Differential Calculus

Functions of Single Variable

Sequence and Series

- ❑ **Sequence**
- ❑ Series
- ❑ Power Series

Sequence: A sequence is an ordered set of real numbers or *a list of numbers in a given order*.

It may be thought as a function $f: N \subseteq \mathbb{N} \to \mathbb{R}$.

If the number of elements in the sequence is infinite (or domain of f is $\mathbb{N}$), it is called an *infinite sequence*.

An infinite sequence $a_1, a_2, \dots, a_n, \dots$ is generally written as $\{a_n\}_{n=1}^{\infty}$.

Example: $1, 4, 7, \cdots, 1 + 3(n-1), \cdots$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots$$

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5} \cdots, (-1)^{n+1}\frac{n}{n+1}, \cdots$$

Convergence of Sequence:

A sequence $\{a_n\}_0^\infty$ **converges** to the number L if to every positive number ϵ, however small, there exists an integer *N* such that for all n,

$$n > N \implies |a_n - L| < \epsilon$$

In other words, the sequence $\{a_n\}_0^\infty$ approaches the limit L, if by taking n large enough, we can make $|a_n - L|$ as small as we please.

Notation: $\lim_{n\to\infty} a_n = L$ OR $a_n \to L$

If no such number L exists, we say that the sequence **diverges**.

Remark: A convergent sequence has a (unique) limit.

Divergence of Sequence:

The sequence $\{a_n\}$ **diverges** to ∞ $(a_n \to \infty)$ if for every number M there is an integer N such that for all $n > N$, we have $a_n > M$.

The sequence $\{a_n\}$ **diverges** to ∞ $(a_n \to -\infty)$ if for every number m there is an integer N such that for all $n > N$, we have $a_n < m$.

A sequence $\{a_n\}_0^\infty$ has an infinite limit (divergent series), if, no matter how large the number M may be, an index N can be found such that $a_n > M$ for all $n > N$. Similar argument for $-\infty$.

Example: Show that $\lim_{n\to\infty} \frac{1}{n} = 0$

Consider: $\left|\frac{1}{n} - 0\right| < \epsilon \Rightarrow \frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon}$

Hence N is any natural number such that $N > \frac{1}{\epsilon}$

Thus for any $\epsilon > 0$, there is a natural number N such that $|x_n| < \epsilon$ for every $n \geq N$.

Example: Let $a_n = \frac{2n+1}{3n+5}$. Show that $\lim_{n\to\infty} a_n = \frac{2}{3}$

$$\left|a_n - \frac{2}{3}\right| = \left|\frac{2n+1}{3n+5} - \frac{2}{3}\right| = \left|\frac{2n+1}{3n+5} - \frac{2}{3}\right| = \frac{7}{3(3n+5)} < \epsilon \quad \Rightarrow 3n+5 > \frac{7}{3\epsilon} \quad \Rightarrow n > \frac{7}{9\epsilon} - \frac{5}{3}$$

Hence N is any natural number such that $N > \frac{7}{9\epsilon} - \frac{5}{3}$

L'Hôpital's Rule: If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ where n is an integer, then $a_n \to L$.

Example: Find the limit of the sequence $\left\{\frac{n}{e^n}\right\}_{n=1}^{\infty}$

Consider $\lim_{x\to\infty} \frac{x}{e^x}$

Using L'Hospitals' rule $\lim_{x\to\infty} \frac{1}{e^x} = 0$ Thus we conclude $\lim_{n\to\infty} \frac{n}{e^n} = 0$

Example: Find the limit of the sequence $\sqrt[n]{n}$

Consider $y = f(x) = \sqrt[x]{x}$

$\ln y = \frac{1}{x} \ln x \quad \Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{x} \ln x = \lim_{x\to\infty} \frac{1}{x} = 0 \quad \Rightarrow \lim_{x\to\infty} y = 1$

Thus we conclude $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Limit Calculations

