Series solution of second order linear eq^n

- To deal with larger class of equations with variable coefficients if necessary to search our solution.

- Principal tool: Representation of a given function as a Power series Integration:

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ is called power series about } x = x_0.$$

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad \text{OR} \quad y'' + r_1(x)y + r_2(x)y = 0$$

$$e^x = \sum_{x=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(x-0)^n}{n!}$$

- convergence : A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to be converge at a pt. x if $\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$ exists for that x.

- Absolute convergence:

$$\sum_{n=0}^{\infty} |a_n| \, |(x-x_0)|^n \text{ converge}$$

$$\text{Convergence} \;\not\Rightarrow\; \text{Absolute convergence}$$
$$\Leftarrow$$

* Ratio test for convergence

If $a_n \neq 0$ & if for a fixed value of x,

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| \quad < 1 \text{ the series absolutely converges}$$

$$> 1 \text{ diverges}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x-x_0| \quad = 1 \text{ the test is incendusive.}$$

- If the power series converge absolutely for $x = x_0$,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$|x - x_0| <$$

Q. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(x+1)^{n+1}}{(n+1)\,2^{n+1}}}{\dfrac{(x+1)^n}{n\,2^n}} \right| = \lim_{n\to\infty} \left| \frac{(x+1)\,n}{(n+1)\times 2} \right| = \left| \frac{(x+1)}{2} \right|$$

$\left|\frac{x+1}{2}\right| < 1 \Rightarrow |x+1| < 2$ for absolute convergence.

$-3 < x < 1$

$|x+1| < 2$ is $-3 < x < 1$ converges

$|x+1| > 2$ $x < -3,\ x > 1$ diverges

$|x+1| = 2$ $x = 1$, $x = -3$

$$\sum_{n=0}^{\infty} \frac{2^n}{n\,2^n} = \sum_{n=0}^{\infty} \frac{1}{n} \qquad \sum_{n=0}^{\infty} \frac{\cancel{2^n}(-1)^n}{n\,\cancel{2^n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

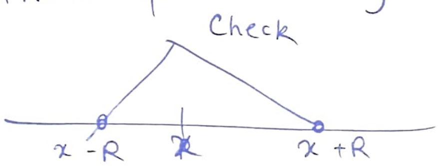
* Radius of convergence

There is a radius of con. as R s.t $\sum_{n=} a_n (x - x_0)^n$ converges absolutely for $|x - x_0| < R$ & diverges for $|x - x_0| > R$

1) For a series that converges only at $x = x_0$, we say that $R = 0$

2) For a series that converges for all x, we say $R = \infty$

3) $|x - x_0| < R$ for those pts. it may converge or diverge



The function as f is continuous & radius of converges of all order for $|x-x_0|<R$

f', f'' can be computed

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$$a_1 = f'(x_0) \leftarrow f'(x) = a_1 + 2a_2(x-x_0) + \cdots + n a_n (x-x_0)^{n-1} + \cdots$$

$$a_2 = \frac{f''(x_0)}{\lfloor 2} \leftarrow f''(x) = 2a_2 + \cdots + n(n-1) a_n (x-x_0)^{n-2} + \cdots$$

The value of a_n is given by $a_n = \frac{f^n(x_0)}{\lfloor n}$

Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{\lfloor k} (x-x_0)^k$$

Maclaurian series

$x_0 = 0$ $$\sum_{k=0}^{\infty} \frac{f^k(0)}{\lfloor k} x^k$$

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$
$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots$$
$$f''(x) = 2a_2 + 6a_3(x-x_0) + \cdots$$
$$a_n = \frac{f^n(x_0)}{n!}$$

Taylor Series $\Rightarrow \sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!}(x-x_0)^k$

$x_0 = 0 \Rightarrow$ Maclaurian series

*Analytive function

- A function f is s.t be analytic at x_0 if it's taylor Series about the point x_0

$$\sum_{n=0}^{\infty} \frac{f^{n}(x_0)\,(x-x_0)^n}{n!}$$ exists with radius of convergence R

- Ex. 1) All polynomial functions are analytic everywhere.
2) Rational functions are analytic except there is a value of x for which denominator is zero
3) $f(x) = \dfrac{1}{x^2 - 3x + 2}$ is analytic except $x = 1$ & $x = 2$

$$= \frac{1}{(x-1)(x-2)}$$

* SHIFT OF INDEX OF SUMMATION

Q.] Write $\sum_{n=2}^{\infty} a_n x^n$ as series where first term corresponds to $n = 0$

$n = m+2$

$$\sum_{m=0}^{\infty} a_{m+2}\, x^{m+2}$$

$$\sum_{n=0}^{\infty} a_{n+2}\, x^{n+2}$$

Hint: Let $m = n - 2$ ie $m + 2 = n$

$n = 2$ & $m = 0$

$$\sum_{n=0}^{\infty} a_{m+2}\, x^{m+2} \longrightarrow \sum_{n=0}^{\infty} a_{n+2}\, x^{n+2}$$

Q.] Write $x^2 \sum_{n=0}^{\infty} (r + n)\, a_n x^{r+n-1}$ as a series whose generic term involves x^{r+n}.

