Continuity of a Functions of One Variable

A function $y = f(x)$ is said to be continuous at a point x_0 if

I. $f(x)$ is defined at x_0

II. $\lim_{x \to x_0} f(x)$ exists

III. $\lim_{x \to x_0} f(x) = f(x_0)$

Mathematically:

A function $y = f(x)$ is said to be continuous at a point x_0, if for a given $\epsilon > 0$, there exist a real number $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

Example: Discuss the continuity of $y = 2^{1/x}$

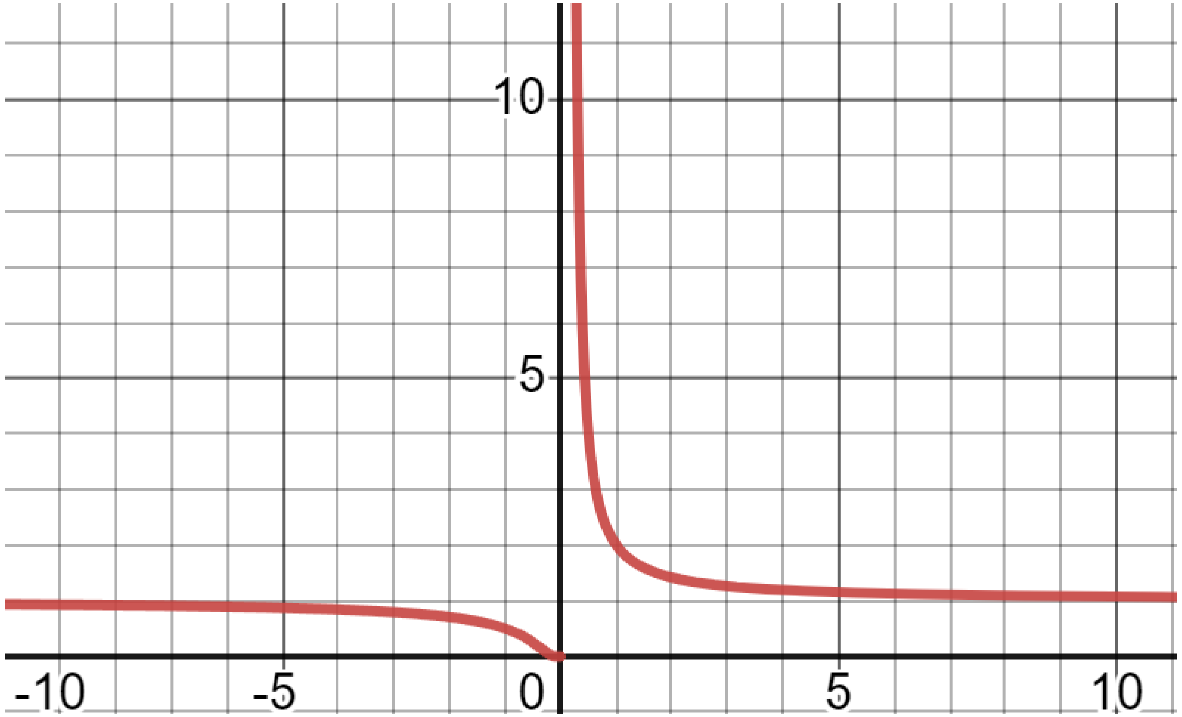
The given function is discontinuous at $x = 0$.

There are two reasons:

1. The function is not defined at $x = 0$

2. Compute limits:

$$\lim_{x \to 0-} 2^{\frac{1}{x}} = 0 \qquad \lim_{x \to 0+} 2^{\frac{1}{x}} = +\infty$$



Note:

- If the function $f(x)$ fails to be continuous at x_0, then we say that $f(x)$ is discontinuous at x_0.

- We say that $f(x)$ is continuous on D if $f(x)$ is continuous at every point of D.

Basic Properties:

1. Let $f, g: D \to \mathbb{R}$ be functions, where $D \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$.

   Suppose f and g are continuous at x_0. Then

(a) $f + g$, $f - g$, $f.g$, and cf (for any $c \in \mathbb{R}$) are continuous at x_0

(b) Further, if $g \neq 0, \forall x \in D$, and g is continuous at x_0, then f/g is continuous at x_0.

