MA 102: Linear Algebra, Integral Transforms & Special Functions
Tutorial Sheet - 7
Second Semester of the Academic Year: 2023-2024

Notation : Field $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

---

1. Label the following statements as true or false, and justify your answer.
   (a) Every change of coordinate matrix is invertible.
   (b) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β' -coordinate into β -coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

2. Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

3. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by $T(f(x)) = f'(x)g(x) + 2f(x)$ and $U(a + bx + cx^2) = (a + b, c, a - b)$. Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively. Compute $[UT]_\beta^\gamma$.

4. Label the following statements as true or false, and justify your answer.
   (a) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.
   (b) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

5. Obtain the inverse of each of the following matrices using row reduced echelon form.
   (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$

6. Let $T : M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ be the mapping defined by $T(A) = A^t$, the transpose of A.
   (a) Show that ± 1 are the only eigenvalues of T.
   (b) Describe the eigenvectors corresponding to the eigenvalues 1 and -1 respectively.

7. Prove that a linear operator T on a finite dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

8. Find the dimension of A, and a basis for row space of A consisting of original rows of A

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

9. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 0 \end{bmatrix}$ be a given matrix. Find row reduced echelon form R of A and an invertible matrix U such that $R = UA$.

10. Let $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$ be a given matrix.

(a) Find a basis of the row space of A.

(b) Find a basis for the column space of A.

(c) Find a basis for the null space of A.

11. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ be a given matrix.

(a) Find eigenvalues of the matrix A.

(b) Find eigenvectors corresponding to each eigenvalues.

12. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose range space $R(T) = V$, where $V = \{(x_1, x_2, x_3) : x_1 + 2x_2 + x_3 = 0\}$. Find the eigenvalues and eigenvectors for a matrix representation of the linear transformation T.

13. Let A be an $n \times n$ triangular matrix over the field $\mathbb{F}$. Prove that the eigenvalues of A are the diagonal entries of A.

14. Let V be the vector spaces of all the continuous functions from $\mathbb{R}$ into $\mathbb{R}$. Let T be the linear operator on V defined by:

$$(Tf)(x) = \int_0^x f(t)dt,$$

prove that T has no eigenvalues.

15. Let T be a linear operator on $\mathbb{F}^n$, let A be the matrix of T in the standard ordered basis for $\mathbb{F}^n$, let W be the subspace of $\mathbb{F}^n$ spanned by the column vectors of A. What does W have to do with T.

16. Let V be a two dimensional vector space over the field $\mathbb{F}$, and let β be an ordered basis for V. If T is a linear operator on V and

$$[T]_\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

prove that $T^2 - (a + d)T + (ad - bc)I = 0$.

17. For the given ordered basis β and β' find the change of coordinate matrix.

(a) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{(1, 0), (0, 1)\}$

(b) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$

18. Show that every non-singular matrix is a product of elementary matrices.

19. Suppose $T \in \mathcal{L}(V)$ is an invertible linear transformation.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T iff $\dfrac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

*********END*********