Indian Institute of Technology Ropar
Department of Mathematics
MA102 - Linear Algebra and Integral Transforms and Special Functions
Second Semester of Academic Year 2023-24

Notation :

- Field $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.
- $N(T)$:=Null space of T and $R(T)$:=Range space of T.

1. Consider $P_3[x]$ be space of all polynomials of degree ≤ 3, over the field $\mathbb{R}$.
   Define $T(P(x)) = xP'(x) - P(x)$, for all $P(x) \in P_3[x]$.
   (a) Show that T is a linear transformation on $P_3[x]$.
   (b) Find $N(T)$ and $R(T)$.

2. Let $T : P(\mathbb{R}) \rightarrow P(\mathbb{R})$,(where $P(\mathbb{R})$ is space of all ploynomials over the field $\mathbb{R}$),be defined by $T(f(x)) = f'(x)$.Prove that
   (a) T is a linear map.
   (b) T is onto, but not one-to-one.

3. Let V and W be vector spaces over the field $\mathbb{F}$ with subspaces V_1 and W_1 , respectively. If $T : V \rightarrow W$ is a linear map, prove that $T(V_1)$ is a subspace of W and that $\{x \in V : T(x) \in W_1\}$is a subspace of V.

4. Let $T : \mathbb{R}^3(\mathbb{R}) \rightarrow \mathbb{R}(\mathbb{R})$ be a linear map. Describe geometrically the possibilities for the null space of T.

5. Let $T : \mathbb{R}^3(\mathbb{R}) \rightarrow \mathbb{R}^3(\mathbb{R})$ be the linear map that reflects a vector in the xy plane. Find the Linear map.

6. Let $P(\mathbb{R})$ be space of all polynomials over the field $\mathbb{R}$. Define $T : P(\mathbb{R}) \rightarrow P(\mathbb{R})$ by
   $$T(f)(x) = \int_0^x f(t)dt$$
   for all $f \in P(\mathbb{R})$.

   Prove that
   (a) T is a linear map .
   (b) T is one-to-one.
   (c) T is not onto.

7. Let $V = C(\mathbb{R})$, the vector space of continuous real-valued functions over the field $\mathbb{R}$. Define $T : V \rightarrow \mathbb{R}$ by
   $$T(f) = \int_{-1}^{1} f(t)dt$$
   for all $f \in V$.
   (a) Show that T is a linear map.
   (b) What can you say about injectivity of T ?

8. Give an example of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $N(T) = R(T)$.

9. Suppose V and W are finite-dimensional vector spaces over the field $\mathbb{F}$.
   (a) If $\dim(V) > \dim\,(W)$. Then, show that there is no injective linear map from V to W.
   (b)If $\dim(V) < \dim\,(W)$. Then, show that there is no surjective linear map from V to W.

10. Prove that there does not exist a linear map $T : \mathbb{R}^5(\mathbb{R}) \to \mathbb{R}^5(\mathbb{R})$ such that $R(T) = N(T)$.

11. Let $T : V \to W$ be a linear map (where V and W are vector spaces over the field $\mathbb{F}$) and $\{v_1, v_2, ..., v_n\}$ is a set of vectors in V such that $\{T(v_1), ..., T(v_n)\}$ is a linearly independent set in W. Prove that $\{v_1, v_2, ..., v_n\}$ is linearly independent set in V. Is the converse true? (If true prove it otherwise give a counterexample.)

12. Let V and W be vector spaces over the $\mathbb{F}$ and $T : V \to W$ be a linear map.
(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.
(b) Suppose $B = \{v_1, v_2, ..., v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(B) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W.

$* * * * * * *$ END $* * * * * * *$