Mean Value Theorems

- Rolle’s Theorem
- Proof of Rolle’s Theorem
- Worked Problems

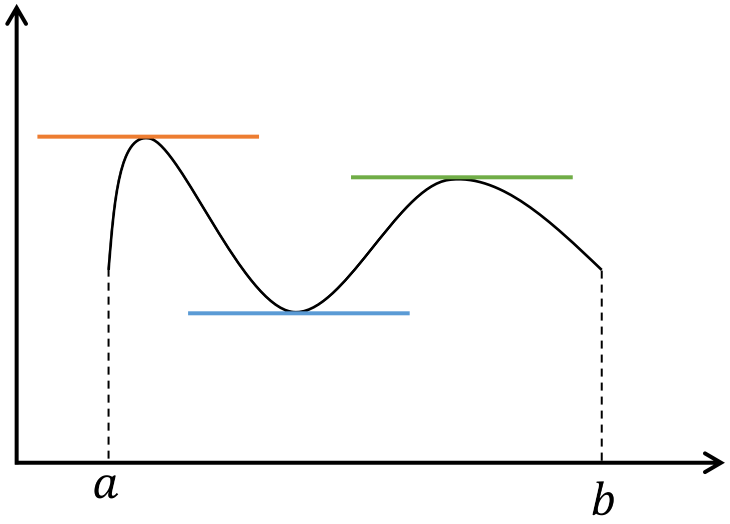
Rolle's Theorem

If a function f is

a) Continuous in $[a, b]$

b) Differentiable in (a, b)

c) $f(a) = f(b)$



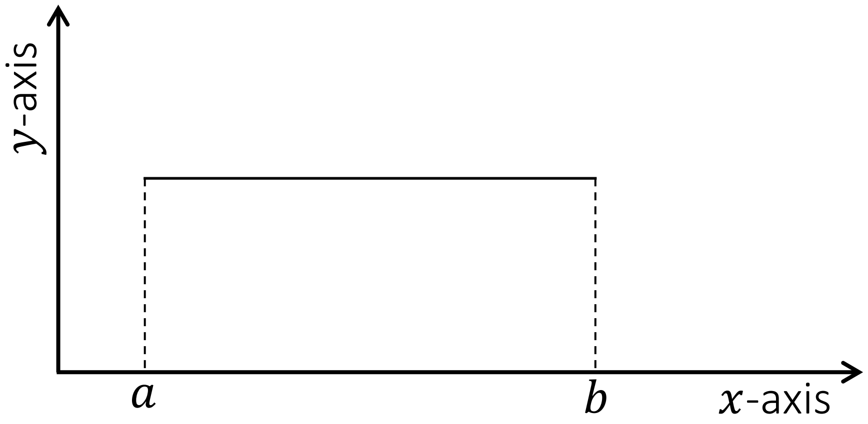
Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$

Proof of Rolle's Theorem

Suppose M & m are maximum and minimum of f in $[a, b]$

(Extreme value theorem: A continuous function on $[a,\ b]$ reaches its maximum and minimum)

Case –I ($M = m$)



