

The method of undetermined coefficients can be used when right side of eqn is polynomial

NOTE:- When the R.H.S of ODE is a polynomial

$$y'' - 3y' - 4y = 4t^2 - 1$$

we assume PT $y(t) = At^2 + Bt + C$
same degree poly as the non homogeneous.

eqn
- const

SUMMARY

Consider ODE $ay'' + by' + cy = g(t)$

(1) If $g(t) = e^{\alpha t}$ then $y(t) = Ae^{\alpha t}$

(2) If $g(t) = \sin \alpha t$ or $\cos \alpha t$, then
 $y(t) = A \sin \alpha t + B \cos \alpha t$

(3) If $g(t)$ is a polynomial, then $y(t)$ is a polynomial of same degree.

- The same principle extends to the case when $g(t)$ is a product of any two, or all three, of these types of functions

Q Find a ps. of $y'' - 3y' - 4y = -8e^t \cos 2t$

PT $y(t) = Ae^t \cos 2t + Be^t \sin 2t$

($y(t)$ is the product of e^t & a linear combination of $\cos 2t$ & $\sin 2t$)

Remark Suppose $g(t)$ is sum of two terms.

$$g(t) = g_1(t) + g_2(t)$$

Suppose y_1 & y_2 are solutions of eqns

$$ay'' + by' + cy = g_1(t)$$

$$\& ay'' + by' + cy = g_2(t)$$

Then $y_1 + y_2$ is a solution of

$$ay'' + by' + cy = g(t)$$

↳ $g(t)$ any finite no. of terms

Ex 1

Q Find a PI of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t$$

↓

$$y'' - 3y' - 4y = 3e^{2t}$$

$$y'' - 3y' - 4y = 2\sin t$$

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

Q

Find a PI of

$$y'' - 3y' - 4y = 2e^{-t} \quad \text{--- (1)}$$

Ans Assume $y(t) = Ae^{-t}$

$$\therefore y'(t) = -Ae^{-t}$$

$$\therefore y''(t) = Ae^{-t}$$

Putting y , y' & y'' into (1), we get

$$Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$$

$$\Rightarrow 0 = 2e^{-t} \rightarrow \text{---}$$

Why such a situation arise?

Corresponding homogeneous eqn

$$y'' - 3y' - 4y = 0$$

$$\text{CE } m^2 - 3m - 4 = 0$$

$$\Rightarrow (m+1)(m-4) = 0$$

$$\Rightarrow m_1 = -1 \quad m_2 = 4$$

$\therefore y_1(t) = e^{-t}$ $y_2(t) = e^{4t}$ they form a fundamental set of solutions of $Ly = 0$

- Thus our assumed PI is a sol of corresponding $Ly = 0$
To find a particular sol of $\textcircled{v} y(t) = Ate^{-t}$

$$y'(t) = -Ate^{-t} + Ae^{-t}$$

$$y''(t) = Ate^{-t} - Ae^{-t} - Ae^{-t} \\ = -2Ae^{-t} + Ate^{-t}$$

$$-5A = 2 \Rightarrow A = -2/5$$

$$\text{Hence PI is } y(t) = -\frac{2}{5}te^{-t}$$

NOTE If the assumed form of PI duplicates a sol of corresponding eqn, then modify the assumed PI by multiplying t . If this modification is insufficient then multiply by t second time.

Summary

$$ay'' + by' + cy = g(t)$$

a, b, c are constants

- 1) Find g 's of corresponding homogeneous eqn
2. $g(t)$ — exp, sine, cosine, poly, sum or prod
3. $g(t) = g_1(t) + \dots + g_n(t)$
form n subproblem

a. $y_i'(t)$

any duplication multiply t , or by t^2

b. $y_1(t) + \dots + y_n(t)$

c. Form sum $\overset{CF}{\cancel{CF}} + PI$

7) Use initial cond

Summary

$$ay'' + by' + cy = g(t)$$

$$g(t) = \sum_{k=1}^{n_0} g_k(t)$$

$$ay'' + by' + cy = g_1(t)$$

$$ay'' + by' + cy = g_2(t)$$

$$ay'' + by' + cy = g_n(t)$$

The particular soln of

$ay'' + by' + cy = g(t)$ is of the below form

$$g(t) \\ P_n(t) = a_0 t^n + \dots + a_n$$

$$PI \ y(t) \\ t^s (A_0 t^n + A_1 t^{n-1} + \dots + A_n)$$

$$P_n(t) = e^{\alpha t}$$

$$t^s (A_0 t^n + \dots + A_n) e^{\alpha t}$$

$$P_n(t) e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$$

$$t^s [A_0 t^n + \dots + A_n] e^{\alpha t} \begin{cases} \cos \beta t \\ \sin \beta t \end{cases} \\ + [B_0 t^n + B_1 t^{n-1} + \dots + B_n] e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$$