2. Let $D \subseteq \mathbb{R}$, $f: D \to \mathbb{R}$, and let $|f|$ be defined by $|f|(x) = |f(x)|$ for $x \in D$.

a) If f is continuous at a point $x_0 \in D$, then $|f|$ it continuous at x_0.

b) If $f(x) \geq 0\ \forall x \in D$ & f is continuous at x_0, then $\sqrt{f}$ is also continuous at x_0.

3. If f is continuous at x_0 and $f(x_0) \neq 0$, then $\exists \delta > 0 \ni f(x)$ and $f(x_0)$ have the same sign.

4. (Composites of continuous functions)
Let $D, E \subseteq \mathbb{R}$ and let $f: D \to \mathbb{R}$, $g: E \to \mathbb{R}$ be functions $\ni f(D) \subseteq E$.

If f it continuous at a point $x_0 \in D$ & g is continuous at $y_0 = f(x_0) \in E$, then the composition $gof: D \to \mathbb{R}$ is continuous at x_0.

Derivative

Let $y = f(x)$ be a function of single variable.

If the ratio

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}, \qquad \Delta x \neq 0$$

tends to a definite limit as Δx tends to 0.

Then this limit is called the derivative of $f(x)$ at the point x.

It is usually denoted by $f'(x)$ or $y'(x)$ or $\dfrac{dy}{dx}$

Differentiability & Differentials

A function $f(x)$ is said to be *differentiable* at the point x, if when x is given the increment Δx (arbitrary increment), the increment Δy can be expressed in the form

$$\Delta y = A\,\Delta x + \epsilon\,\Delta x$$

where A is independent of Δx and $\epsilon \to 0$ as $\Delta x \to 0$.

The first term on the right hand side ($A\,\Delta x$) is called **differential** (or Total differential) of y and is denoted by dy. Thus

$$dy = A\,\Delta x$$

Differentiability & Derivative

*The necessary and sufficient condition that the function $y = f(x)$ is **differentiable** at the point x is that it possesses a finite definite **derivative** at this point.*

Differentiability $\Longrightarrow$ Existence of Derivative

Suppose the function $y = f(x)$ is differentiable. This implies $\Delta y = A\,\Delta x + \epsilon\,\Delta x$.

Taking limit $\Delta x \to 0$, we get $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = A + \lim\limits_{\Delta x \to 0} \epsilon \;\; \Longrightarrow f'(x) = A$

$\Longrightarrow$ if $f(x)$ is differentiable then $f'(x)$ exists and has definite value A

