CONCEPTS COVERED

INTEGRAL CALCULUS

Improper Integrals

- ❑ Introduction to Improper Integrals
- ❑ Evaluation of Improper Integrals

Proper Integral

The Integral $\int_a^b f(x)dx$ is **proper** if

the range of integration is finite and the integrand is bounded.

Improper Integral

The Integral $\int_a^b f(x)dx$ is **improper** if

- $a = -\infty$ and/or $b = \infty$ and $f(x)$ is bounded. (First kind)
- $f(x)$ is unbounded at one or more points of $a \leq x \leq b$. (Second kind)
- Both 1 and 2 type. (Third kind or mixed kind)

Examples - Proper Integrals

$$\int_0^2 \sqrt{(x^2+1)}\,\mathrm{d}x \qquad \int_0^1 \frac{\sin x}{x}\,\mathrm{d}x$$

Examples - Improper Integrals

$$\int_0^\infty \cos x\, dx$$ (First Kind)

$$\int_0^1 \frac{1}{x-1}\, dx$$ (Second Kind)

$$\int_0^\infty \frac{1}{(1-x)^2}\, dx$$ (Third Kind)

Evaluation of Improper Integrals of First Kind

- $$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx$$

- $$\int_{-\infty}^b f(x)dx = \lim_{R\to\infty} \int_{-R}^b f(x)dx$$

- $$\int_{-\infty}^\infty f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^c f(x)dx + \lim_{R_2\to\infty} \int_c^{R_2} f(x)dx$$

$$= \lim_{\substack{R_1\to\infty \\ R_2\to\infty}} \int_{-R_1}^{R_2} f(x)dx$$

Evaluation of Improper Integrals of First Kind

- $$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{c} f(x)dx + \lim_{R_2\to\infty}\int_{c}^{R_2} f(x)dx = \lim_{\substack{R_1\to\infty \\ R_2\to\infty}}\int_{-R_1}^{R_2} f(x)dx$$

Note: Do not use

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\left[\int_{-R}^{R} f(x)\,dx\right]$$ Cauchy Principal Value

CPV may exist, even though the integral does now exist.

$$\int_{-\infty}^{\infty}\frac{2x}{1+x^2} = ?$$

$$\int_{-\infty}^{\infty}\frac{2x}{1+x^2} = \lim_{R\to\infty}\int_{-R}^{R}\frac{2x}{1+x^2}dx$$

$$= \lim_{R\to\infty}\left(\ln(1+R^2) - \ln(1+R^2)\right)$$

$$= 0$$

$$\int_{-\infty}^{\infty}\frac{2x}{1+x^2} = \lim_{\substack{R_1\to\infty \\ R_2\to\infty}}\int_{-R_1}^{R_2}\frac{2x}{1+x^2}dx = \lim_{\substack{R_1\to\infty \\ R_2\to\infty}}\left(\ln(1+R_2^2) - \ln(1+R_1^2)\right)$$ Does not exist

Evaluation of Improper Integrals of First Kind

- $$\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \frac{\pi}{2} - \tan^{-1} 2 + \frac{1}{2}\ln 3$$

$$\frac{2x^2}{x^4-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \frac{1}{x^2+1}$$

- $$\int_0^\infty \sin x\,dx = \lim_{R\to\infty}(1-\cos R)$$

Does not exist

Evaluation of Improper Integrals of Second Kind

$$\int_a^b f(x)\,dx$$

$f(x)$ is unbounded

- If $f(x) \to \infty$ as $x \to b$ then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx$$

- If $f(x) \to \infty$ as $x \to a$ then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)dx$$

Evaluation of Improper Integrals of Second Kind

- If $f(x) \to \infty$ as $x \to a$ and $x \to b$, then

$$\int_a^b f(x)dx = \lim_{\substack{\varepsilon_1 \to 0^+ \\ \varepsilon_2 \to 0^+}} \int_{a+\varepsilon_1}^{b-\varepsilon_2} f(x)dx$$

