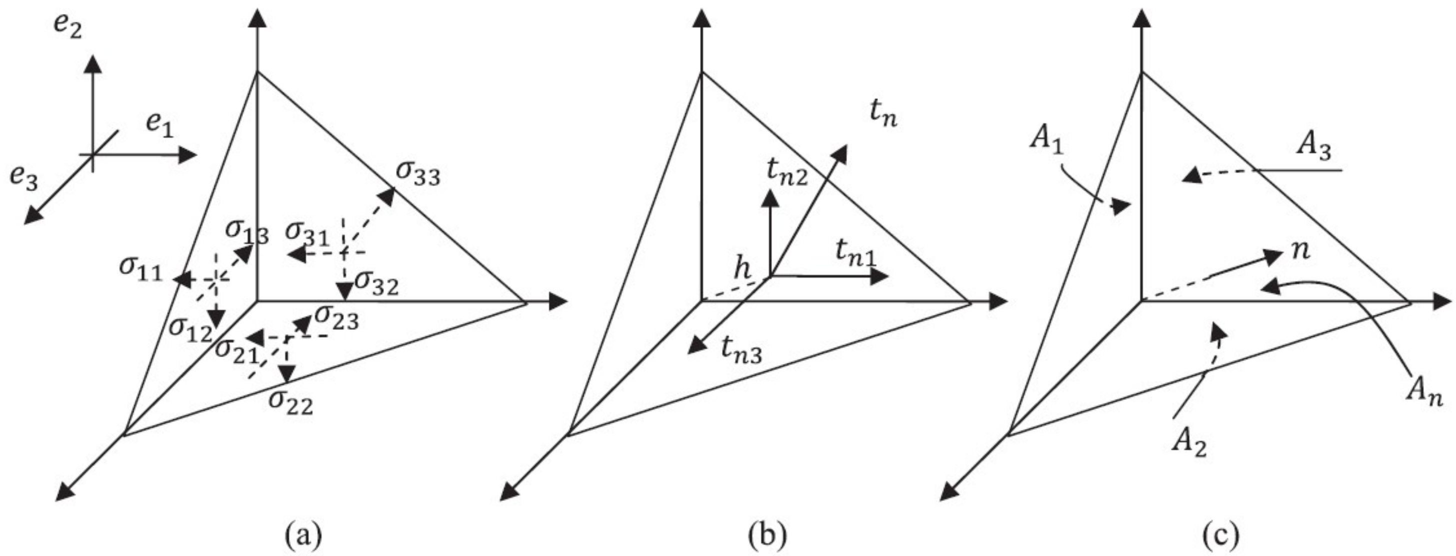
$$\boldsymbol{n} = n_x \boldsymbol{e}_1 + n_y \boldsymbol{e}_2 + n_z \boldsymbol{e}_3$$

- Consider a tetrahedra at P
- Three of its faces are normal
- Let h is the dist from P to ABC
- If we take out the tetrahedra

  The FBD will look like fig



Stress on an arbitrary plane



A1 = nx.A
A2 = ny.A
A3 = nz.A

Stress on an arbitrary plane

- Resultant stress on plane ABC

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \overset{n}{\boldsymbol{T}}_x^{\,2} + \overset{n}{\boldsymbol{T}}_y^{\,2} + \overset{n}{\boldsymbol{T}}_z^{\,2}$$

$$\left|\overset{n}{\boldsymbol{T}}\right|^2 = \sigma_n^2 + \tau_n^2$$

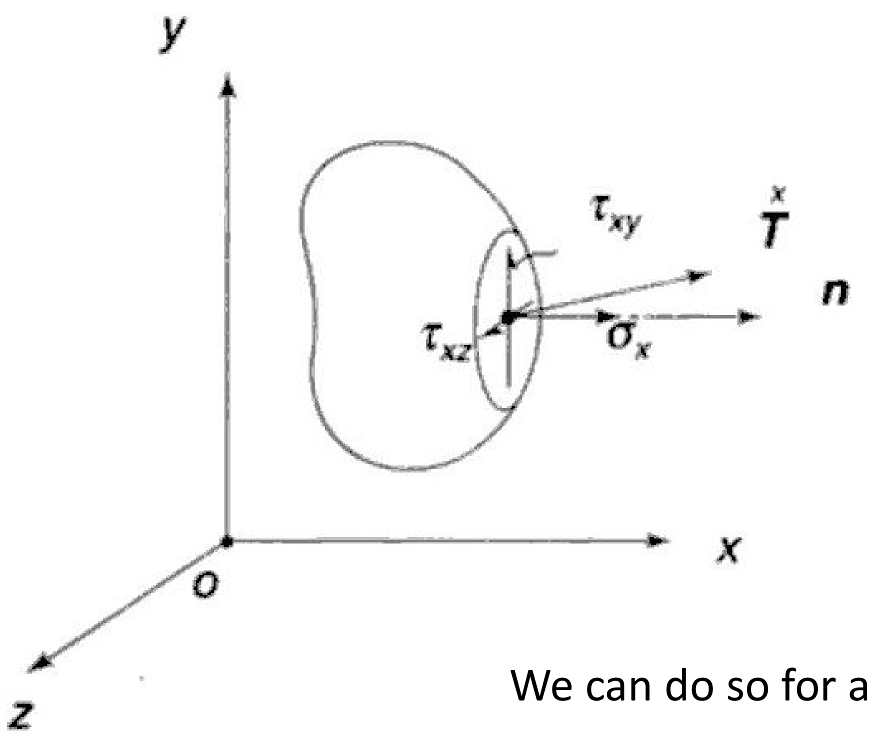
- Normal stress can be obtained as

$$\sigma_n = n_x \overset{n}{\boldsymbol{T}}_x + n_y \overset{n}{\boldsymbol{T}}_y + n_z \overset{n}{\boldsymbol{T}}_z$$

$$= n_x^2 \sigma_x + n_y^2 \sigma_y + n_z^2 \sigma_z + 2 n_x n_y \, \tau_{xy} + 2 n_y n_z \, \tau_{yz} + 2 n_z n_x \, \tau_{zx}$$