Existence of Derivative $\Longrightarrow$ Differentiability

Differential of an independent variable x:
$dx = 1 \cdot \Delta x = \Delta x$

Conversely, if $f'(x)$ has definite value A then

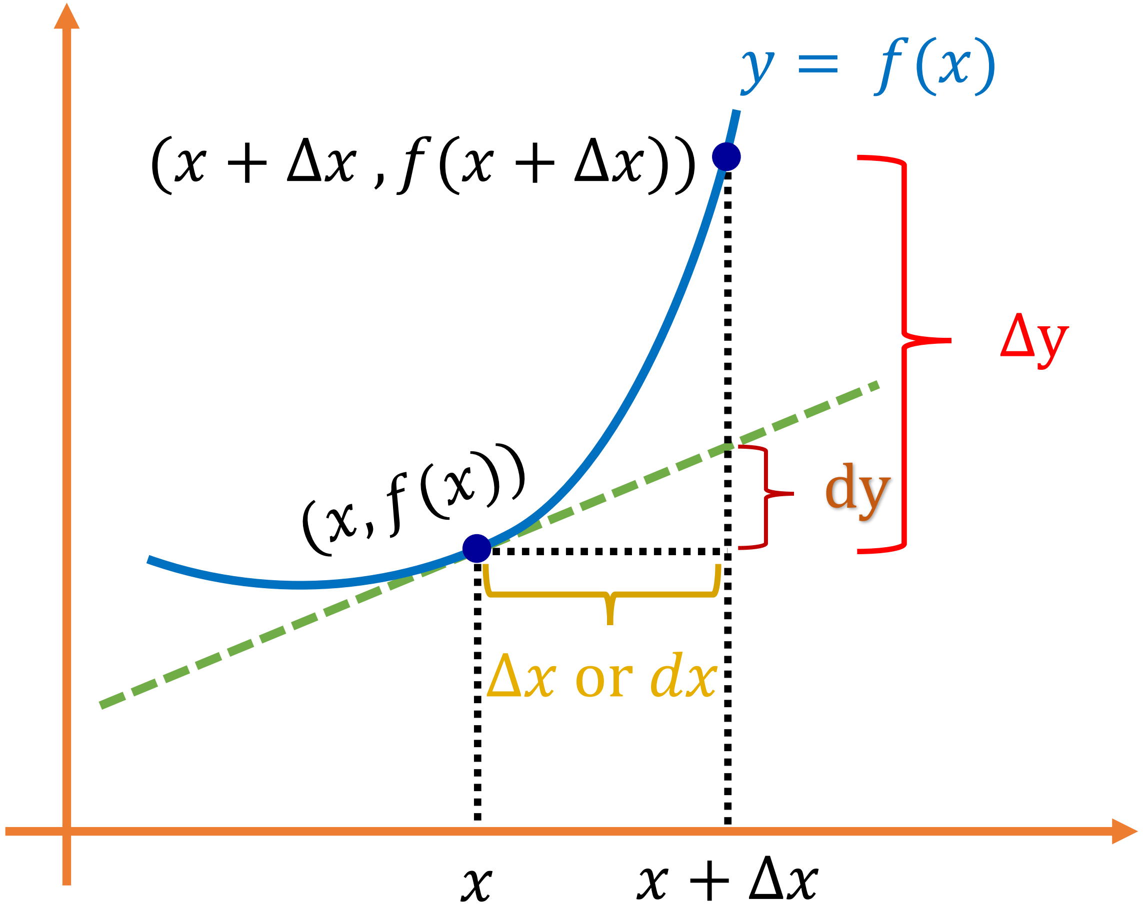
$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = A \quad \Longrightarrow \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = A + \epsilon, \qquad \epsilon \to 0, \text{ as } \Delta x \to 0$$

$$\Longrightarrow \quad f(x+\Delta x) - f(x) = A\,\Delta x + \epsilon\,\Delta x, \qquad \epsilon \to 0,$$

This implies, f is differentiable

REMARK: The differential of a function is the product of its derivative and an (arbitrary) increment Δx of the independent variable x, i.e., $dy = f'(x)\,\Delta x$

Geometrical Interpretation of Differentials



$$\Delta y = A\,\Delta x + \epsilon\,\Delta x$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = A$$