- If $f(x) \to \infty$ as $x \to c$ where $a < c < b$, then

$$\int_a^b f(x)dx = \lim_{\varepsilon_1 \to 0^+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to 0^+} \int_{c+\varepsilon_2}^{b} f(x)\,dx$$

Note: Do not use

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right]$$

Cauchy Principal Value

$$c.v. \int_{-1}^{2} \frac{1}{x} dx = \lim_{\varepsilon \to 0^+} \left[\int_{-1}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{2} \frac{1}{x} dx\right]$$

$$= \lim_{\varepsilon \to 0^+} [\ln \epsilon - \ln 1 + \ln 2 - \ln \epsilon]$$

$$= \ln 2$$

Remark: One needs to be careful to evaluate the improper integral where the integrand is not defined or not bounded at an interior point of the of the range of the integral.

Consider $\int_a^b f(x)\,dx$ Suppose $f(x)$ is unbounded at a point c, where $a < c < b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\epsilon \to 0^+}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\right] c.v$$

$$= \lim_{\epsilon_1 \to 0^+} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2 \to 0^+} \int_{c+\epsilon_2}^b f(x)\,dx$$

Consider $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_{0}^{1} \frac{1}{x^3}\,dx$

$$= \lim_{\epsilon \to 0^+} \left[\int_{-1}^{-\epsilon} \frac{1}{x^3} dx + \int_{\epsilon}^{1} \frac{1}{x^3} dx \right] = \lim_{\epsilon \to 0^+} \left[\left(-\frac{1}{2}\right)\left(\frac{1}{\epsilon^2} - 1\right) + \left(-\frac{1}{2}\right)\left(1 - \frac{1}{\epsilon^2}\right) \right] = 0$$

$$= \lim_{\epsilon_1 \to 0} \int_{-1}^{-\epsilon_1} \frac{1}{x^3} dx + \lim_{\epsilon_2 \to 0} \int_{\epsilon_2}^{1} \frac{1}{x^3} dx$$ Both improper integrals do not exist!

Evaluation of Improper Integrals of Second Kind

Example - 1

$$\int_0^1 \frac{dx}{\sqrt{(1-x)}} = \lim_{\epsilon \to 0^+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{(1-x)}} = \lim_{\epsilon \to 0^+} \left[-2\sqrt{1-x}\right]_0^{1-\epsilon}$$

$$= -2 \lim_{\epsilon \to 0^+} \left(\sqrt{\epsilon} - 1\right)$$

$$\int_0^1 \frac{dx}{\sqrt{(1-x)}} = 2$$ Integral converges

Evaluation of Improper Integrals of Second Kind

Example - 2

$$\int_0^2 \frac{dx}{(2x-x^2)} = \lim_{\epsilon_1\to 0^+} \int_{\epsilon_1}^{1} \frac{dx}{(2x-x^2)} + \lim_{\epsilon_2\to 0^+} \int_{1}^{2-\epsilon_2} \frac{dx}{(2x-x^2)}$$

$$= \lim_{\epsilon_1\to 0^+} \frac{1}{2}\left[\ln\frac{x}{2-x}\right]_{\epsilon_1}^{1} + \lim_{\epsilon_2\to 0^+} \frac{1}{2}\left[\ln\frac{x}{2-x}\right]_{1}^{2-\epsilon_2}$$

$$= -\lim_{\epsilon_1\to 0^+} \frac{1}{2}\left[\ln\frac{\epsilon_1}{2-\epsilon_1}\right] + \lim_{\epsilon_2\to 0^+} \frac{1}{2}\left[\ln\frac{2-\epsilon_2}{\epsilon_2}\right]$$

$$\int_0^2 \frac{dx}{(2x-x^2)} = \infty$$ Integral Diverges

Test Integral - I

$$\int_a^R \frac{1}{x^p}\,dx = \begin{cases} \ln\left(\frac{R}{a}\right), & p = 1 \\ \frac{1}{1-p}\left[\frac{1}{R^{p-1}} - \frac{1}{a^{p-1}}\right], & p \neq 1 \end{cases}$$

