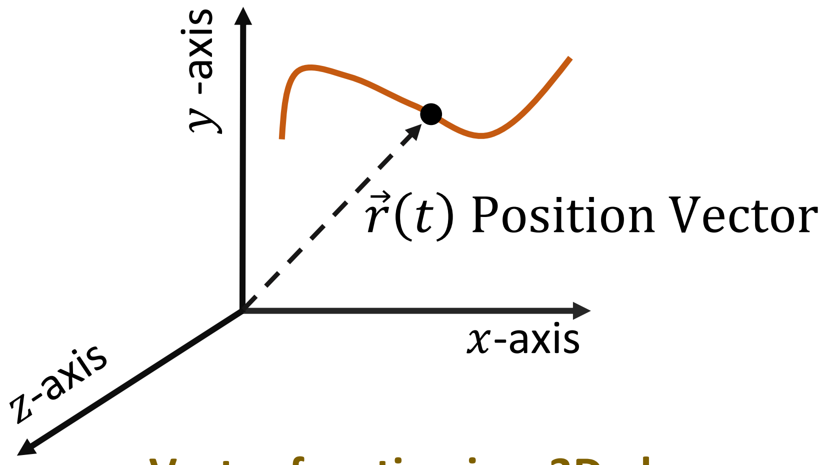
VECTOR CALCULUS

- Vector Functions
- Limit, Continuity and Differentiability
- Gradient of a Scalar Function

Vector Functions of One Variable - functions that map a real number to a vector

A vector function, say $\vec{r}(t)$, is written in the form

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \qquad a \le t \le b.$$



Vector function in a 3D plane

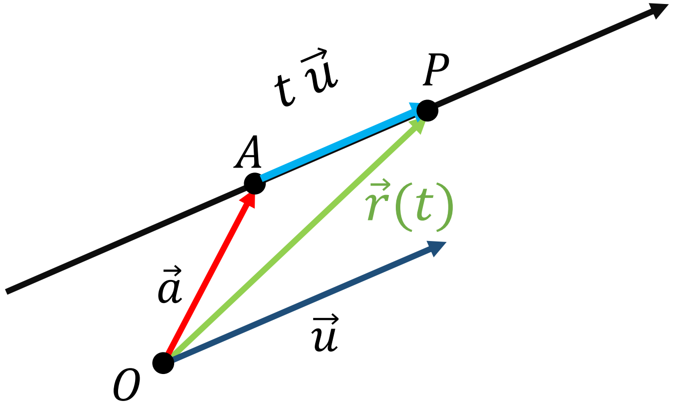
Here x, y and z are real-valued functions of the parameter t

and $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are unit vectors along x, y and z-axes, respectively.

In 2D plane, $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, \qquad a \le t \le b.$

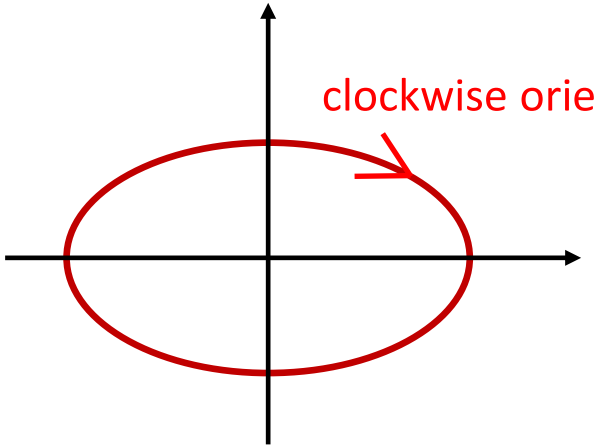
Vector Functions of one Variable

Example 1: Equation of a straight line passing through A with position vector $\vec{a}$ parallel to the vector $\vec{u}$



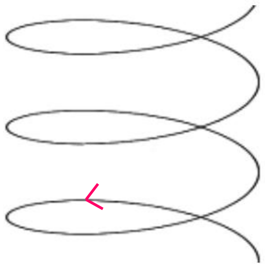
$$\vec{r}(t) = \vec{a} + t\,\vec{u}, \qquad t \in \mathbb{R}$$

Example 2: Consider $\vec{r}(t) = 3\cos t\;\hat{\imath} - 2\sin t\;\hat{\jmath},\ 0 \le t \le 2\pi$



Example 3: $\vec{r}(t) = 2\cos t\;\hat{\imath} + 2\sin t\;\hat{\jmath} + t\,\hat{k},\ 0 \le t \le 2\pi$

helix



Limit and Continuity of Vector Functions

- **Limit :** $\lim_{t\to a} \vec{r}(t) = \left[\lim_{t\to a} x(t)\right]\hat{\imath} + \left[\lim_{t\to a} y(t)\right]\hat{\jmath} + \left[\lim_{t\to a} z(t)\right]\hat{k}$

  provided $x(t)$, $y(t)$, and $z(t)$ have limits as $t \to a$.

- **Continuity :** A vector-valued function $\vec{r}(t)$ is continuous at $t = a$ if and only if each of its component functions is continuous at $t = a$

Example: Discuss continuity of $\vec{r}(t) = t\,\hat{\imath} + \hat{\jmath} + (2 - t^2)\hat{k}$

Since each component of $\vec{r}(t)$ is continuous for all $t \in \mathbb{R}$

The given vector function of one variable is continuous for all $t \in \mathbb{R}$

Example: Discuss continuity of $\vec{r}(t) = \dfrac{1}{t-2}\,\hat{\imath} + t\,\hat{\jmath} + \ln(t)\,\hat{k}$

The given vector is continuous for all $t > 0$ except $t = 2$

Differentiability of Vector Functions

- **Differentiability :** $\vec{r}(t)$ is said to be differentiable if

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}\,(t)}{\Delta t} \text{ exists.}$$

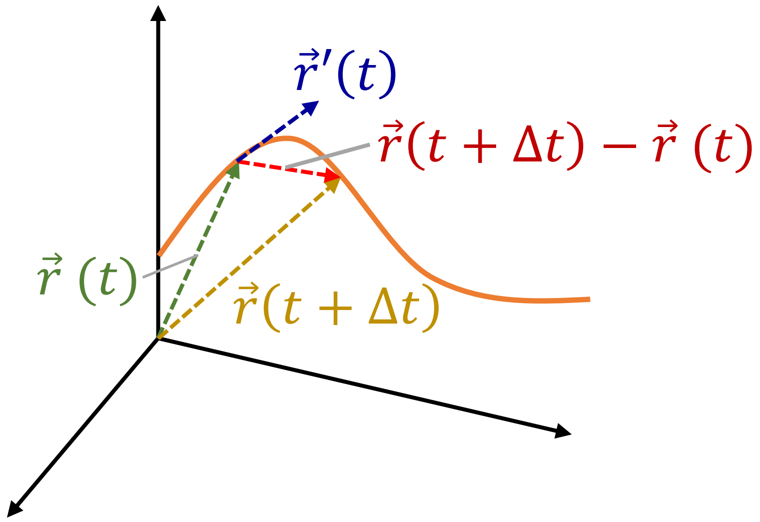
Similar to limit evaluation, differentiation of vector-valued functions can be done on a component-wise as

$$\frac{d\vec{r}(t)}{dt} = \frac{dx(t)}{dt}\,\hat{\imath} + \frac{dy(t)}{dt}\,\hat{\jmath} + \frac{dz(t)}{dt}\,\hat{k}$$

Geometrical Interpretation

$\vec{r}\,'(t)$ is a vector tangent to the curve given by $\vec{r}(t)$ and pointing in the direction of increasing values of t.

Unit tangent vector: $\vec{u} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$



Arc Length of a Curve

Let a curve be given by the vector function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k},\ a \le t \le b$

Recalls from integral calculus – Parametric equation of the curve $x = x(t), y = y(t), z = z(t)$:

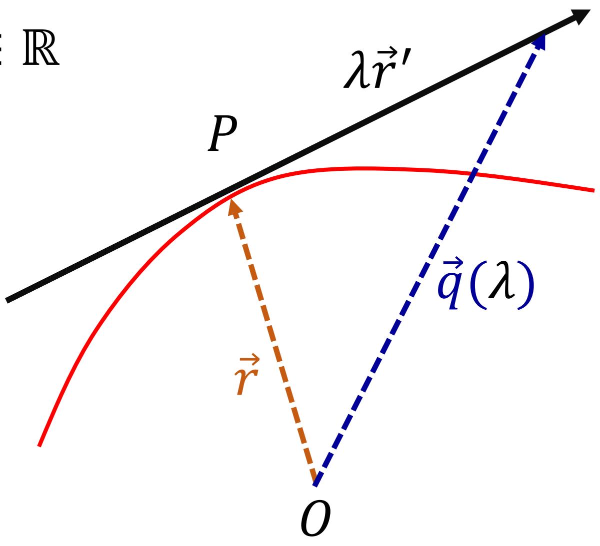
$$\text{Length} = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\ \mathrm{dt}$$

Note that $|\vec{r}'(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}$ (length of the tangent vector)

Length in terms of position vector $\vec{r}(t) = \int_a^b |\vec{r}'(t)|\ \mathrm{dt}$

Equation of a Tangent to a Curve C at Point P

$$\vec{q}(\lambda) = \vec{r} + \lambda \vec{r}', \qquad \lambda \in \mathbb{R}$$



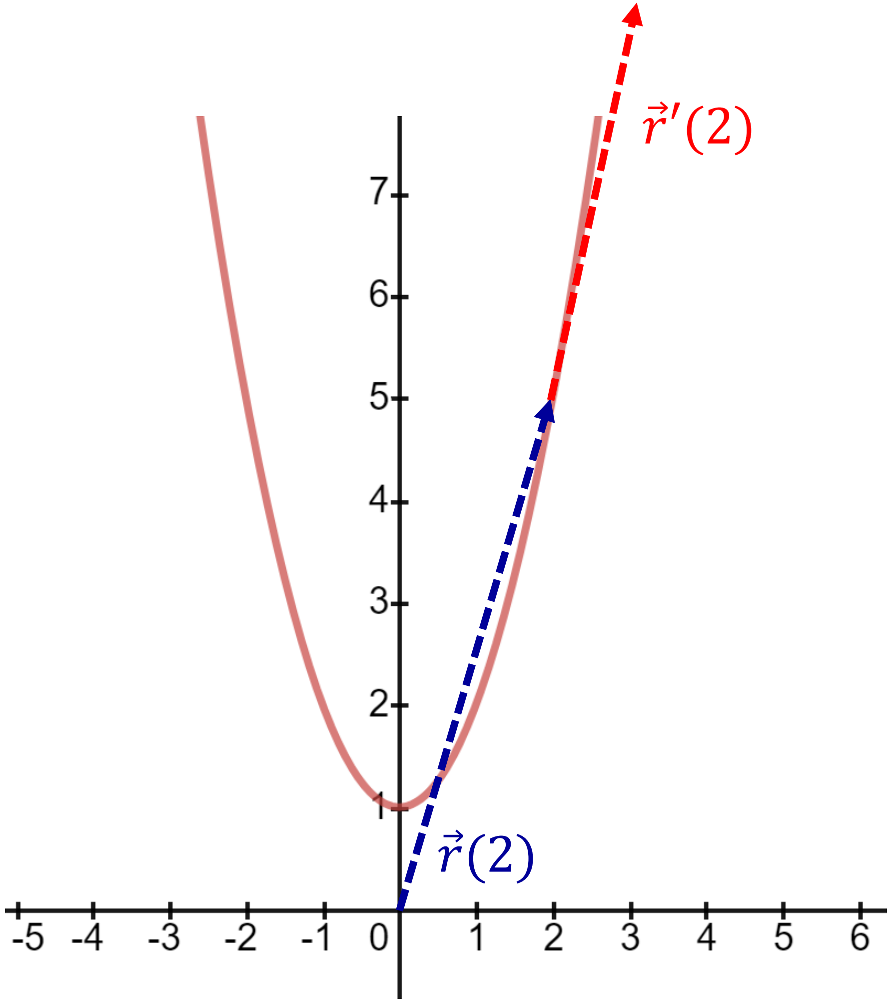
Example: Consider $\vec{r} = t\,\hat{\imath} + (t^2 + 1)\hat{\jmath}$

