TRIPLE INTEGRALS

Triple Integral

Divide the region V into n sub regions of respective volumes $\delta V_1, \delta V_2, \dots, \delta V_n$

Let (x_j, y_j, z_j) be an arbitrary point in the jth sub-region.

Consider the sum $\displaystyle\sum_{j=1}^{n} f\left(x_j, y_j, z_j\right) \delta V_j$

$\boxed{\text{Represent Volume if } f = 1}$

If the limit exists as $n \to \infty$ and $\delta V_j \to 0$ then

$$\int\int\int_V f(x, y, z) dV = \lim_{n \to \infty} \sum_{j=1}^{n} f\left(x_j, y_j, z_j\right) \delta V_j$$

Triple Integral - Evaluation

$$\int\int\int_V f(x,y,z)\,dV = \int_{z=a}^{b}\left\{\int_{y=\psi_1(z)}^{\psi_1(z)}\left\{\int_{x=f_1(y,z)}^{f_2(y,z)} f(x,y,z)\,dx\right\}dy\right\}dz$$

Note: Similar to double integrals, the order of integration is immaterial if the limits of integration are constants.

$$\int_a^b\int_c^d\int_e^f f(x,y,z)\,dx\,dy\,dz = \int_e^f\int_c^d\int_a^b f(x,y,z)\,dz\,dy\,dx$$

$$= \int_c^d\int_e^f\int_a^b f(x,y,z)\,dz\,dx\,dy$$

Problem -1: Evaluate $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

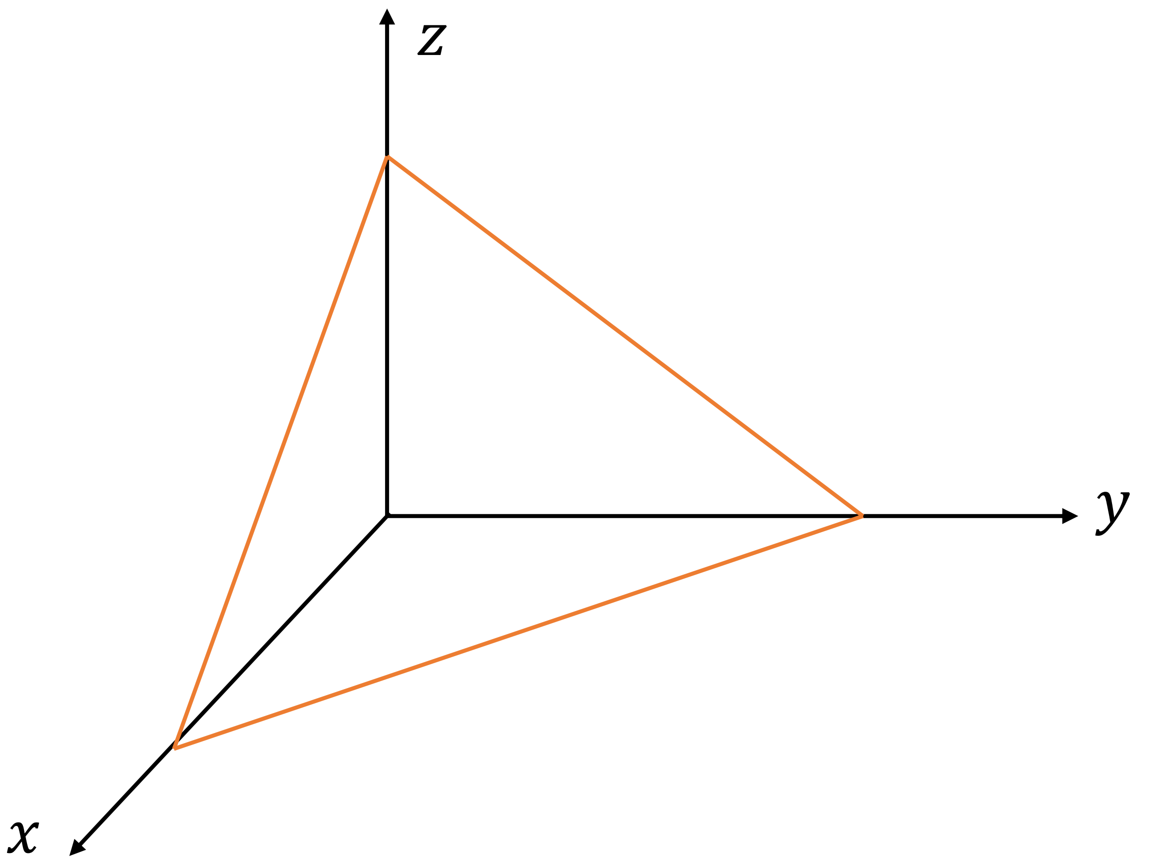
$$I = \int_0^a \int_0^x e^{x+y+z} \Big|_0^{x+y} dydx = \int_0^a \int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dydx$$