Concept of stress

- We can choose rectangular Cartesian axes to define our problem and the component we get will be called as rectangular components



The component of stress vector along x-axis and plan x:

$$\sigma_x \quad \tau_{xy} \text{ and } \tau_{xz}$$

We can do so for all three axes.

Concept of stress

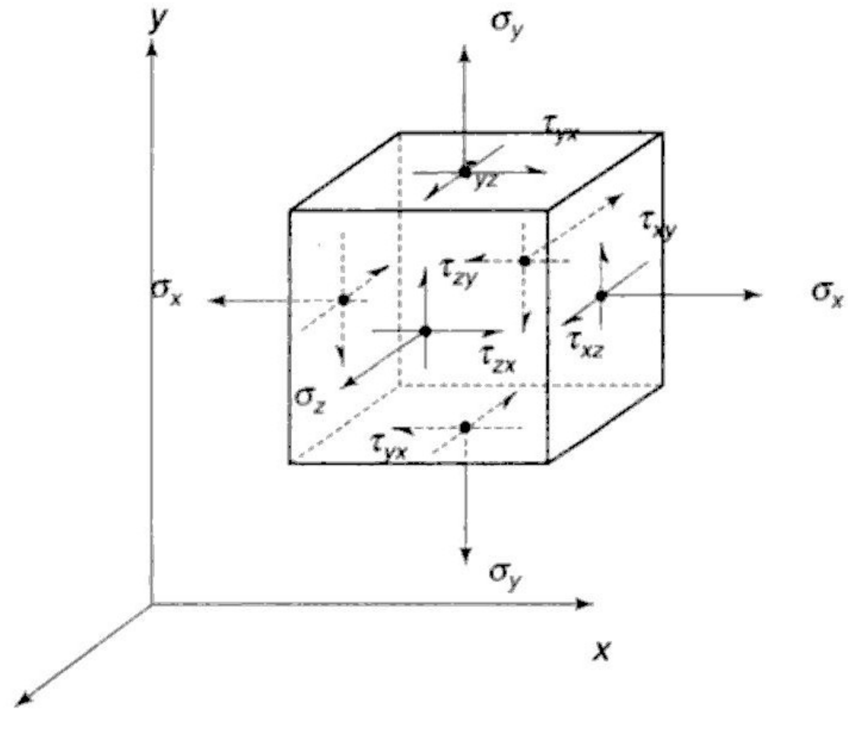
- All components

$\sigma_x,\ \tau_{xy},\ \tau_{xz}$ on x plane

$\sigma_y,\ \tau_{yx},\ \tau_{yz}$ on y plane

$\sigma_z,\ \tau_{zx},\ \tau_{zy}$ on z plane

$$\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_1) = \sigma_x \boldsymbol{e}_1 + \tau_{xy} \boldsymbol{e}_2 + \tau_{xz} \boldsymbol{e}_3$$
$$\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_2) = \tau_{yx} \boldsymbol{e}_1 + \sigma_y \boldsymbol{e}_2 + \tau_{yz} \boldsymbol{e}_3$$
$$\boldsymbol{T}^n(\boldsymbol{x}, \boldsymbol{n} = \boldsymbol{e}_3) = \tau_{zx} \boldsymbol{e}_1 + \tau_{zy} \boldsymbol{e}_2 + \sigma_z \boldsymbol{e}_3$$



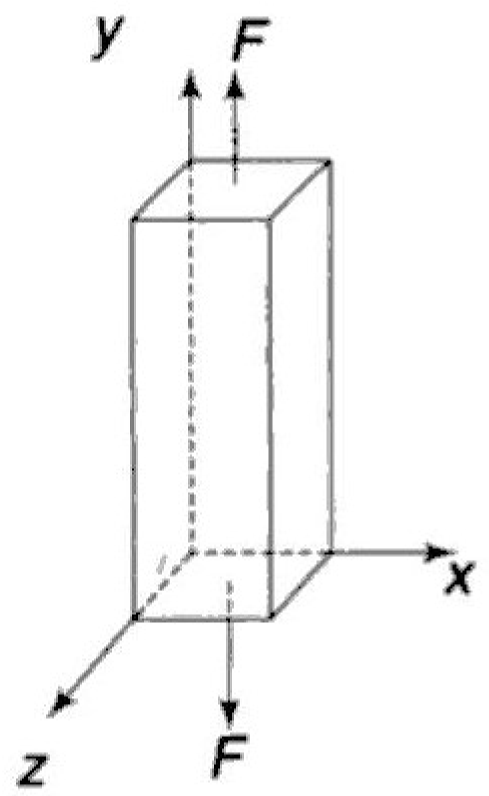
- Components of stress tensor
- Meaning of 2 subscripts