Tangent vector $\vec{r}' = \hat{\imath} + 2t\,\hat{\jmath}$

Equation of the tangent at $t = 2$:

$$\vec{q}(\lambda) = (2\hat{\imath} + 5\,\hat{\jmath}) + \lambda(\hat{\imath} + 4\,\hat{\jmath})$$

$$= (2 + \lambda)\hat{\imath} + (5 + 4\lambda)\hat{\jmath}$$



Gradient of a Scalar Function (Function of Several Variables)

Let $f(x, y, z)$ be a function of x, y, and z such that f_x, f_y and f_z exist.

The gradient of f, denoted by grad f, is the vector

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

Vector Function

Nabla or Del operator

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

$$\Longrightarrow \text{grad } f = \nabla f$$

Tangent Plane and Normal Line to a Surface

Let a surface S be given by $z = g(x, y)$. Define the function $f(x, y, z) = g(x, y) - z$.

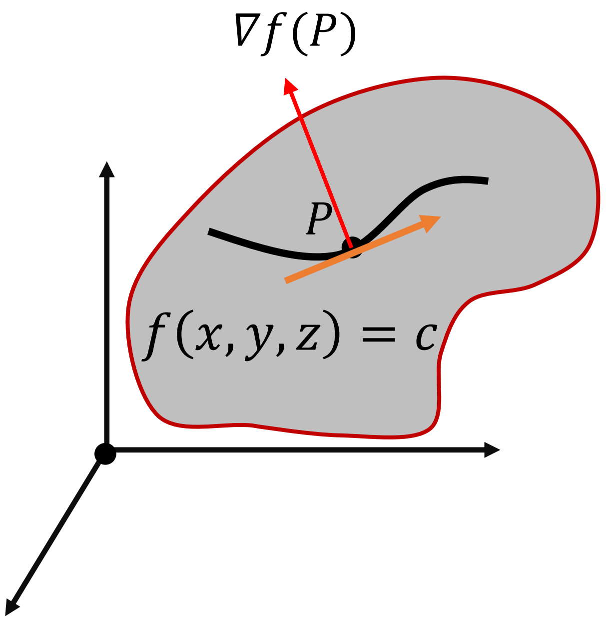
Then the given surface $z = g(x, y)$ can be treated as the level surface of $f(x, y, z)$ given by $f(x, y, z) = 0$.

Note that level surfaces of a function $f(x, y, z)$ are given by $f(x, y, z) = c$

Example: Let $f(x, y, z) = x^2 + y^2 + z^2$

The Level surfaces are concentric spheres centred at the origin.

Let $P(x_0, y_0, z_0)$ be a point on S and let C be a curve on S through P that is defined by the vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$



Since, the curve lies on the surface, we have $f\big(x(t), y(t), z(t)\big) = c, \ \forall t$

$$\Longrightarrow \ \frac{d}{dt} f\big(x(t), y(t), z(t)\big) = 0 \Longrightarrow f_x(x, y, z)\, x' + f_y(x, y, z)\, y' + f_z(x, y, z)\, z' = 0$$

At (x_0, y_0, z_0) we have $\nabla f(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$

$\Rightarrow$ The gradient at P is orthogonal to the tangent vector of every curve on S through P.

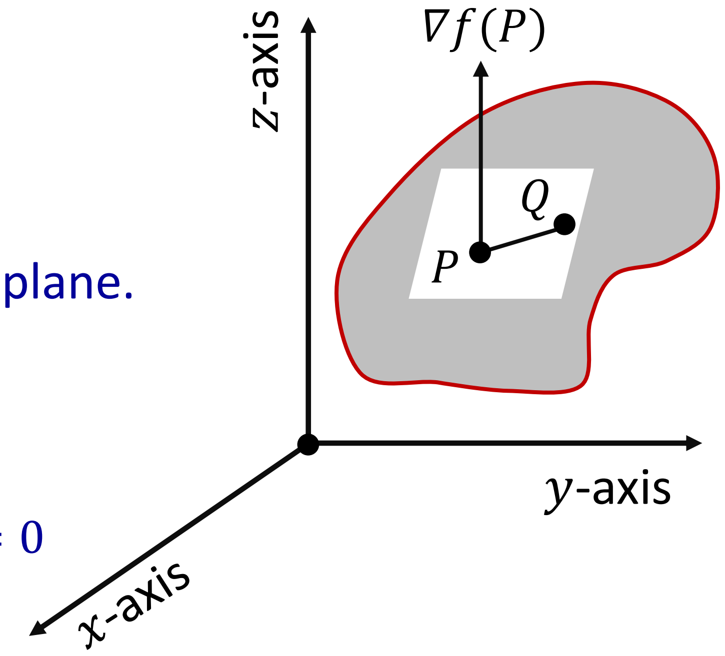
Unit normal vector to a surface $f(x, y, z) = c$: $\dfrac{\nabla f}{|\nabla f|}$

The plane through $P(x_0, y_0, z_0)$ that is normal to $\nabla f(x_0, y_0, z_0)$ is called the **tangent plane** to S at P

Let $Q(x, y, z)$ be an arbitrary point in the tangent plane.

Then the vector $(x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} + (z - z_0)\hat{k}$ lies in the tangent plane.

$$\Rightarrow \left((x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} + (z - z_0)\hat{k}\right) \cdot \left(f_x(P_0)\,\hat{\imath} + f_y(P_0)\,\hat{\jmath} + f_z(P_0)\,\hat{k}\right) = 0$$

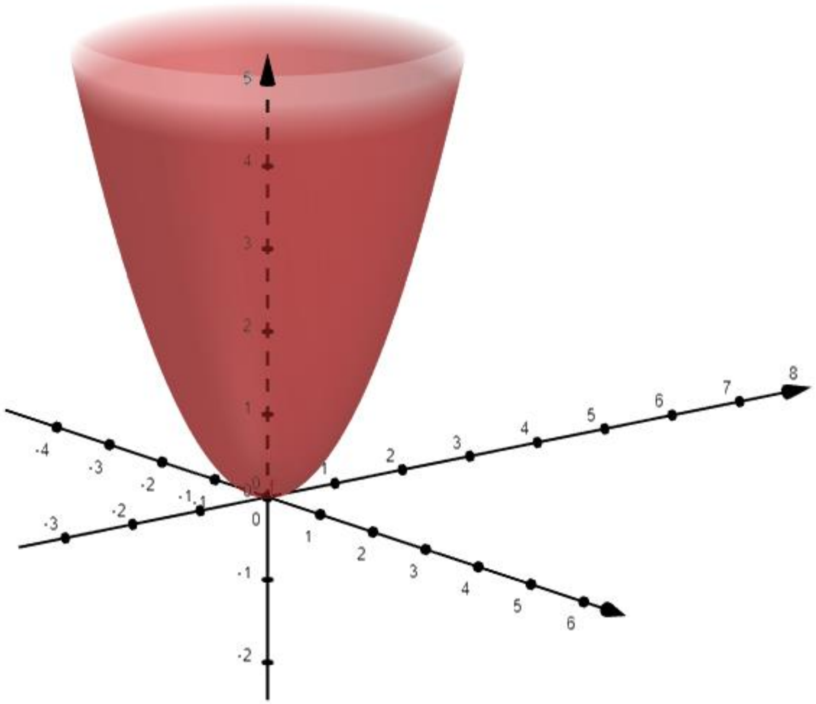


$$\boxed{(x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0) = 0}$$

Example : Find the unit normal to the surface $x^2 + y^2 - z = 0$ at the point $(1,1,2)$.

Define $f = x^2 + y^2 - z \quad \Rightarrow \nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} - \hat{k}$

$$\nabla f(1,1,2) = 2\,\hat{\imath} + 2\,\hat{\jmath} - \hat{k}$$



Unit normal vector $\hat{n} = \dfrac{1}{\sqrt{4+4+1}}(2\,\hat{\imath} + 2\,\hat{\jmath} - \hat{k})$

$$= \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{1}{3}\hat{k}$$

The other unit normal vector is $-\hat{n} = -\dfrac{2}{3}\hat{\imath} - \dfrac{2}{3}\hat{\jmath} + \dfrac{1}{3}\hat{k}$

KEY TAKEAWAY

- Vector valued functions $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \qquad a \leq t \leq b.$
- $\vec{r}'(t)$ is a vector tangent to the curve given by $\vec{r}(t)$
- $\text{grad } f = \dfrac{\partial f}{\partial x}\hat{\imath} + \dfrac{\partial f}{\partial y}\hat{\jmath} + \dfrac{\partial f}{\partial z}\hat{k}$
- $\text{grad } f$ is the normal vector to a surface $f(x, y, z) = c$

- Vector and Scalar Fields
- Directional Derivatives

Vector Field   Function that maps a point in space/plane to a vector

A vector field over a solid region (or a plane) **R** is a function that assigns a vector $\vec{F}(x,y,z)$ $\left(\text{or } \vec{F}(x,y)\right)$ to each point in **R**: $\vec{F}(x,y,z) = f(x,y,z)\hat{\imath} + g(x,y,z)\hat{\jmath} + h(x,y,z)\hat{k}$

Example: Velocity of the air inside a room is defined by a vector field.

