Book Royce & Prina

~~Prob~~ Problem 41

The equation $P(x)y'' + Q(x)y' + R(x)y = 0$
is said to be exact if it can be written in the form
$$[P(x)y']' + [f(x)y]' = 0.$$
where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$ and $R(x)$.

The latter equation can be integrated resulting in a first order linear equation for y.

Ques Show that a necessary & sufficient condition for exactness is $P''(x) - Q'(x) + R(x) = 0$.

Let the linear differential equation of order 2 be
$$P_0\left(\frac{d^2y}{dx^2}\right) + P_1\left(\frac{dy}{dx}\right) + P_2 y = \phi(x) \quad \text{— (1)}$$
where P_0, P_1 & P_2 are functions of x alone.

Let (1) be exact.

$\mapsto$ It can be obtained from an equation of first order simply be differentiation.

We assume that (1) can be obtained by differentiating once the equation
$$P_0\left(\frac{dy}{dx}\right) + Q_1 y = \int \phi(x)\,dx + C \quad \text{— (2)}$$
where Q_1 is some function of x

Differentiating (2) w.r.t. x,
$$\left(P_0\frac{d^2y}{dx^2} + P_0'\frac{dy}{dx}\right) + \left(Q_1\frac{dy}{dx} + Q_1'y\right) = \phi(x)$$

or $$P_0\frac{d^2y}{dx^2} + (P_0' + Q_1)\frac{dy}{dx} + Q_1'y = \phi(x) \quad \text{— (3).}$$

Now (1) & (3) must be same, so equaling coefficients

$$P_1 = P_0' + Q_1 \;; \; \& \; Q_1' = P_2$$

$$\Rightarrow P_1' = P_0'' + Q_1'$$

$$\Rightarrow P_1' = P_0'' + P_2$$

$$\Rightarrow \boxed{P_0'' - P_1' + P_2 = 0}$$

Q. Show that equation

$$(1+x^2)y'' + 3xy' + y = 1 + 3x^2$$

is exact & hence solve the equation

Ans Given $(1+x^2)y'' + 3xy' + y = 1+3x^2$ —(1)

Comparing (1) with

$$P_0 y'' + P_1 y' + P_2 y = \phi(x)$$

Now we have

$$P_2 - P_1' + P_0'' = 1 - 3 + 2 = 0$$

Shows that (1) is exact & its first integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\, y = \int (1+3x^2)\,dx + c_1$$

or. $$(1+x^2)\left(\frac{dy}{dx}\right) + (3x - 2x)y = x + x^3 + c_1$$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{1+x^2}\, y = x \cdot + \frac{c_1}{1+x^2}$$

This is a linear eqn.

I.F $e^{\int \frac{x}{1+x^2}dx} = (1+x^2)^{-1/2}$

& soln is $y(1+x^2)^{\frac{1}{2}} = \int (1+x^2)^{\frac{1}{2}} \{x + c_1/(1+x^2)\}dx$

$+ c_1$

Q Test for exactness & solve

$$(1+x^2)y'' + 4xy' + 2y = \sec^2 x$$

given that $y = 0$, $y' = 1$ when $x = 0$.