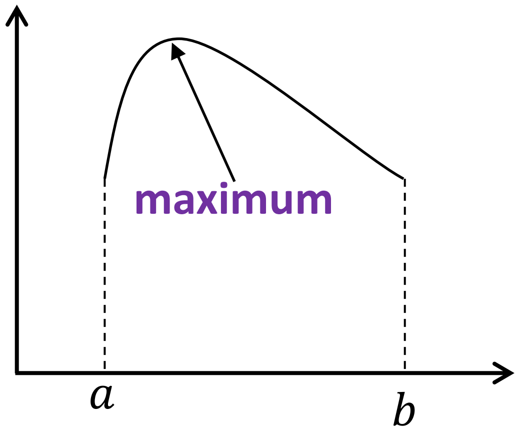
In this case:
$f(x) = M = m =$ constant

This implies $f'(x) = 0, \quad \forall\, x \in (a, b)$

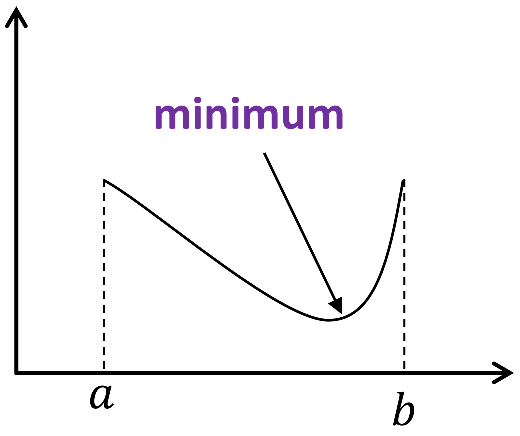
Proof of Rolle's Theorem

Suppose M & m are maximum(local) and minimum(local) of f in $[a, b]$

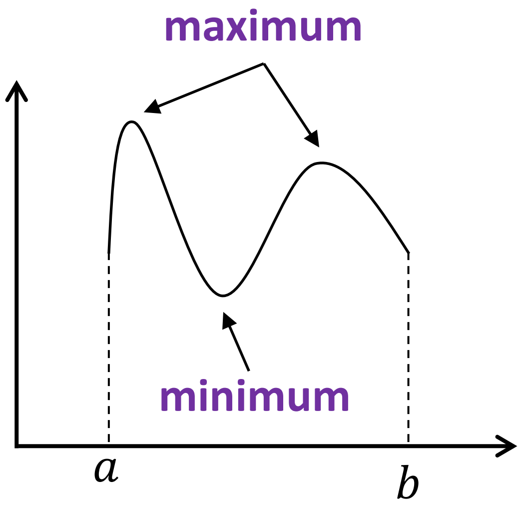
Case –II ($M \neq m$)



maximum is different



minimum is different



both are different

Proof of Rolle's Theorem: Case –II ($M \neq m$)

Suppose M is different from the equal values $f(a)$ & $f(b)$ and let $f(c) = M$

Since $f(c)$ is the maximum value, we have

$$f(c+\Delta x) - f(c) \leq 0, \quad \text{for} \quad \Delta x > 0 \text{ or } \Delta x < 0$$

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \leq 0 \quad \text{for} \quad \Delta x > 0 \Rightarrow \lim_{\substack{\Delta x \to 0 \\ \Delta x > 0}} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c) \leq 0$$

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \geq 0 \quad \text{for} \quad \Delta x < 0 \quad \Rightarrow f'(c) \geq 0$$

$f'(c) = 0$

Remark - 1

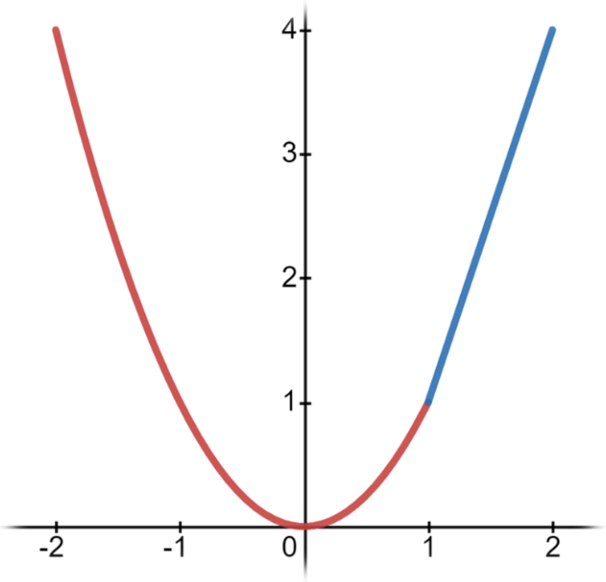
The hypotheses of Rolle's theorem are sufficient but not necessary for the conclusion.

$$Continuity\ in\ [a\ b], differentiability\ in\ (a,b), \big(f(a)=f(b)\big) \Longrightarrow f'(c)=0$$

Meaning, if all three hypotheses are met then conclusion is guaranteed. However, if the hypotheses are not met then you may or may not reach the conclusion.

Consider:

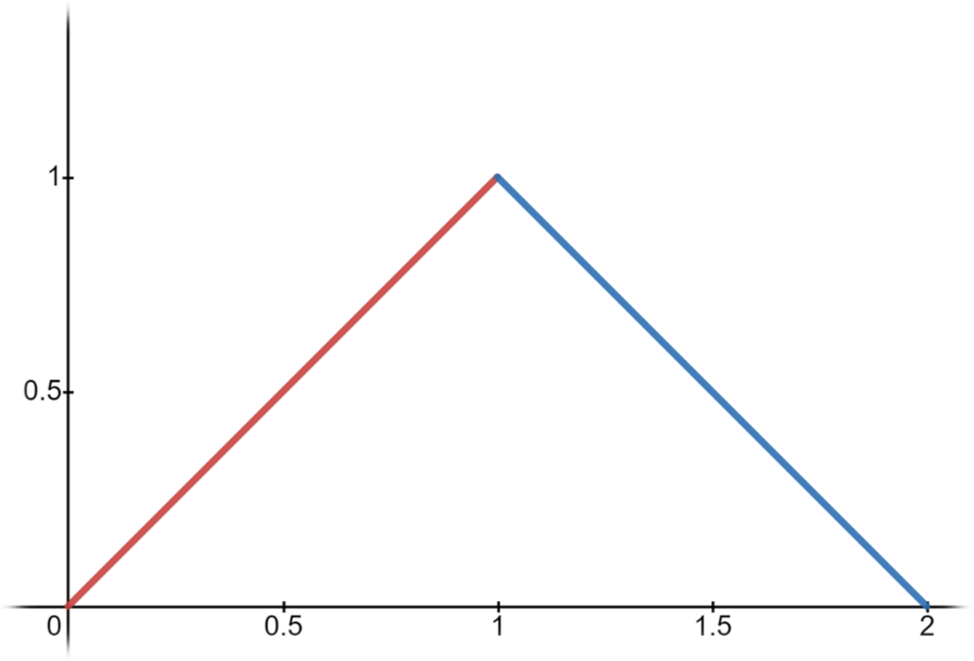
$$f(x) = \begin{cases} x^2; & -2 \le x \le 1 \\ 3x-2; & 1 < x \le 2 \end{cases}$$



$f'(0) = 0$

Further, consider:

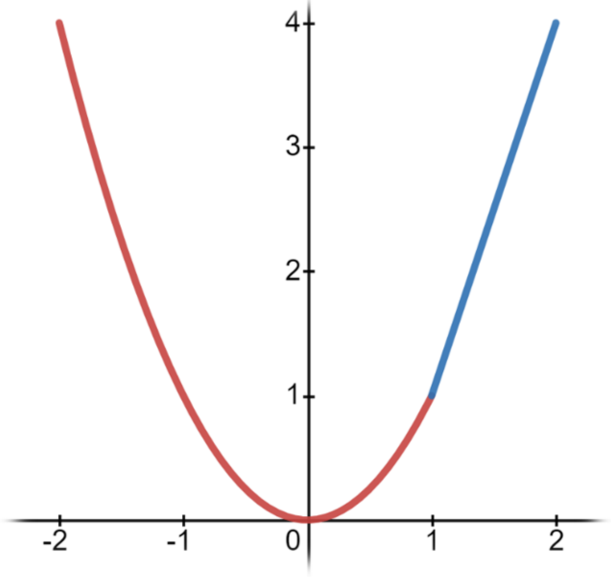
$$f(x) = \begin{cases} x; & 0 \le x \le 1 \\ 2 - x; & 1 < x \le 2 \end{cases}$$



$f'(x) \neq 0$, for any $x \in (0, 2)$

Previous Example:

$$f(x) = \begin{cases} x^2; & -2 \le x \le 1 \\ 3x - 2; & 1 < x \le 2 \end{cases}$$



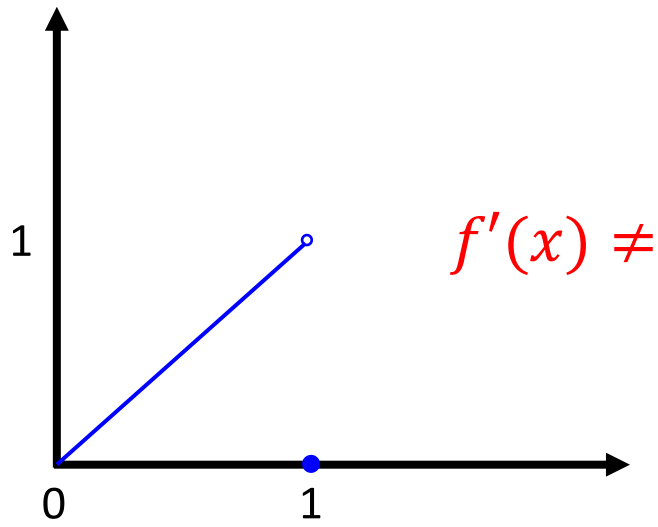
$f'(0) = 0$

Remark - 2

The continuity condition for the function on the closed interval $[a, b]$ *is essential.*

Consider:

$$f(x) = \begin{cases} x; & 0 \leq x < 1 \\ 0; & x = 1 \end{cases}$$

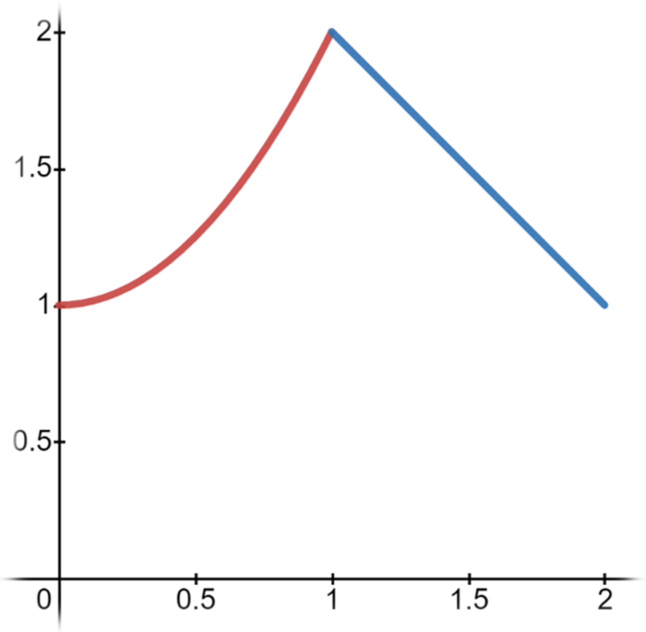


$f'(x) \neq 0$ for any $x \in (0, 1)$.

Note that f is continuous and differentiable on $(0, 1)$, and also $f(0) = f(1)$.

Example - 1 Discuss the applicability of Rolle's theorem to the function

$$f(x) = \begin{cases} x^2 + 1\,, & x \in [0, 1] \\ 3 - x, & x \in (1, 2] \end{cases}$$



Example - 2 Using Rolle 's Theorem, show that the equation $x^{13}+7x^3-5=0$ has exactly one real root in $[0,1]$.

Suppose $f(x) = x^{13}+7x^3-5$ has more than one real root in $[0,1]$.

Take any two roots, say α and β, that is, we have $f(\alpha)=0=f(\beta)$, $0<\alpha<\beta<1$

Rolle 's Theorem implies $f'(c)=0$ for some $c\in(\alpha,\beta)$

This implies $13\mathrm{c}^{12}+21\mathrm{c}^2=0$ for some $c\in(\alpha,\beta)$

Note that $c>0$ and therefore $13\mathrm{c}^{12}+21\mathrm{c}^2\neq 0$.

It contradicts our assumption of more than one real root.

On the other hand $f(0)=-5$ and $f(1)=3$.

It confirms the existence of at least one root.