Example: Gradient of a function is an example of a vector field:

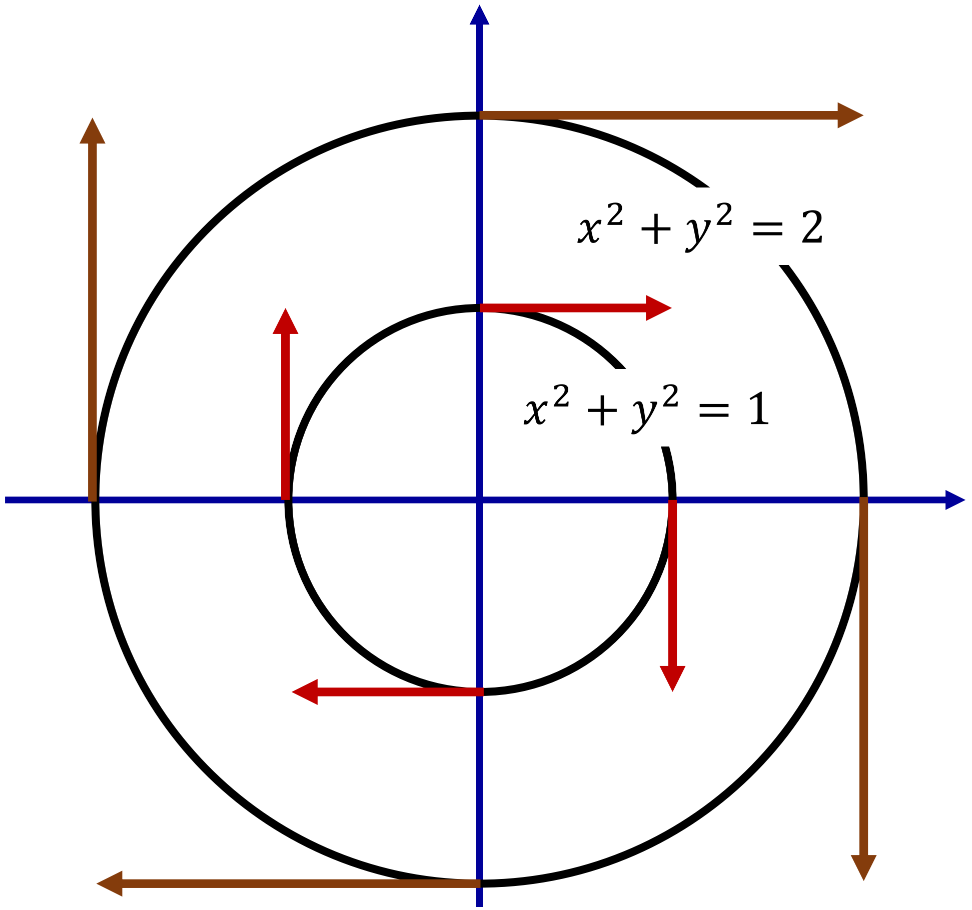
Suppose $f(x,y) = 3x^2y + 2\,xy^3$

$\text{grad } f = \nabla f = (6xy + 2y^3)\,\hat{\imath} + (3x^2 + 6\,xy^2)\,\hat{\jmath}$ **Vector Field (in the plane)**

Example: $\vec{F}(x,y) = y\hat{\imath} - x\hat{\jmath}$

Magnitude of $\vec{F}(x,y)$: $x^2 + y^2$ $\Rightarrow$ vectors of equal magnitude lie on circles $x^2 + y^2 = c$

(level curves)



$\vec{F}(1,0) = -\hat{\jmath}$

$\vec{F}(0,1) = \hat{\imath}$

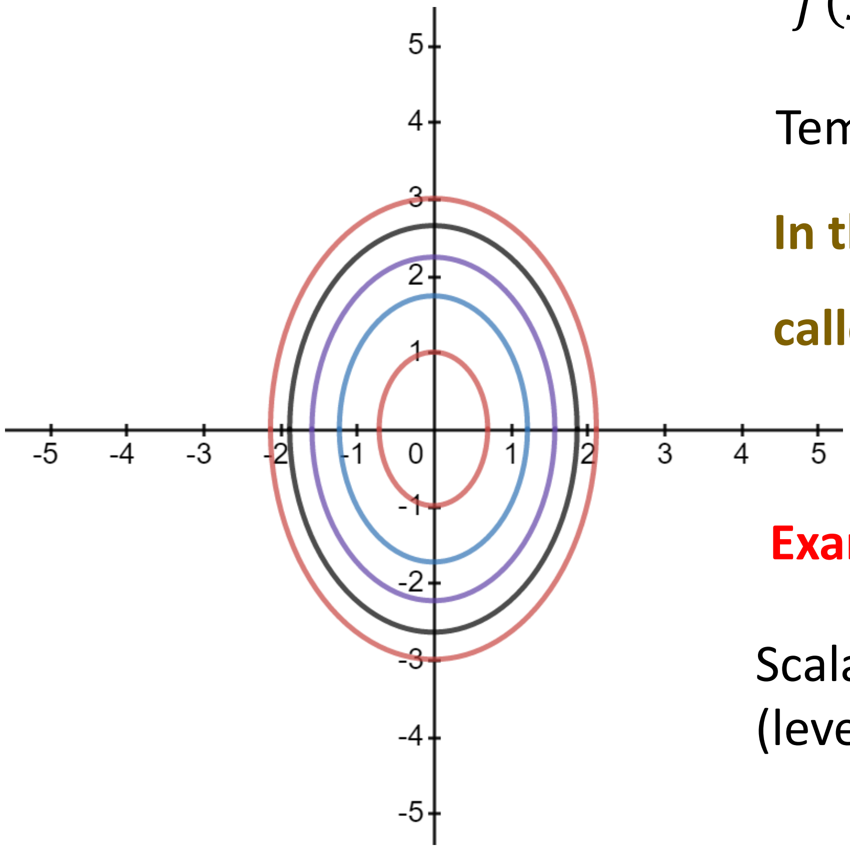
$\vec{F}(-1,0) = \hat{\jmath}$

$\vec{F}(0,-1) = -\hat{\imath}$

Scalar Field

Function that maps a point in space/plane to a scalar

A scalar field over a solid region (or a plane) **R** is a function that assigns a scalar to each point in **R**:



$$f(x,y,z) = 3x^2 + 2y^2 + z^2$$

Temperature inside a room is defined by a scalar field.

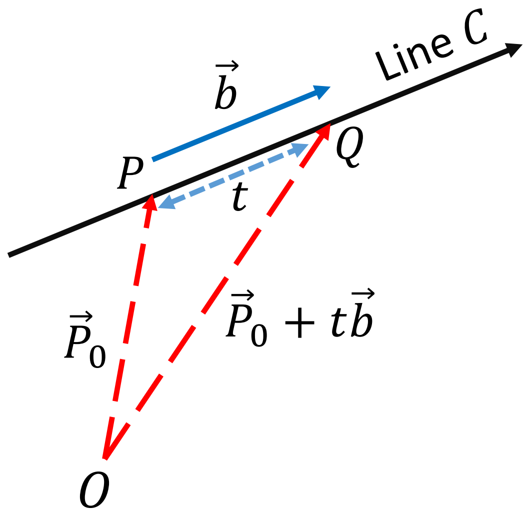
In the context of vectors, a real valued function of several variables is called a scalar field.

Example: Consider $F(x,y) = 2x^2 + y^2$

Scalar filed may be visualize using level curves of $F(x,y)$
(level surface in case of $F(x,y,z)$)

Directional Derivative of a Scalar Field $\boldsymbol{f(x,y,z)}$ at $\boldsymbol{P(x_0,y_0,z_0)}$ along a Vector $\vec{\boldsymbol{b}}$

Let $|\vec{b}| = 1$. Let C be the line passing through P and parallel to $\vec{b}$



Position vector of the line C is : $\vec{r}(t) = \vec{P}_0 + t\vec{b} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Rate of change of f in the direction $\vec{b}$ is given as

$$\lim_{t\to 0}\frac{f(Q)-f(P)}{t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right) = \nabla f\cdot\frac{d\vec{r}}{dt} = \nabla f(x_0,y_0,z_0)\cdot\vec{b}$$

At any point P, the directional derivative of f represents the rate of change in f along $\vec{b}$ at the point P, it is denoted by $D_b f = \nabla f\big|_P \cdot \vec{b}$

Example 1: Find the directional derivative of $f(x,y) = 4 - x^2 - \frac{1}{4}y^2$ at $(1,2)$ in the direction $\vec{u} = \hat{\imath} + \sqrt{3}\,\hat{\jmath}$

$\nabla f = -2x\,\hat{\imath} - \frac{1}{2}y\,\hat{\jmath} \;\Rightarrow\; \nabla f(1,2) = -2\,\hat{\imath} - \hat{\jmath}$ Gradient of f at $(1,2)$

$\vec{b} = \frac{1}{2}\,\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$ Unit vector in the direction of $\vec{u}$

$D_b f = (-2\,\hat{\imath} - \hat{\jmath}) \cdot \left(\frac{1}{2}\,\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}\right) = -1 - \frac{\sqrt{3}}{2}$ Directional Derivative

Example 2: Find the directional derivative of the scalar field $f = 2x + y + z^2$ in the direction of the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ and evaluate this at the origin.

$$\nabla f = 2\hat{\imath} + \hat{\jmath} + 2z\,\hat{k}$$

$$D_{(1,1,1)}f = \nabla f \cdot \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}} = \left(2\hat{\imath} + \hat{\jmath} + 2z\,\hat{k}\right) \cdot \left(\frac{\hat{\imath}}{\sqrt{3}} + \frac{\hat{\jmath}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}z$$

Value at the origin: $\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$

Maximum Rate of Change of a Scalar Field

Rate of change of f in the direction of a unit vector $\vec{b}$: $D_b f = \nabla f \cdot \vec{b} = |\nabla f||\vec{b}| \cos\theta = |\nabla f| \cos\theta$

$\Rightarrow$ Rate of change is maximum when θ is 0, i.e., in the direction of ∇f

$\Rightarrow$ Rate of change is minimum when θ is π, i.e., in the opposite direction of ∇f

$\Rightarrow$ Gradient vector ∇f points in the direction in which f increases most rapidly and

$-\nabla f$ points in the direction in which f decreases most rapidly.

Example: Let $f(x, y, z) = x^2 + y^2 - 2z$. Find the direction of maximum increase of f at $(2, 1, -1)$.

Gradient of f: $2x\,\hat{\imath} + 2y\,\hat{\jmath} - 2\,\hat{k}$

Direction of maximum increase at $(2, -1, 1)$: $4\,\hat{\imath} - 2\,\hat{\jmath} - 2\,\hat{k}$

Note: The above concept of maximum increase/decrease is very useful for optimization problems. Gradient ascent/descent approach is very popular for finding local maximum/minimum.