1- orientation of area
2 – direction of action

Example

- Find the normal and shear stresses on
  1. Plane with normals

$$\boxed{n_x = n_y = \frac{1}{\sqrt{2}}, n_z = 0}$$



cross-section 2 cm × 3 cm

Average stress on the plane

$$\sigma_{y'} = 1000 \text{ N/cm}^2$$

$$\overset{n}{T}_x = 0, \qquad \overset{n}{T}_y = \frac{1000}{\sqrt{2}}, \qquad \overset{n}{T}_z = 0$$

$$\sigma_n = \frac{1000}{2} = 500 \text{ N/cm}^2$$

$$\tau_n = 500 \text{ N/cm}^2$$

Example

- Given that

$$\sigma_x = 10{,}000\ N/cm^2 \quad \sigma_y = -5{,}000\ N/cm^2$$

$$\sigma_z = -5{,}000\ N/cm^2,\ \tau_{xy} = \tau_{yz} = \tau_{zx} = 10{,}000\ N/cm^2.$$

- Determine normal and shear stress on plane which is equally inclined to all three axes.

- Direction cosines of area vector??

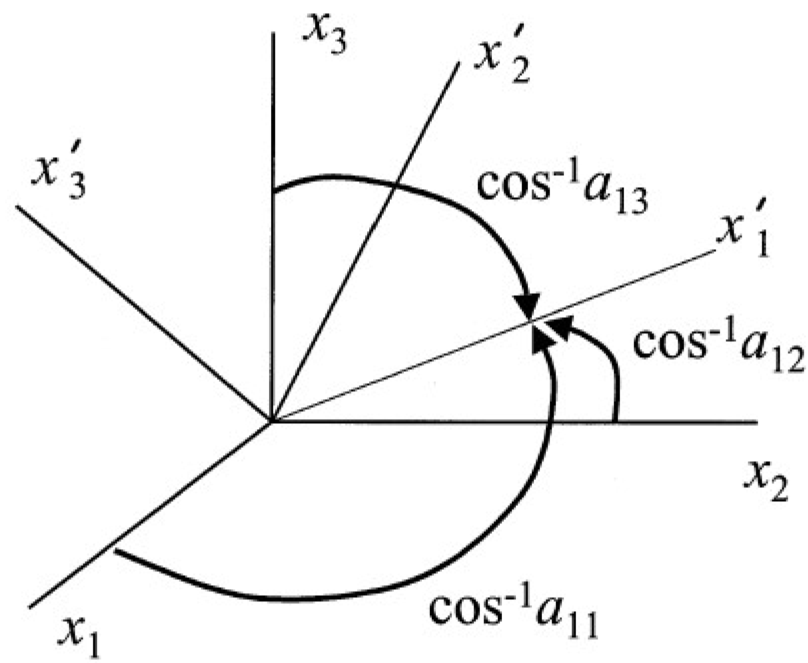
Co-ordinate transformation

- Transformation matrix

$$\hat{\mathbf{e}}'_i = a_{ij}\hat{\mathbf{e}}_j$$

$$\hat{\mathbf{e}}_i = a_{ji}\hat{\mathbf{e}}'_j$$

$$a_{iq}a_{jq} = \delta_{ij}$$



Direction cosines

		$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
		x_1	x_2	x_3
$\hat{\mathbf{e}}'_1$	x'_1	a_{11}	a_{12}	a_{13}
$\hat{\mathbf{e}}'_2$	x'_2	a_{21}	a_{22}	a_{23}
$\hat{\mathbf{e}}'_3$	x'_3	a_{31}	a_{32}	a_{33}

$$\begin{bmatrix}\hat{\mathbf{e}}'_1\\ \hat{\mathbf{e}}'_2\\ \hat{\mathbf{e}}'_3\end{bmatrix} = \underbrace{\boxed{\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}}}_{\mathbf{A}}\begin{bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\\ \hat{\mathbf{e}}_3\end{bmatrix}$$

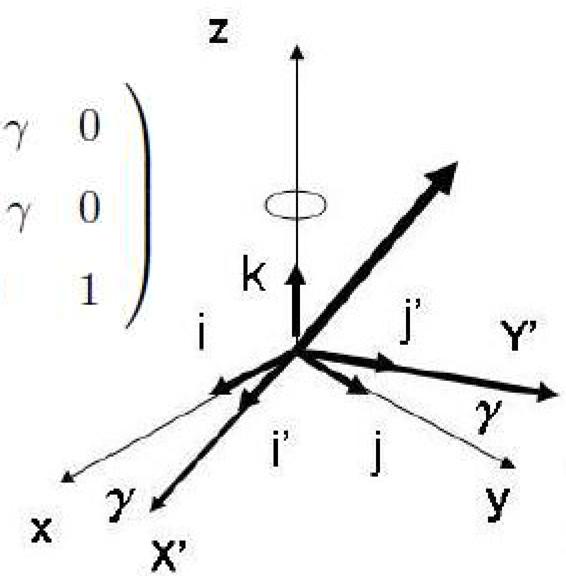
Co-ordinate transformation

det (**A**) = 1 proper orthogonal transformation.

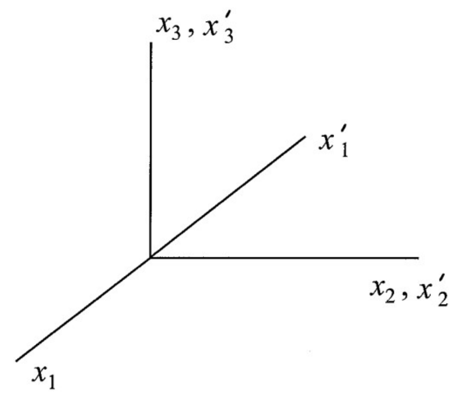
det (**A**) = -1 improper orthogonal transformation.