$$= \int_0^a \frac{e^{2(x+y)}}{2} \Big|_0^x dx - \int_0^a e^{x+y} \Big|_0^x dx = \frac{1}{2}\int_0^a \{(e^{4x} - e^{2x}) - 2(e^{2x} - e^{x})\} dx$$

$$= \frac{1}{2}\int_0^a (e^{4x} - 3\, e^{2x} + 2e^{x})\, dx = \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}$$

Problem -2: Evaluate $I = \iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$

R is the region bounded by
$x = 0, y = 0, z = 0 \;\&\; x + y + z = 1$



$$I = \int \int \int \frac{1}{(x+y+z+1)^3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dydx$$

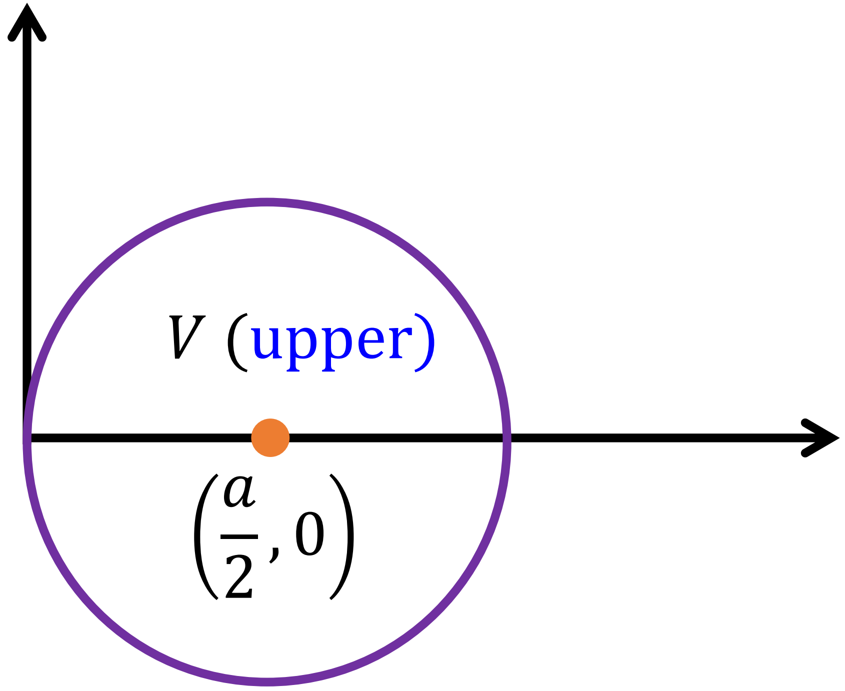
$$I = \int_0^1 \int_0^{1-x} \left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dydx \;= \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$$

Problem -3: Using triple integral find the volume common to a sphere $x^2 + y^2 + z^2 = a^2$ and a circular cylinder $x^2 + y^2 = ax$.

$$V = 4 \int\int\int_V dx\, dy\, dz = 4 \int\int\int_V dz\, dy\, dx$$

$$= 4 \int \int \qquad \int \qquad dz\, dy\, dx$$

$$= 4 \int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \sqrt{a^2 - x^2 - y^2}\; dy\, dx$$