KEY TAKEAWAY

- Vector Field – Function that maps a point to a vector
- Scalar Field - Function that maps a point to a scalar
- Directional Derivative $D_b f = \nabla f|_P \cdot \vec{b}$

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- Vector Field – Function that maps a point to a vector

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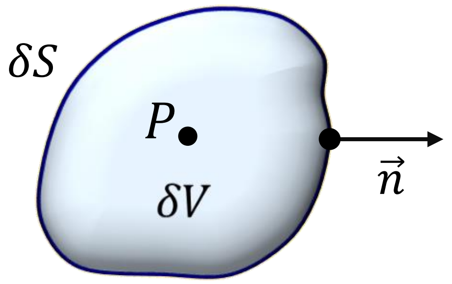
- **Divergence of a Vector Field**
- **Curl of a Vector Field**
- **Conservative Field**

Divergence of a Vector Field



Flux: Surface integral of the perpendicular component of a vector field over a surface

The divergence of a vector field $\vec{v}$ at a point P is defined as

$$\operatorname{div} \vec{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_{\delta S} \vec{v} \cdot \vec{n} \; d\sigma$$

$\vec{v} \cdot \vec{n}$: component of $\vec{v}$ in the direction of $\vec{n}$

Flux of the vector field $\vec{v}$ out of a small closed surface

div: Flux density (flux entering or leaving at a point)

where δV is a small volume enclosing P with surface δS and $\vec{n}$ is the outward pointing normal to δS.

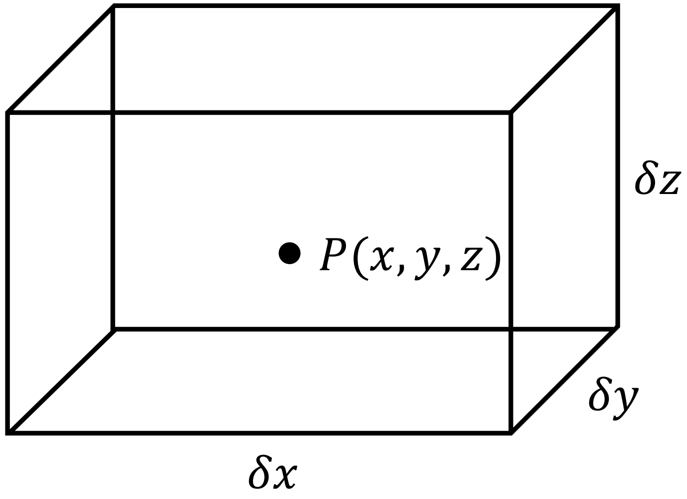
Computation of Divergence

The divergence of a vector field $\vec{v} = v_1 \hat{\imath} + v_2 \hat{\jmath} + v_3 \hat{k}$ is the scalar field given by

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Physical Interpretation of Divergence of a Vector Field

Suppose $\vec{v}(x, y, z)$ is the velocity of a fluid at a point $P(x, y, z)$.

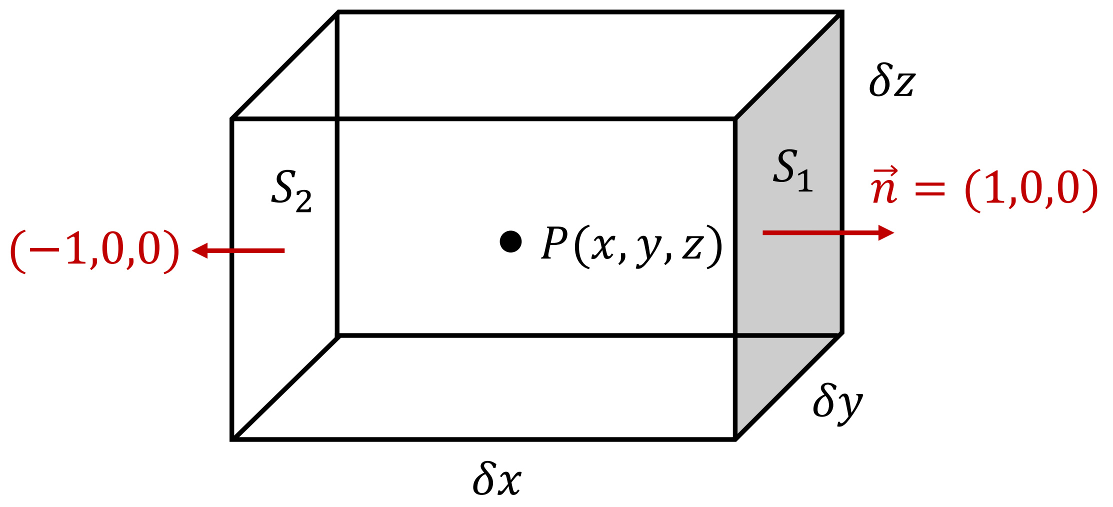


Measure the rate per unit volume at which fluid flows out of this box across its faces:

$$\operatorname{div} \vec{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_S \vec{v} \cdot \vec{n}\, d\sigma = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \frac{1}{\delta x\, \delta y\, \delta z} \left(\sum_{i=1}^{6} \iint_{S_i} \vec{v} \cdot \vec{n}\; d\sigma \right)$$

Flux outward across S_1:

$$\iint_{S_1} \vec{v}\cdot\vec{n}\,d\sigma \approx v_1\left(x+\frac{\delta x}{2},y,z\right)\delta y\,\delta z$$



Flux outward across S_2:

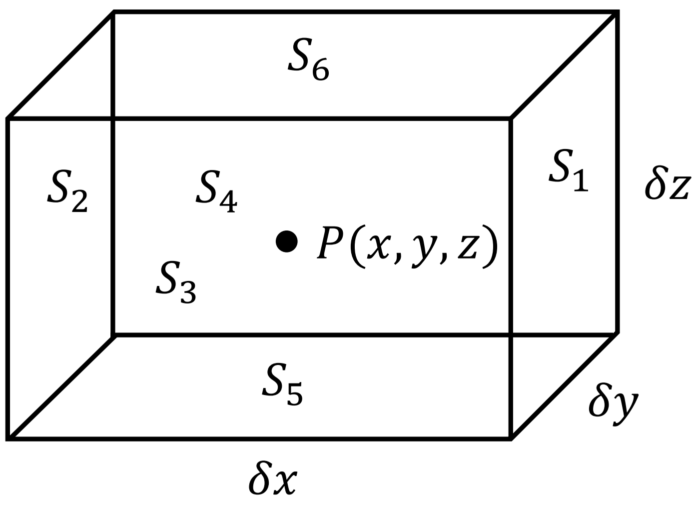
$$\iint_{S_2} \vec{v}\cdot\vec{n}\,d\sigma \approx -v_1\left(x-\frac{\delta x}{2},y,z\right)\,\delta y\,\delta z$$

Flux outward across S_1 & S_2:

$$\iint_{S_1+S_2} \vec{v}\cdot\vec{n}\,d\sigma \approx \left(v_1\left(x+\frac{\delta x}{2},y,z\right)-v_1\left(x-\frac{\delta x}{2},y,z\right)\right)\delta y\delta z \approx \frac{\partial v_1}{\partial x}\delta x\,\delta y\,\delta z$$

Flux outward across S$_1$ & S_2:

$$\iint_{S_1+S_2} \vec{v}\cdot\vec{n}\,d\sigma \approx \frac{\partial v_1}{\partial x}\delta x\,\delta y\,\delta z \;=\frac{\partial v_1}{\partial x}\delta V$$



Similarly from other faces:

$$\iint_{S_3+S_4} \vec{v}\cdot\vec{n}\,d\sigma \approx \frac{\partial v_2}{\partial y}\delta V \qquad \iint_{S_5+S_6} \vec{v}\cdot\vec{n}\,d\sigma \approx \frac{\partial v_3}{\partial z}\delta V$$

Flux per unit volume out of the box $\approx \dfrac{\partial v_1}{\partial x}+\dfrac{\partial v_2}{\partial y}+\dfrac{\partial v_3}{\partial z}$

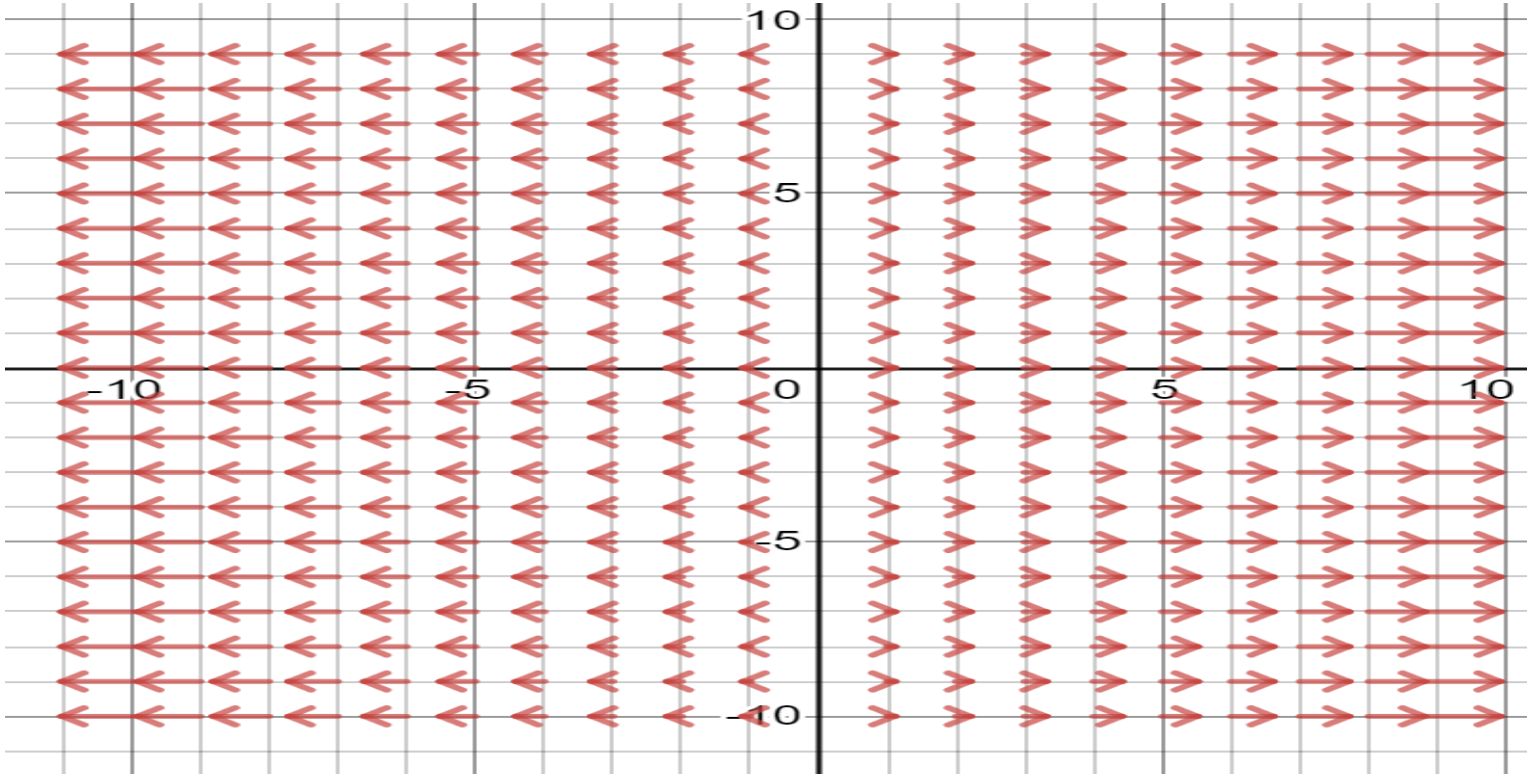
Flux per unit volume at $P(x,y,z)=\dfrac{\partial v_1}{\partial x}+\dfrac{\partial v_2}{\partial y}+\dfrac{\partial v_3}{\partial z}\;=\operatorname{div}\vec{v}$

Divergence can be interpreted as the rate of expanssion or compression of the vector field.

