General theory for nth order linear equations

An nth order ODE is of the form:-

$$P_0(t)y^n + P_1(t)y^{n-1} + \dots + P_{n-1}(t)y' + P_n(t)y = G(t) \quad \text{--- (1)}$$

We assume that the functions $P_0, P_1, \dots, P_n$ & G are continuous real valued functions on some interval $I: \alpha < t < \beta$ and that P_0 is nowhere zero in this interval.

$$L[y] = y^n + p_1(t)y^{n-1} + \dots + p_n(t)y = g(t) \quad \text{--- (2)}$$

is linear differential operator of order n.

THEOREM Consider the IVP.

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = g(t) \quad \text{--- (2)}$$

with $y(t_0) = y_0, y'(t_0) = y_0', \dots, y^{(n-1)}(t_0) = y_0^{(n-1)}$ --- (3)

where $p_1(t), \dots, p_n(t), g(t)$ are continuous on an open interval I. Then $\exists$ a unique solution $y = \phi(t)$ of the ODE that also satisfies I.C. (3). The solution exists throughout the interval

Analogous to second order ordinary differential eqn:-

Theorem If $p_1, p_2, \dots, p_n$ are continuous on I, if $y_1, y_2, \dots, y_n$ are solutions of $L[y] = 0$ & if $W(y_1, y_2, \dots, y_n) \neq 0$ for atleast a pt in I, then every solution of $L[y] = 0$ can be expressed as linear combination of solutions $y_1, y_2, \dots, y_n$

Def A set of solutions $y_1, y_2, \ldots, y_n$ of $L[y]=0$ whose Wronskian is nonzero is known as fundamental set of solutions

Q $y^{(4)}+y^{(3)}-7y''-y'+6y=0$

with Initial conditions

$$y(0)=1,\ y'(0)=0,\ y''(0)=-2,\ y'''(0)=-1$$

Q $y^{(4)}-y=0$

$$y(0)=\frac{7}{2},\ y'(0)=-4,\ y''(0)=\frac{5}{2},\ y'''(0)=-2$$

Soln The characteristic equation is

$$m^4-1=0$$

$$\Rightarrow (m^2-1)(m^2+1)=0$$

$$\Rightarrow m=\pm 1,\ \pm i$$

$\therefore$ The general solution is

$$y=c_1e^{t}+c_2e^{-t}+c_3\cos(t)+c_4\sin(t)$$

$$c_1=0,\ c_2=3,\ c_3=\frac{1}{2},\ c_4=-1.$$

$$y=3e^{-t}+\frac{1}{2}\cos t-\sin t$$

REPEATED ROOTS

→ say $r=r_1$ roots have multiplicity s (where $s \leq n$), then

$$e^{r_1t},\ te^{r_1t},\ t^2e^{r_1t},\ \ldots,\ t^{s-1}e^{r_1t}$$

are corresponding soln of the ODE.

Q $y^{(4)} + 2y'' + y = 0$

Characteristic eqn is $m^4 + 2m^2 + 1 = 0$

$\Rightarrow (m^2+1)^2 = 0$

$\Rightarrow (m^2+1)(m^2+1) = 0$

$\therefore r = \pm i, \pm i$

$\therefore$ The general soln is

$$y = (c_1 \cos t + c_2 \sin t) + t(c_3 \cos t + c_4 \sin t)$$

METHOD OF UNDETERMINED COEFFICIENTS

Ex $y''' - 3y'' + 3y' - y = 4e^t$ —(1)

Soln $(m-1)^3 = 0$

$m = 1$

$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$

To find a PI $Y(t)$ of (1), we assume

$y(t) = Ae^t$.

But e^t, te^t & t^2e^t are all solutions of $L[y] = 0$

Hence, we assume

$Y(t) = At^3 e^t$

Put Y, Y', Y'' & Y''' into (1) we get

$A = \frac{2}{3}$.