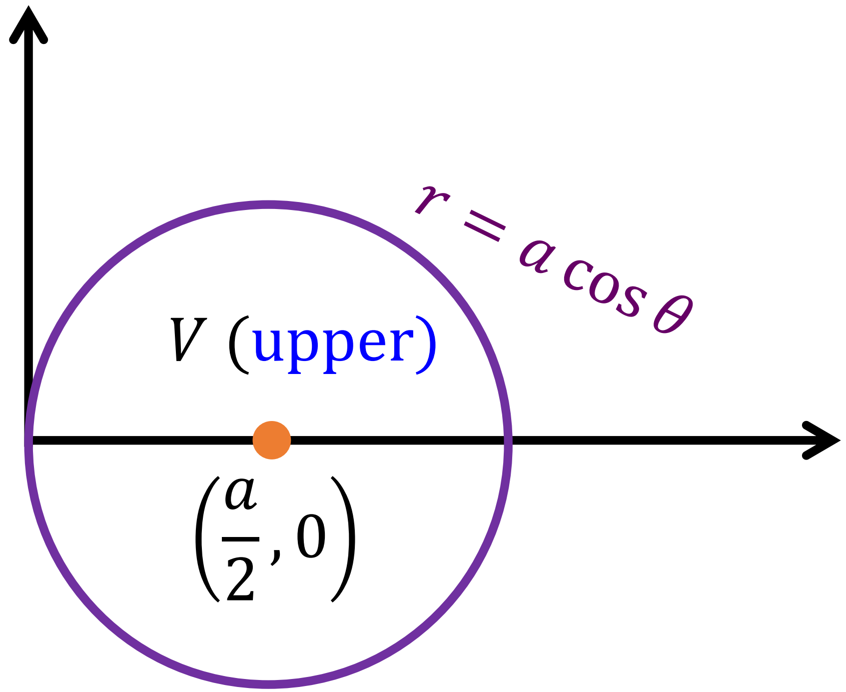
$$= 4\int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \sqrt{a^2 - x^2 - y^2}\; dy\, dx$$

$$= 4\int_0^{\frac{\pi}{2}} \int_{r=0}^{a\cos\theta} \sqrt{a^2 - r^2}\; r\, dr\, d\theta$$

$$= 4\left(-\frac{1}{2}\cdot\frac{2}{3}\right)\int_0^{\frac{\pi}{2}} \left[(a^2 - r^2)^{\frac{3}{2}}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{4}{3}a^3 \int_0^{\frac{\pi}{2}} (\sin^3\theta - 1)\, d\theta$$

$$= \frac{2}{3}a^3\left(\pi - \frac{4}{3}\right)$$



$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Triple Integrals – Change of Variables

Change of Variables in Triple Integrals:

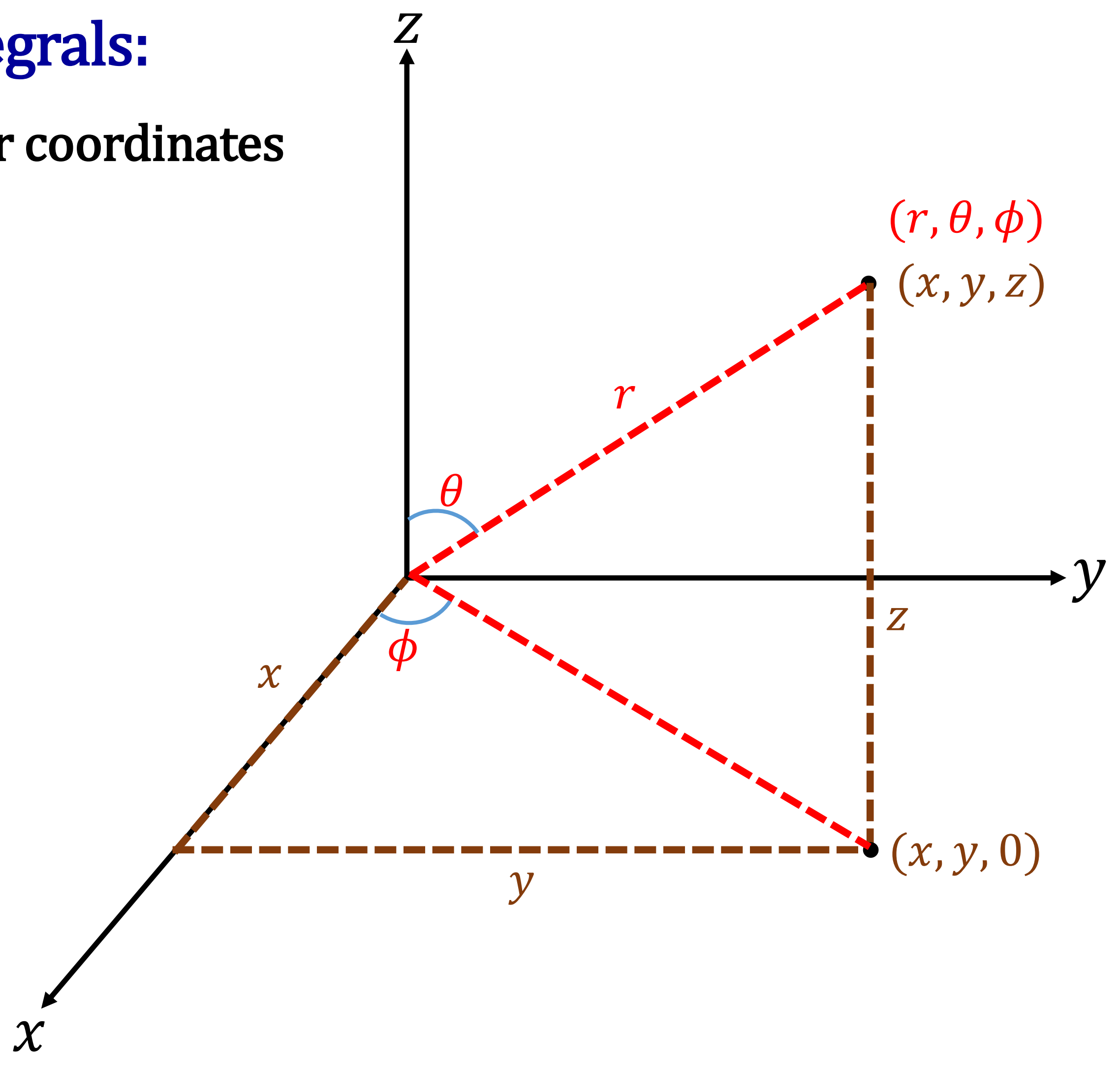
Cartesian coordinates to spherical polar coordinates