Example 1 : Consider $\vec{v} = (x, 0, 0)$

div $\vec{v} = 1$ (positive)

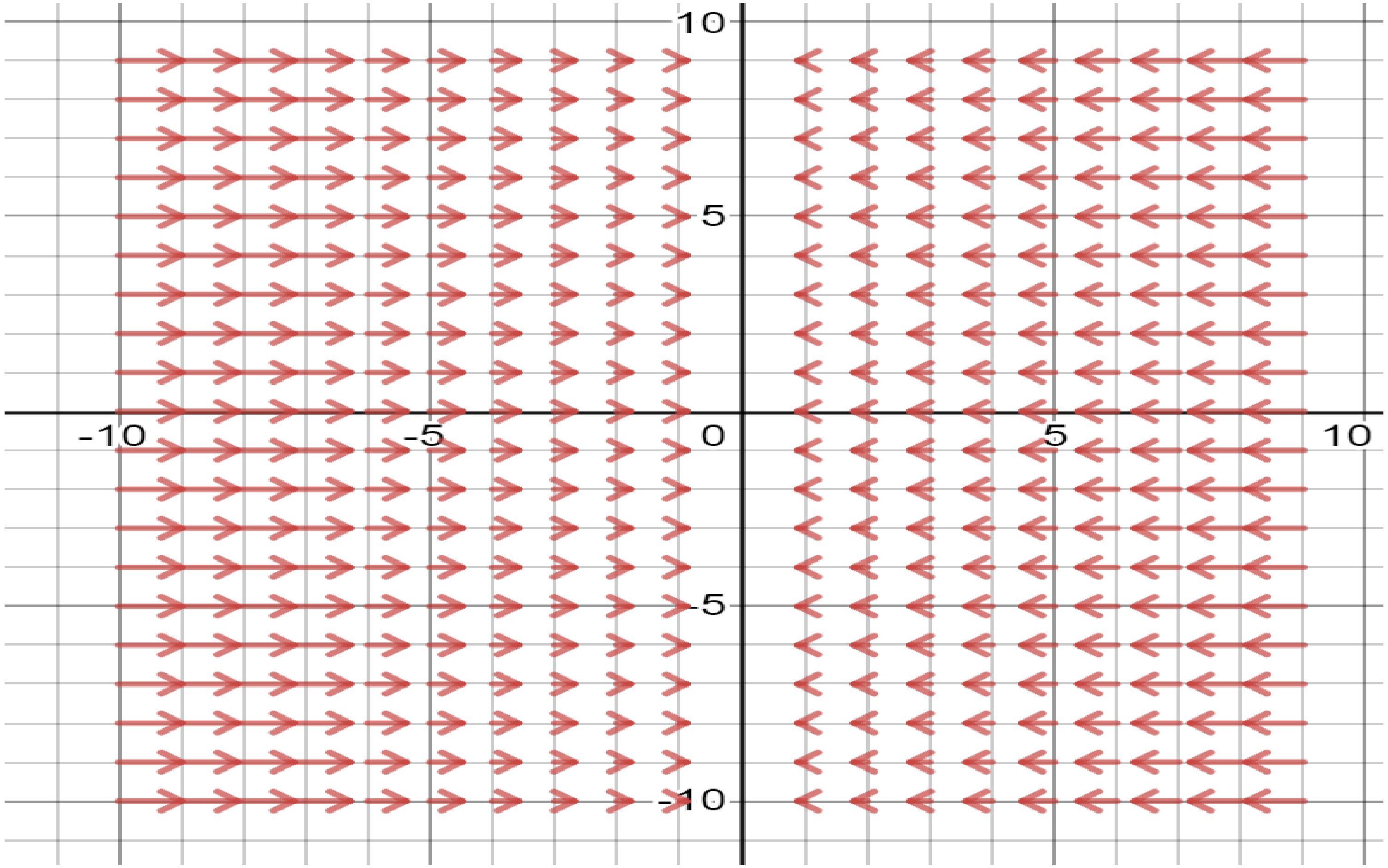
Tendency of fluid is EXPANSION.

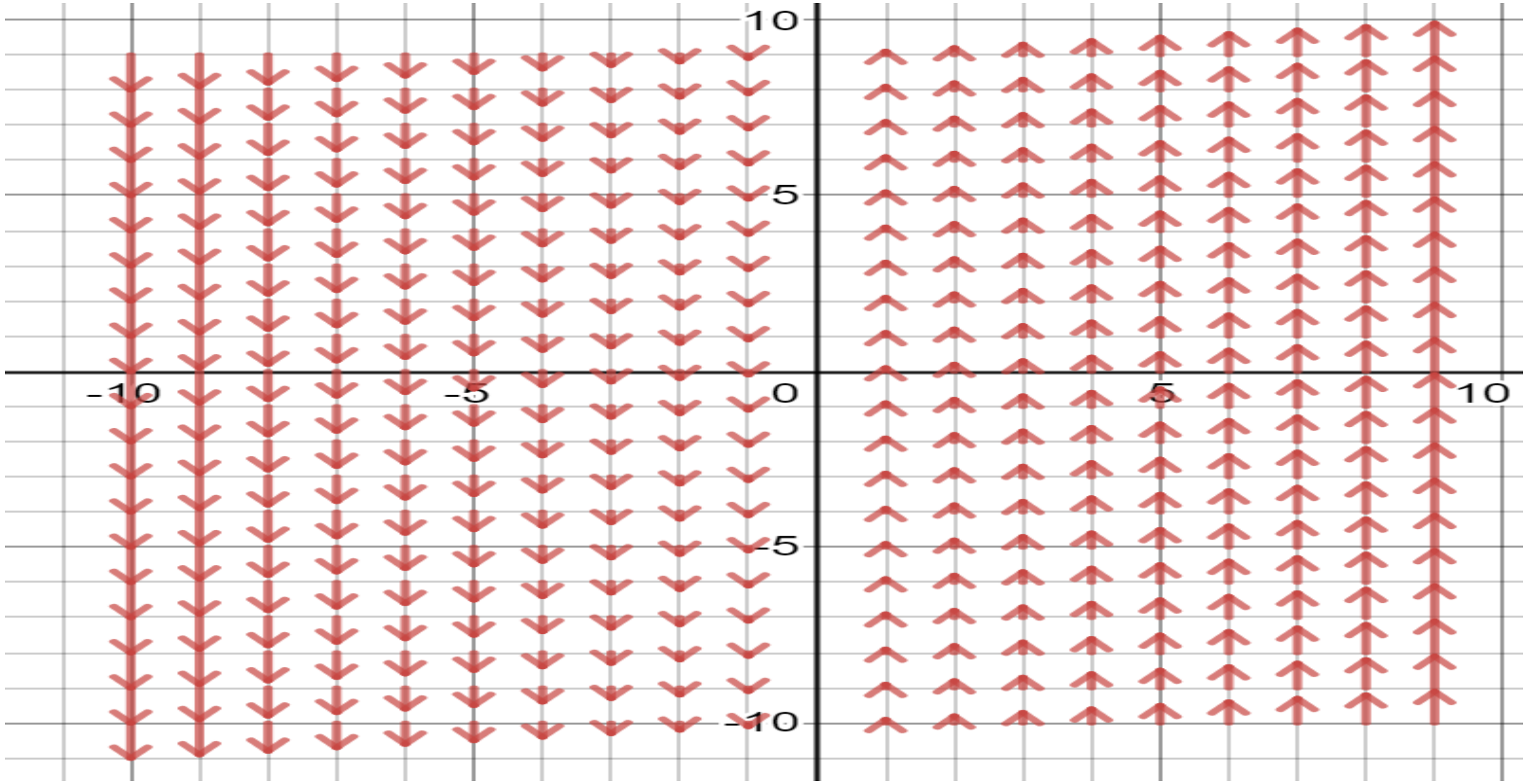


Example 2: Consider $\vec{v} = (-x, 0,0)$

div $\vec{v} = -1$ (negative)

Tendency of fluid is COMPRESSION.





Example 3: Consider $\vec{v} = (0, x, 0)$

div $\vec{v} = 0$

Neither expanding nor contracting.

A vector field $\vec{v}$ for which $\nabla \cdot \vec{v} = 0$ everywhere is said to be **solenoidal**.

The relation div $\vec{v} = 0$ is also known as the **condition of incompressibility**.

Curl of a Vector Field

Curl of a vector $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ field is given by

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

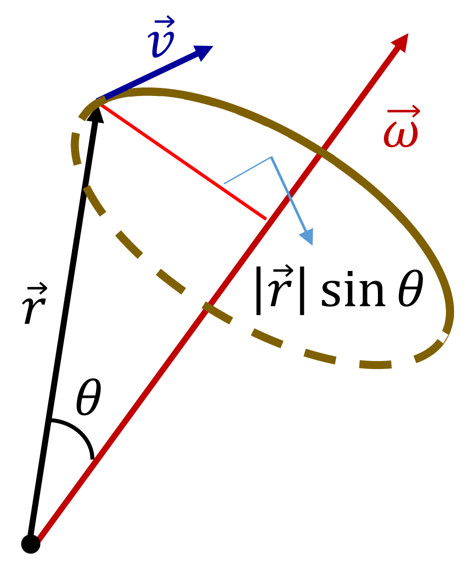
$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

Example: Let $\vec{v} = y\,\hat{\imath} + 2xz\,\hat{\jmath} + ze^x\,\hat{k}$

$$\text{curl } \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 2xz & ze^x \end{vmatrix} = -2x\,\hat{\imath} - ze^x\,\hat{\jmath} + (2z-1)\,\hat{k}$$

Physical Interpretation of Curl of a Vector Field

Suppose an object rotates with uniform angular velocity $\vec{\omega}$



$$\text{tangential speed} \;=\; \text{angular speed} \times \text{radius}$$

$$|\vec{v}| = |\vec{\omega}|\,|\vec{r}|\sin\theta \;=\; |\vec{\omega}\times\vec{r}|$$

Note that the direction of $\vec{v}$ is perpendicular to both $\vec{r}$ and $\vec{\omega}$