$a > 0$

$$\boxed{\int_a^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty} \int_a^R \frac{1}{x^p}\,dx = \begin{cases} \infty, & p \leq 1 \\ \frac{1}{p-1}\frac{1}{a^{p-1}}, & p > 1 \end{cases}}$$

Test Integral - II

$$\int_{a+\epsilon}^{b} \frac{1}{(x-a)^p}\,dx = \begin{cases} \dfrac{1}{1-p}\left[\dfrac{1}{(b-a)^{p-1}} - \dfrac{1}{\epsilon^{p-1}}\right], & p \neq 1 \\ \ln(b-a) - \ln\epsilon\,, & p = 1 \end{cases}$$

$$\boxed{\int_{a}^{b} \frac{1}{(x-a)^p}\,dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} \frac{1}{(x-a)^p}\,dx = \begin{cases} \infty, & p \geq 1 \\ \dfrac{1}{1-p}\dfrac{1}{(b-a)^{p-1}}, & p < 1 \end{cases}}$$

CONCLUSIONS

Improper Integral $\int_a^b f(x)dx$

1. $a = -\infty$ and/or $b = \infty$ and $f(x)$ is bounded.

2. $f(x)$ is unbounded at one or more points of $a \leq x \leq b$.

Evaluation of Improper Integrals

CONCEPTS COVERED

INTEGRAL CALCULUS

Improper Integrals

- Convergence of Type – I improper Integrals
- Worked Problems

Recall (Previous Lecture)

Test Integral

$$\int_a^\infty \frac{1}{x^p}\,dx\ , \qquad a > 0, \qquad \text{converges for } p > 1 \quad \& \quad \text{diverges if } \ p \le 1$$

Convergence: Type - I Integrals

$$\int_a^b f(x)dx$$

$a = -\infty$ and/or $b = \infty$ and $f(x)$ is bounded

Comparison Test-I:

Suppose f and g are integrable over $[a, c]$, $\forall\ c \geq a$ and that $0 \leq f \leq g$, $\forall x > a$, then

i. $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges

ii. $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges

Comparison Test-II (limit Comparison test):

Suppose f and g are integrable over $[a, c]$, $\forall\, c \geq a$ and $f \geq 0, g > 0$ $\forall x > a$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = k(\neq 0)$$

Then both the integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together

Further, If $k = 0$ and $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ converges

If $k = \infty$ and $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges

REMARK: $\boldsymbol{\mu}$ **− test** Comparison test (II) with $g(x) = \dfrac{1}{x^{\mu}}$

$$\boxed{\int_a^{\infty} \frac{1}{x^{\mu}}\, dx = \begin{cases} \text{diverges,} & \mu \le 1 \\ \text{converges,} & \mu > 1 \end{cases}}$$

Let $f(x) \ge 0$ in the interval $[a, \infty)$, $a > 0$. (OR $f(x) \le 0$)

a) If $\exists\ \mu > 1$ such that $\lim_{x\to\infty} x^{\mu} f(x)$ exists then $\int_a^{\infty} f(x)\ dx$ is convergent.

b) If $\exists\ \mu \le 1$ such that $\lim_{x\to\infty} x^{\mu} f(x)$ exists and $\ne 0$ then $\int_a^{\infty} f(x)\ dx$ is divergent

and the same is true if $\lim_{x\to\infty} x^{\mu} f(x)$ is $+\infty$ (OR $-\infty$)

Problem – 1: Test the convergence of $\int_1^{\infty} \frac{dx}{x\sqrt{x^2+1}}$

Note that $\frac{1}{x\sqrt{x^2+1}} \sim \frac{1}{x^2}$

Let $f(x) = \frac{1}{x\sqrt{x^2+1}}$ and $g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1 \ (\neq 0)$$

As $\int_1^{\infty} \frac{1}{x^2}\, dx$ converges $\Longrightarrow \int_1^{\infty} \frac{dx}{x\sqrt{x^2+1}}$ converges