$$x = r\sin\theta\cos\phi\,,$$

$$y = r\sin\theta\sin\phi\,,$$

$$z = r\cos\theta$$

Note that $x^2 + y^2 + z^2 = r^2$



Change of Variables in Triple Integrals:

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Cartesian coordinates to spherical polar coordinates

$$\int\int\int_D f(x,y,z)\,dxdydz = \int\int\int_{\widehat{D}} f(r\sin\theta\cos\phi\,, r\sin\theta\sin\phi\,, r\cos\theta)\,|J|\,dr\,d\theta\,d\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Change of Variables in Triple Integrals:

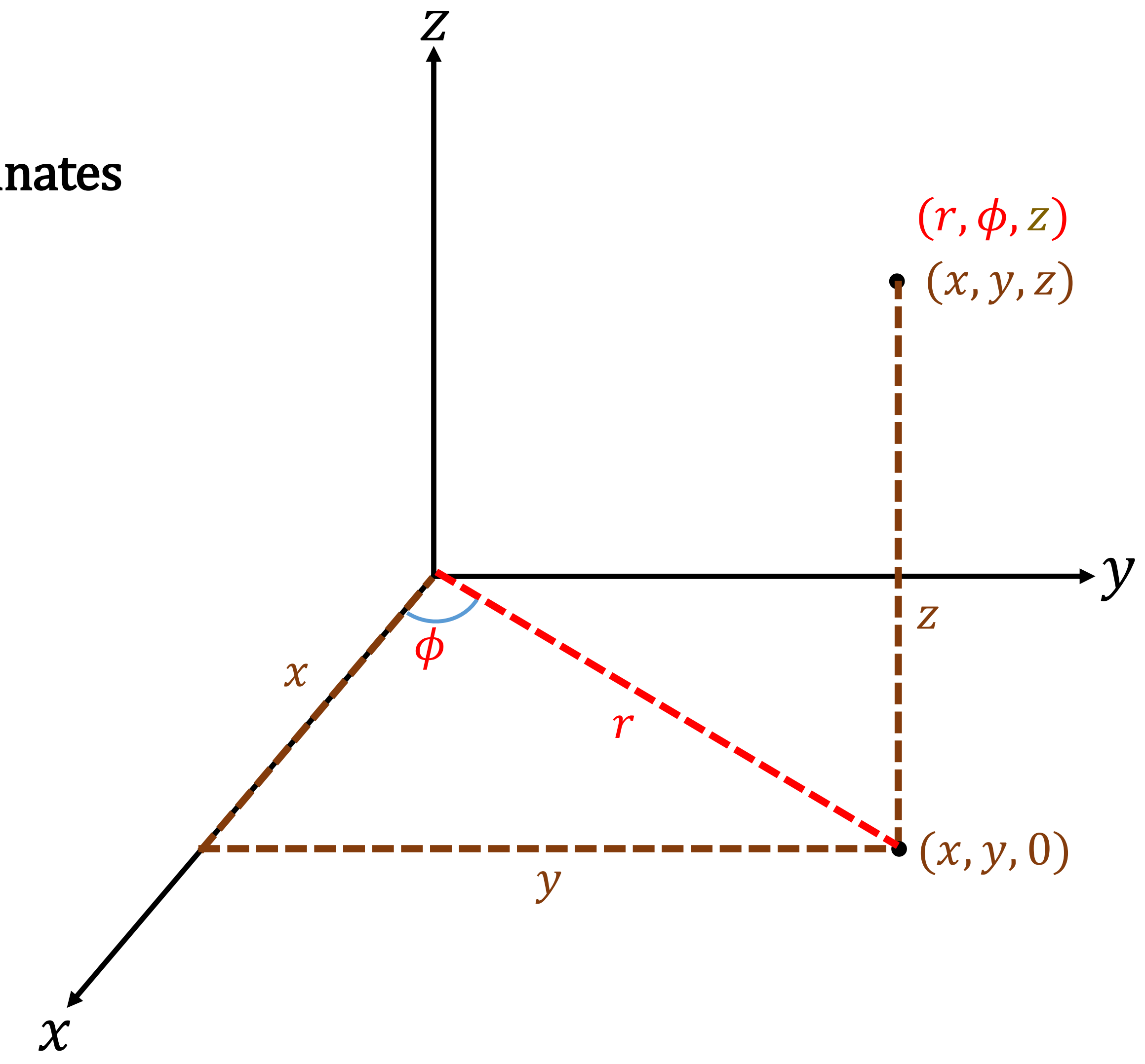
Cartesian coordinates to cylindrical coordinates

$$x = r\cos\phi\,,$$

$$y = r\sin\phi\,,$$

$$z = z$$

Note that $x^2 + y^2 = r^2$



Change of Variables in Triple Integrals:

$x = r\cos\phi \quad y = r\sin\phi \quad z = z$

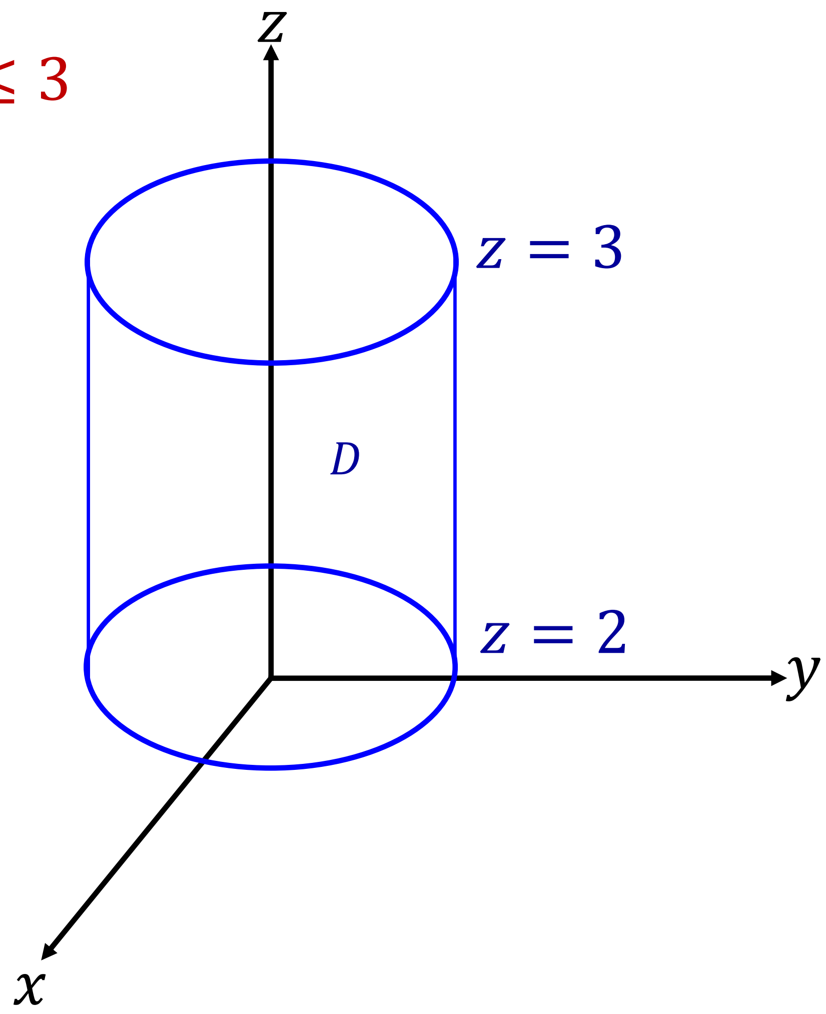
Cartesian coordinates to cylindrical coordinates