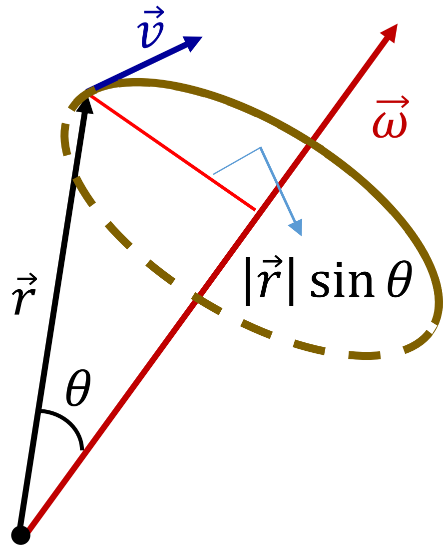
Since $\vec{v}$ and $\vec{r}\times\vec{\omega}$ both have same direction and same magnitude, we conclude

$$\vec{v} = \vec{\omega}\times\vec{r}$$

Let $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and $\vec{\omega} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$

$$\vec{v} = \vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\,\hat{\imath} + (cx - az)\,\hat{\jmath} + (ay - bx)\,\hat{k}$$

$$\vec{v} = (bz - cy)\,\hat{\imath} + (cx - az)\,\hat{\jmath} + (ay - bx)\,\hat{k}$$



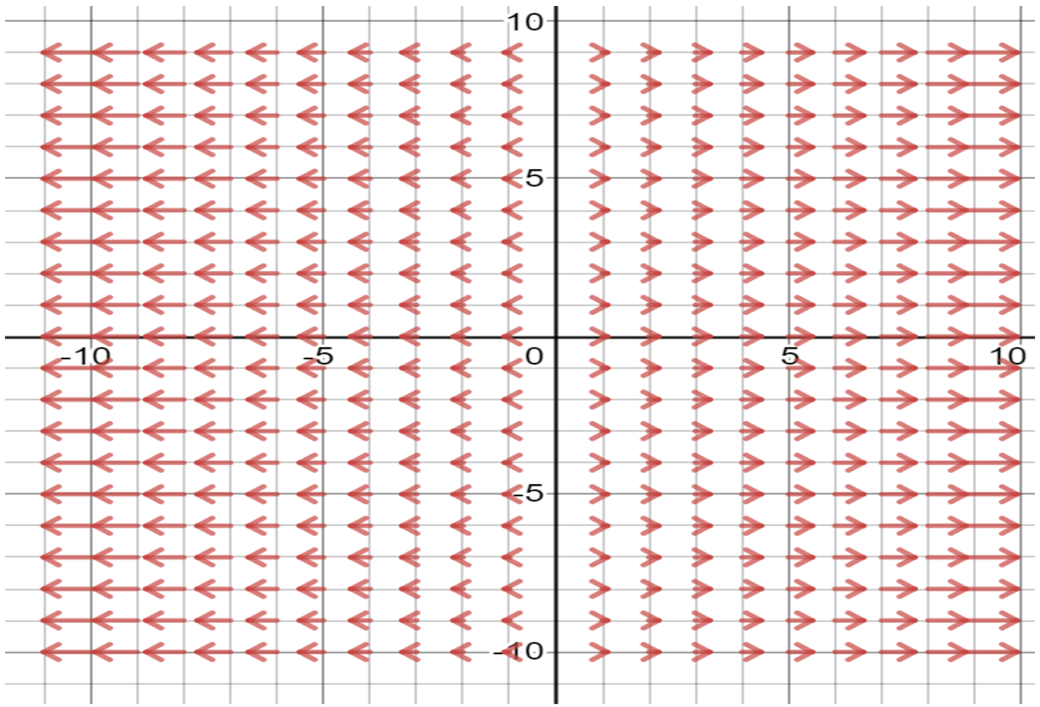
$$\nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (bz - cy) & (cx - az) & (ay - bx) \end{vmatrix}$$

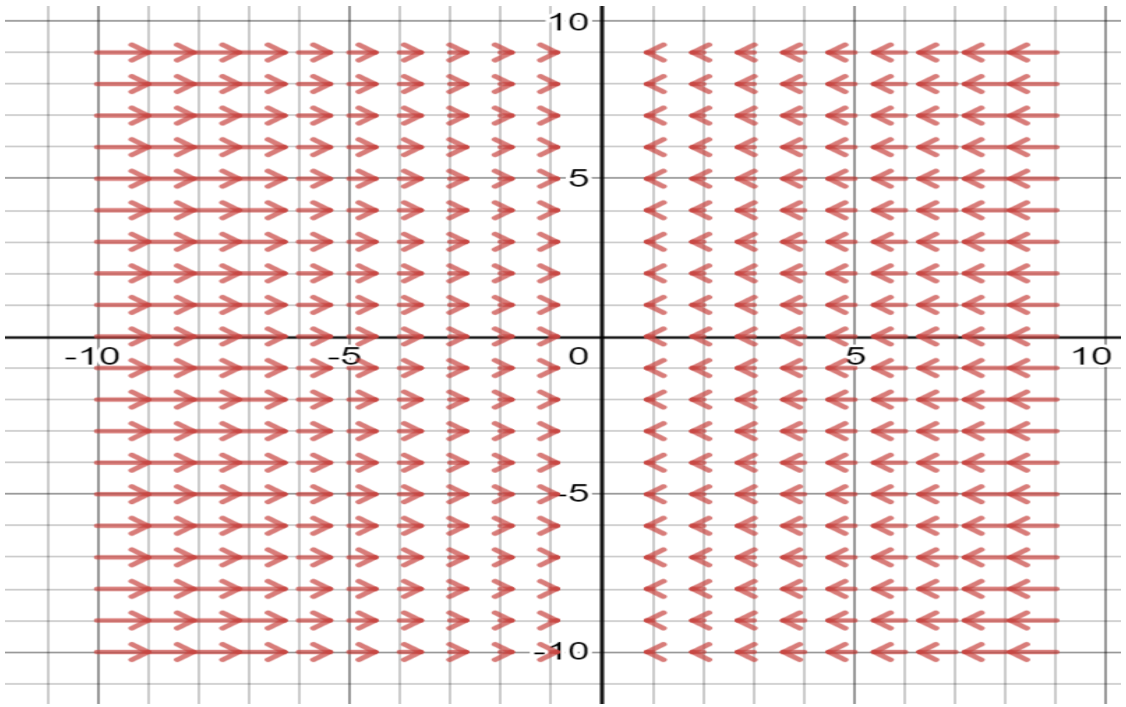
$$= 2a\,\hat{\imath} + 2b\,\hat{\jmath} + 2c\,\hat{k} \;= 2\,\vec{\omega}$$

curl $\vec{v}$ signifies the tendency of **ROTATION**.

The vector curl $\vec{v}$ is directed along the axis of rotation with magnitude twice the angular speed.

A vector filed $\vec{v}$ for which $\nabla \times \vec{v}$ is zero everywhere is said to be IRROTATIONAL.



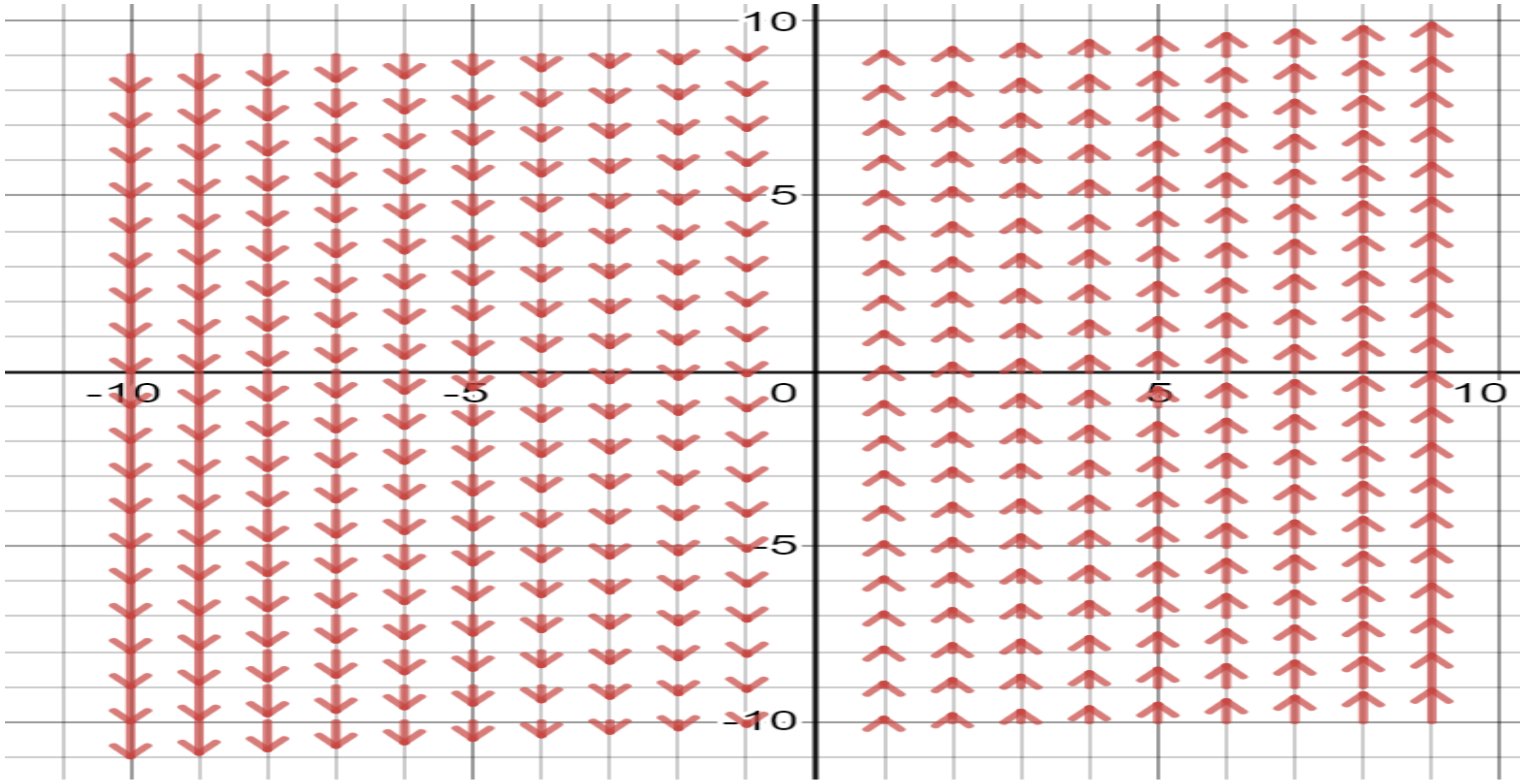


Example 1: $\vec{v} = (\pm x, 0, 0)$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = \hat{\imath}.0 - \hat{\jmath}.0 + \hat{k}.0 = \vec{0}$$

No sense of rotation. **IRROTATIONAL**

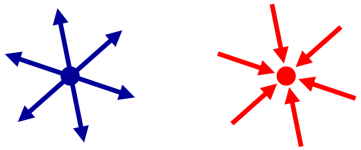
Example 2: $\vec{v} = (0, x, 0)$ $\nabla \times \vec{v} = \hat{k}$ Rotation is about an axis in the $z -$ direction.