Here s is the smallest non^{neg} integer $(0, 1, 2)$ that will ensure that no term in $y_i(t)$ is a solution of corresponding homogeneous eqn

Q Solve the ODE

$$y'' + 2y' + 3y = \begin{cases} 1 & 0 \leq t \leq \pi/2 \\ 0 & t > \pi/2 \end{cases}$$

with initial conditions $y(0) = 0$ & $y'(0) = 0$.

Ans Characteristic eqn is

$$m^2 + 2m + 3 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 12}}{2}$$

$$= -1 \pm 2i$$

$$\therefore \text{CF is } c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$$

Now, by inspection, P.I. is

$$y_p(t) = \begin{cases} 1/5 & 0 \leq t \leq \pi/2 \\ 0 & t > \pi/2 \end{cases}$$

For $0 \leq t \leq \pi/2$, the general soln is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= e^{-t} (c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{5} \end{aligned}$$

$$\text{I.C.s. is } y(0) = 0 = y'(0)$$

$$c_1 = -\frac{1}{5}, \quad c_2 = -\frac{1}{10}$$

$$\therefore y(t) = \frac{1}{5} - \frac{1}{10} (2e^{-t} \cos 2t + e^{-t} \sin 2t)$$

on $0 \leq t \leq \pi/2$

$$y(t) = d_1 e^{-t} \cos 2t + d_2 e^{-t} \sin 2t$$

$$y(\pi/2) = -d_1 e^{-\pi/2} = \frac{1 + e^{-\pi/2}}{5}$$

$$y'(\pi/2) = d_1 e^{-\pi/2} - 2d_2 e^{-\pi/2} = 0$$

$$d_1 = \frac{-(1 + e^{\pi/2})}{5} e^{-t} \cos(2t) - \frac{(1 + e^{\pi/2})}{10} e^{-t} \sin(2t)$$

Q. 10

Find soln of

$$y'' + y = \begin{cases} t & 0 \leq t \leq \pi \\ \pi e^{\pi-t} & t > \pi \end{cases}$$

satisfying the ICs $y(0) = 0$ & $y'(0) = 1$

Assume that y & y' are also continuous at $t = \pi$.

Method of undetermined coefficients can't be applied to
ODE $y'' + y = t \tan x$.

Now introduce variation of parameters

It is a general method

Can be used to determine a formula for particular soln of an arbitrary 2nd order linear non homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

Assume C.F. of (1)

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Soln corresponding to

$$y'' + p(t)y' + q(t)y = 0$$

KEY

$$y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$y' = u_1'(t)y_1(t) + u_2'(t)y_2(t) + u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

SET $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$

Then,

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

Now put y, y', y'' into (1)

$$u_1'(t)y_1''(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

Continue after pg 3

Now $|A| \neq 0$ as y_1 & y_2 are fundamental set of solutions

$$|A| = W(y_1, y_2)(t)$$

Solving (6) & (10), we get

$$\left. \begin{aligned} u_1'(t) &= \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} \\ u_2'(t) &= \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \end{aligned} \right\} \text{ (11)}$$

where $W(y_1, y_2)$ is non zero, as y_1 & y_2 are fundamental set of solutions.

By integrating (11) we get:-

$$u_1(t) = - \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt + c_1$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt + c_2 \quad \text{--- (12)}$$

Putting $u_1(t)$ & $u_2(t)$ in (4) we get the general soln of (1) as.

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{CF} + \underbrace{\left[-y_1 \int \frac{y_2(t)g(t)}{W} dt + y_2 \int \frac{y_1(t)g(t)}{W} dt \right]}_{PI}$$

Q Apply the method of variation of parameters to solve the following ODE:-

$$y'' + a^2 y = \sec(ax) \quad \text{--- (1)}$$

Ans Characteristic eqn is $m^2 + a^2 = 0$
 $m = \pm ai$

CF of (1) is

$$y_c(x) = A \cos(ax) + B \sin(ax)$$

where A & B are constant.