(OR apply $\mu -$ test as $\mu = 2$)

Problem – 2: Test the convergence of $\int_1^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$

Let $f(x) = \frac{x^2}{\sqrt{x^5+1}} \left(\sim \frac{1}{\sqrt{x}}\right)$ and $g(x) = \frac{1}{\sqrt{x}}$

Note that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2\sqrt{x}}{\sqrt{x^5+1}} = 1$

As $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, by comparison test $\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$ diverges

(OR apply $\mu-$ test as $\mu = 0.5$)

Problem – 3: Show that the integral $\int_0^{\infty} e^{-x^2}\, dx$ converges

$$\int_0^{\infty} e^{-x^2}\, dx = \int_0^{1} e^{-x^2}\, dx + \int_1^{\infty} e^{-x^2}\, dx$$

Note that: $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots > x^2, \quad \forall x > 0 \,\&\, x < 0 \Longrightarrow e^{-x^2} < \frac{1}{x^2}$

Since $\int_1^{\infty} \frac{1}{x^2}\, dx$ converges, the integral $\int_1^{\infty} \mathrm{e}^{-x^2}\, dx$ converges

Problem – 4: Show that the integral $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$ converges

Note that $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \int_0^1 \frac{\sin^2 x}{x^2}\, dx + \int_1^\infty \frac{\sin^2 x}{x^2}\, dx$

Since $\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2} dx$ converges

$\int_1^\infty \frac{\sin^2 x}{x^2}\, dx$ converges

Problem – 5: Show that the integral $\int_1^\infty \frac{x \tan^{-1} x}{(1+x^4)^{\frac{1}{3}}} \mathrm{d}x$ diverges

$$\text{Let } f(x) = \frac{x \tan^{-1} x}{(1+x^4)^{\frac{1}{3}}} = \frac{\tan^{-1} x}{x^{\frac{1}{3}}(1+x^{-4})^{\frac{1}{3}}} \qquad \left(\sim x^{-\frac{1}{3}} \text{ at } \infty\right)$$

$$\text{Let } g(x) = \frac{1}{x^{\frac{1}{3}}} \qquad \text{Note that } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\pi}{2}$$

This follows the result. (OR apply $\mu -$ test as $\mu = 1/3$)

Problem – 6: Discuss convergence of $I_1 = \int_1^\infty \frac{\sqrt{4+\sqrt{x}}}{x}\,dx$ & $I_2 = \int_2^\infty \frac{4-3\sin(2x)}{x^2}\,dx$

$$f(x) = \frac{\sqrt{4+\sqrt{x}}}{x} = \frac{\sqrt{\frac{4}{\sqrt{x}}+1}}{x^{\frac{3}{4}}} \quad \left(\sim x^{-\frac{3}{4}} \text{ at } \infty\right)$$

Let $g(x) = \frac{1}{x^{\frac{3}{4}}}$ Note that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

The integral I_1 diverges

$$f(x) = \frac{4-3\sin(2x)}{x^2} \le \frac{7}{x^2}$$

$\int_2^\infty \frac{7}{x^2}\,dx$ converges

$\Rightarrow I_2$ converges

CONCLUSIONS

Comparison Test -I: Let $0 \le f(x) \le g(x)$

$$\int_a^{\infty} g(x)dx \ \ \text{conveges} \implies \int_a^{\infty} f(x)dx \ \ \text{conveges}$$

$$\int_a^{\infty} f(x)dx \ \ \text{diverges} \implies \int_a^{\infty} g(x)dx \ \ \text{diverges}$$

CONCLUSIONS

Comparison Test -II:

Let $0 \leq f(x) \leq g(x)$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \mathrm{k}$$

if $k \neq 0$ then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ behave the same

if $k = 0$ & $\int_a^\infty g(x)dx$ conveges $\Longrightarrow \int_a^\infty f(x)dx$ conveges

if $k = \infty$ & $\int_a^\infty g(x)dx$ diverges $\Longrightarrow \int_a^\infty f(x)dx$ diverges