KEY TAKEAWAY

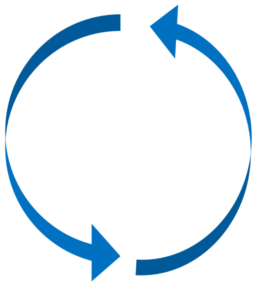
- Divergence of $\vec{v}$: $\text{div}\,\vec{v} = \nabla \cdot \vec{v} = \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z}$

- Expansion or Compression



- curl of $\vec{v}$: $\text{curl}\,\vec{v} = \nabla \times \vec{v}$

- Sence of Rotation



- **Smooth and Piecewise Smooth Curves**
- **Simple Closed Curves**
- **Line Integrals**

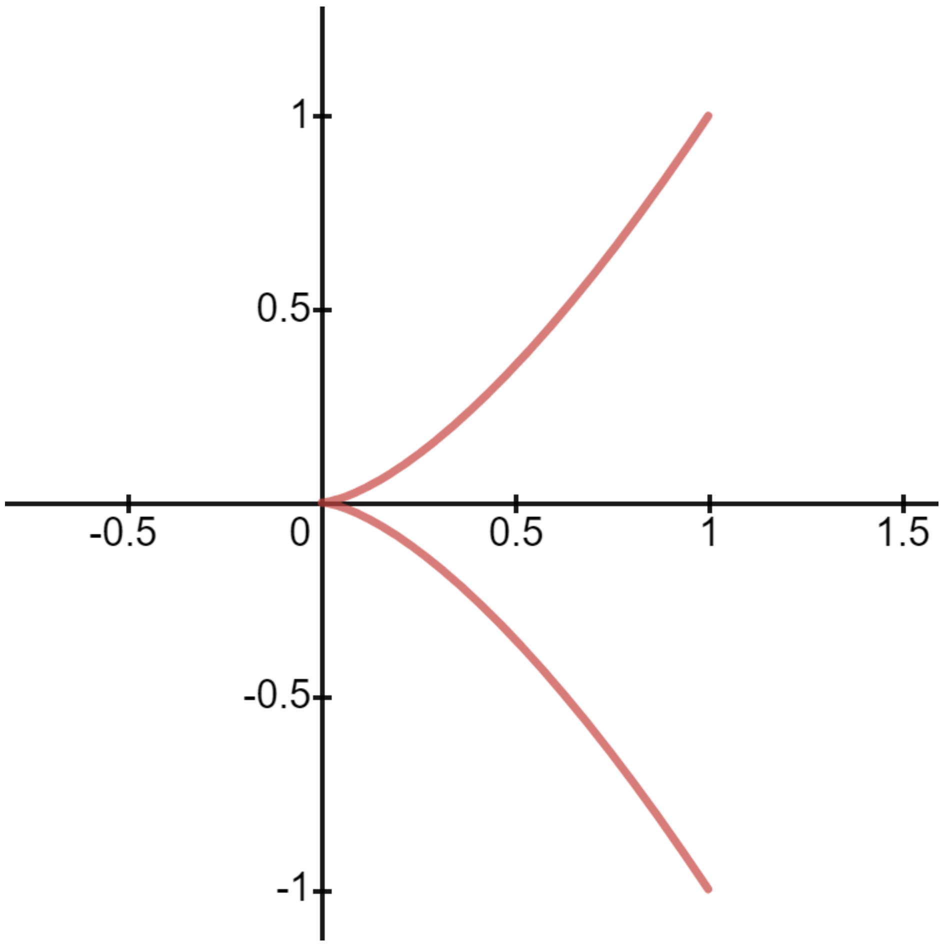
Smooth Curves : Let $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k},\ t \in [a, b]$ denote a curve in space.

If $\vec{r}(t)$ posses a continuous first order derivative (nowhere zero) for the given values of t then the curve is known as smooth.

In other words, the space curve $\vec{r}(t)$ is smooth when $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$ are continuous on $[a, b]$ and not simultaneously zero on (a, b)

Note that the condition **nowhere zero** ensures that the curve has no sharp corners or cusps.

Graph of $\vec{r}(t) = t^2\,\hat{\imath} + t^3\,\hat{\jmath}$



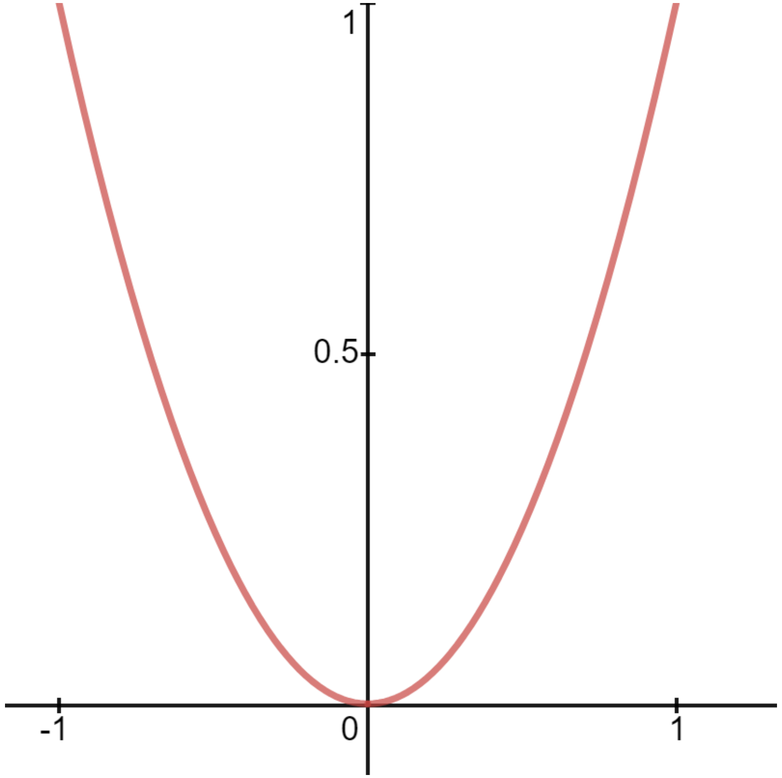
Consider $\vec{r}(t) = t^2\,\hat{\imath} + t^3\,\hat{\jmath},\ \ t \in [-1, 1]$

Compute $\dfrac{d\vec{r}(t)}{dt} = 2t\,\hat{\imath} + 3t^2\,\hat{\jmath}$

$$\Longrightarrow \frac{d\vec{r}(t)}{dt} = 0 \ \text{ for } \ t = 0$$

(Indicate non-smoothness)

Graph of $\vec{r}(t) = t^3\,\hat{\imath} + t^6\,\hat{\jmath}$



Note that $\dfrac{d\vec{r}(t)}{dt} = 0$ does not necessarily implies non-smoothness.

However, $\dfrac{d\vec{r}(t)}{dt} \neq 0$ always implies smoothness.

Consider $\vec{r}(t) = t^3\,\hat{\imath} + t^6\,\hat{\jmath},\ t \in [-1, 1] \Longrightarrow \dfrac{d\vec{r}(t)}{dt} = 0 \text{ for } t = 0$

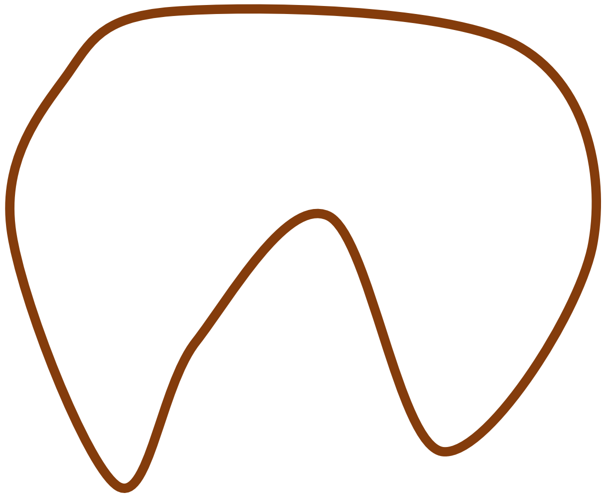
But the curve is smooth

Alternate parameterization: $\vec{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath},\ t \in [-1, 1]$

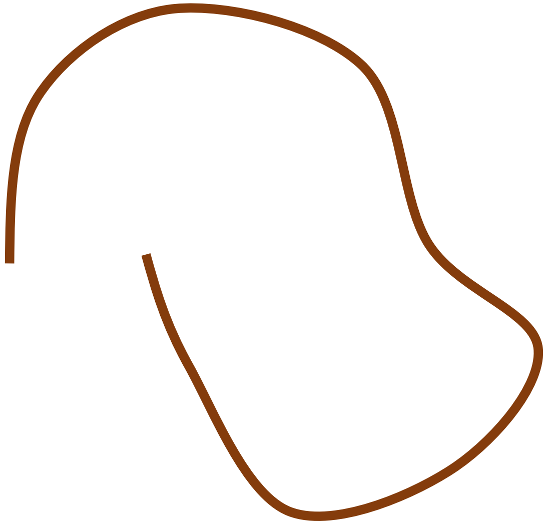
$$\Longrightarrow \frac{d\vec{r}(t)}{dt} \neq 0, \qquad \forall t$$

Piecewise Smooth Curve: If it is made up of a finite number of smooth curves.

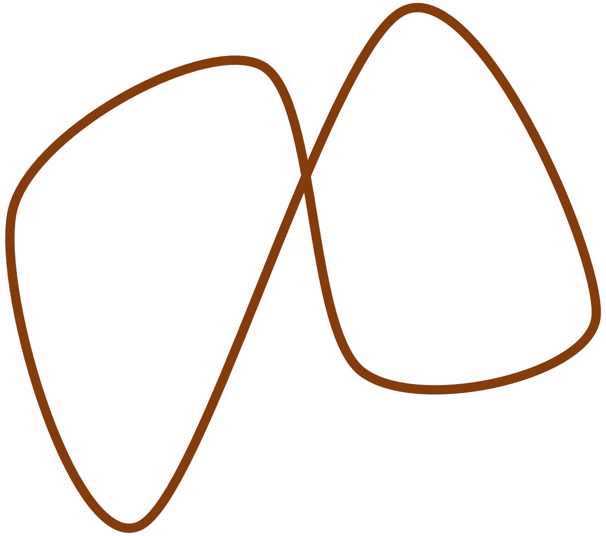
Simple Closed Curve : A curve which does not intersect itself anywhere and initial and end points are same is known as simple closed curve.



