Concepts Covered

Differential Calculus

Functions of Single Variable

- Taylor Polynomial
- Taylor Series
- Worked Examples

Taylor Formula (Generalization of MVT)

Assume that the function f has all derivatives up to the order $(n+1)$ in some interval containing the point $x = x_0$.

We wish to find a polynomial $P_n(x)$ of degree n, such that

$$P_n(x_0) = f(x_0) \qquad P_n'(x_0) = f'(x_0) \qquad P_n''(x_0) = f''(x_0) \quad \ldots \quad P_n^{(n)}(x_0) = f^{(n)}(x_0)$$

What do we expect with such a polynomial ?

Close to the function f at least in the neighborhood of $x = x_0$

How to construct such a polynomial ?

Polynomial Construction

Consider $P_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \cdots + c_n(x - x_0)^n$

We find the undermined coefficients c_i so that $P_n^{(k)}(x_0) = f^{(k)}(x_0),\ k = 0, 1, 2\ \ldots\ n$

Note that $P_n'(x) = 1\,c_1 + 2c_2(x - x_0) + 3\,c_3(x - x_0)^2 + \cdots + n\,c_n(x - x_0)^{n-1}$

$$P_n''(x) = 2 \cdot 1\,c_2 + 3 \cdot 2\,c_3(x - x_0) + n(n-1)\,c_n(x - x_0)^{n-2}$$

$$\vdots$$

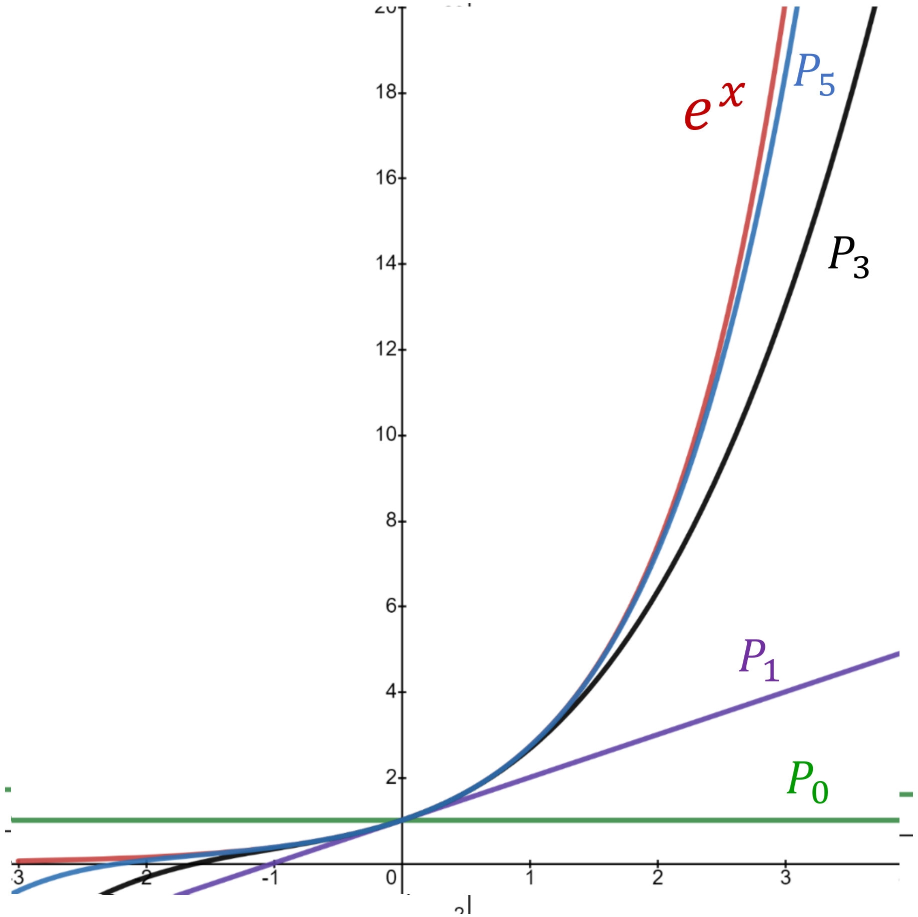
$$P_n^{(n)}(x) = n(n-1)\ldots 2 \cdot 1 \cdot c_n(x - x_0)^0$$

We get $c_0 = f(x_0)$ $\quad c_1 = f'(x_0)$ $\quad c_2 = \dfrac{f''(x_0)}{2.1}$ $\quad \cdots \quad c_n = \dfrac{f^{(n)}(x_0)}{n!}$

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Taylor's Polynomial of order *n*

Example - 1 Taylor's Polynomial of e^x around $x = 0$.



$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24};$$

$$\mathrm{P}_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_2(x) = 1 + x + \frac{x^2}{2};$$

$P_0(x) = 1;$ $P_1(x) = 1 + x;$

Relation: Taylor Polynomial and the Function

Denoting $R_n(x)$ the deference between the values of the given function $f(x)$ and the constructed polynomial $P_n(x)$

$$R_n(x) = f(x) - P_n(x)$$

The function $R_n(x)$ is called remainder.

How to evaluate $R_n(x)$?