- Rotation about *Z*

$$\begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



- Reflection about *X*



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation rules

- Vector

$$\mathbf{v} = v'_j \hat{\mathbf{e}}'_j = v_i \hat{\mathbf{e}}_i = v_i a_{ji} \hat{\mathbf{e}}'_j$$

$$v'_j = a_{ji} v_i \quad \text{or} \quad \mathbf{v}' = \mathbf{A}\mathbf{v} = \mathbf{v}\mathbf{A}^{\mathrm{T}}$$

$$v_k = a_{jk} v'_j \quad \text{or} \quad \mathbf{v} = \mathbf{v}'\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{v}'$$

- Tensor

$$T_{ij} = a_{qi} a_{mj} T'_{qm} \qquad \mathbf{T} = \mathbf{A}^{\mathrm{T}}\mathbf{T}'\mathbf{A}$$

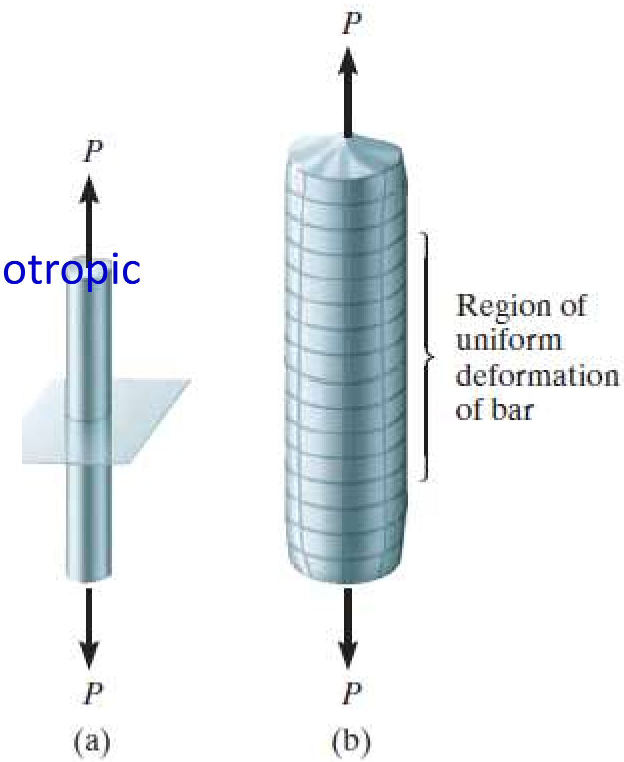
$$T'_{ij} = a_{iq} a_{jm} T_{qm} \qquad \mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^{\mathrm{T}}$$

$$C'_{ijkl} = R_{ip} R_{jq} R_{kr} R_{ls} C_{pqrs}$$

$$C_{ijkl} = R_{pi} R_{qj} R_{rk} R_{sl} C'_{pqrs}$$

Some special/simple cases

- Axial Loading
  - axially loaded **prismatic** bar
  - load P is applied to the bar through the centroid of its cross-sectional area
  - Arbitrary plane is normal to loading
  - the bar will deform uniformly
    - material of the bar is both homogeneous and isotropic
    - Continuous medium



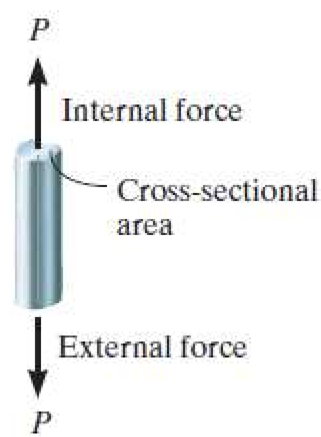
Average normal stress

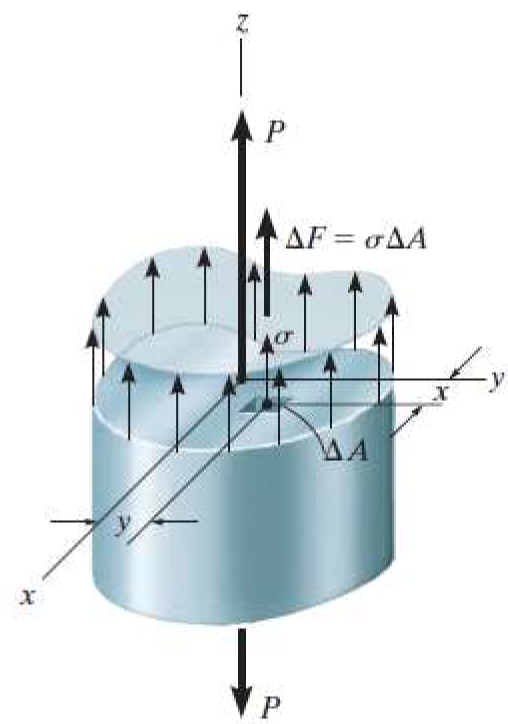
- Pass a section through the bar, and separate it into two parts

- The force intensity on that section is defined as the normal stress.

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \qquad \sigma_{ave} = \frac{P}{A}$$

Relation between traction and stress!!