$$\int\int\int_D f(x,y,z)\, dxdydz = \int\int\int_{\hat{D}} f(r\cos\phi\,, r\sin\phi\,, z)\, |J|\, dr\, d\phi\, dz$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Problem -1: Changing to cylindrical coordinate, evaluate

$$\iiint_D z(x^2+y^2)\,dx\,dy\,dz \quad D:\ x^2+y^2 \le 1,\ 2 \le z \le 3$$



$$x = r\cos\phi\,,\quad y = r\sin\phi\,,\quad z = z$$

Note that $x^2+y^2 = r^2$ and $J = r$

$$I = \int_{z=2}^{3}\int_{\phi=0}^{2\pi}\int_{r=0}^{1} z\ r^2\, r\, dr\, d\phi\, dz$$

$$I = \int_{z=2}^{3} \int_{\phi=0}^{2\pi} \int_{r=0}^{1} z \;\; r^2 \, r \, dr \, d\phi \, dz$$

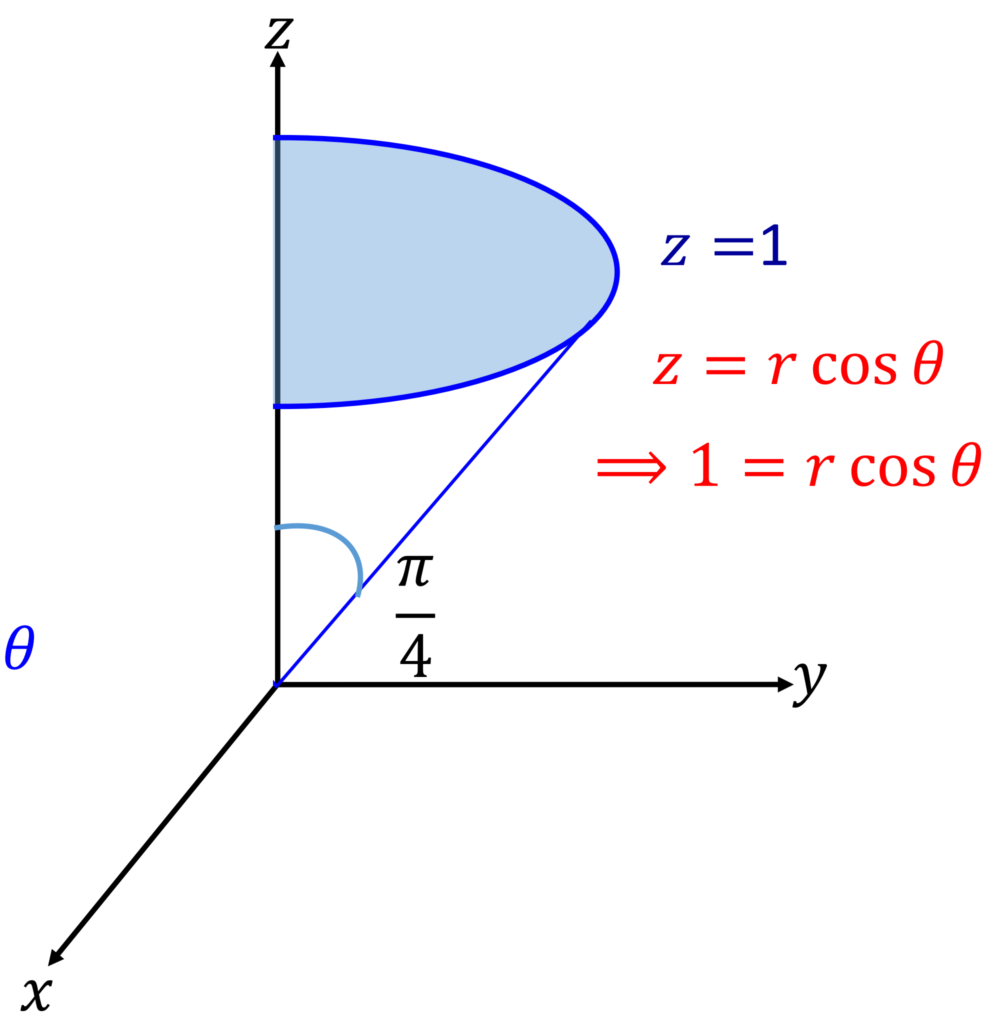
$$= \int_{z=2}^{3} \int_{\phi=0}^{2\pi} \quad \frac{1}{4} z \quad d\phi \, dz$$