Note that

$$R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0$$

We have $R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0$

Consider

$$g(x) = (x - x_0)^{n+1}, \qquad \forall\, x \in I$$

This implies:

$$g^{(k)}(x_0) = 0, k = 0,1, \dots, n \quad \& \quad g^{(n+1)}(x_0) = (n+1)!$$

Let x be a point in I and suppose $x > x_0$. Apply Cauchy's MVT for R_n & g in $[x_0, x]$

$$\frac{R_n(x) - R_n(x_0)}{g(x) - g(x_0)} = \frac{R_n'(\xi_1)}{g'(\xi_1)} \implies \frac{R_n(x)}{g(x)} = \frac{R_n'(\xi_1)}{g'(\xi_1)}, \qquad x_0 < \xi_1 < x$$

Apply Cauchy's MVT for R_n' and g' in $[x_0, \xi_1]$

$$\frac{R_n(x)}{g(x)} = \frac{R_n''(\xi_2)}{g''(\xi_2)}, \qquad x_0 < \xi_2 < \xi_1 < x$$

Continuing applying Cauchy's MVT

$$\Longrightarrow \quad \frac{R_n(x)}{g(x)} = \frac{R_n^{(n+1)}(\xi_{n+1})}{g^{(n+1)}(\xi_{n+1})}, \qquad x_0 < \xi_{n+1} < \xi_n < \cdots < \xi_1 < x$$

$$\boxed{R_n(x) = \frac{R_n^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}, \qquad x_0 < \xi < x}$$

$$R_n(x) = \frac{R_n^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \qquad x_0 < \xi < x$$

Also note that $R_n(x) = f(x) - P_n(x)$

$$R_n^{(n+1)}(x) = f^{(n+1)}(x) - P_n^{(n+1)}(x) = f^{(n+1)}(x)$$

$$\mathrm{R}_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \ x_0 < \xi < x$$

Lagrange form of Remainder

$$\mathrm{R}_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}\big(x_0 + \theta(x-x_0)\big), 0 < \theta < 1$$

Different forms of Remainders

$$R_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}\big(x_0 + \theta(x - x_0)\big), \qquad 0 < \theta < 1,$$ Lagrange Form

$$R_n(x) = \frac{(x - x_0)^{n+1}(1 - \theta)^n}{n!} f^{(n+1)}\big(x_0 + \theta(x - x_0)\big),$$ Cauchy Form

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x - t)^n f^{n+1}(t)\, dt$$ Integral Form

$$R_n(x) = \int_{x_0}^{x}\int_{x_0}^{t_{n+1}}\int_{x_0}^{t_n} \dots \int_{x_0}^{t_2} f^{n+1}(t_1)\, dt_1 \dots dt_n dt_{n+1}$$ Integral Form

Taylor's Theorem or Taylor's Formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

$$\mathrm{R}_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1},\ x_0 < \xi < x$$

Special case $n = 0$

$$f(x) = f(x_0) + \frac{f'(\xi)}{1!}(x - x_0), \quad x_0 < \xi < x$$

$$\Longrightarrow \frac{f(x) - f(x_0)}{(x - x_0)} = f'(\xi), \qquad x_0 < \xi < x$$

Lagrange Mean Value Theorem

Remarks

- *If we set* $x_0 = 0$ *in the Taylor's formula of the function* $f(x)$, *the it is called* *Maclaurin's formula*.

- *In the Taylor's formula, if the remainder* $R_n(x) \to 0$ *as* $n \to \infty$, *then*

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \cdots$$

is called Taylor's series. For $x_0 = 0$, *it is called Maclaurin's series.*

Remarks

- *There are examples of smooth functions whose Taylor's series diverges everywhere other than the point of expansion.*

- *There are examples of smooth functions whose Taylor series converges to some other function.*

- *Consider* $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. *One can easily show that* $f^{(n)}(0) = 0, \forall\, n$

  Hence, its Maclaurin's series $0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots$

  The series converges but it does not converge to $f(x)$

Example - 2 Maclaurin's Series of e^x

Note that $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for all values of n.

Maclaurin's Theorem

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

$$\mathrm{R}_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}\big(x_0 + \theta(x - x_0)\big)$$

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} e^{\theta x}, 0 < \theta < 1$$

Does $R_n(x) \to 0$ *as* $n \to \infty$?

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} e^{\theta x} \implies |R_n(x)| = \frac{|x|^{n+1}}{(n+1)!} e^{\theta x} \rightarrow \text{is finite for given } x$$

For a fixed x we can always find a natural number N such that $|x| < N$

Consider $n > N$

$$\frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{1 \cdot 2 \cdot \dots \cdot (n+1)} = \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \dots \cdot \frac{|x|}{N-1} \cdot \overbrace{\frac{|x|}{N}}^{=: q < 1} \cdot \frac{|x|}{N+1} \cdots \frac{|x|}{n+1}$$

$$\implies \frac{|x|^{n+1}}{(n+1)!} < \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \dots \cdot \frac{|x|}{N-1} \cdot q \cdot q \cdots q = \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \dots \cdot \frac{|x|}{N-1} \cdot q^{(n+1)-(N-1)}$$

$$\implies \frac{|x|^{n+1}}{(n+1)!} < \frac{|x|^{N-1}}{(N-1)!} \cdot q^{n-N+2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\boxed{\lim_{n\to\infty} R_n = 0}$$

KEY TAKEAWAY

Taylor's Formula

$$f(x) = \overbrace{f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}^{\text{Taylor's Polynomial}} + R_n$$

Remainder: $R_n = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1},\ x_0 < \xi < x$

$R_n(x) \to 0 \ as\ n \to \infty$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

Taylor's Series