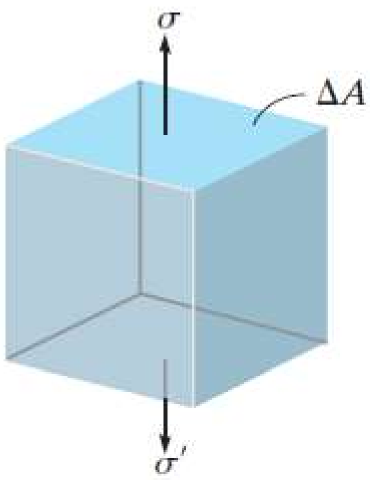


Equilibrium

- For equilibrium

$$\sigma(\Delta A) - \sigma'(\Delta A) = 0$$

$$\boxed{\sigma = \sigma'}$$



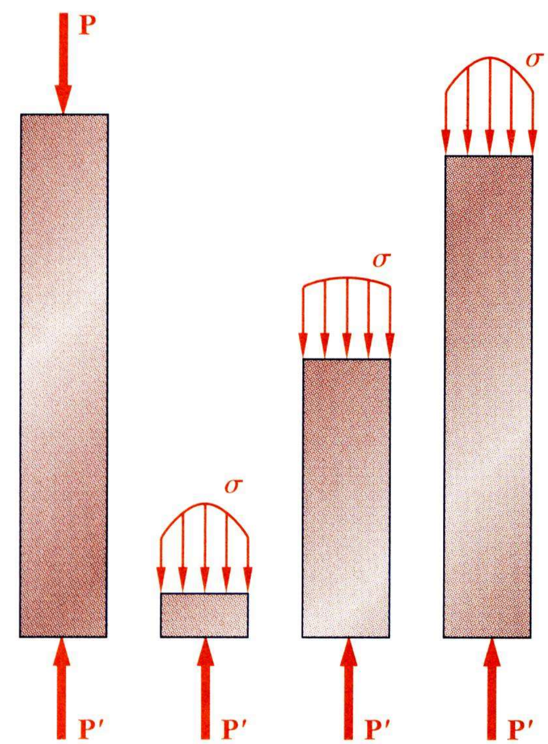
- What does it imply for an uni-axial tensile test??

Average normal stress distribution

- The normal stress at a particular point may not be equal to the average stress but the resultant of the stress distribution must satisfy

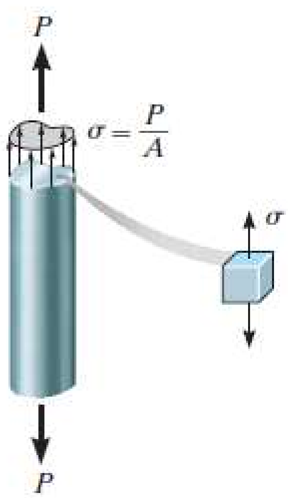
$$P = \sigma_{ave} A = \int dF = \int_A \sigma \, dA$$

- The detailed distribution of stress is statically indeterminate, i.e., can not be found from statics alone.

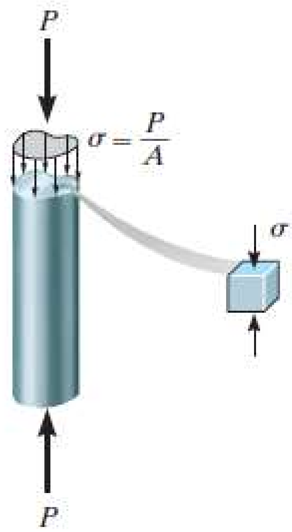


Tension and compression

- The analysis performed is valid for compression test as well.



Tension



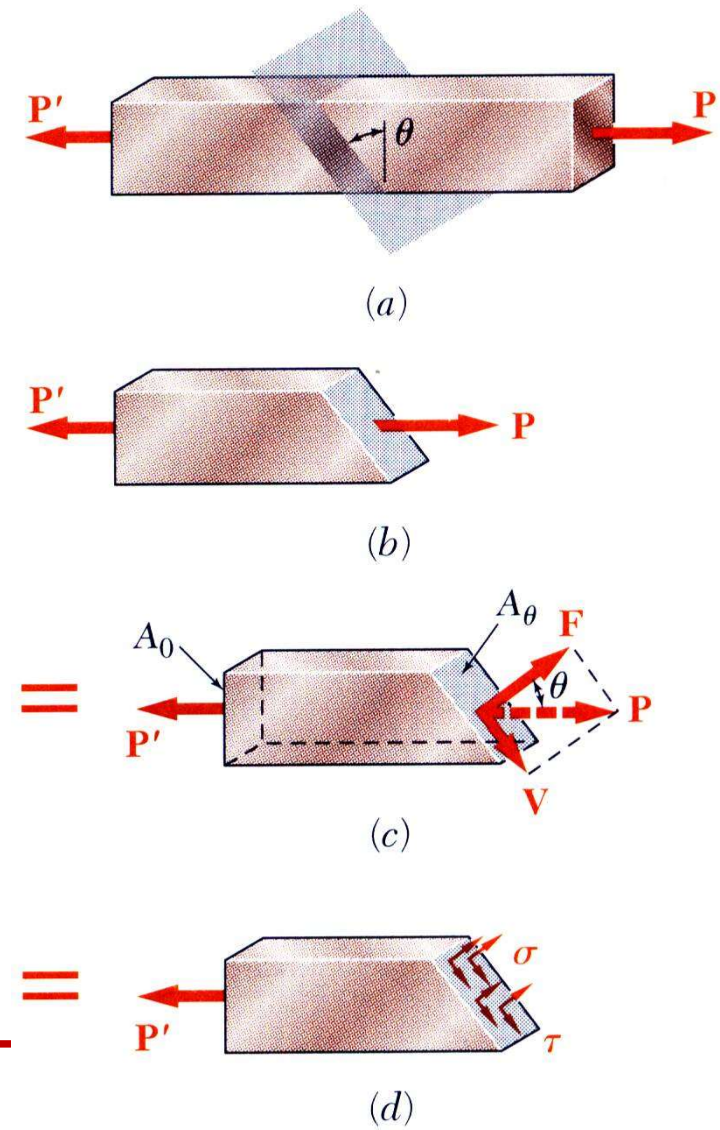
Compression

- Maximum average normal stress
  - Due to complex loading or non-prismatic section
  - P/A maximum