$$\sum_{n=0}^{\infty} (r+n)\, a_n\, x^{r+n+1} \qquad (n+1 = m$$

$$\sum_{n=0}^{\infty} (r+m-1)\, a_{m-1}\, x^{m+r}$$

$m = 1$

~~r+n+1 = m~~

$$\cancel{x^2 \sum_{n=0}^{\infty} (r+n)\, a_n x^{r+n-1}}$$

$$\cancel{\sum_{n=0}^{\infty} (r+n)\, a_n\, x^{r+n+1}}$$

$$\cancel{\sum_{n=0}^{\infty} (m-1)\, a_{m-1-r}\, x^m}$$

* Ordinary and secondary point of

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

$$y'' + \frac{a_1(x)}{a_0(x)}y' + \frac{a_2(x)}{a_0(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \longrightarrow (1)$$

- Ordinary point : A point is called an ordinary point of the DE (1) if $p(x)$ & $q(x)$ in the equivalent normalized diff eq^n are analytical at x_0.

- Singular point: If at least there $p(x)$ & $q(x)$ are not analytical at x_0. ($p(x)$ / $q(x)$ should not be analytic)

Ex. 1) $y'' + xy' + (x^2+2)y = 0$

$\underbrace{x}_{p(x)}$, $\underbrace{(x^2+2)}_{q(x)}$ $\rightarrow$ analytical at all point

$\therefore$ All the points are ordinary.

2) $(x-1)y'' + xy' + \frac{1}{x}y = 0$

$$y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

$\frac{1}{x(x-1)} \longrightarrow$ It's analytical at every point except $x=0$, $x=1$

$\frac{x}{x-1} \rightarrow$ It's analytical at every point except $x=1$

$\therefore$ $x = 0, 1$ are singular pts and other pts are ordinary.

- Singular points

$\rightarrow$ Regular singular pts: A singular pt $x=x_0$ of DE (1) $(x-x_0)p(x)$ & $(x-x_0)^2 q(x)$ are analytic at $x=x_0$

- $\lim_{x\to x_0}(x-x_0)p(x)$ a finite
- $\lim_{x\to x_0}(x-x_0)^2 q(x)$ a finite

$\rightarrow$ Irregular singular pts:

If either or both of these functions are not analytic at $x=x_0$.

* Ordinary and secondary point of the ODE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

$$y'' + \frac{a_1(x)}{a_0(x)}y' + \frac{a_2(x)}{a_0(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \longrightarrow (1)$$

- Ordinary point: A point is called an ordinary point of the DE (1) if $p(x)$ & $q(x)$ in the equivalent normalized diff eq^n are analytical at x_0.

- Singular point: If at least there $p(x)$ & $q(x)$ are not analytical at x_0. ($p(x)$ / $q(x)$ should not be analytic)

Ex. 1) $y'' + xy' + (x^2+2)y = 0$

$\underset{p(x)}{x}$, $\underset{q(x)}{(x^2+2)}$ → analytical at all point

∴ All the points are ordinary.

2) $(x-1)y'' + xy' + \frac{1}{x}y = 0$

$$y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

$\frac{x}{x-1}$ → It's analytical at every point except $x=1$

$\frac{1}{x(x-1)}$ → It's analytical at every point except $x=0, x=1$

∴ $x = 0, 1$ are singular pts and other pts are ordinary.

- Singular points

→ Regular singular pts: A singular pt $x = x_0$ of DE (1) $(x-x_0)p(x)$ & $(x-x_0)^2q(x)$ are analytic at $x = x_0$

- $\lim_{x\to x_0}(x-x_0)p(x)$ a finite
- $\lim_{x\to x_0}(x-x_0)^2q(x)$ a finite

→ Irregular singular pts: If either or both of these functions are not analytic at $x = x_0$

In ex.(2)

$$p(x) = \frac{x}{x-1} \qquad q(x) = \frac{1}{x(x-1)}$$

~~Let~~ At $x=0$

$$\lim_{x\to 0} (x-0)\frac{x}{x-1} = 0, \quad \lim_{x\to 0} (x-0)^2 \frac{1}{(x-1)x} = 0$$

$\therefore$ $x=0$ is a regular singular point

At $x=1$

$$\lim_{x\to 1} (x-1)\frac{x}{x-1} = 1, \quad \lim_{x\to 1} (x-1)^2 \frac{1}{(x-1)x} = 0$$

$\therefore$ $x=1$ is a regular singular point.

Q] $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$

* Series solution of differential eqn near are ordinary point

$$y'' + p(x)y' + q(x)y = 0 \longrightarrow (1)$$

Problem of solving eq (1) in of an ordinary pt x_0 We try to seek solutions of the form.

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

- Find series solution of D F

$$y'' + y = 0 \qquad -\infty < x < \infty$$

$p(x) = 0,\ q(x) = 1 \Rightarrow$ analytic at every pt.

$\therefore$ ~~ord~~ all the pts are ordinary pts.

check at $x=0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad |x-0| < R$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$y''+y=0 \Rightarrow \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n (x)^n = 0$$