$$\{F'(x)\}^2\frac{d^2y}{dt^2} + F''(x)\frac{dy}{dt} + p(x)\frac{dy}{dt}F'(x) + q(x)y = 0$$

$$\Rightarrow \{F'(x)\}^2\frac{d^2y}{dt^2} + [F''(x) + p(x)F'(x)]\frac{dy}{dt} + q(x)y = 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} + \frac{F''(x)+p(x)F'(x)}{\{F'(x)\}^2}\frac{dy}{dt} + \frac{q(x)}{\{F'(x)\}^2}y = 0} \quad -(2)$$

The eqn (2) will have constant coefficients if & only if

$$\frac{F''(x)+p(x)F'(x)}{\{F'(x)\}^2} \text{ is constant}$$

$$\& \frac{q(x)}{\{F'(x)\}^2} = \text{const}$$

$$\Rightarrow \frac{q(x)}{(F'(x))^2} = \text{const.} = 1 \text{ (say)}$$

$$\Rightarrow q(x) = \{F'(x)\}^2$$

$$\Rightarrow \boxed{F(x) = \int q^{\frac{1}{2}}dx}$$

$$F''(x) = \frac{1}{2}q^{-\frac{1}{2}}q'(x)$$

from relation (3)

$$\frac{F''(x)+p(x)F'(x)}{\{F'(x)\}^2} = \text{const}$$

$$\Rightarrow \frac{\frac{1}{2}q^{-\frac{1}{2}}q' + p(x)q^{\frac{1}{2}}}{q} = \text{const} \Rightarrow \frac{q^{-\frac{1}{2}}[q' + 2pq]}{2q} = \text{c[...]}$$

$$\Rightarrow \frac{q' + 2pq}{q^{3/2}} = \text{const}$$

Remark

1) If $q(x) < 0$ then eqn (5) must be modified

2) Limitation :- If $\frac{q'(x) + 2pq}{q^{3/2}} = \text{constant}$

then you can proceed to get the transformation as

$t = F(x) = \int q^{\frac{1}{2}} dx$, otherwise you can't proceed

Ex Solve $y'' - y' \cot x + \sin^2 x \, y = 0$

By change of independent variable -

Sol By change of independent variable

$p(x) = -\cot x \quad q(x) = \sin^2 x$

$$\frac{q'(x) + 2p(x)q(x)}{q^{3/2}} = \frac{2\sin x \cos x + 2\left(\frac{-\cos x}{\sin x}\right)\sin^2 x}{(\sin^2 x)^{3/2}}$$

$$= 0 = \text{const}$$

Hence $t = F(x) = \int q^{\frac{1}{2}} dx$

$= \int \sin x \, dx$

$= -\cos x$

$$e^{x}A'(x)-e^{-x}B'(x)=\frac{2}{1+e^{x}}.$$

Also $e^{x}A'(x)+e^{-x}B'(x)=0$

Hence $\dfrac{dA}{dx}=\dfrac{1}{e^{x}(1+e^{x})}$

$$\frac{dB}{dx}=-\frac{e^{x}}{(1+e^{x})}$$

$\therefore A=\displaystyle\int\frac{e^{-x}}{1+e^{x}}\,dx$ Put $e^{x}=z$

$e^{x}dx=dz$

$$=\int\frac{dz}{z^{2}(1+z)}$$

$$=-e^{-x}+\log\left(\frac{1+e^{x}}{e^{x}}\right)+c_1$$

$$B=-\int\frac{e^{x}}{1+e^{x}}\,dx=-\int\frac{dz}{1+z}$$

$$=-\log(1+e^{x})+c_2$$

Hence, GS is

$$y=A(x)\,e^{x}+B(x)\,e^{-x}$$

Q Given that $y=x$ is a solution of

$$(x^{2}+1)y''-2xy'+2y=0,$$

solve the given ODE

Then solve the below ODE by method of variation of parameters

We introduce symbol $D^k = \frac{d^k}{dx^k}$, for any positive integer k.

Problem 20th 22 Pg237 For eg e^{-x} is a solution of

$$(D+1)y = 0$$

The differential operator $D+1$ is said to annihilate, or to be an annihilator of e^{-t}.

Def An annihilator of a function f is the differential map A s.t. $Af = 0$.

For $f := e^{mx}$ the annihilator is $D-m$.

$f := x^n$ the annihilator is D^{n+1}

$f := \sin(ax+b)$ the annihilator is D^2+a^2

Same for $f := \cos(ax+b)$

$f := x^n e^{mx}$ the annihilator is $(D-m)^{n+1}$.