Stress on oblique plane

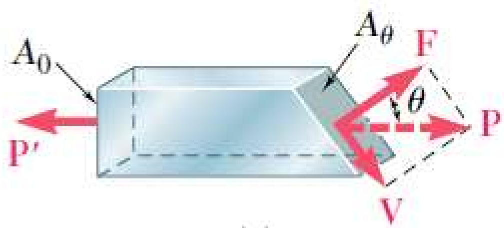
- Pass a section through the member forming an angle θ with the normal plane
- From equilibrium conditions, the distributed forces (stresses) on the plane must be equivalent to the force *P*
- Resolve *P* into components normal and tangential to the oblique section,

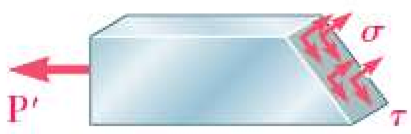
$$F = P\cos\theta \qquad V = P\sin\theta$$



Average stress

- The average normal and shear stresses on the oblique plane are





$$\sigma = \frac{F}{A_\theta} = \frac{P\cos\theta}{A_0 / \cos\theta} = \frac{P}{A_0}\cos^2\theta$$

$$\tau = \frac{V}{A_\theta} = \frac{P\sin\theta}{A_0 / \cos\theta} = \frac{P}{A_0}\sin\theta\cos\theta$$

Using generalized method

$$\overset{n}{T}_x = n_x\,\sigma_x + n_y\,\tau_{yx} + n_z\,\tau_{zx}$$
$$\overset{n}{T}_y = n_x\,\tau_{xy} + n_y\,\sigma_y + n_z\,\tau_{zy}$$
$$\overset{n}{T}_z = n_x\,\tau_{xz} + n_y\,\tau_{yz} + n_z\,\sigma_z$$

Direction cosines – cosθ, sinθ, 0

Only non zero stress = P/A = σ_{xx}

$$\boxed{\begin{aligned} \left|\overset{n}{T}\right|^2 &= \overset{n}{T}_x^{\,2} + \overset{n}{T}_y^{\,2} + \overset{n}{T}_z^{\,2} \\ \left|\overset{n}{T}\right|^2 &= \sigma_n^2 + \tau_n^2 \end{aligned}}$$

Cross shears

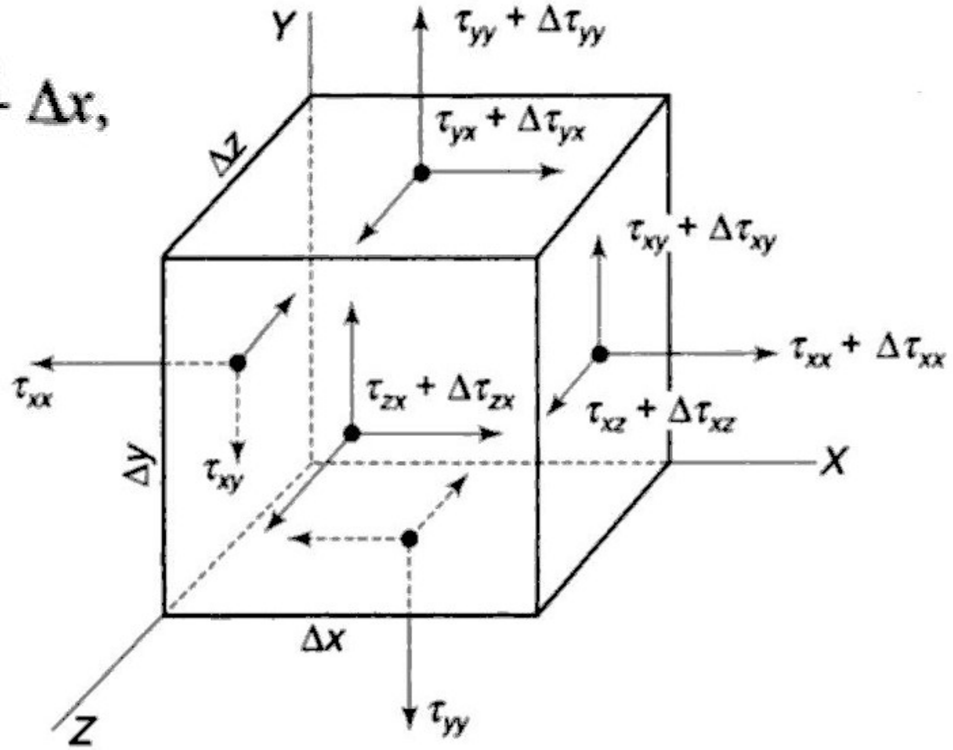
- It can be proved that cross shears are equal
  - We can replace τ with σ for notation consistency

$$\Delta\tau_{xx} = \frac{\partial\tau_{xx}}{\partial x}\Delta x, \qquad \Delta\tau_{xy} = \frac{\partial\tau_{xy}}{\partial x}\Delta x,$$

