MA 102: Linear Algebra, Integral Transforms and Special Functions
Tutorial Sheet - 1
Second Semester of the Academic Year 2023-2024

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1. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define
$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$
$$c(x, y) = (cx, y).$$
Is V, with these operations, a vector space over the field of real numbers?

2. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define
$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0).$$
Is V, with these operations, a vector space over the field of real numbers?

3. On $\mathbb{R}^n$, define two operations
$$a \oplus b = a - b$$
$$ca = -ca.$$
The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, .)$?

4. Let V be the set C^2 with the usual vector addition, but with scalar multiplication defined by $\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha y \\ \alpha x \end{pmatrix}$. Determine whether or not V is a vector space with these operations.

5. Let V be the set C^2 with the usual scalar multiplication, but with vector addition defined by $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} y + w \\ x + z \end{pmatrix}$. Determine whether or not V is a vector space with these operations.

6. For the vector space $V = \mathbb{R}^3$ over $\mathbb{R}$, check whether $W \subseteq V$ as given below, is a subspace or not:
   (a) $W = \{(a, b, c) : a, b, c \in \mathbb{R} \mid a + b + c = 1\}$.
   (b) $W = \{(a, b, c) : a, b, c \in \mathbb{R} \mid b = 0\}$.
   (c) $W = \{(a, b, c) : a, b, c \in \mathbb{R} \mid a = b = c\}$.

7. Let $V = \mathbb{M}_{m,n}(\mathbb{R})$ be the vector space containing all $m \times n$ matrices with entries in $\mathbb{R}$. Then,
   (a) for $m = n$, prove that the set $W_1 \subseteq V$ consisting of all antisymmetric matrices forms a subspace of V.
   (b) for $m = n$, show that the set $W_1 \subseteq V$ of all matrices with $trace(M) = 0$ for all $M \in W_1$ is a subspace of V.

8. Prove that the intersection $W_1 \cap W_2$ of two subspaces $W_1, W_2 \subseteq V$ is again a subspace of V.

9. Let A be a 2x3 matrix.
   (a) Let $U = \{x \in \mathbb{R}^3 : Ax = 0\}$. Show that U is a subspace of $\mathbb{R}^3$.
   (b) Is $W = \{x \in \mathbb{R}^3 : Ax = b\}$ a subspace when $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$? Explain.

10. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

11. Let V be the vector space of the functions $f : \mathbb{R} \to \mathbb{R}$. Show that W is a subspace of V, where:
    (a) $W = \{f(x) : f(1) = 0\}$, all functions whose value at 1 is 0.
    (b) $W = \{f(x) : f(3) = f(1)\}$, all functions assigning the same value to 3 and 1.
    (c) $W = \{f(x) : f(-x) = -f(x)\}$; the set of odd functions.

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