For product of functions annihilator composes,

Eg $f = x^n \sin(ax+b)$

annihilator is $(D^2+a^2)^{n+1}$

Same $f = x^n \cos(ax+b)$

$f := e^{mx}\sin(ax+b)$ $\quad (D-m)^2+a^2$.

$f := e^{mx}\cos(ax+b)$. Product by exponential gives translation in diff operator D.

$f := x^n e^{mx} \sin(ax+b)$

annihilator is $[(D-m)^2 + a^2]^{n+1}$

- Given an inhomogeneous ODE $Ly = f$
  if f admits an annihilator A
  ie $Af = 0$,
  then we have new homogeneous ODE
  $$(A \circ L)\, y = 0$$
  possibly of higher order than L.

- Consider linear, constant coefficient, second order, inhomogeneous ODE
  $$(a_2 D^2 + a_1 D + a_0)\, y = f$$
  A is annihilator of f then we get
  $$A(a_2 D^2 + a_1 D + a_0)\, y = 0$$
  y is obtained as a linear combination of linearly solutions of $A(a_2 D^2 + a_1 D + a_0)$ subject to its solvability

Consider ODE

$$a_2 y'' + a_1 y' + a_0 y = e^{mx}$$

Apply annihilator of f

$$(D-m)(a_2 D^2 + a_1 D + a_0) y = 0$$

CE of new homogeneous eqn

$$(\mu - m)(a_2 \mu^2 + a_1 \mu + a_0) = 0$$

Three roots m_1, m_2 original CE

& addition root m.

- If m is distinct from m_1 & m_2

  then $y = y_c + y_p$

  where y_c is the solution of homogeneous ODE

  & $y_p = Ae^{mx}$

  where A is to be determined & called undetermined coefficient

  If m is equal to one of m_1 & m_2 then

  $$y = y_c + y_p$$

  where $y_p = Axe^{mx}$ where A is to be determined

To obtain A in y_p use the y in given ODE & equate like variable both sides

Consider

$$y'' - 2y' - 3y = 2e^{4x}$$

$f(x) = 2e^{4x}$.

Annihilator $\quad 2(D-4)$

$$2(D-4)(D^2-2D-3)y = 0$$

Roots are $3, -1, 4$

$y(x) = y_c + \alpha_3 e^{4x}$.

$y_c = \alpha_1 e^{3x} + \alpha_2 e^{-x}$.

Using y in given ODE

$$(D^2-2D-3)(y_c + \alpha_3 e^{4x}) = 2e^{4x}$$

or $(D^2-2D-3)(\alpha_3 e^{4x}) = 2e^{4x}$

$\alpha_3 e^{4x}[16-8-3] = 2e^{4x}$

$$\alpha_3 = \frac{2}{5}.$$

$y_p = \frac{2}{5} e^{4x}$

$\boxed{\text{GS} \quad \alpha_1 e^{3x} + \alpha_2 e^{-x} + \frac{2}{5} e^{4x}}$

Consider ODE

$$y'' - 3y' + 2y = x^2 e^x.$$

$$f(x) = x^2 e^x.$$

annihilator of $x^2 e^x$ is $(D-1)^3$.

$$(D-1)^3 (D^2 - 3D + 2) y = 0.$$

roots are $1, 2, 1, 1, 1$ Three repeated roots with new root same as one of roots of original CE.

$$y(x) = y_c + (\alpha_3 x + \alpha_4 x^2 + \alpha_5 x^3) e^x.$$

$$y_c = \alpha_1 e^x + \alpha_2 e^{2x}$$

$$(D^2 - 3D + 2)(\alpha_3 x + \alpha_4 x^2 + \alpha_5 x^3) e^x = x^2 e^x.$$

$$-3\alpha_5 = 1, \quad 6\alpha_5 - 2\alpha_4 = 0$$

$$2\alpha_4 - \alpha_3 = 0.$$

$$\alpha_5 = \frac{-1}{3}, \quad \alpha_4 = -1, \quad \alpha_3 = -2$$

$$y_p(x) = \left(-2x - x^2 - \frac{1}{3} x^3\right) e^x$$

GS $y(x) := \alpha_1 e^x + \alpha_2 e^{2x} + y_p(x)$

Consider ODE

$$y'' - 2y' + y = x^2 e^x$$

$$\boxed{f(x) = x^2 e^x}$$

annihilator of $x^2 e^x$ is $(D-1)^3$.

New homogeneous ODE is

$$(D-1)^3 (D^2 - 2D + 1) y = 0$$

$$y(x) = y_c + (\alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4) e^x$$

where $y_c = (\alpha_1 + \alpha_2 x) e^x$ General soln of original CE

$$(D^2 - 2D + 1)(\alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4) e^x = x^2 e^x$$

$$2\alpha_3 = 0, \quad 6\alpha_4 = 0 \quad 12\alpha_5 = 1$$

$$y_p(x) = \frac{1}{12} x^4 e^x.$$

<u>GS</u> $y(x) := (\alpha_1 + \alpha_2 x) e^x + \frac{1}{12} x^4 e^x$

Sine/cosine in RHS

Consider ODE

$$a_2 y'' + a_1 y' + a_0 y = \sin mx$$

where $f(x) = \sin mx$

Use annihilator of f,

$$(D^2+m^2)(a_2D^2+a_1D+a_0)y = 0.$$

CE $(\mu^2+m^2)(a_2\mu^2+a_1\mu+a_0) = 0.$

four roots

m_1, m_2 original CE

& additional complex roots $\pm im$.

$y = y_c + y_p$.

$y_p = A\sin mx + B\cos mx$

If complex roots are equal to m_i's

$y_p = x(A\sin mx + B\cos mx)$

Consider $y'' - 2y' - 3y = 2e^x - 10\sin x$

CE $m^2 - 2m - 3 = 0$ $m_1 = 3, m_2 = -1$

CF $y_c(x) = \alpha_1 e^{3x} + \alpha_2 e^{-x}$.

$2(D-1)(-10(D^2+1))$

$m_3 = 1$ $m_4 = \pm i$

e^x & $\sin x$ are linearly independent from solns of hom eqn. $y_p(x) = \alpha_3 e^x + \alpha_4 \sin x + \alpha_5 \cos x$

...near homogeneous, kth order ODE with constant ...

...ficients is given as

$$Ly = \sum_{i=0}^{k} a_i y^{(i)}$$

CE kth degree polynomial of m

$$\sum_{i=0}^{k} a_i m^i = 0.$$

- If (CE) admits k distinct real roots $(m_i)_{i=1}^{k}$ then $y(x) = \sum_{i=1}^{k} \alpha_i e^{m_i x}$

l repeated real roots & rest are distinct

$$y(x) = \left(\sum_{i=1}^{l} \alpha_i x^{i-1}\right) e^{mx} + \sum_{i=l+1}^{k} \alpha_i e^{m_i x}$$

non-repeated pair of complex roots $a \pm ib$
rest are distinct real roots

$$y(x) = e^{ax}(a_1 \sin bx + a_2 \cos bx) + \sum_{i=3}^{k} \alpha_i e^{m_i x}$$

repeated pair of l complex roots $a \pm ib$

$$e^{ax}\left[\sum_{i=1}^{l} \alpha_i x^{i-1} \sin bx + \sum_{i=1}^{l} \alpha_{l+i} x^{i-1} \cos bx\right]$$

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Consider

$$y^{(4)} + y'' = 3x^2 + 4\sin x - 2\cos x$$

CE $m^4 + m^2 = 0$

Two repealed roots 0

Pair of conjugate complex roots $\pm i$

$$y_c(x) := \alpha_1 + \alpha_2 x + \alpha_3 \sin x + \alpha_4 \cos x$$

~~CE~~ D^3 annihal for $3x^2$

also has zero as roots.

$$\left(Ax^4 + Bx^3 + Cx^2\right)$$

CE of annihilator of $4\sin x \quad -2\cos x$

$D^2+1 \qquad \pm i$

$Dx\sin x + Ex\cos x$

$$y_p(x) = Ax^4 + Bx^3 + Cx^2.$$

is the required Transformation

Corresponding reduced diff eqn

$$[F'(x)]^2 \frac{d^2y}{dt^2} + [F''(x) + p(x)F'(x)]\frac{dy}{dt} + q(x)y = 0$$

$$\Rightarrow \sin^2 x \frac{d^2y}{dt^2} + \left[\cos x + \left(\frac{-\cos x}{\sin x}\right)\sin x\right]\frac{dy}{dt} + (\sin^2 x)y = 0$$

$$\Rightarrow \sin^2 x \frac{d^2y}{dt^2} + (\sin^2 x)y = 0$$

$$\boxed{\frac{d^2y}{dt^2} + y = 0}$$

Q Solve the following ODE by variation of parameters:-

$$y'' - y = \frac{2}{1+e^x}$$

Ans Characteristic eqn is $m^2 - 1 = 0$

$$m = \pm 1$$

CF $\quad y = Ae^x + Be^{-x}$

$$\frac{dy}{dx} = A(x)e^x - B(x)e^{-x} + e^x A'(x) + e^{-x}B(x)$$

We have chosen A & B s.t.

$$\boxed{e^x A'(x) + e^{-x} B'(x) = 0}$$

Therefore $\frac{dy}{dx} = A(x)e^x - B(x)e^{-x}$.

$$\frac{d^2y}{dx^2} = \text{—}$$