CONCEPTS COVERED

INTEGRAL CALCULUS

Improper Integrals

- ❑ Convergence of Type – I improper Integrals
- ❑ Convergence Test: Dirichlet's Test

Dirichlet's Test: Let $f, g: [a, \infty) \to \mathbb{R}$ be such that

- f is integrable on each interval $[a, b]$, $b > a$

- The integrals $\displaystyle\int_a^b f(x)dx$ are uniformly bounded

$$\left\{ \exists\, C > 0, \text{ s.t. } \left|\int_a^b f(x)dx\right| \le C \text{ for all } b > a \ (b < \infty) \right\}$$

- g is monotone and bounded on $[a, \infty)$ and $\lim_{x\to\infty} g(x) = 0$

Then, the improper integral $\displaystyle\int_a^\infty f(x)\, g(x)\, dx$ converges

Problem – 1: The Integral $\int_1^\infty \frac{\sin x}{x^p}\,dx$ is convergent for $p > 0$.

Let $f(x) = \sin x$ and $g(x) = \frac{1}{x^p}$

Note that $\left|\int_1^b \sin x \; dx\right| = |\cos 1 - \cos b| \le |\cos 1| + |\cos b| < 2,$ for $1 \le b < \infty$.

Also note that

$g(x) = \frac{1}{x^p}$ is monotone decreasing function tending to 0 as $x \to \infty$, for $p > 0$.

Using Dirichlet's test $\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges for $p > 0$.

Problem – 2: Test the convergence of $\int_0^\infty \frac{\sin x}{x}\, e^{-x}\, dx$

$$\int_0^\infty \frac{\sin x}{x}\, e^{-x}\, dx = \int_0^1 \frac{\sin x}{x}\, e^{-x}\, dx + \int_1^\infty \frac{\sin x}{x}\, e^{-x}\, dx$$

$\left|\int_1^b \sin x\, dx\right| < 2$ for $1 \le b < \infty$.

Note that e^{-x}/x is monotone and bounded as well as $\lim_{x\to\infty} e^{-x}/x = 0$

Hence by Dirichlet's test $\int_0^\infty \frac{\sin x}{x}\, e^{-x}\, dx$ converges

Absolute Convergence

The integral $\int_0^\infty f(x)\, dx$ converges absolutely $\Leftrightarrow$ $\int_0^\infty |f(x)|\, dx$ converges

The integral $\int_0^\infty f(x)\, dx$ converges conditionally $\Leftrightarrow$ It converges but not absolutely

Problem – 3: The Integral $\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges absolutely for $p > 1$.

Note that $\frac{|\sin x|}{x^p} \leq \frac{1}{x^p}, \qquad p > 1$

Recall that $\int_1^\infty \frac{1}{x^p}\,dx$ converges

By comparison test $\int_1^\infty \left|\frac{\sin x}{x^p}\right| dx$ converges

Theorem: $\int_a^\infty f(x)dx$ converges if $\int_a^\infty |f(x)|dx$ converges but the converse is not true.

Example: $\int_0^\infty \frac{\sin x}{x}dx$ converges conditionally

Note that $$\int_0^\infty \frac{\sin x}{x}dx = \underbrace{\int_0^1 \frac{\sin x}{x}dx}_{\text{Proper}} + \underbrace{\int_1^\infty \frac{\sin x}{x}dx}_{\text{Example -1}}$$

$\Rightarrow$ The integral $\int_0^\infty \frac{\sin x}{x}dx$ conveges

Now we will show that $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ does not converge

$\sin(n\pi + y) = (-1)^n \sin y$

Subst. $x = n\pi + y$

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx = \sum_{n=0}^{\infty} \int_0^\pi \frac{|\sin(n\pi + y)|}{n\pi + y} dy$$

$$= \sum_{n=0}^{\infty} \int_0^\pi \frac{|(-1)^n \sin y|}{(n\pi + y)} dy = \sum_{n=0}^{\infty} \int_0^\pi \frac{\sin y}{(n\pi + y)} dy \geq \sum_{n=0}^{\infty} \int_0^\pi \frac{\sin y}{(n\pi + \pi)} dy$$

$$= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+1}$$

divergent series

Hence the improper integral $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges

REMARKS

Integral of the type: $$\int_{-\infty}^{b} f(x)\, dx$$

Substitute $x = -t$:

$$\int_{-b}^{\infty} f(-t)\, dt$$

CONCLUSIONS

Dirichlet's Test:

$$\left|\int_a^b f(x)dx\right| \leq C \quad \text{for all } b > a,$$

g is monotone decreasing to zero as $x \to \infty$

Then $\int_a^\infty f(x)g(x)dx$ converges.

Absolute Convergence

$$\int_0^\infty \frac{\sin x}{x} dx \ \text{ does not convege absolutely}$$

CONCEPTS COVERED

INTEGRAL CALCULUS

Improper Integrals

- Convergence of Type – II improper Integrals
- Worked Problems

Recall (Previous Lectures)

Test Integral

$$\int_a^b \frac{1}{(x-a)^p}\,dx \quad \text{converges for } p < 1 \quad \& \quad \text{diverges if } p \geq 1$$

Convergence: Type - II Integrals

$$\int_{a^+}^{b} f(x)dx$$

$f(x)$ becomes unbounded at $x = a$

For the case

$$\int_{a}^{b^-} f(x)\, dx$$

We can set $x = b - t$ and get

$$\int_{0+}^{b-a} f(b-t)dt$$

Comparison Test-I

Suppose $0 \leq f \leq g, \quad a < x \leq b$, then

- $\int_{a^+}^{b} f(x)dx$ converges if $\int_{a^+}^{b} g(x)\, dx$ converges

- $\int_{a^+}^{b} g(x)dx$ diverges if $\int_{a^+}^{b} f(x)\, dx$ diverges

Comparison Test-II (limit Comparison test):

Suppose $0 \leq f \;\&\; 0 < g, \quad a < x \leq b$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = k$$

If $k \neq 0$ then both the integrals $\int_{a^+}^{b} f(x)dx$ and $\int_{a^+}^{b} g(x)\,dx$ behave the same

Further, if $k = 0$ and $\int_{a^+}^{b} g(x)dx$ converges then $\int_{a^+}^{b} f(x)dx$ converges

If $k = \infty$ and $\int_{a^+}^{b} g(x)dx$ diverges then $\int_{a^+}^{b} f(x)dx$ diverges

$\boldsymbol{\mu}$ − test Comparison test (II) with $g(x) = \dfrac{1}{(x-a)^{\mu}}$

- if $\exists\ 0 < \mu < 1$ such that $\lim\limits_{x \to a+} (x-a)^{\mu} f(x)$ exsits then $\int_{a+}^{b} f(x)\, dx$ conveges absolutely

- if $\exists\ \mu \geq 1$ such that $\lim\limits_{x \to a+} (x-a)^{\mu} f(x)$ exsits $(\neq 0, \text{it may be} \pm \infty)$

  then $\displaystyle\int_{a+}^{b} f(x)\, dx$ diverges

Dirichlet's Test:

- $$\left|\int_{a+\epsilon}^{b} f(x)dx\right| < C, \qquad \forall \quad b > a,$$
- g is monotone, bounded and $\lim_{x \to a^+} g(x) = 0$

Then $$\int_{a^+}^{b} f(x)g(x)\,dx \quad \text{conveges}$$

Problem – 1: Test the convergence of $\displaystyle\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$

Note that the integrand is unbounded at upper end.

Set $3-x=t$ implies $dx=-dt$

$$\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}} = \int_0^3 \frac{dt}{t\sqrt{(3-t)^2+1}}$$

Convergence of $\displaystyle\int_0^3 \frac{dt}{t\sqrt{(3-t)^2+1}}$

Take $g(t) = \dfrac{1}{t}$

Note that $\displaystyle\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{t\to 0}\frac{1}{\sqrt{(3-t)^2+1}} = \frac{1}{\sqrt{10}}$

$\displaystyle\Rightarrow \int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$ diverges since $\displaystyle\int_0^3 \frac{1}{t}\,dt$ diverges.

Problem – 2: Test the convergence of $\displaystyle\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\, dx$

Notice: $\left|\dfrac{\sin x}{\sqrt[3]{x-\pi}}\right| \leq \dfrac{1}{\sqrt[3]{x-\pi}}$

and $\displaystyle\int_{\pi}^{4\pi} \frac{1}{\sqrt[3]{x-\pi}}\, dx$ converges

$\Rightarrow \displaystyle\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\, dx$ converges absolutely.

Problem – 2: Consider the following improper Integral $\int_0^1 \frac{2+\sin(\pi x)}{(1-x)^p}\,dx$

Find all the values of p for which the above integral converges or diverges.

Using comparison test $0 < \frac{2+\sin(\pi x)}{(1-x)^p} \leq \frac{3}{(1-x)^p}, \qquad 0 \leq x < 1$

Since $\int_0^1 \frac{3}{(1-x)^p}\,dx$ convergs for $p < 1,$ the integral $\int_0^1 \frac{2+\sin(\pi x)}{(1-x)^p}\,dx$ converges

Again, using comparison test $0 < \frac{1}{(1-x)^p} \leq \frac{2+\sin(\pi x)}{(1-x)^p}, \qquad 0 \leq x < 1$

Since $\int_0^1 \frac{1}{(1-x)^p}\,dx$ diverges for $p \geq 1,$ the given integral diverges for $p \geq 1$

Note: Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kind.

Problem – 3: Test the convergence of $\int_1^\infty \frac{1}{x\sqrt{x-1}}\,dx$

$$\int_1^\infty \frac{1}{x\sqrt{x-1}}\,dx = \int_1^2 \frac{1}{x\sqrt{x-1}}\,dx + \int_2^\infty \frac{1}{x\sqrt{x-1}}\,dx$$

Functions for comparison

$$g_1 = \frac{1}{\sqrt{x-1}} \qquad g_2 = \frac{1}{x^{3/2}}$$

Both converge by comparison test

Evaluation of $$\int_{1}^{\infty} \frac{1}{x\sqrt{x-1}}\,dx$$

$$\int_{1+\epsilon}^{R} \frac{1}{x\sqrt{x-1}}\,dx = 2\left(\tan^{-1}\sqrt{R-1} - \tan^{-1}\sqrt{\epsilon}\right)$$

subst. $\sqrt{x-1} = t$

$$\int_{1}^{\infty} \frac{1}{x\sqrt{x-1}}\,dx = \lim_{\substack{\epsilon\to 0^+ \\ R\to\infty}} \int_{1+\epsilon}^{R} \frac{1}{x\sqrt{x-1}}\,dx = \pi$$

CONCLUSIONS

Comparison Test -I:

Let $0 \le f(x) \le g(x), a < x \le b$

$$\int_{a^+}^{b} g(x)dx \ \ \text{conveges} \implies \int_{a^+}^{b} f(x)dx \ \ \text{conveges}$$

$$\int_{a^+}^{b} f(x)dx \ \ \text{diverges} \implies \int_{a^+}^{b} g(x)dx \ \ \text{diverges}$$

CONCLUSIONS

Comparison Test -II:

Let $0 \leq f(x), 0 < g(x), a < x \leq b$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = k$$

if $k \neq 0$ then $\int_{a^+}^{b} f(x)dx$ and $\int_{a^+}^{b} g(x)dx$ behave the same

if $k = 0$ & $\int_{a^+}^{b} g(x)dx$ conveges $\Longrightarrow \int_{a^+}^{b} f(x)dx$ conveges

if $k = \infty$ & $\int_{a^+}^{b} g(x)dx$ diverges $\Longrightarrow \int_{a^+}^{b} f(x)dx$ diverges