Key Takeaway

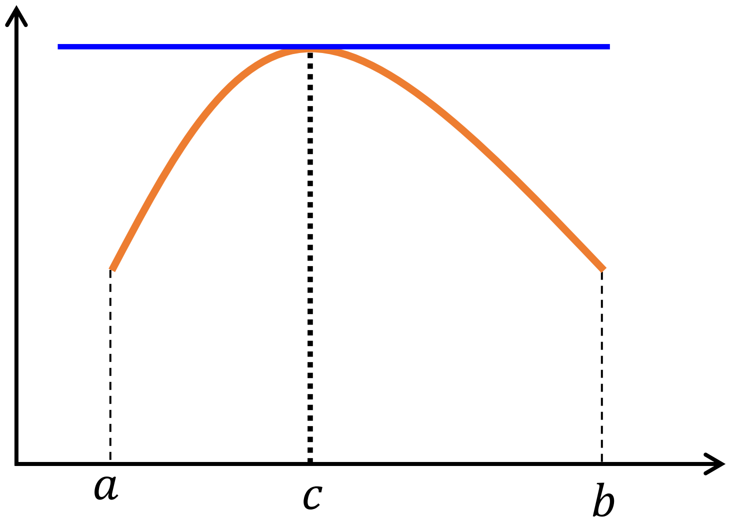
Rolle’s Theorem

If a function f is

$a)$ Continuous in $[a, b]$

$b)$ Differentiable in (a, b)

$c)$ $f(a) = f(b)$



Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$

Mean Value Theorems

- Lagrange's Mean Value Theorem
- Cauchy's Mean Value Theorem

Lagrange's Mean Value Theorem

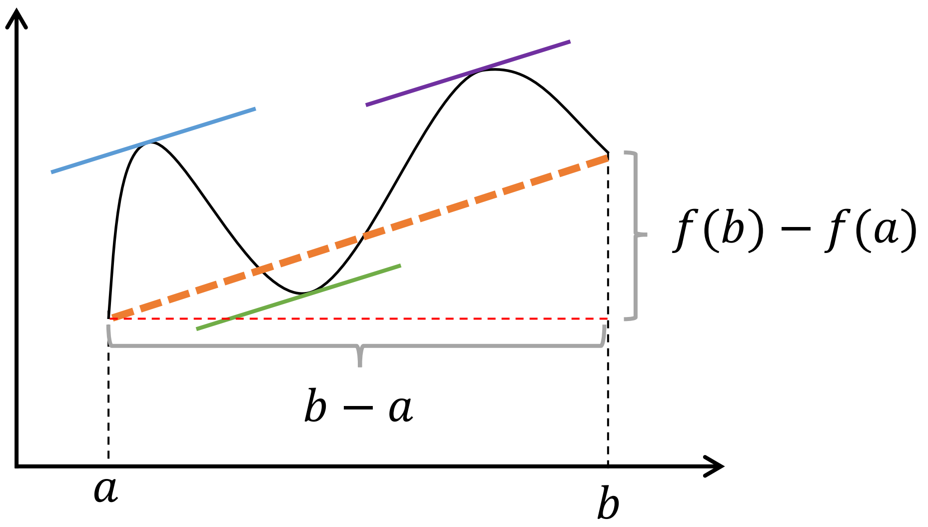
If a function f is

a) Continuous in $[a, b]$

b) Differentiable in (a, b)

Then there exists a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



In other words, there is at least one tangent line in the interval that is parallel to the line segment that goes through the endpoints of the curve.

Proof of Lagrange's Mean Value Theorem

Define a function

$$\phi(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right] x$$

Note that the function $\phi(x)$ satisfies all the conditions of Rolle's theorem.