$\therefore Y_p(t) = \frac{2}{3} t^3 e^t$

Generalsoln :-

$Y(t) = Y_c(t) + Y_p(t)$

$= c_1 e^t + c_2 te^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t$

The method of solution of second order linear equation can be extended to higher order equations

Theorem A linear homogeneous kth order ODE with constant coefficients is given by $Ly = \sum_{i=0}^{k} a_i y^{(i)} = 0$ and its (CE) characteristic equation is the kth degree polynomial of m,

$$\sum_{i=0}^{k} a_i m^i = 0$$

a) If (CE) admits k distinct real roots $\{m_i\}_{i=1}^{k}$ then

$$y(x) := \sum_{i=1}^{k} \alpha_i e^{m_i x}$$ is a general solution of ODE $Ly=0$

b) If (CE) admits l repeated real roots m & the rest are distinct then

$$y(x) := \left(\sum_{i=1}^{l} \alpha_i x^{i-1}\right) e^{mx} + \sum_{i=l+1}^{k} \alpha_i e^{m_i x}$$

$$= (\alpha_1 e^{mx} + \alpha_2 x e^{mx} + \quad + \alpha_l x^{l-1} e^{mx}) + \sum_{i=l+1}^{k} \alpha_i e^{m_i x}$$

c) If (CE) admits non-repealed pair of complex roots $a \pm ib$, & rest are distinct real roots then

$$y(x) := e^{ax}(\alpha_1 \sin bx + a_2 \cos bx) + \sum_{i=3}^{k} \alpha_i e^{m_i x}$$

d) If (CE) admits repeated pair of l complex roots $a \pm ib$ then the corresponding part of general solution is written as

$$e^{ax}\left[\sum_{i=1}^{l} \alpha_i x^{i-1} \sin bx + \sum_{i=1}^{l} \alpha_{l+1} x^{i-1} \cos bx\right].$$

Problem Solve

$$y^{(3)} - 4y'' + y' + 6y = 0$$

(CE) is $m^3 - 4m^2 + m + 6 = 0$

✓ Roots are $m_1 = -1, m_2 = 2$ & $m_3 = 3$

✓ Roots are real & distinct

✓ Corresponding solutions are e^{-x}, e^{2x} & e^{3x};

✓ They are linearly independent

✓ General solution is

$$y(x) = \alpha_1 e^{-x} + \alpha_2 e^{2x} + \alpha_3 e^{3x}.$$

Problem $y^{(3)} - 4y'' - 3y' + 18y = 0$

CE is $m^3 - 4m^2 - 3m + 18 = 0$

$m_1 = m_2 = 3$ and $m_3 = -2$

The two of the roots are repeated

Corresponding solutions are e^{3x} & e^{-2x}.

General solutions is $y(x) := (\alpha_1 + \alpha_2 x)e^{3x} + \alpha_3 e^{-2x}$

Problem $y^{(4)} - 5y^{(3)} + 6y'' + 4y' - 8y = 0$

(CE) is $m^4 - 5m^3 + 6m^2 + 4m - 8 = 0$

$m_1 = m_2 = m_3 = 2 \quad m_4 = -1$

General solution $y(x) := (\alpha_1 + \alpha_2 x + \alpha_3 x^2)e^{2x} + \alpha_4 e^{-x}$

Problem

$$y^{(4)}-4y^{(3)}+14y''-20y'+25y=0$$

(CE) is $m^4-4m^3+14m^2-20m+25=0$

$m_1=m_2=1+2i$ & $m_3=m_4=1-2i$

Complex pair of roots are repeated

General solution is

$$y(x)=e^{x}[(\alpha_1+\alpha_2 x)\sin 2x+(\alpha_3+\alpha_4 x)\cos 2x].$$

Using method of annihilator:-

$$y^{(4)}+y''=3x^2+4\sin x-2\cos x.$$

(CE) $m^4+m^2=0$

Two repeated roots 0

Pair of conjugate complex roots $\pm i$

Complementary function $y_c(x):=\alpha_1+\alpha_2 x+\alpha_3\sin x+\alpha_4\cos x$

The characteristic equation of the annihilator of $3x^2$ in f

D^3 also has zero as its roots

Corresponding PI is $Ax^4+Bx^3+Cx^2$

Similarly (CE) of annihilator of

$4\sin x$ & $-2\cos x$ in f,

D^2+1, also has $\pm i$ as its roots.