Let $a_n \to a$ and $b_n \to b$. Then

$$\boxed{a_n \pm b_n \to a \pm b}$$

$$\boxed{a_n b_n \to ab}$$

$$\boxed{\frac{a_n}{b_n} \to \frac{a}{b} \quad \text{if } b \neq 0 \text{ and } b_n \neq 0 \text{ for all } n.}$$

Example: $\displaystyle \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}(1) + \lim_{n\to\infty}\left(\frac{1}{n}\right) = 1$

Sandwich Theorem: Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \le b_n \le c_n$ holds for all n beyond some index N. If $a_n \to L$ and $c_n \to L$ then $b_n \to L$.

Example: $\dfrac{\cos n}{n} \to 0 \quad \text{because} \quad -\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$

Example: $\dfrac{1}{2^n} \to 0 \quad \text{because} \quad 0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$

Example: $(-1)^n \dfrac{1}{n} \to 0 \quad \text{because} \quad -\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}$

Remark: If $|b_n| \le c_n$ and $c_n \to 0$, then $b_n \to 0$.

Remark: If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Continuous function theorem for Sequence:

If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \to f(L)$.

Example: $\sqrt{\dfrac{n+1}{n}} \to 1$

Taking $a_n = \frac{n+1}{n}, f(x) = \sqrt{x}$ and $L = 1$

Example: $2^{\frac{1}{n}} \to 1$

Taking $a_n = \frac{1}{n}, f(x) = 2^x$ and $L = 0$

Bounded sequence: A sequence $\{a_n\}$ is said to be bounded if there exists numbers m and M such that $m \leq a_n \leq M$ for every n.

If there exists no $\widetilde{M}$ such that $\widetilde{M} < M$ and $a_n \leq \widetilde{M}$ for every n, then M is called *least upper bound of the set* $\{a_n : n \in \mathbb{N}\}$.

Similarly, if there exists no $\widetilde{m}$ such that $\widetilde{m} > m$ and $a_n \geq \widetilde{m}$ for every n, then m is called *greatest lower bound of the set* $\{a_n : n \in \mathbb{N}\}$.

Theorem: Every convergent sequence is bounded, but the converse fails.

There are bounded sequences that do not converge.

Example: The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is convergent and $\left|\frac{n}{n+1}\right| < 1$

The sequence $1,2,3,4,1,2,3,4,1,2,3,4, \ldots$ is bounded but not convergent

Boundedness + ? $\Rightarrow$ Convergence

- A sequence $\{a_n\}$ is said to be **monotonically increasing** **(nondecreasing)** if $a_{n+1} \geq a_n$ for every n
- A sequence $\{a_n\}$ is said to be **monotonically decreasing** **(nonincreasing)** if $a_{n+1} \leq a_n$ for every n
- A sequence $\{a_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing

Theorem: Every bounded monotonic sequence is convergent.

Note: A monotonic sequence has always a limit, either finite or infinite; the sequence is convergent provided that $|a_n|$ is less than a number M independent of n; otherwise the sequence diverges.

Theorem: Suppose $\{a_n\}$ is a bounded and increasing sequence. Then the *least upper bound* of the set $\{a_n : n \in \mathbb{N}\}$ is the limit of the sequence $\{a_n\}$.

Theorem: Suppose $\{a_n\}$ is a bounded and decreasing sequence. Then the *greatest lower bound* of the set $\{a_n : n \in \mathbb{N}\}$ is the limit of the sequence $\{a_n\}$.

RECALL:

Sequence: A sequence is an ordered set of real numbers or *a list of numbers in a given order*.

Convergence: the sequence $\{a_n\}_0^\infty$ approaches the limit L, if by taking n large enough, we can make $|a_n - L|$ as small as we please. If no such number L exists, we say that the sequence **diverges**.

If $\{a_n\}_0^\infty$ tends to $\pm\infty$ as $n \to \infty$, the sequence $\{a_n\}_0^\infty$ is said to be divergent.