$$dy = A\,dx$$

Note: dy and dx measure changes along the tangent line

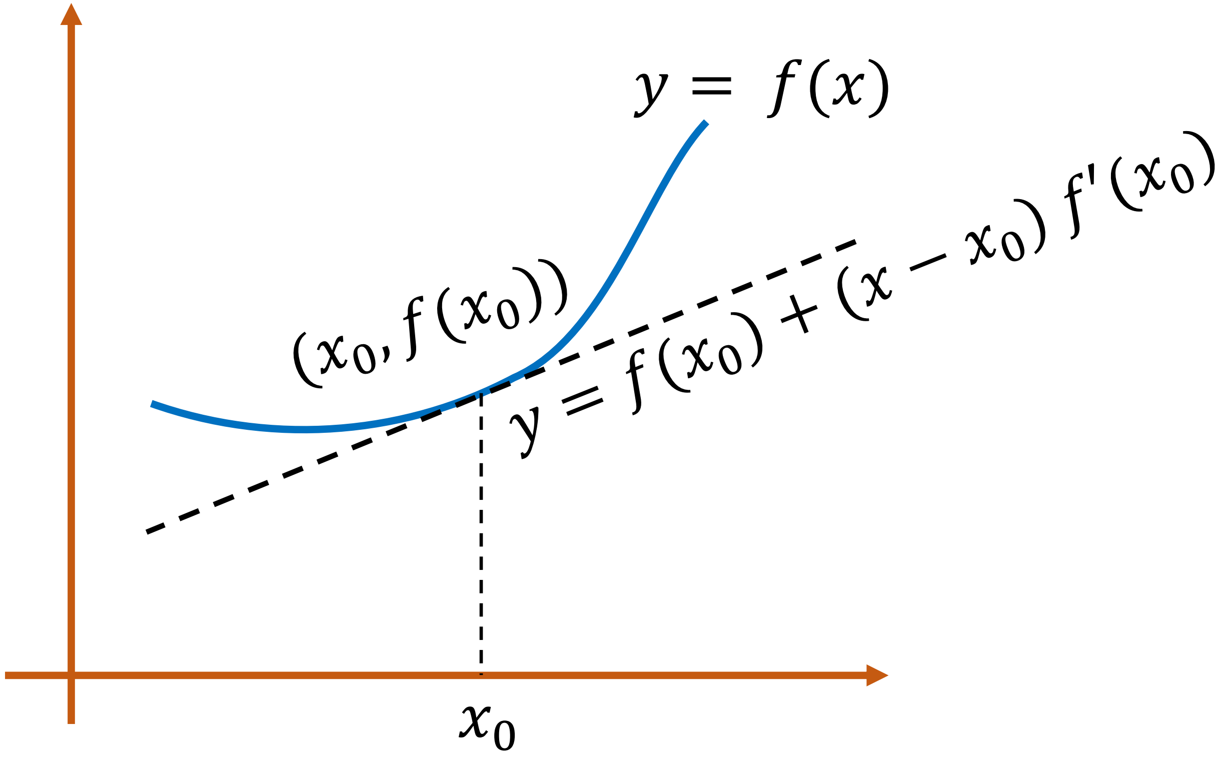
While Δy and Δx measure changes for the function $f(x)$

Geometrical Interpretation of Differentiability

A function $y = f(x)$ is said to be differentiable at the point $P(x_0, y_0)$ if it can be approximated in the neighborhood of this point by a linear function.

Mathematically,

$$f(x) = \underbrace{f(x_0) + (x - x_0)\ A}_{\text{linear function of } x} + \epsilon\,(x - x_0)$$



Equation of the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$

Testing Differentiability

- Existence of $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} =: f'(x)$

- $\Delta y = dy + \epsilon\, \Delta x, \qquad dy = A\, \Delta x$

- $\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0$

Example 1: Show that the function $f(x) = x^2$ is differentiable.

Let $y = f(x) = x^2$

$$\Delta y = f(x + \Delta x) - f(x) = \underbrace{2x}_{f'(x)} \; \Delta x + \underbrace{\Delta x}_{\epsilon} \; \Delta x$$

This implies the given function is differentiable and its derivative is $2x$.

Alternatively,

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x \qquad \text{OR} \qquad \lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0$$

Example 2: Given the function $y = x^2$, find Δy and dy at $x = 2$ and $\Delta x = 1, \Delta x = 0.1, \Delta x = 0.01$.

$$\Delta y = f(x + \Delta x) - f(x) \qquad \& \qquad dy = f'(x)dx$$

$\boldsymbol{\Delta x}$	$\boldsymbol{\Delta y}$	$\boldsymbol{dy}$
1		
0.1		
0.01		

Example 3: Find the derivative of the function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

First compute its derivative at $x = 0$: $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^2 \sin\left(\frac{1}{\Delta x}\right) - 0}{\Delta x} = 0$

Now compute its derivative at $x \neq 0$: $-\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$

$$\Rightarrow f'(x) = \begin{cases} -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Is f' continuous?

KEY TAKEAWAY

The function $y = f(x)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta y = A\,\Delta x + \epsilon\,\Delta x$$

where A is independent of Δx and ϵ is a function of Δx such that $\epsilon \to 0$ as $\Delta x \to 0$.

The linear function $A\,\Delta x$ is called the total differential of y at the point (x, y) and is denoted by dy.

The value of A is the derivative of f at x.

KEY TAKEAWAY

We call a function $y = f(x)$ differentiable at the point $P(x, y)$ if

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{exists.}$$

The value of the above limit is called the derivative of f at x.

Remark: *Note that* $\frac{dy}{dx}$ *is not just a notation for* $f'(x)$ *but it is a ratio of the two differentials. Therefore writing* dx *and* dy *alone makes sense.*