$$\Delta\tau_{xz} = \frac{\partial\tau_{xz}}{\partial x}\Delta x$$

Similarly we can find other increments

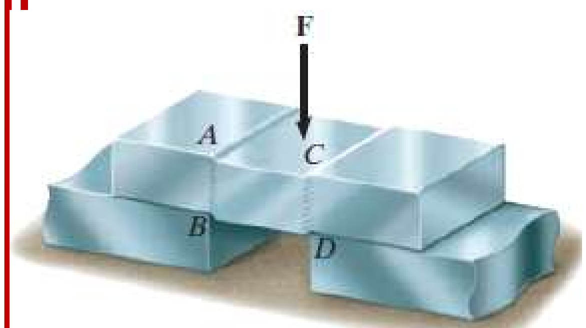
For equilibrium, the moments about x, y and z-axes must vanish.

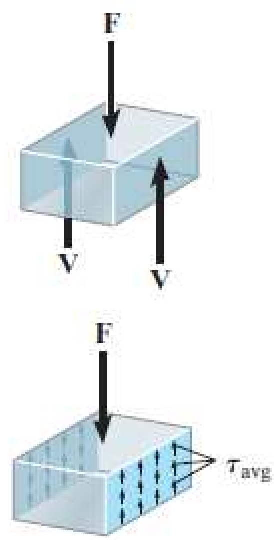


Taking moment about z-axis

Shearing Stress

- Stress component that acts in the plane of sectioned area
- The area vector of arbitrary plane is perpendicular to applied loading





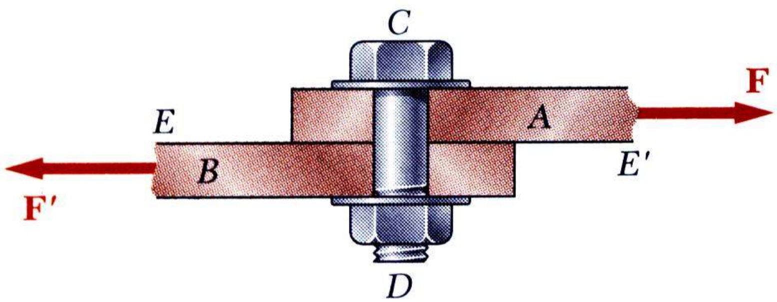
$$\tau_{avg} = \frac{V}{A}$$

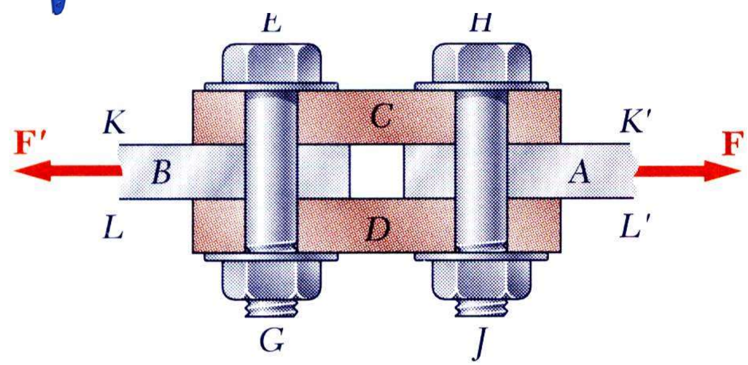
Check the resisting area.

