Differential Calculus
Functions of Single Variable

Sequence and Series

- ❑ Sequence
- ❑ Series
- ❑ Power Series

Series: Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of ream numbers, then $a_1 + a_2 + \cdots + a_n + \cdots$ is called a series.

Notation: $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

The sum of first n terms $\sum_{i=1}^{n} a_i$ is denoted by S_n

S_n: nth partial sum

$\{S_n\}$: Sequence of partial sums

- A series is the sum of a sequence
- The partial sum of a series again forms a sequence

Convergence of Series

- If S_n tends to a finite limit as $n \to \infty$, the series $\sum a_n$ is said to be convergent.

- If S_n does not tend to a finite limit as $n \to \infty$, the series $\sum a_n$ is said to be divergent.

In some literature, divergence is referred as

- If S_n tends to $\pm\infty$ as $n \to \infty$, the series $\sum a_n$ is said to be divergent.

- If S_n does not tend to a unique limit as $n \to \infty$ then the series $\sum a_n$is said to be oscillatory or non-convergent.

Example: Determine whether the series $1 - 1 + 1 - 1 + \cdots$ converge or diverge.

$S_1 = 1$ $\quad$ $S_2 = 0$ $\quad$ $S_3 = 1$ $\quad$ $S_4 = 0$

Thus the sequence of partial sums is $1, 0, 1, 0, 1, \ldots$

Divergent (Oscillatory) sequence

Geometric Series:

A geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \quad (a \neq 0)$

converges if

If the series converges then the sum is

Case $r = 1$: The series $a + a + a + \cdots$ $\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} a = \pm\infty$

Case $r = -1$: The series $a - a + a - \cdots$ The sequence of partial sum: $a, 0, a, 0, \ldots$

diverges

Case $r \neq 1$: The series $a + ar + ar^2 + ar^3 + \cdots$

$$S_n = \frac{a(1 - r^{n+1})}{1 - r} = \frac{a}{1 - r} - \frac{ar^{n+1}}{1 - r}$$

Case $|r| < 1$: $r^n \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} S_n = \dfrac{a}{1 - r}$ converges

Case $|r| > 1$: $\begin{cases} r^n \to \infty \text{ as } n \to \infty, \text{if } r > 1 \\ r^n \text{oscillates and grow in magnitude as } n \to \infty, \text{if } r < -1 \end{cases}$ diverges

Test for Convergence:

Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with non-negative terms and suppose that

$$a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3, \ldots \quad (\text{OR } a_n \leq b_n \text{ for all } n > N)$$

- If the "*bigger series*" $\sum b_n$ converges then the "*smaller series*" $\sum a_n$ also converges.
- If the "*smaller series*" $\sum a_n$ diverges then the "*bigger series*" $\sum b_n$ also diverges.

Example: Consider the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \cdots$

Compare it with the series

$$S:\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots = 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

Note that each term of the harmonic series is greater or equal than the corresponding term of "S".

Clearly the series "S" is divergent and hence by comparison test the Harmonic series.

Standard Series (Useful for Comparison):

1. Geometric series

2. p − series

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$ converges for and diverges for

Case 1: $p<1$: $$1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots>1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$ divergent

Case 1: $p>1$: $$\text{Compare } 1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}+\frac{1}{9^p}\ldots+\frac{1}{15^p}\cdots \text{with}$$

("Bigger series") $$1+\frac{1}{2^p}+\frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{8^p}+\frac{1}{8^p}\ldots+\frac{1}{8^p}\cdots \text{with}$$

$$=1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\cdots$$ Geometric series

Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n > N$

If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \rho > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge

If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges then $\sum a_n$ converges

If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges

Example: Test whether the following series converge or diverge $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$

Consider $b_k = \dfrac{1}{2k^2}$ $\qquad \lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = 1$

The given series converges

Ratio Test

Let $\sum a_n$ be a series with $a_n > 0$ for $n > N$ and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- If $\rho < 1$, the series converges
- If $\rho > 1$ or $\rho = \infty$, the series diverges
- If $\rho = 1$, test fails

Example: Test whether the following series converge or diverge $\displaystyle\sum_{i=1}^{\infty} \frac{1}{k!}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

The given series converges

Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n > N$ and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

Then

- If $\rho < 1$, the series converges
- If $\rho > 1$ or $\rho = \infty$, the series diverges
- If $\rho = 1$, test fails

Example: Test whether the following series converge or diverge $\displaystyle\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$

$$\rho = \lim_{k\to\infty} \frac{4k-5}{2k+1} = 2 > 1$$

The series diverges

Example: consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Root or Ratio test fails in both the cases.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

Divergent Test (Necessary Condition for Convergence)