Assume

$$y = A(x) \cos(ax) + B(x) \sin(ax) \quad \text{--- (2)}$$

is a Q.S of (1)

$$\frac{dy}{dx} = -aA(x) \sin(ax) + aB(x) \cos(ax) \\ + \cos(ax) A'(x) + \sin(ax) B'(x)$$

We choose A & B s.t.

$$\cos(ax) \frac{dA}{dx} + \sin(ax) \frac{dB}{dx} = 0 \quad \text{--- (3)}$$

$$\text{Hence } \frac{dy}{dx} = -aA(x) \sin(ax) + aB(x) \cos(ax) \quad \text{--- (4)}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -a^2 A(x) \cos(ax) - a^2 B(x) \sin(ax) \\ - aA'(x) \sin(ax) + aB'(x) \cos(ax)$$

Pulling y, y' & y'' from (2), (4) & (5) into (1) we get

$$\begin{aligned} -a^2 A(x) \cos(ax) - a^2 B(x) \sin(ax) \\ - a A'(x) \sin(ax) + a B'(x) \cos(ax) \\ + a^2 A(x) \cos(ax) + a^2 B(x) \sin(ax) = \sec(ax) \end{aligned}$$

$$\Rightarrow -a A'(x) \sin(ax) + a B'(x) \cos(ax) = \sec(ax) \quad (6)$$

(3) \Rightarrow

$$\cos(ax) A'(x) + \sin(ax) B'(x) = 0$$

Solving it, we get-

$$A'(x) = \frac{-\sin(ax) \sec(ax)}{W(y_1, y_2)(x)} = -\frac{\tan(ax)}{a}$$

$$B'(x) = \frac{\cos(ax) \sec(ax)}{W(y_1, y_2)(x)} = \frac{1}{a}$$

$$\text{Now, } W(y_1, y_2) = a$$

$$\text{As } B'(x) = \frac{1}{a} \Rightarrow B(x) = \frac{x}{a} + C_2$$

$$A'(x) = -\frac{\tan(ax)}{a} \Rightarrow A(x) = \frac{1}{a^2} \log(\cos(ax)) + C_1$$

GS of (1) is

$$y = A(x) \cos(ax) + B(x) \sin(ax)$$

Q Find a particular soln of

$$y'' - 3y' - 4y = 2\sin t$$

$y(t) = A\sin t$, A is a constant to be determined

$$y'(t) = A\cos t, \quad y''(t) = -A\sin t$$

$$\therefore -A\sin t - 3A\cos t - 4A\sin t = 2\sin t$$

$$\Rightarrow (2 + 5A)\sin t + 3A\cos t = 0 \quad \text{--- (1)}$$

We want (1) to hold $\forall t$.

Thus it must hold for $t = 0$ & $t = \pi/2$

$$\begin{aligned} \text{(1)} &\Rightarrow 3A = 0 \Rightarrow A = 0 \\ &\quad 2 + 5A = 0 \end{aligned}$$

Hence, our assumption about $y(t)$ is not proper.

Modify

Let $y(t) = A\sin t + B\cos t$, where A & B are constants to be determined

$$y'(t) = A\cos t - B\sin t$$

$$y''(t) = -A\sin t - B\cos t$$

$$\begin{aligned} (-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) \\ = 2\sin t \end{aligned}$$

$$\Rightarrow (-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t$$

Equating coeffs. of \sin & \cos

$$\begin{cases} -5A + 3B = 2 \\ -5B - 3A = 0 \end{cases} \Rightarrow \begin{aligned} A &= -5/17 \\ B &= 3/17 \end{aligned}$$

$$\text{PI is } y(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$$

$$y(x) = \alpha_1 \sin x + \alpha_2 \cos x + x$$

is the general solution for non homogeneous ODE

- ✓ To find general solution of non homogeneous ODE it's enough to find a particular integral of the non homogen.
- ✓ Method of undetermined coefficients helps in finding a particular integral of the special cases of f .
- ✓ Method is generally applicable to problems in which homogeneous eqn $LY = 0$ has constant coefficients & the non homogeneous terms consists polynomials, exponential functions, sines & cosines

Q Find a particular soln of

$$y'' - 3y' - 4y = 3e^{2t}$$

Ans Assume PI $y(t) = Ae^{2t}$

$$y'(t) = 2Ae^{2t} \quad y''(t) = 4Ae^{2t}$$

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$\Rightarrow -6A = 3 \Rightarrow A = -\frac{1}{2}$$

$$\therefore \boxed{y(t) = -\frac{1}{2} e^{2t}}$$

the desired PI