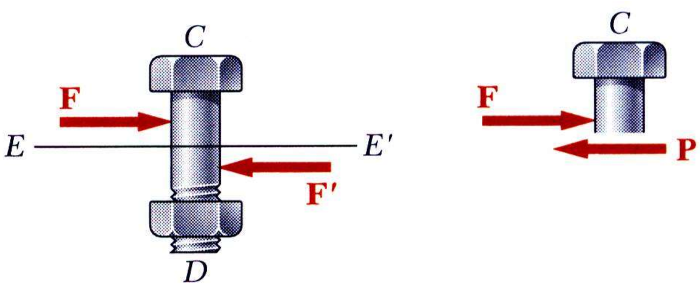
Shearing stress example

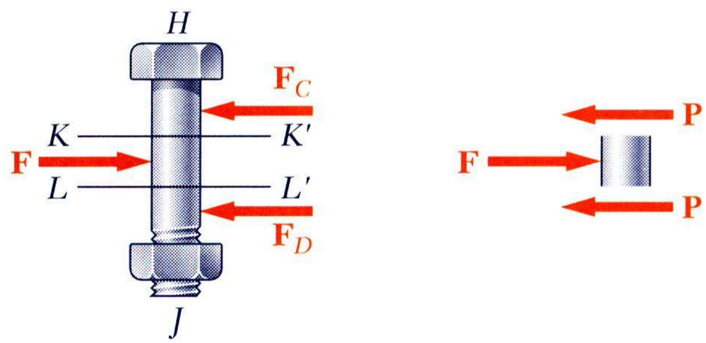
Double Shear

Single Shear





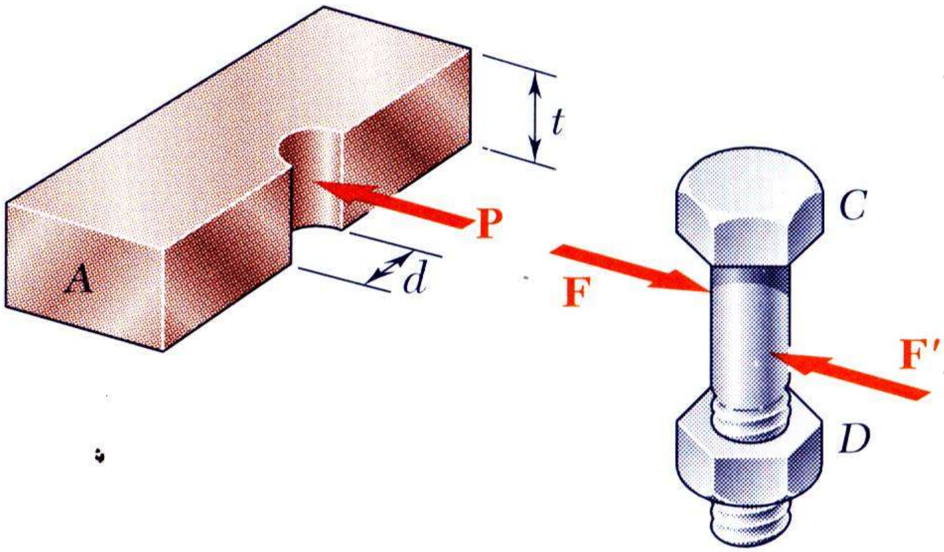


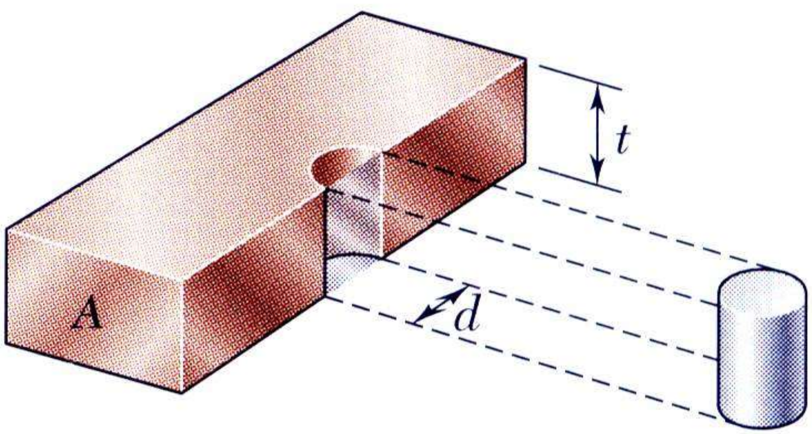


$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A}$$

$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{2A}$$

Bearing Stress in Connections





- Bolts, rivets, and pins create stresses on the points of contact or bearing surfaces of the members they connect
- The resultant of the force distribution on the surface is equal and opposite to the force exerted on the pin.
- Corresponding average force intensity is called the bearing stress,

$$\sigma_b = \frac{P}{A} = \frac{P}{t\,d}$$

Check the resisting area.

Allowable Stress

- How to ensure the safety of a structural member or machine element??
- The stress being developed due to applied loading should be well below the stress a mechanical element can bear
- In other words the design stress must be lower than the critical stress
- Or the design load must be smaller than the failure load
- One method of specifying the allowable load for a member is to use a number called the factor of safety. The ***factor of safety (F.S.)***

$$\boxed{\text{F.S.} = \frac{F_{\text{fail}}}{F_{\text{allow}}}} \qquad = \frac{\sigma_{\text{fail}}}{\sigma_{\text{allow}}} \qquad = \frac{\tau_{\text{fail}}}{\tau_{\text{allow}}}$$

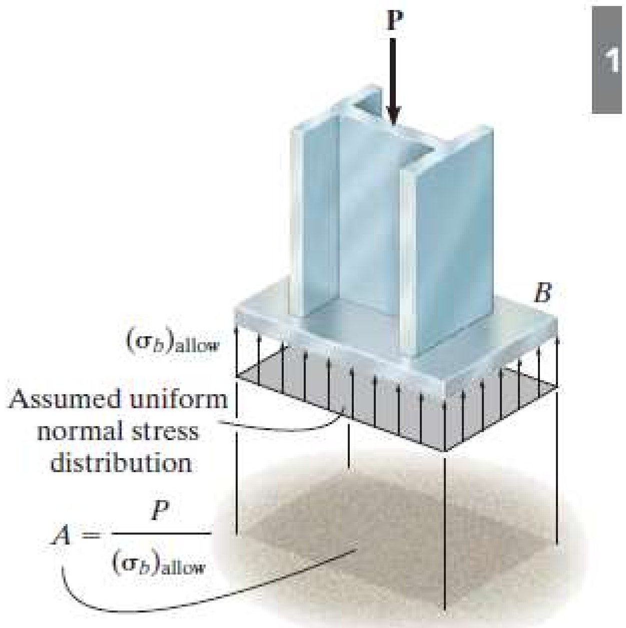
If the load applied to the member is *linearly related to the stress* developed within the member.

Allowable Stress

- The factor of safety must be *greater than 1* in order to avoid the potential for failure
  - depend on the types of materials to be used and the intended purpose of the structure or machine
  - F.S. used in the design of aircraft or space vehicle components may be close to 1 in order to reduce the weight of the vehicle
  - In nuclear power plant, the factor of safety for some of its components may be as high as 3 due to uncertainties in loading or material behavior

Design of Simple Connections

- Required area at the section



The area of the column base plate B is determined from the allowable bearing stress for the concrete.

The embedded length l of this rod in concrete can be determined using the allowable shear stress of the bonding glue.

The area of the bolt for this lap joint is determined from the shear stress, which is largest between the plates.