$$= \frac{2\pi}{4} \int_{z=2}^{3} z \quad dz$$

$$= \frac{5\pi}{4}$$

Problem -2: Changing to spherical coordinate, evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$



$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

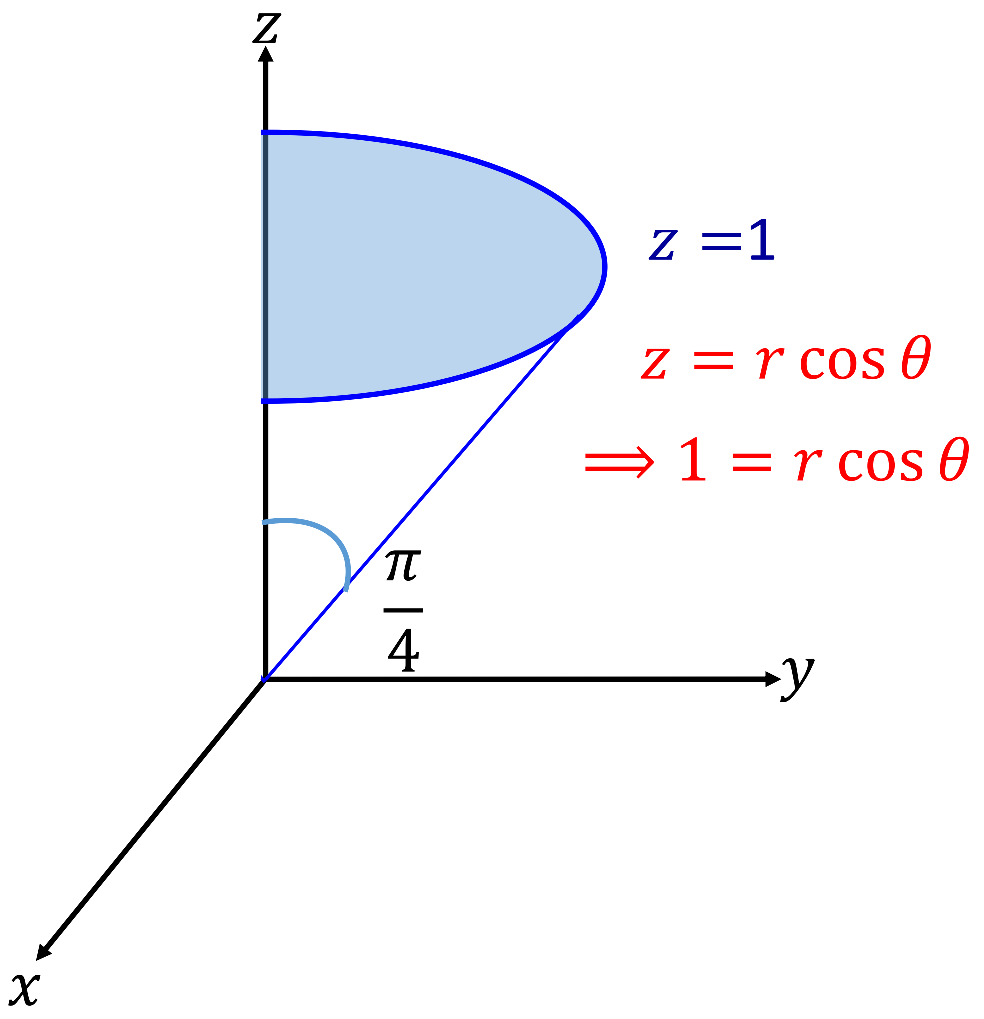
$J = r^2\sin\theta\,, \quad x^2+y^2+z^2 = r^2$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_{r=0}^{\sec\theta} \frac{1}{r}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{1}{2} \sec^2\theta\ \sin\theta\ \ d\theta\, d\phi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec\theta \Big|_0^{\frac{\pi}{4}}\ d\phi\ = \frac{(\sqrt{2}-1)\pi}{4}$$



Problem -3: Changing to spherical coordinate, evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2+y^2+z^2}}\, dz\, dy\, dx$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

$$J = r^2\sin\theta\,, \quad x^2+y^2+z^2 = r^2$$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{r=0}^{1} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\, dr\, d\phi\, d\theta$$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{r=0}^{1} \frac{r^2 \sin\theta}{\sqrt{1-r^2}}\, dr\, d\phi\, d\theta$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\theta\, d\phi\, d\theta$$

$$= \frac{\pi}{4}\frac{\pi}{2}\left[-\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{8}$$

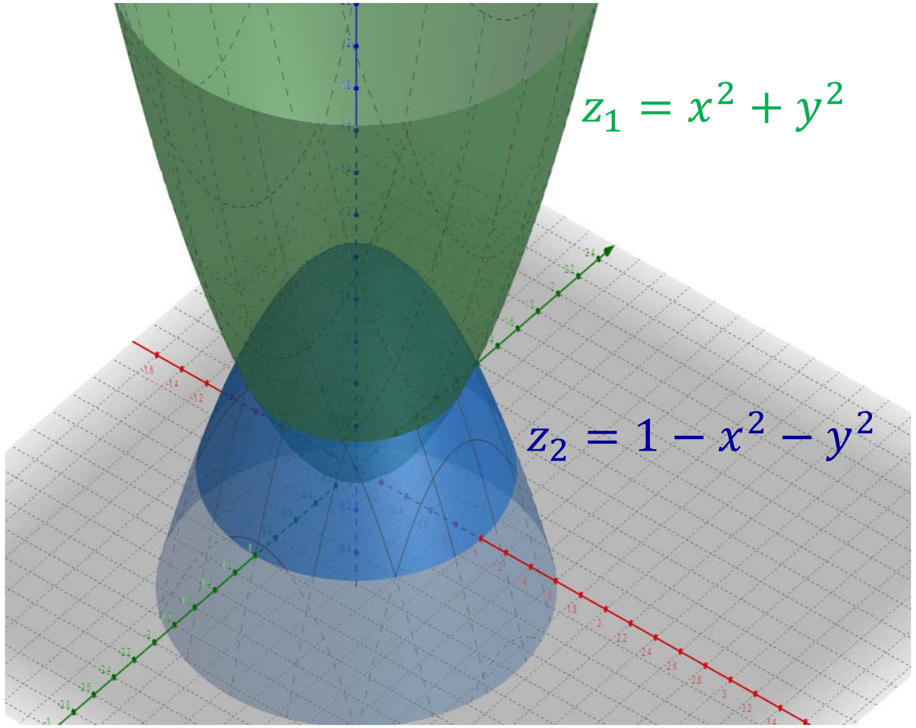
First evaluate $\displaystyle\int_{r=0}^{1} \frac{r^2}{\sqrt{1-r^2}}\, dr$

$$= \int_0^{\frac{\pi}{2}} \sin^2 t\ \ dt \quad (\text{sub. } r = \sin t)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2t)\, dt$$

$$= \frac{\pi}{4}$$

Problem -4: Find the volume of the solid formed by two Paraboloids: $z = x^2 + y^2$ & $z = 1 - x^2 - y^2$



Intersecting curve:

$$x^2 + y^2 = 1 - x^2 - y^2 \quad \Rightarrow \quad x^2 + y^2 = \frac{1}{2}$$

(Projection on xy plane)

$$V = \iiint_V dx\,dy\,dz = \int \quad \int \quad \int \quad dz\,dy\,dx$$

Changing to Cylindrical coordinates

$$V = 4\int \int \int \quad r\,dz\,dr\,d\theta = \frac{\pi}{4}$$

Conclusion:

Triple Integrals – Change of Variables

- **Spherical coordinates**
- **Cylindrical coordinates**

Thank You