If$\{a_n\}_0^\infty$ does not tend to a unique limit as $n \to \infty$ then the sequence $\{a_n\}_0^\infty$ is said to be oscillatory or non-convergent.

Every convergent sequence is bounded, but the converse fails.

Every bounded monotonic sequence is convergent

Subsequence: A subsequence of a sequence $\{a_n\}$ is any sequence of the form $\{b_n\}$, where $b_m = a_{n_m}$ and the n_n are integers with $n_1 < n_2 < n_3 < \cdots$

In other words, a subsequence is formed by considering any infinite subcollection of the terms of a sequence without changing their order.

Example: Consider the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

Then one subsequence of this sequence would be $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$

Some properties of sequences and subsequences

1. Changing a finite number of terms in a sequence has no effect on convergence, divergence, or the limit, if it exists.

Example: The sequences $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \cdots, \frac{1}{n}, \cdots$ and $2, 8, 5, \frac{1}{10}, \frac{1}{5}, \frac{1}{6}, \cdots, \frac{1}{n}, \cdots$ both converge to 0

2. Any subsequence of a convergent sequence converges and its limit is the limit of the original sequence.

Example: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$ coverges to 0.

And so also its subsequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots, \frac{1}{2n}, \cdots$

3. Any subsequence of a sequence that diverges to ∞ also diverges to ∞

Example: $1, 2, 3, \ldots, n, \ldots$ diverges to ∞.

Consider its subsequence $1, 8, 27, 64, \ldots, n^3, \ldots$ Its also diverges to ∞.

4. Note: If $\{a_n\}$ converges to 0 and $\{b_n\}$ converges, then $\{a_n b_n\}$ converges to zero.

Example: Consider $a_n = \frac{1}{n}, b_n = \frac{n}{n+1}$

5. If two subsequences of a given sequence converge to distinct limits, then the sequence diverges.

Example: Consider $1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$

It contains subsequences $1, 1, 1, \ldots$ and $2, 2, 2, \ldots$, for example,

and these subsequences converge to 1 & 2

6. The sum, difference, product and quotient of two divergent sequences need not diverge.

Example: Let $a_n = (-1)^{n+1},\ b_n = (-1)^n,\ c_n = (-1)^n$

Note that the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ diverge.

However the sequences $\{a_n + b_n\}, \{a_n b_n\}, \left\{\frac{a_n}{b_n}\right\}$ and $\{b_n - c_n\}$ converge.

7. $\lim\limits_{n\to\infty} a_n = 0 \ \Leftrightarrow\ \lim\limits_{n\to\infty} |a_n| = 0$

Note: $\lim\limits_{n\to\infty} a_n = a \ \Rightarrow\ \lim\limits_{n\to\infty} |a_n| = |a|$ $\qquad \lim\limits_{n\to\infty} |a_n| = |a| \ \not\Rightarrow\ \lim\limits_{n\to\infty} a_n = a$ $\qquad a \neq 0$

Example: $a_n = \dfrac{(-1)^n\, n}{n+1}$ $\qquad |a_n| \to 1$ $\qquad$ but a_n does not converge either to 1 or -1

8. If $a_n \leq M$ for all n and $a_n \to a$, then $a \leq M$. However, even if $a_n < M$ for all M. We may not have $a < M$.

Examples: Consider $a_n = 1 + \frac{1}{n} \leq 2$ we have $\lim_{n\to\infty} a_n = 1 \leq 2$ Consider $a_n = \frac{n}{n+1} < 1$ but $\lim_{n\to\infty} a_n = 1$

9. Suppose $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is sequence of all real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \Longrightarrow \lim_{n\to\infty} a_n = L$$

Note that the converse of the above result is not true, i.e., $\lim_{n\to\infty} a_n = L$ may not imply $\lim_{x\to\infty} f(x) = L$