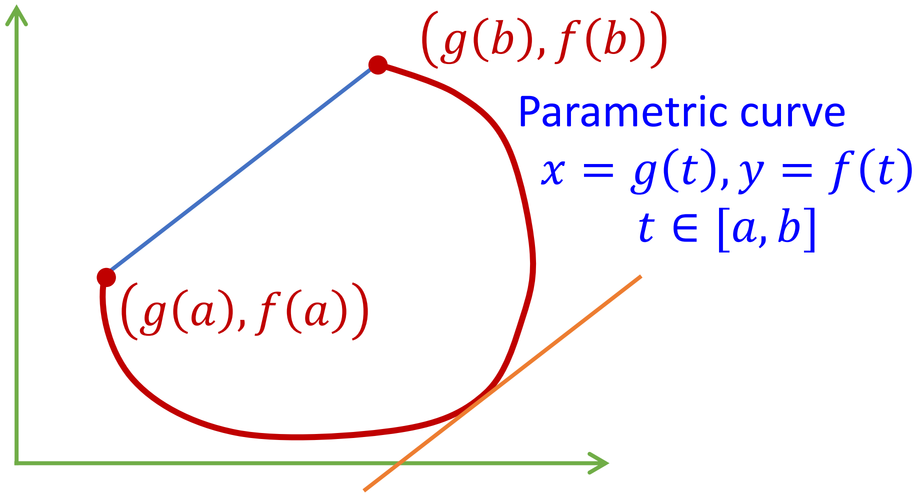
Therefore, Rolle's Theorem gives

$$\phi'(c) = 0 \text{ for some } c \in (a, b) \quad \Rightarrow f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right] = 0$$

Generalized Mean Value Theorem (Cauchy's MVT)

If $f(x)$ and $g(x)$ are two functions continuous in $[a,b]$ and differentiable in (a,b), and $g'(x)$ does not vanish anywhere inside the interval then $\exists$ a point c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$



Proof:

Define a function

$$\phi(x) = \big(f(x) - f(a)\big) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right]\big(g(x) - g(a)\big)$$

Note that $g(b) \neq g(a)$ **?**

Application of Rolle's theorem follows the result.

Generalized MVT

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

What if $g(x) = x$?

Lagrange's MVT

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

What if $f(b) = f(a)$?

Rolle's MVT

$$0 = f'(c)$$

Example - 1 Using Mean Value Theorem, show that

$$|\cos e^x - \cos e^y| \leq |x - y| \qquad x, y \leq 0$$

Consider $f(t) = \cos e^t$ in the interval $[x, y]$. Also assume $x \neq y$ otherwise equality holds.

Apply Lagrange's mean value theorem

$$\frac{\cos e^x - \cos e^y}{x - y} = f'(c), \qquad c \in (x, y)$$

$$|\cos e^x - \cos e^y| = |x - y|\,|e^c \sin e^c|$$

This further implies

$$|\cos e^x - \cos e^y| \leq |x - y| \max_{c \in (x,y)} |e^c \sin e^c| < |x - y|$$

Example - 2

Let f be a differentiable function on $[-2, 2]$ such that $f(-2) = 1,\ f(2) = 5$ and $|f'(x)| \leq 1$ for all $x \in [-2,2]$. Using mean value theorem, find the value of $f(0)$.

Using Lagrange's Mean Value Theorem on $[-2, 0]$, we get

$$\frac{f(0) - f(-2)}{0 - (-2)} = f'(c_1), \text{for some } c_1 \in (-2,0) \Rightarrow -1 \leq \frac{f(0) - 1}{2} \leq 1, \text{since } |f'(x)| \leq 1$$

$$\Rightarrow -1 \leq f(0) \leq 3$$

Using Lagrange's Mean Value Theorem on $[0, 2]$, we get

$$\frac{f(2) - f(0)}{2 - 0} = f'(c_2) \quad \Rightarrow -7 \leq -f(0) \leq -3 \Rightarrow 7 \geq f(0) \geq 3$$

This implies: $f(0) = 3$

Example - 3

The function $f: [0, 1] \to \mathbb{R}$ satisfies $f'(x) = \dfrac{1}{5 - x^2}$ and $f(0) = 2$. Use Lagrange's Mean Value Theorem to estimate the bounds of $f(1)$.

Using Lagrange's Mean Value Theorem on $[0, 1]$, we get

$$\frac{f(1) - f(0)}{1} = f'(c), \text{for some } c \in (0, 1) \qquad \Rightarrow f(1) - 2 = f'(c)$$

The derivative $f'(c)$ can be estimated as $\dfrac{1}{5} < \dfrac{1}{5 - c^2} < \dfrac{1}{4}$

$$\Rightarrow \frac{1}{5} < f(1) - 2 < \frac{1}{4} \qquad \Rightarrow \frac{11}{5} < f(1) < \frac{9}{4}$$