Simple closed curve



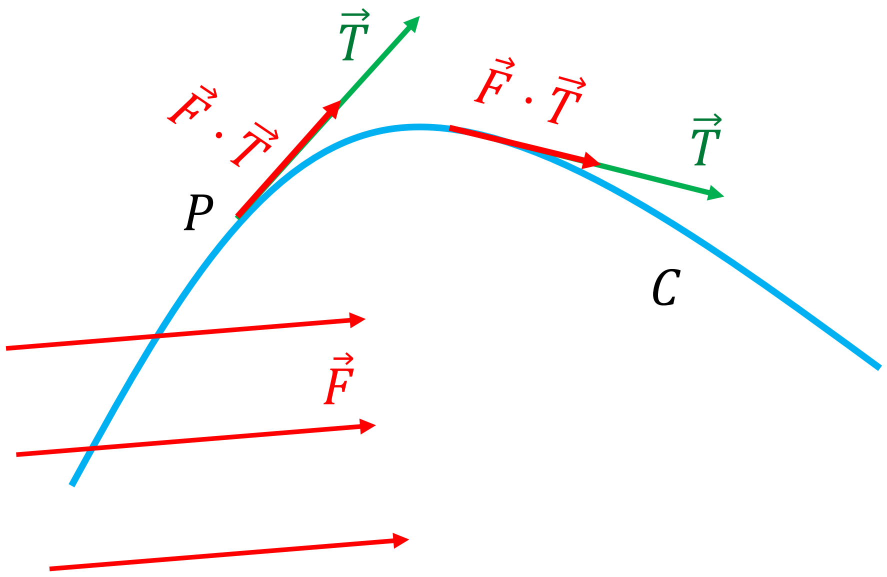
Simple but not closed curve



Closed but not simple

Line Integrals Let a force $\vec{F}$ act upon a particle which is displaced along a given curve C in space.

Let $\vec{T}$ be the unit tangent vector at the point $P(x_i, y_i, z_i)$.



On a small subarc of length Δs_i the work done is

$$\Delta w_i \approx \vec{F}(x_i, y_i, z_i) \cdot \vec{T}(x_i, y_i, z_i)\ \Delta s_i$$

Total work done: $W = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta w_i$

$$= \int_C \vec{F}\cdot\vec{T}\ ds$$

Line Integrals Let the curve C be given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Note that $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ and $ds = |\vec{r}\,'(t)|\, dt$

$$\vec{F} \cdot \vec{T}\ ds = \vec{F} \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\ |\vec{r}\,'(t)|\, dt = \vec{F} \cdot \vec{r}\,'(t)\ dt = \vec{F} \cdot d\vec{r}$$

$$W = \int_C \vec{F} \cdot \vec{T}\ ds = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}\big(x(t), y(t), z(t)\big) \cdot \vec{r}\,'(t)\ dt$$

Evaluation of Line Integrals $\int_C \vec{F} \cdot d\vec{r}$

In Vector form: Note that $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, a \leq t \leq b$ and $d\vec{r} = \dfrac{d\vec{r}}{dt} dt$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \frac{d\vec{r}}{dt} dt}$$

In Component form: Suppose $\vec{F} = F_1(x, y, z)\,\hat{\imath} + F_2(x, y, z)\,\hat{\jmath} + F_3(x, y, z)\,\hat{k}$ and $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$

$$\implies d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz}$$

Problem 1: Find the work done by $\vec{F} = (y - x^2)\,\hat{\imath} + (z - y^2)\,\hat{\jmath} + (x - z^2)\,\hat{k}$ over the curve

$\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k},\ \ 0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$.

Solution: $$\frac{d\vec{r}}{dt} = \hat{\imath} + 2t\hat{\jmath} + 3t^2\hat{k}$$

$$\vec{F}\big(\vec{r}(t)\big) = (t^2 - t^2)\,\hat{\imath} + (t^3 - t^4)\,\hat{\jmath} + (t - t^6)\,\hat{k} = (t^3 - t^4)\,\hat{\jmath} + (t - t^6)\,\hat{k}$$

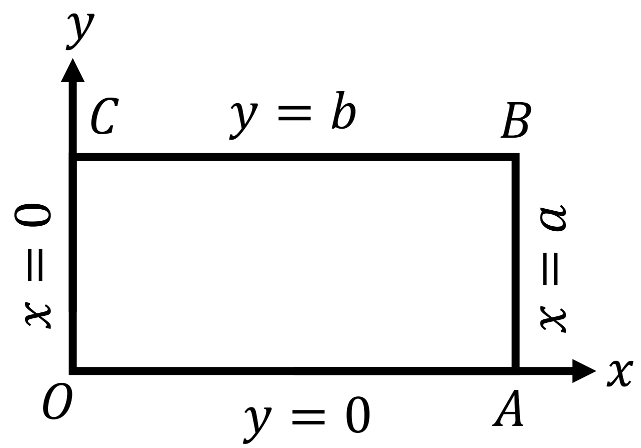
$$\vec{F} \cdot \frac{d\vec{r}}{dt} = 2t(t^3 - t^4) + 3t^2(t - t^6) = 2t^4 - 2t^5 + 3t^3 - 3t^8$$

$$\int \vec{F} \cdot d\vec{r} = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt = \frac{29}{60}$$

Problem 2: Evaluate $\int_C \vec{F}\cdot d\vec{r}$, $\quad \vec{F} = (x^2+y^2)\,\hat{\imath} - 2xy\,\hat{\jmath}$

C : rectangle in xy plane bounded by $y=0,\ x=a;\ \ y=b,\ x=0.$

Solution: $\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+y^2)\,dx\ - 2xy\,dy$



Along OA: $y=0,\ dy=0\ \&\ x$ varies from 0 to a.

$$\int \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$$

Along AB: $x=a,\ dx=0\ \&\ y$ varies from 0 to b: $\int \vec{F}\cdot d\vec{r} = \int_0^b -2ay\,dy = -ab^2$

Along BC: $\int \vec{F}\cdot d\vec{r} = \int_a^0 (x^2+b^2)dx = -\left[\frac{a^3}{3} + ab^2\right]$

Along CO: $\int \vec{F}\cdot d\vec{r} = 0$

$$\int_C \vec{F}\cdot d\vec{r} = -2ab^2$$

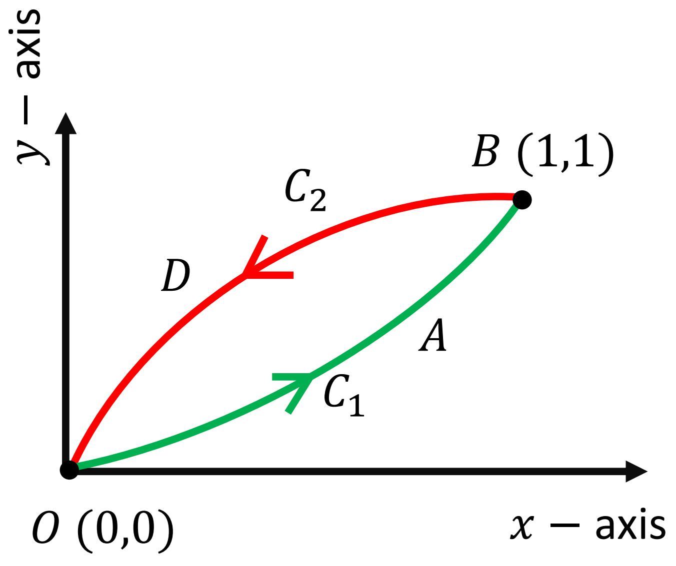
Line Integral as Circulation Let C be an oriented closed curve.

We call the line integral $\oint_C \vec{F} \cdot d\vec{r}$ the circulation of $\vec{F}$ around C.

Problem 3: Find the circulation of $\vec{F}$ around C where

$\vec{F} = (2x + y^2)\hat{\imath} + (3y - 4x)\hat{\jmath}$ and C is the curve

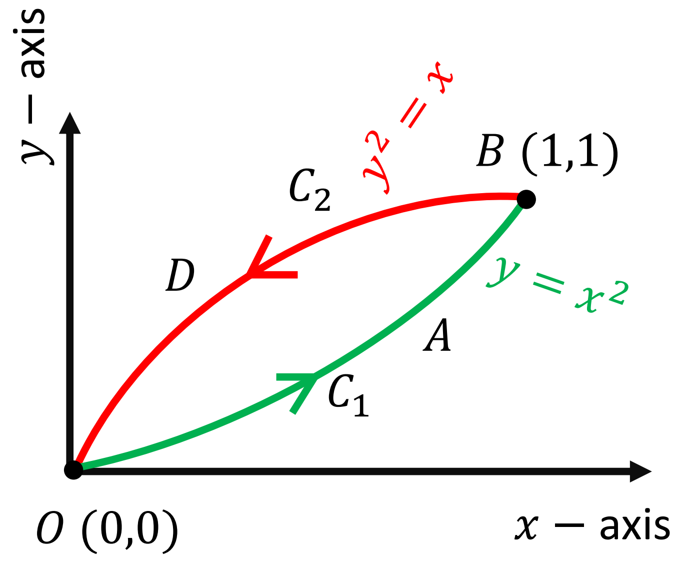
$y = x^2$ from $(0,0)$ to $(1,1)$ and the curve $y^2 = x$ from $(1,1)$ to $(0,0)$.



Solution: $\vec{F} \cdot d\vec{r} = (2x + y^2)dx + (3y - 4x)dy$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = (2x+y^2)dx + (3y-4x)dy$$



Along OAB: $x^2 = y \Rightarrow 2x\,dx = dy$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 (2x + x^4)\,dx + \int_0^1 (3x^2 - 4x)\,2x\,dx = \frac{1}{30}$$

Along BDO: $x = y^2 \Rightarrow dx = 2ydy$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_0^1 (2y^2 + y^2)2y\,dy - \int_0^1 (3y - 4y^2)\,dy = -\frac{6}{4} - \frac{3}{2} + \frac{4}{3} = -\frac{5}{3}$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{1}{30} - \frac{5}{3} = -\frac{49}{30}$$

Problem 4: Evaluate $\oint_C \vec{F} \cdot d\vec{r}\,,\qquad \vec{F} = y\,\hat{\imath} - 2x\,\hat{\jmath},\qquad C: x^2 + y^2 = 9$

Solution: Parametric equation of the circle: $x = 3\cos t\,,\ \ y = 3\sin t\,,\qquad 0 \le t \le 2\pi$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_0^{2\pi} (-9\sin^2 t - 18\cos^2 t)\,dt = -9\oint_0^{2\pi} (\sin^2 t + 2\cos^2 t)\,dt$$

$$= -9\oint_0^{2\pi} (1 + \cos^2 t)\,dt \ = -9\oint_0^{2\pi} \left(1 + \frac{1}{2}(1 + \cos 2t)\right) dt$$

$$= -9\left(\frac{3}{2}\,2\pi + 0\right) = -27\,\pi$$

KEY TAKEAWAY

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a continuous vector field on a smooth curve C given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

The line integral of $\vec{F}$ on C is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \frac{d\vec{r}}{dt}\,dt$$