Corresponding (PI) is $Dx\sin x+Ex\cos x$.

Thus, we seek $y_p(x)=Ax^4+Bx^3+Cx^2$

$+Dx\sin x+Ex\cos x$

Using it in ODE, we get

$$24A + Dx\sin x - 4D\cos x + Ex\cos x + 4E\sin x + 12Ax^2 + 6Bx + 2C - Dx\sin x + 2D\cos x - Ex\cos x - 2E\sin x = 3x^2 + 4\sin x - 2\cos x$$

Thus $A = \frac{1}{4}$, $B = 0$, $C = -3$, $D = 1$ & $E = 2$

General solution

$$y(x) = \alpha_1 + \alpha_2 x + \alpha_3 \sin x + \alpha_4 \cos x + \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x$$

Method of variation of parameter for higher order.

Consider the ODE

$$y^{(3)} - 6y'' + 11y' - 6y = e^x.$$

✓ Complementary function $y_c(x) = \alpha_1 e^x + \alpha_2 e^{2x} + \alpha_3 e^{3x}$

✓ We seek a particular integral of the form

$$y_p(x) = v_1(x)e^x + v_2(x)e^{2x} + v_3(x)e^{3x}$$

Then

$$y_p'(x) = v_1(x)e^x + 2v_2(x)e^{2x} + 3v_3(x)e^{3x}.$$

with additional condition that

$$\boxed{v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x} = 0}$$

Similarly

$$y_p''(x) = v_1(x)e^x + 4v_2(x)e^{2x} + 9v_3(x)e^{3x}$$

with additional condition

$$\boxed{v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x} = 0}$$

Since we need to choose v_1 & v_2 such that y_p is a solution to ODE, we get third identity

$$v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x} = e^x.$$

Thus, for y_p to satisfy ODE,

v_1 & v_2 should be chosen such that

$$\begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^x \end{pmatrix}$$

Determinant is $2e^{6x} \neq 0$ hence invertible

Thus $v_1' = \frac{1}{2}$, $v_2' = -e^{-x}$ $v_3' = \frac{1}{2}e^{-2x}$

Thus $v_1(x) = \frac{x}{2} + c_1$

$v_2(x) = e^{-x} + c_2$

$v_3(x) = -\frac{1}{4}e^{-2x} + c_3$.

We choose $c_1 = c_2 = c_3 = 0$ & obtain

$$y(x) = \left(\alpha_1 + \frac{3}{4}\right)e^x + \alpha_2 e^{2x} + \alpha_3 e^{3x} + \frac{1}{2}xe^x.$$

Q Consider ODE

$$x^3 y^{(3)} - 4x^2 y'' + 8xy' - 8y = 4\ln x$$

Assume $x > 0$. Set $s = \ln x$.

Then $$x^3 y^{(3)} = \frac{d^3y}{ds^3} - 3\frac{d^2y}{ds^2} + 2\frac{dy}{ds}.$$

and ODE in s variable becomes

$$\frac{d^3y}{ds^3} - 7\frac{d^2y}{ds^2} + 14\frac{dy}{ds} - 8y = 4s.$$

The complementary function is

$$y_c = \alpha_1 e^{s} + \alpha_2 e^{2s} + \alpha_3 e^{4s}$$

We seek

$$y_p = As + B$$

using method of undetermined coefficients

Then $y_p' = A$, $y_p'' = y_p^{(3)} = 0$

$$14A - 8As - 8B = 4s.$$

Thus $-8A = 4$ & $14A - 8B = 0$

$$\Rightarrow A = -\frac{1}{2} \text{ \& } B = -\frac{7}{8}.$$

<u>General solution in s variable</u>

$$y_s = \alpha_1 e^{s} + \alpha_2 e^{2s} + \alpha_3 e^{4s} - \frac{1}{2}s - \frac{7}{8}.$$

$$y(x) := \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^4 - \frac{1}{2}\ln x - \frac{7}{8}.$$