The series $a_1 + a_2 + a_3 \cdots$ is convergent, only when $\lim_{n\to\infty} a_n = 0$

$$S_n = S_{n-1} + a_n \qquad \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \qquad \Rightarrow \lim_{n\to\infty} a_n = 0$$

- If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum a_n$ diverges

Example: The series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$ diverges since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

$$\left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots\right)$$

Divergent Test (Necessary Condition for Convergence)

$\boxed{\text{The series } a_1 + a_2 + a_3 \cdots \text{ is convergent, only when } \lim_{n\to\infty} a_n = 0}$

- If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ may either converge or diverge (the condition is not sufficient)

Example: Consider the Harmonic series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} + \cdots$

Note that $\lim_{n\to\infty} a_n = 0$ but the series is divergent!

Divergent Test (Necessary Condition for Convergence)

The series $a_1 + a_2 + a_3 \cdots$ is convergent, only when $\lim_{n\to\infty} a_n = 0$

Alternating Series Test

For a decreasing series ($a_{n+1} \leq a_n$) (*non-increasing*) whose terms are alternatively positive and negative (alternating series), the condition $\lim_{n\to\infty} a_n = 0$ is also **sufficient for convergence**

Example: The alternating Harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ is convergent

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

Example: Test the following alternating series convergnece

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}$$

$$\frac{a_{k+1}}{a_k}=\frac{k(k+4)}{(k+2)(k+3)}=\frac{k^2+4k}{k^2+5k+6}=\frac{k^2+4k}{(k^2+4k)+(k+6)}<1$$

The given series is *decreasing series*.

Now check $\lim_{k\to\infty}\frac{k+3}{k(k+1)}=\lim_{k\to\infty}\frac{\frac{1}{k}+\frac{3}{k^2}}{1+\frac{1}{k}}=0$

Using alternating series test, the given series is convergent.

Alternating Series (terms are alternately positive and negative)

Absolutely Convergent: An alternating series $\sum a_n$ is called absolutely convergent if the series of the absolute values of the terms, i.e., $\sum |a_n|$ converges.

Conditionally Convergent: A series that converges but does not converge absolutely **converges conditionally**.

Example: The alternating Harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ is conditionally convergent

The alternating series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots$ is absolutely convergent

NOTE: Absolute convergence implies convergence!

Example: Consider $1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots \qquad (*)$

If $p > 1$, the series converges **absolutely**. If $0 < p \leq 1$, the series converges **conditionally**.

If $p > 1$: it is known that the series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ converges

If $p > 0$: the sequence $\left\{\frac{1}{n^p}\right\}$ is a decreasing sequence with $\frac{1}{n^p} \to 0$, hence the series (*) converges for $p > 0$

Ratio Test & Root Test

Let $\sum a_n$ be a series (with non-zero terms for ratio test) and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho$$

Then

- If $\rho < 1$, the series converges absolutely
- If $\rho > 1$ or $\rho = \infty$, the series diverges
- If $\rho = 1$, test fails

Example: Test the following series for convergence $\sum_{k=1}^{\infty}(-1)^k \frac{2^k}{k!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{k+1}\,k!}{2^k(k+1)!} = \lim_{n\to\infty}\frac{2}{(k+1)} = 0$$

The given series converges absolutely.

POWER SERIES

A series of the form $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.

is called a power series in $(x - x_0)$ or a power series at x_0.

Here x is a variable and a_i's & x_0 are constants.

Example: Consider the power series centered at $x = 2$: $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|(x-2)|\, n}{n+1} = |(x-2)|$

Thus the power series is convergent when $|x - 2| < 1, \text{i.e.}, 1 < x < 3$

Divergent when $|x - 2| > 1, \text{i.e.}, x < 1 \text{ or } x > 3$

The power series converges for $1 \leq x < 3$

When $x = 3$, the power series is the harmonic series, which diverges

When $x = 1$, the power series is the alternating harmonic series, which converges

For every power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, there exists a non-negative real number R such that for every $|x-x_0| < R$, the series is absolutely convergent and for $|x-x_0| > R$, the series is divergent.

We write $R = \infty$ if the series converges for all x and $R = 0$ if the series converges only for $x = x_0$.

No general statement can be made about the convergence of a power series on end points $x = x_0 + R$ and $x = x_0 - R$.

The interval of convergence of a power series is the interval that consists of all values of x for which the series is convergent

The radius of convergence of the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ can be calculated as

$$R = \frac{1}{\lim_{n\to\infty} \left|\left(\frac{a_{n+1}}{a_n}\right)\right|} = \lim_{n\to\infty} \left|\left(\frac{a_n}{a_{n+1}}\right)\right| \qquad \text{or} \qquad R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

How to Test a Power Series for Convergence

- Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely.
- If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint.