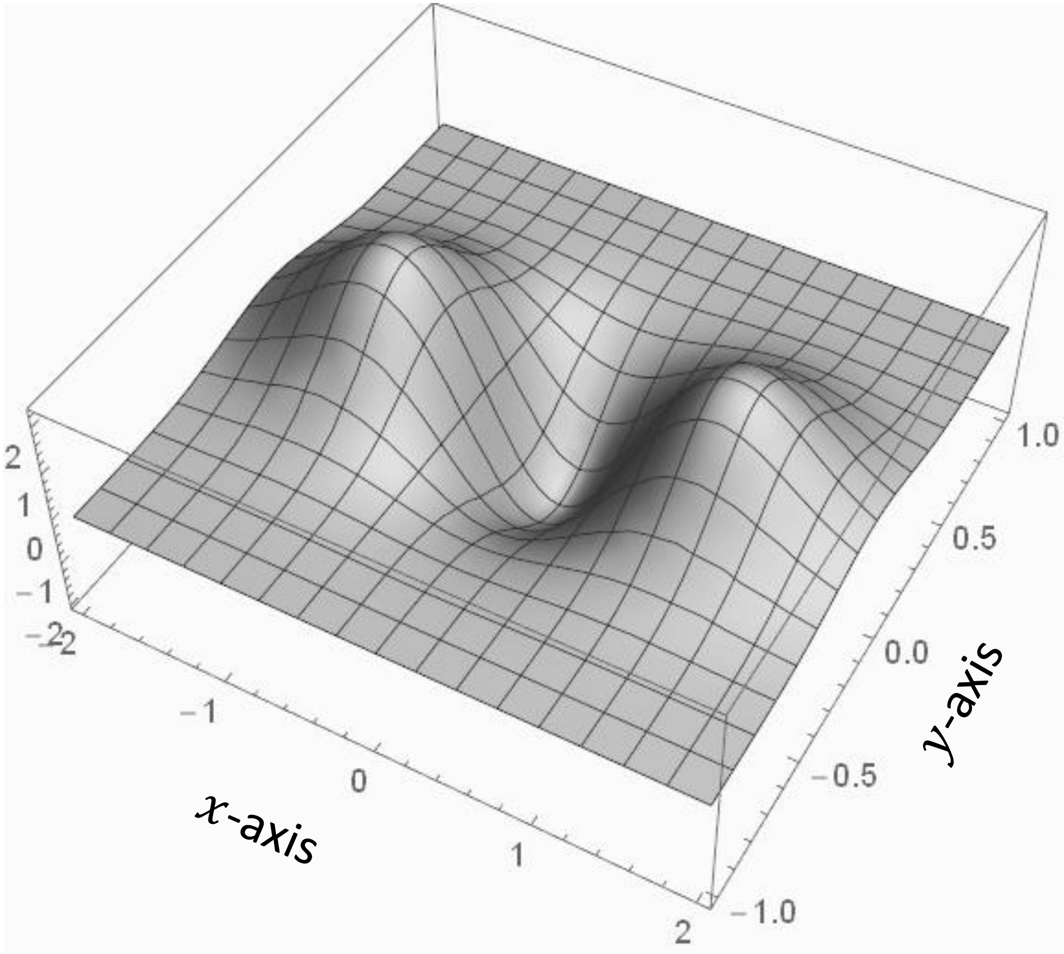
CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Maxima and Minima
  - Necessary Conditions
  - Worked Problems

Local Maximum or Minimum

$$z = (4x^2 + y^2)e^{-x^2-4y^2}$$

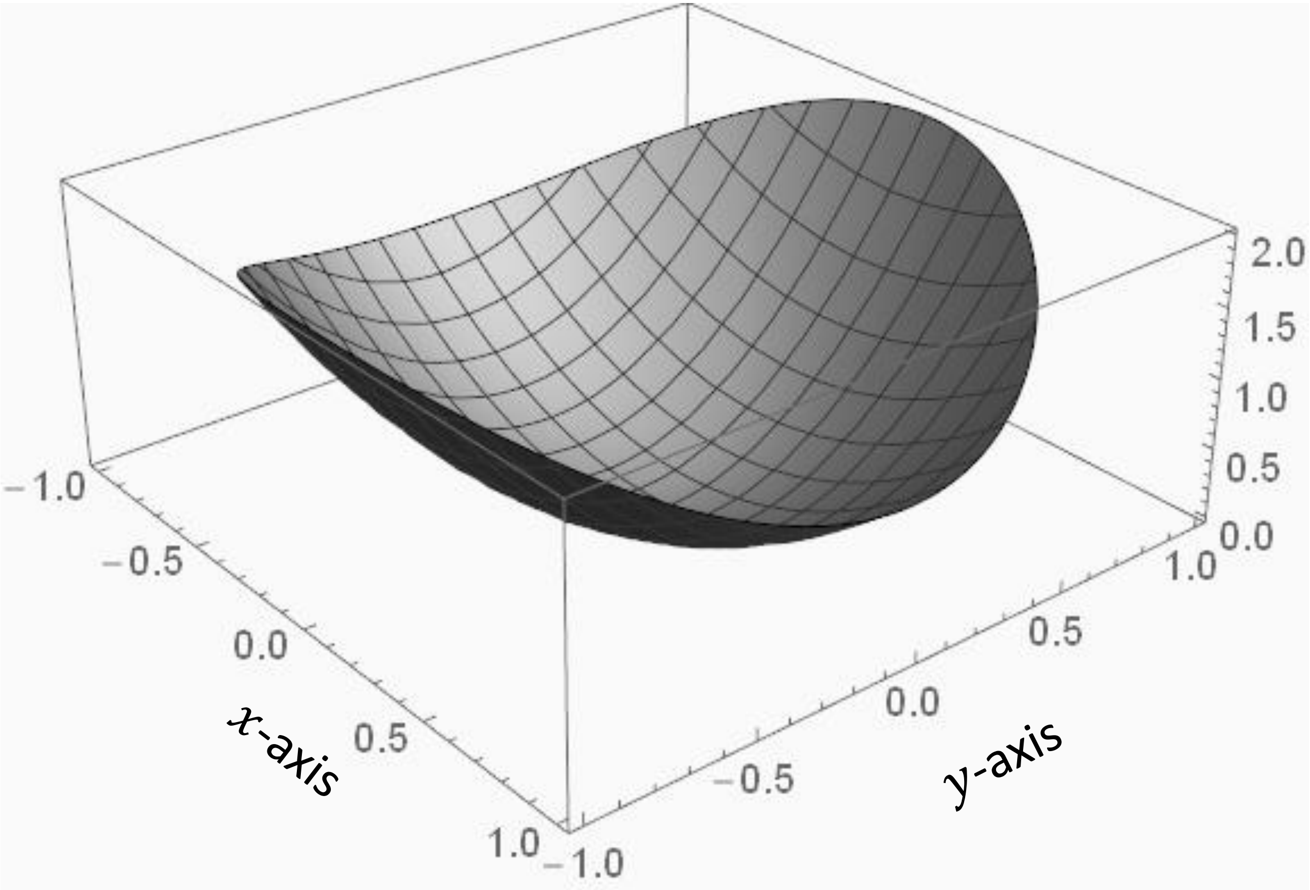


A function $z = f(x, y)$ has a maximum (or a minimum) at the point (x_0, y_0) if at every point in a neighborhood of (x_0, y_0), the function assumes a smaller value (or a larger value) than at the point itself.

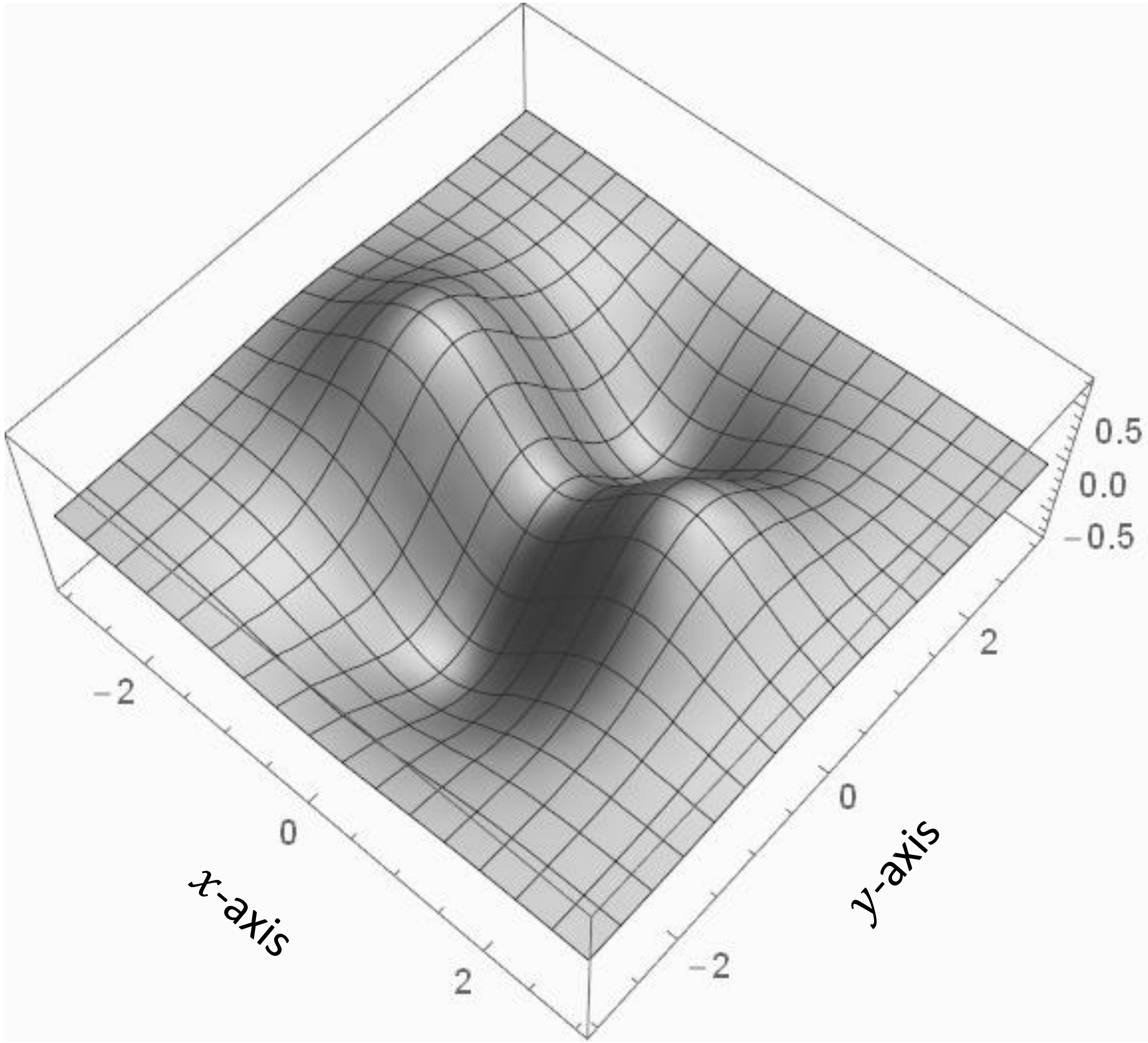
Maximum and minimum values together are called extreme values.

Absolute or Global Maximum/Minimum

$$f(x, y) = x^2 + 2y^2, \qquad x^2 + y^2 \leq 1$$



The smallest and the largest values attained by a function over entire domain including the boundary of the domain are called absolute (or global) minimum and absolute (or global) maximum, respectively.



Critical point & Saddle Points

The point (x_0, y_0) is called critical point (or stationary point) of $f(x, y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

OR $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ do/does not exist.

A critical point where the function has no minimum or maximum is called a **saddle point**.

Necessary condition for a function to have extremum

Let $f(x, y)$ be continuous and have first order partial derivatives at a point $P(a, b)$. Then necessary conditions for the existence of an extreme value of it at the point P are

$$f_x(a, b) = 0 \quad \& \quad f_y(a, b) = 0$$

(The point P is a critical point)

OR

If the point $P(a, b)$ is a relative extremum of the function $f(x, y)$ then $P(a, b)$ is also a critical point of $f(x, y)$.

Necessary Condition for a function to have extremum

Let $(a+h, b+k)$ be a point in the neighborhood of the point $P(a, b)$.

Then P will be point of maximum if

$$\Delta f = f(a+h, b+k) - f(a, b) \leq 0 \qquad \text{for all sufficiently small } h \,\&\, k$$

and a point of minimum if

$$\Delta f = f(a+h, b+k) - f(a, b) \geq 0 \qquad \text{for all sufficiently small } h \,\&\, k$$

Necessary Condition for a function to have extremum

Taylor series expansion about the point (a, b)

$$f(a+h, b+k) = f(a, b) + \left(h\, f_x + k\, f_y\right)_{(a,b)} + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Noting $\Delta f = f(a+h, b+k) - f(a, b)$

$$\Delta f = h\, f_x(a, b) + k\, f_y(a, b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

For sufficiently small h & k, the sign of Δf will depend on the sign of

$$h\, f_x(a, b) + k\, f_y(a, b)$$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Letting $h \to 0$, we get

$$\Delta f = k\, f_y(a,b) + \frac{1}{2}k^2 f_{yy}(a,b) + \cdots$$

h	$h - 1000h^2 - 2000h^3$
0.1	−11.9
0.01	−0.092
0.001	−0.000002
0.0001	0.000089998

Note that the sign of Δf depends on the sign of $k\, f_y(a,b)$. That is

Assuming $f_y > 0$:

$$\Delta f > 0 \ \text{ for } \ k > 0$$

$$\Delta f < 0 \ \text{ for } \ k < 0$$

Assuming $f_y < 0$:

$$\Delta f < 0 \ \text{ for } \ k > 0$$

$$\Delta f > 0 \ \text{ for } \ k < 0$$

Therefore the function cannot have an extremum unless $\boxed{f_y = 0}$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Similarly, letting $k \to 0$ we find that Δf changes sign h:

Assuming $f_x > 0$:

$\Delta f > 0$ for $h > 0$

$\Delta f < 0$ for $h < 0$

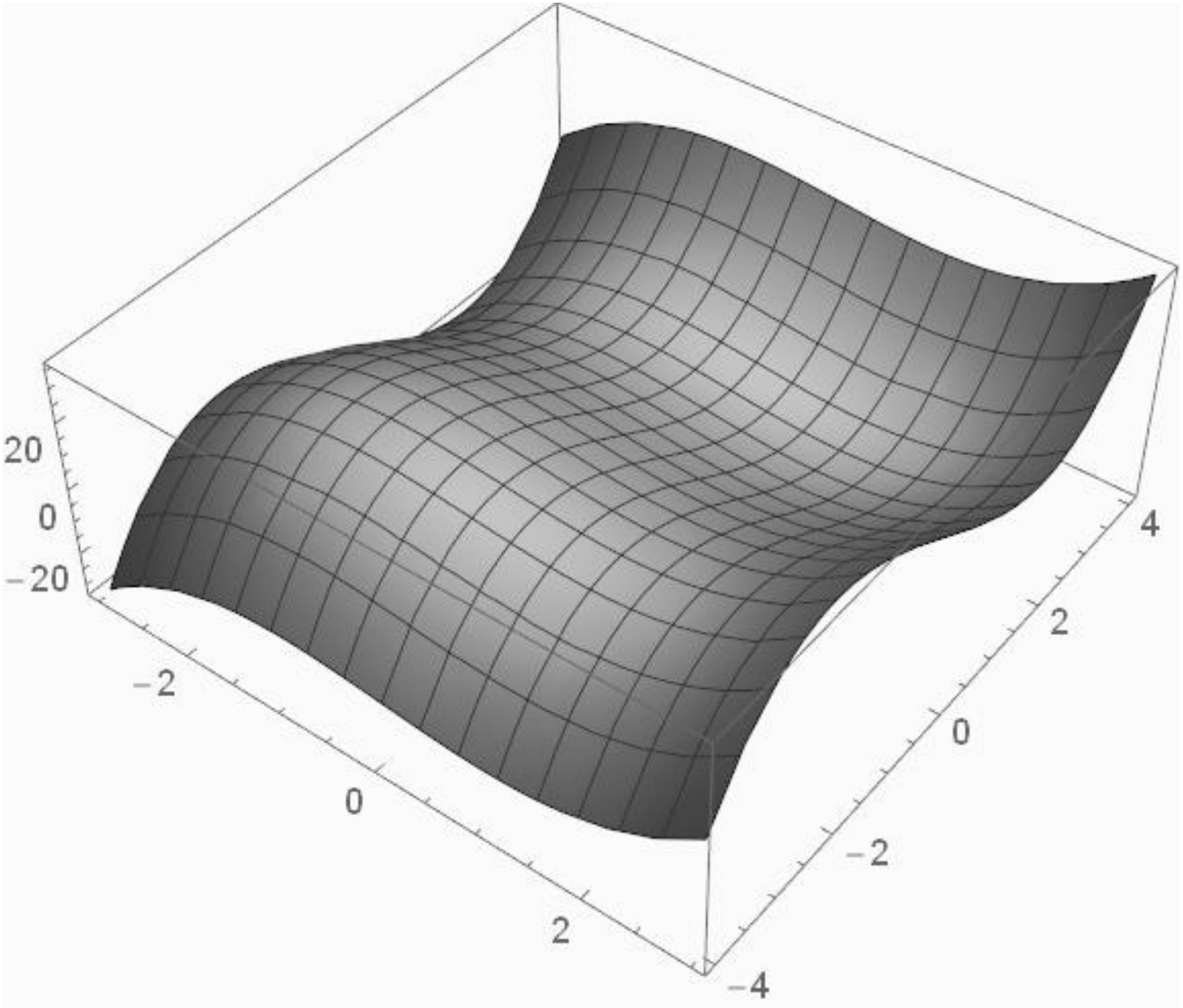
Assuming $f_x < 0$:

$\Delta f < 0$ for $h > 0$

$\Delta f > 0$ for $h < 0$

Therefore the function cannot have an extremum unless $f_x = 0$

Thus, the necessary conditions for the existence of an extremum at the point (a,b) is that $f_x(a,b) = 0$ & $f_y(a,b) = 0$



Problem - 1

Find all critical points of the function
$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$

Critical points are obtained by solving

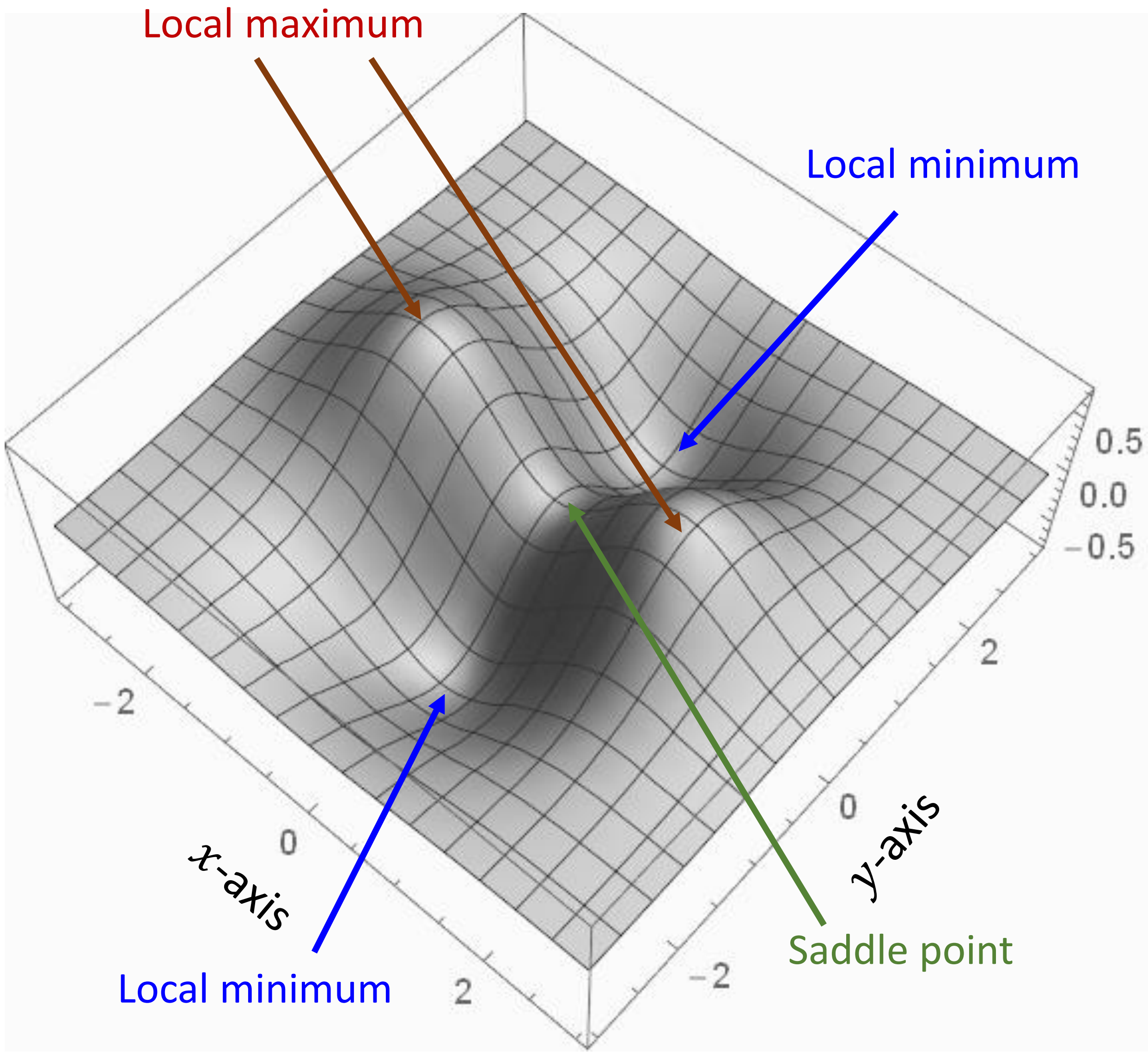
$$f_x(x, y) = 0 \;\; \& \;\; f_y(x, y) = 0$$

$$f_x(x, y) = 0 \Longrightarrow 3x^2 - 3 = 0$$

$$f_y(x, y) = 0 \Longrightarrow 3y^2 - 12 = 0$$

Critical Points are:

$(\pm 1, \pm 2)$



Problem - 2

Find all critical points of the function

$$f(x,y) = (x^2 - y^2)e^{\frac{-x^2-y^2}{2}}.$$

$$f_x = 0 \Longrightarrow \ (2 - \ (x^2 - y^2))\, x = 0$$

$$f_y = 0 \Longrightarrow \ (-2 - \ (x^2 - y^2))\, y = 0$$

Critical Points:

$$(0,0), \left(\pm\sqrt{2}, 0\right), \left(0, \pm\sqrt{2}\right)$$

CONCLUSIONS

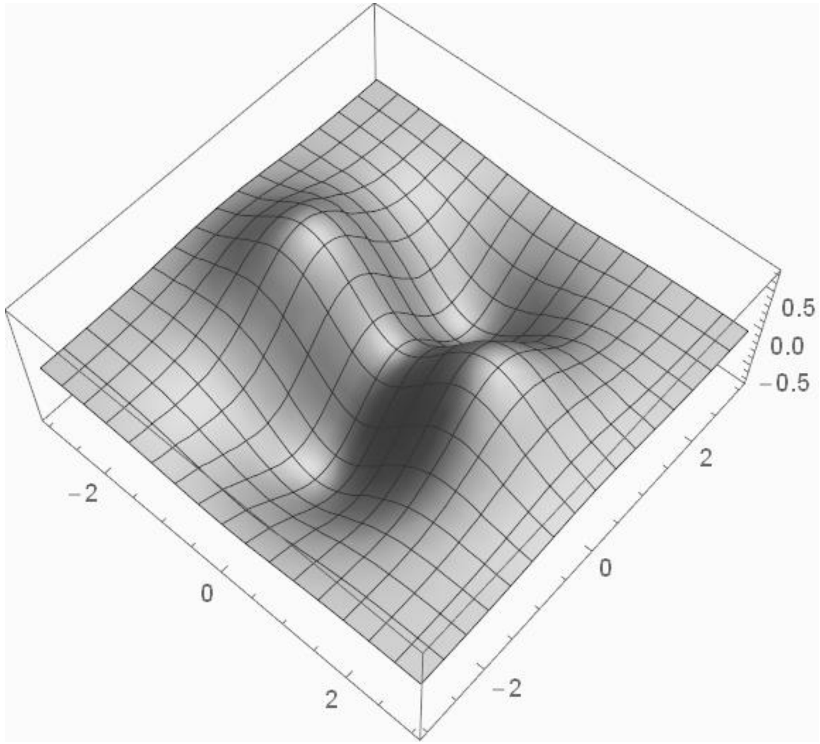
Necessary condition for extrema

$$f_x(a, b) = 0$$

$$f_y(a, b) = 0$$

Critical points are candidates for

- Local extrema
- Saddle points

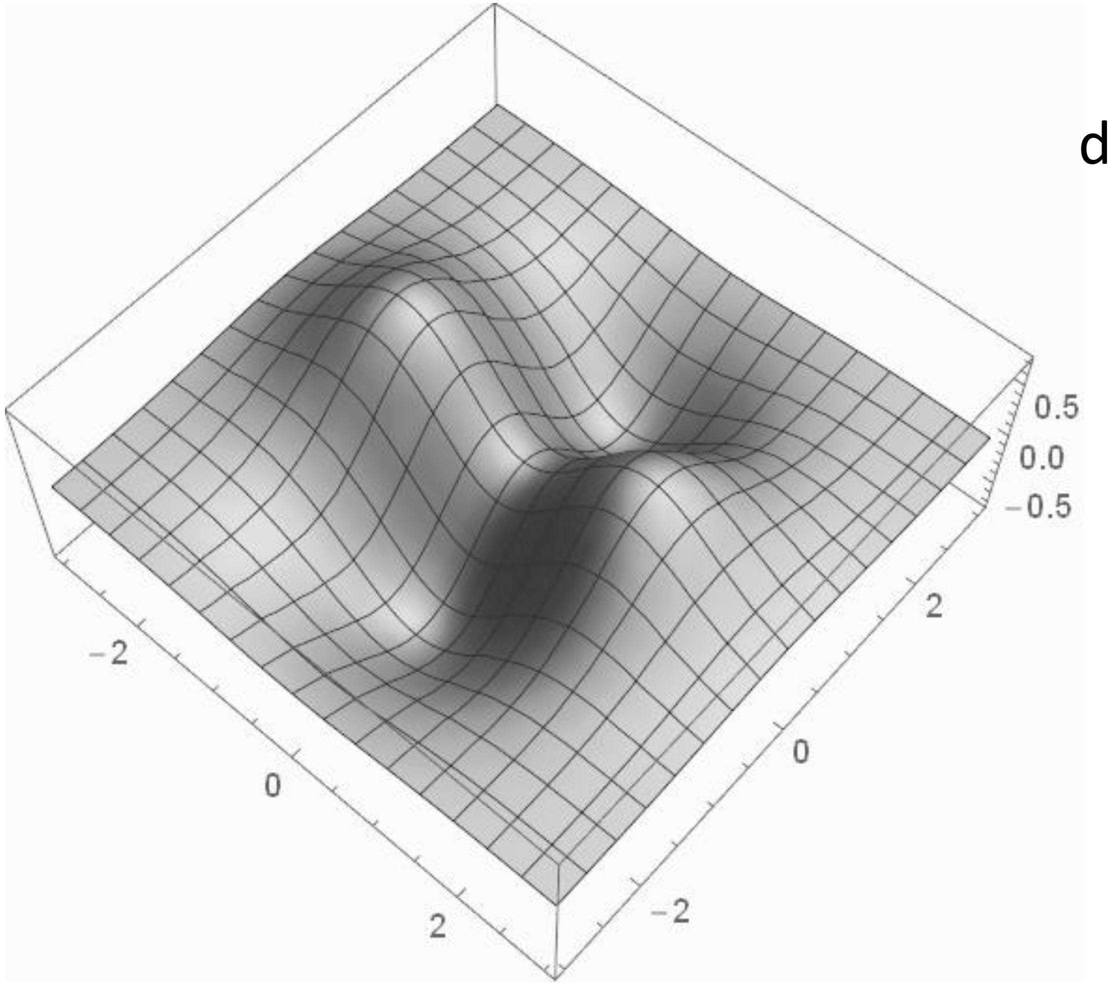


CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Maxima and Minima
  - Sufficient Conditions
  - Worked Problems

Local Extrema (Previous Lecture)



A point (a, b) will be a point of local extrema if

$$\Delta f = f(a+h, b+k) - f(a, b)$$

does not change its sign for all sufficiently small h & k

Taylor′s Series $\Delta f = h\, f_x(a, b) + k\, f_y(a, b)$

$$+\frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Necessary Condition

$$f_x(a, b) = 0 \quad \& \quad f_y(a, b) = 0$$

Sufficient condition for a function to have extremum

Notation: $r = f_{xx}(a,b)$, $s = f_{xy}(a,b)$, $t = f_{yy}(a,b)$

Let a function $f(x,y)$ be continuous and have continuous second order partial derivatives at a point $P(a,b)$. If $P(a,b)$ is a critical point, then the point P is a point of

i. local maximum if $rt - s^2 > 0$ and $r < 0$

ii. local minimum if $rt - s^2 > 0$ and $r > 0$

iii. saddle point if $rt - s^2 < 0$

iv. test fails if $rt - s^2 = 0$ (some other way to characterize)

Sufficient condition for a function to have extremum

Consider $\Delta f = f(a+h, b+k) - f(a,b)$

$$\Delta f = h\, f_x(a,b) + k\, f_y(a,b) + \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)_{(a,b)} + \cdots$$

Since (a,b) is a critical point, $f_x(a,b) = 0$ & $f_y(a,b) = 0$, we have

$$\Delta f = \frac{1}{2}\left(h^2\, f_{xx} + 2hk\, f_{xy} + k^2 f_{kk}\right)_{(a,b)} + \cdots$$

OR

$$\Delta f = \frac{1}{2}(h^2\, r + 2hk\, s + k^2\, t) + \cdots$$

$$\Delta f = \frac{1}{2}(h^2\, r + 2hk\, s + k^2\, t) + \cdots$$

Assuming $r \neq 0$

$$\Delta f = \frac{1}{2r}(h^2\, r^2 + 2hk\, rs + k^2\, rt) + \cdots$$

$$\Delta f = \frac{1}{2r}(h^2\, r^2 + 2hk\, rs + k^2 s^2 - k^2 s^2 + k^2\, rt) + \cdots$$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 - k^2 s^2 + k^2 rt\right) + \cdots$$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right) + \cdots$$

$$\Delta f = \frac{1}{2r}\left((hr + ks)^2 + k^2(rt - s^2)\right) + \cdots$$

Case – I: $rt - s^2 > 0$

$$\Delta f > 0 \quad \text{if } r > 0$$

$$\Delta f < 0 \quad \text{if } r < 0$$

The point (a, b) is a point of minimum if $rt - s^2 > 0,\quad r > 0$

The point (a, b) is a point of maximum if $rt - s^2 > 0,\quad r < 0$

$$\Delta f = \frac{1}{2r}\left((hr+ks)^2 + k^2(rt - s^2)\right) + \cdots$$

Case – II: $rt - s^2 < 0$

Let $k \to 0$ & $h \neq 0 \implies \Delta f > 0$ if $r > 0$

Let $k \neq 0$ & choose h such that $hr + ks = 0 \implies \Delta f < 0$ if $r > 0$

$\implies$ The sign of Δf depends on h & k

Hence no maximum/minimum of f can occur at $P(a, b)$.

$\implies$ The point $P(a, b)$ is a saddle point

$$\Delta f = \frac{1}{2r}\left((hr+ks)^2 + k^2(rt-s^2)\right) + \cdots$$

Case – III: $rt - s^2 = 0$

$$\Delta f = \frac{1}{2r}(hr+ks)^2 + \cdots$$

If we take h & k such that $hr = -ks$, then the whole second order terms of the right hand side will vanish.

Therefore, the conclusion will depend on the higher order terms.

One has to find some other way to investigate such points.

Working rules for investigating local extrema

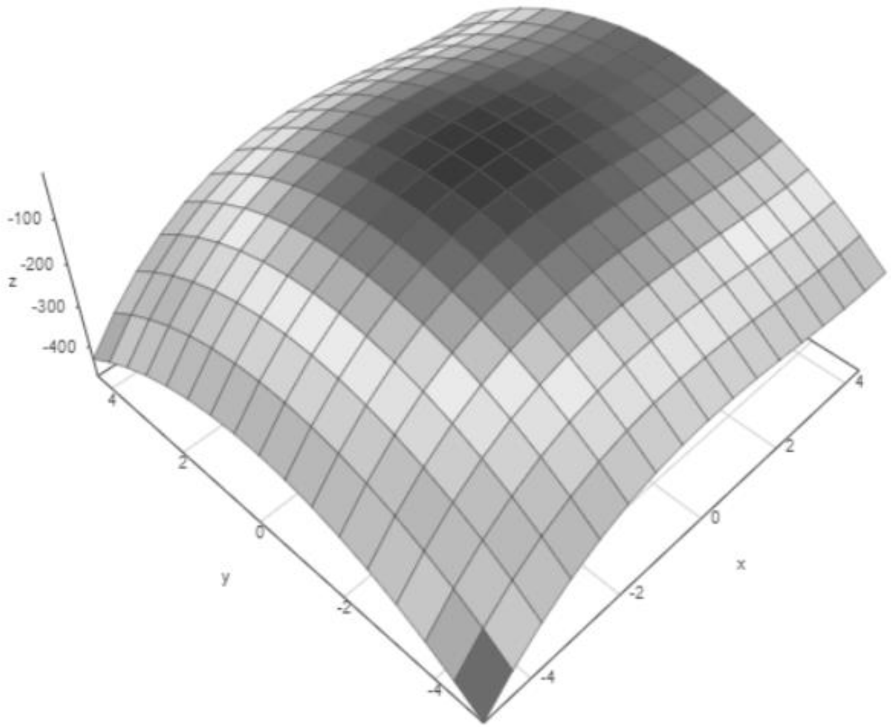
- Find all critical points $\quad f_x = 0 \;\;\&\;\; f_y = 0$

- For each critical point, evaluate

$$r = f_{xx}, \qquad s = f_{xy}, \qquad t = f_{yy}$$

- Identification
  - If $rt - s^2 > 0 \;\&\; r < 0$ maximum
  - If $rt - s^2 > 0 \;\&\; r > 0$ minimum
  - If $rt - s^2 < 0$ Saddle point
  - If $rt - s^2 = 0$ Test Fails

Example: Find all critical points of $f(x,y) = x^3 - 6x^2 - 8y^2$ and investigate their nature for local maximum/minimum and saddle point.



Critical points: $\left.\begin{array}{l} f_x = 0 \\ \\ f_y = 0 \end{array}\right\} \Rightarrow (0,0) \;\; \& \;\; (4,0)$

	$(0,0)$	$(4,0)$
$r = f_{xx}$	-12	12
$s = f_{xy}$	0	0
$t = f_{yy}$	-16	-16
$rt - s^2$	192	-192

$(0,0)$ is a point of local maximum & $(4,0)$ is a saddle point.

CONCLUSIONS

- Necessary Conditions

$$f_x = 0 \quad \& \quad f_y = 0$$

- Sufficient Conditions

  - If $rt - s^2 > 0 \;\&\; r < 0$ maximum
  - If $rt - s^2 > 0 \;\&\; r > 0$ minimum
  - If $rt - s^2 < 0$ saddle point
  - If $rt - s^2 = 0$ needs further investigation

CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Maxima and Minima
  - Worked Problems

Working rules for investigating local extrema (Recall)

- Find all critical points $f_x = 0 \;\; \& \;\; f_y = 0$
- For each critical point, evaluate

$$r = f_{xx}, \qquad s = f_{xy}, \qquad t = f_{yy}$$

- Identification
  - If $rt - s^2 > 0 \;\; \& \;\; r < 0$ maximum
  - If $rt - s^2 > 0 \;\; \& \;\; r > 0$ minimum
  - If $rt - s^2 < 0$ saddle point
  - If $rt - s^2 = 0$ needs further investigation

Problem - 1 Discuss local extrema of the function $f(x,y) = (4x^2 + y^2)e^{-x^2-4y^2}$

$$f_x(x,y) = 2x\, e^{-x^2-4y^2}(4 - 4x^2 - y^2)$$

$$f_y(x,y) = 2y\, e^{-x^2-4y^2}(1 - 16x^2 - 4y^2)$$

Critical Points:

$$(0,0), \left(0, \frac{1}{2}\right), \left(0, -\frac{1}{2}\right), (1,0), (-1,0)$$

$$f_x(x,y) = 2\,e^{-x^2-4y^2}(4x - 4x^3 - xy^2)$$

$$r = f_{xx}(x,y) = 2\,e^{-x^2-4y^2}(4 - 20x^2 + 8x^4 - y^2 + 2\,x^2y^2)$$

$$f_y(x,y) = 2\,e^{-x^2-4y^2}(y - 16y\,x^2 - 4y^3)$$

$$s = f_{xy}(x,y) = 4\,xy\,e^{-x^2-4y^2}(-17 + 16x^2 + 4y^2)$$

$$f_y(x,y) = 2\,e^{-x^2-4y^2}(y - 16y\,x^2 - 4y^3)$$

$$t = f_{yy}(x,y) = 2\,e^{-x^2-4y^2}(1 - 20y^2 - 16x^2 - 128\,x^2y^2 + 32y^4)$$

$$r = 2\, e^{-x^2-4y^2}(4 - 20x^2 + 8x^4 - y^2 + 2\, x^2y^2)$$

$$s = 4\, xy\, e^{-x^2-4y^2}(-17 + 16x^2 + 4y^2)$$

$$t = 2\, e^{-x^2-4y^2}(1 - 20y^2\ - 16x^2 - 128\, x^2y^2 + 32y^4)$$

Identification

$P_1(0,0)$: $r = 8$ $s = 0$ $t = 2$

$$rt - s^2 = 16 > 0$$

$\Longrightarrow$ The point $P_1(0,0)$ is a local minimum.

$$r = 2\, e^{-x^2-4y^2}(4 - 20x^2 + 8x^4 - y^2 + 2\, x^2y^2) \qquad s = 4\, xy\, e^{-x^2-4y^2}(-17 + 16x^2 + 4y^2)$$

$$t = 2\, e^{-x^2-4y^2}(1 - 20y^2 \; - 16x^2 - 128\, x^2y^2 + 32y^4)$$

$P_{2/3}(0, \pm 1/2)$: $\quad r = \dfrac{15}{2e} \qquad s = 0 \qquad t = -\dfrac{4}{\mathrm{e}} \qquad rt - s^2 = -\dfrac{30}{e^2} < 0$

$\Longrightarrow$ The point $P_{2/3}$ are saddle points

$P_{4/5}(\pm 1{,}0)$: $\quad r = -\dfrac{16}{e} \qquad s = 0 \qquad t = -\dfrac{30}{\mathrm{e}} \qquad rt - s^2 = \dfrac{480}{e^2} > 0$

$\Longrightarrow$ The point $P_{4/5}$ are local maxima.

Problem - 2 Discuss local extrema of the function $f(x, y) = y^2 + x^2 y + x^4$

$f_x = 2xy + 4x^3$ $\quad f_y = 2y + x^2$ $\quad$ Stationary points: $(0,0)$

$r = f_{xx}(0,0) = 0$ $\quad s = f_{xy}(0,0) = 0$ $\quad t = f_{yy}(0,0) = 2$

$\Longrightarrow rt - s^2 = 0$ $\quad$ Test fails!

Consider $\Delta f = f(0 + h, 0 + k) - f(0,0) = k^2 + h^2 k + h^4$

$$= \left(\frac{k}{2} + h^2\right)^2 + \frac{3}{4}k^2 > 0, \qquad \forall\, h \neq 0, k \neq 0$$

$\Longrightarrow$ $(0,0)$ is a point of local minimum.

Problem - 3 Discuss local extrema of the function $f(x,y) = 2x^4 - 3x^2y + y^2$

$f_x = 8x^3 - 6xy$ $\quad f_y = -3x^2 + 2y$ $\quad$ Stationary points: $(0,0)$

$r = f_{xx}(0,0) = 0$ $\quad s = f_{xy}(0,0) = 0$ $\quad t = f_{yy}(0,0) = 2$

$\Longrightarrow rt - s^2 = 0$ $\quad$ Test fails!

Consider $\Delta f = f(0+h, 0+k) - f(0,0) = 2h^4 - 3h^2k + k^2$

$= 2h^4 - 2h^2k - h^2k + k^2 = 2h^2(h^2 - k) - k(h^2 - k)$

$= (h^2 - k)(2h^2 - k)$

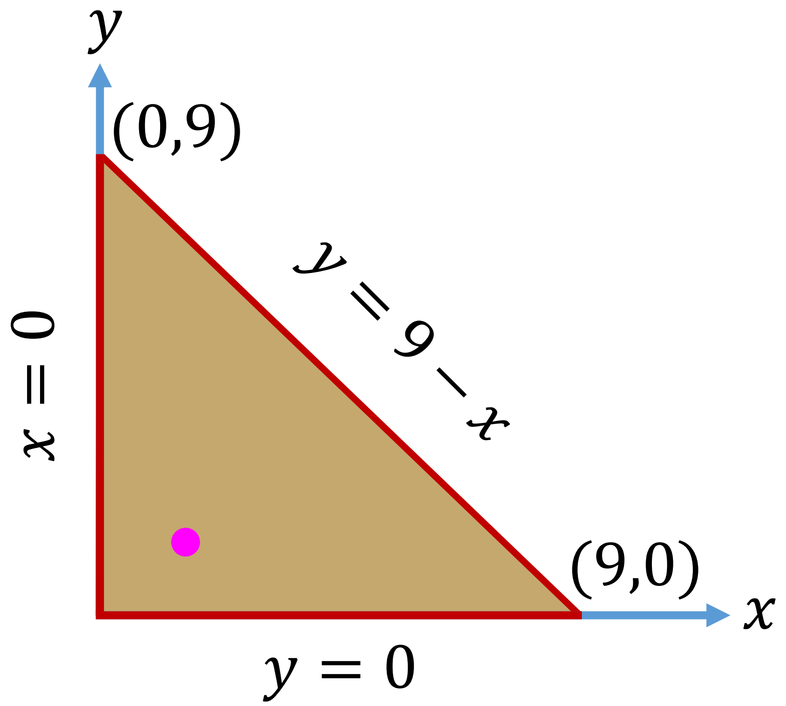
For $k < 0,\ \Delta f > 0$ $\quad$ For $h^2 < k < 2h^2,\ \Delta f < 0$

$\Longrightarrow (0,0)$ is a saddle point

Problem - 4 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular plate in the first quadrant bounded by the lines $x = 0, y = 0, y = 9 - x$



Interior Points: Stationary points

$$\left.\begin{aligned} f_x &= 2 - 2x = 0 \\ f_y &= 2 - 2y = 0 \end{aligned}\right\} (x, y) = (1,1)$$

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

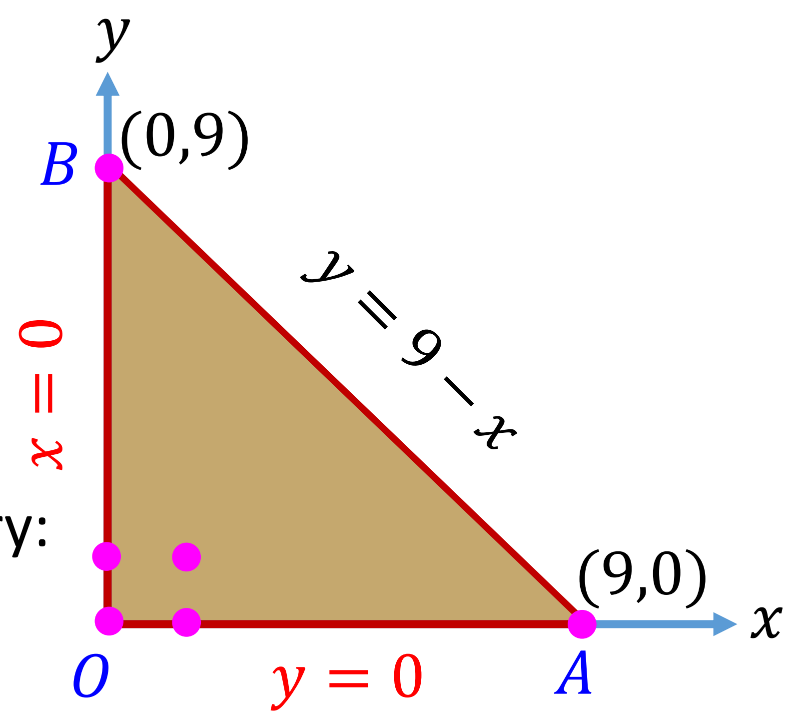
Boundary:

Along OA $\quad f = 2 + 2x - x^2, \qquad x \in [0,9]$

Stationary points $\quad f_x = 0 \Longrightarrow x = 1$

Possible candidates (points) for extrema along this boundary:

$(0,0) \qquad (9,0) \qquad (1,0)$



Along OB $\quad f = 2 + 2y - y^2, \qquad y \in [0,9]$

Possible candidates (points) for extrema along this boundary:

$(0,0) \qquad (0,9) \qquad (0,1)$

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

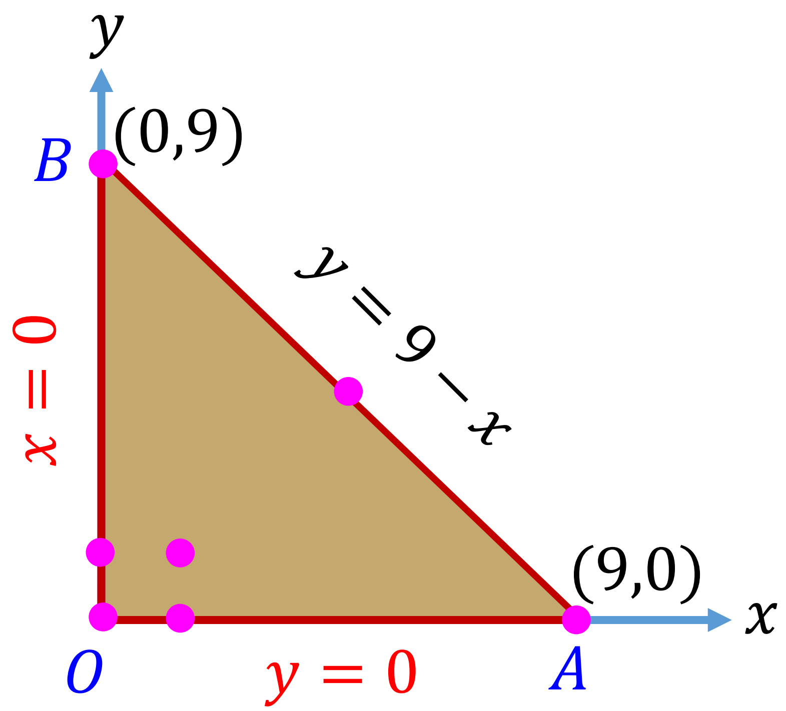
Boundary:

Along AB: $\quad y = 9 - x$

$$f = 2 + 2x + 2(9-x) - x^2 - (9-x)^2, \qquad x \in [0,9]$$

$$f = -61 + 18x, -2x^2, \qquad x \in [0,9]$$

$$f_x = 0 \implies (x,y) = \left(\frac{9}{2}, \frac{9}{2}\right)$$



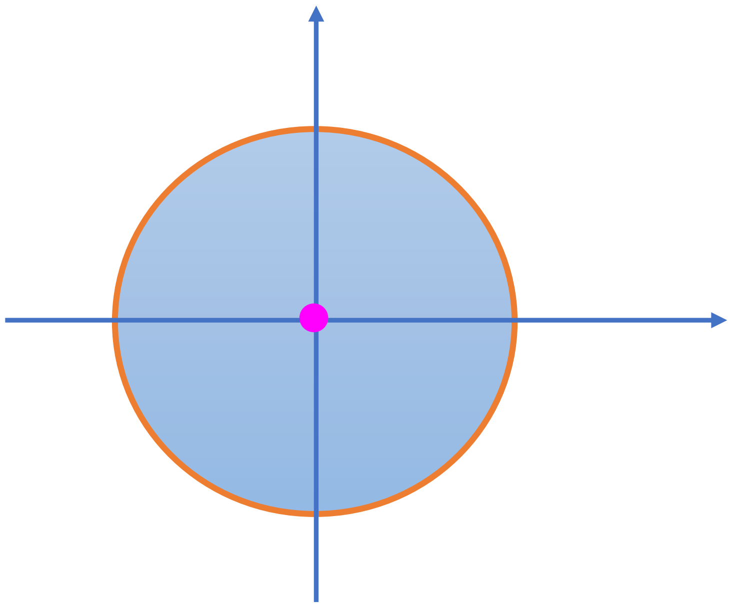
$(\boldsymbol{x},\boldsymbol{y})$	$(1,1)$	$(0,0)$	$(1,0)$	$(9,0)$	$(0,1)$	$(0,9)$	$(9/2,9/2)$
f	$\mathbf{4}$	2	3	$\mathbf{-61}$	3	$\mathbf{-61}$	$-41/2$

The Maximum is $\mathbf{4}$ and the minimum value is $\mathbf{-61}$

Problem - 4 Find the absolute maximum and minimum values of $f(x,y) = x^3 + 3y^2$ on the unit disk $x^2 + y^2 \leq 1$

Interior Points: Stationary points

$$\left.\begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= 6y = 0 \end{aligned}\right\} (x,y) = (0,0)$$



Along the Boundary $x^2 + y^2 = 1$

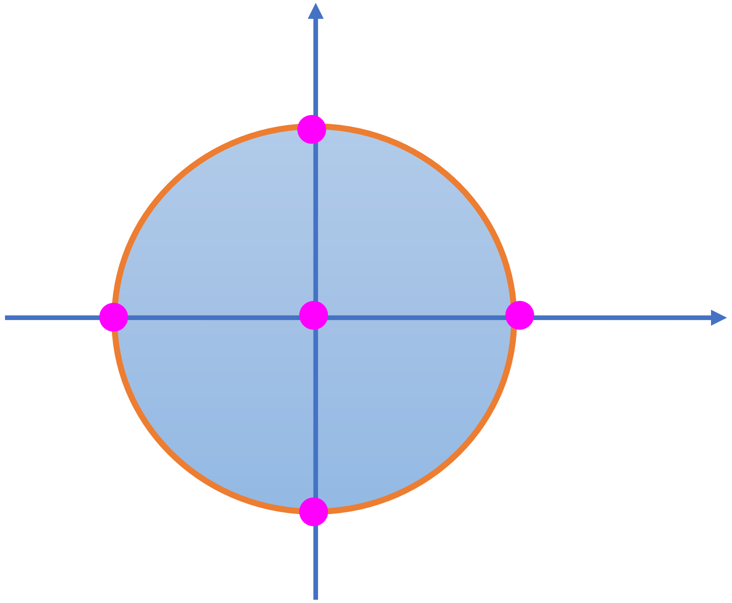
$$\min/\max\ f = x^3 + 3(1 - x^2), -1 \leq x \leq 1$$

Stationary points $(-1 < x < 1)$: $3x^2 - 6x = 0 \Rightarrow x = 0 \,\&\, x = 2 \Rightarrow x = 0$

Possible candidates (points) for extrema along this boundary: $(0,1), (0,-1)$ $(-1,0), (1,0)$

maximum and minimum values of $(x, y) = x^3 + 3y^2$ on $x^2 + y^2 \leq 1$

Possible candidates (points) for extrema: $(0,0), (0,1), (0,-1)\ (-1,0), (1,0)$



(x, y)	$(0,0)$	$(0,1)$	$(0,-1)$	$(-1,0)$	$(1,0)$
f	0	$\mathbf{3}$	3	$\mathbf{-1}$	1

The Maximum is 3 and the minimum value is -1

CONCLUSIONS

Maxima/minima can occur only at

- Boundary points of the domain (closed and bounded domain)
- Critical points $(f_x = 0 = f_y)$

CONCEPTS COVERED

MULTIVARIABLE CALCULUS

- Method of Lagrange's Multiplier
    - Worked Problems

Method of Lagrange's Multiplier

Find the Maxima/Minima of the function

$u = f(x, y)$ with the constraint $\phi(x, y) = 0$

Using chain rule

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

At the point of extrema $\frac{du}{dx} = 0 \quad \Longrightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$

At the point of extrema $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

The equation $\phi(x, y) = 0$ is satisfied at any point and so at the point of extrema

$$\boxed{\phi(x, y) = 0}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$$

$$\boxed{\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0}$$

We eliminate $\frac{dy}{dx}$ from the above equations.

$$\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}\right)\underbrace{\left(-\frac{f_y}{\phi_y}\right)}_{=:\lambda} + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \boxed{\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0}$$

$$\boxed{\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0} \qquad \boxed{\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0} \qquad \boxed{\phi(x, y) = 0}$$

Method of Lagrange's Multiplier (Working Rule)

max/min $u = f(x, y)$ with the constraint $\phi(x, y) = 0$

Define an auxiliary function $F(x, y, \lambda) = f(x, y) + \lambda\, \phi(x, y)$

Necessary conditions for extrema of F

$$F_x = 0 \Longrightarrow \quad f_x + \lambda \phi_x = 0$$

$$F_y = 0 \Longrightarrow \quad f_y + \lambda \phi_y = 0$$

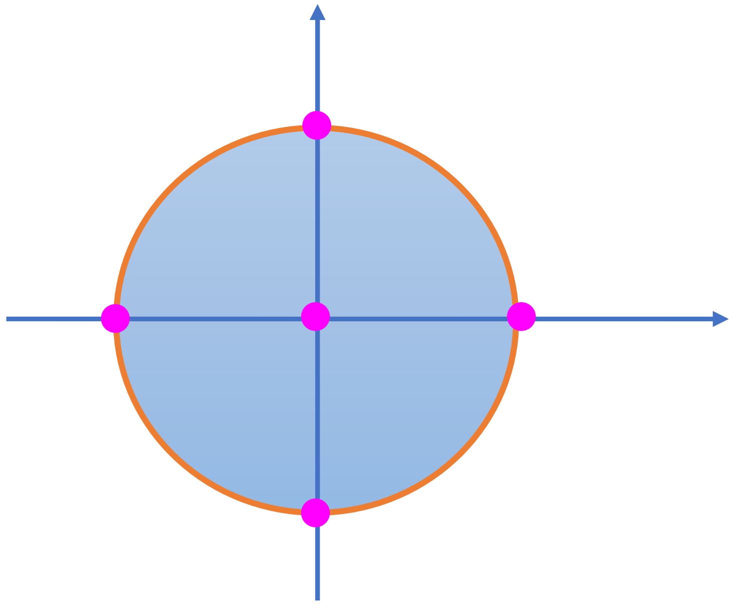
$$F_\lambda = 0 \Longrightarrow \quad \phi = 0$$

REMARK:

Using the method of Lagrange's multiplier, we obtain stationary points (candidates for extrema). We do not determine the nature of the stationary points. In practice we usually are interested in finding max/min value of the function under some given constraint.

Usually there are a few candidates (critical points), so we can evaluate f at all of them and choose the largest and smallest values. Hence, no further test is required if we wish to find only absolute maximum and minimum.

Problem - 1 Find the absolute maximum and minimum values of $f(x, y) = x^3 + 3y^2$ on the unit disk $x^2 + y^2 \leq 1$



Interior Points: Stationary points $\left.\begin{array}{l} f_x = 3x^2 = 0 \\ f_y = 6y = 0 \end{array}\right\}$ $(x, y) = (0, 0)$

Along the Boundary $x^2 + y^2 = 1$

$$\min/\max\ f = x^3 + 3(1 - x^2), -1 \leq x \leq 1$$

Stationary points $(-1 < x < 1)$: $3x^2 - 6x = 0 \Rightarrow x = 0 \;\&\; x = 2 \Rightarrow x = 0$

Possible candidates (points) for extrema along this boundary: $(0, 1), (0, -1)$ $(-1, 0), (1, 0)$

(x, y)	$(0,0)$	$(0,1)$	$(0, -1)$	$(-1,0)$	$(1,0)$
f	0	$\mathbf{3}$	3	$\mathbf{-1}$	1

The Maximum is 3 and the minimum value is -1

maximum and minimum values of $(x, y) = x^3 + 3y^2$ on $x^2 + y^2 \leq 1$

Interior Points: Stationary points $(x, y) = (0, 0)$

Along the Boundary $x^2 + y^2 = 1$ $\quad \max/\min f(x, y) = x^3 + 3y^2$ $\quad$ Subject to: $x^2 + y^2 = 1$

Auxiliary function for the Lagrange multiplier

$$F(x, y, \lambda) = x^3 + 3y^2 + \lambda(x^2 + y^2 - 1)$$

Critical Point:

$$F_x = 0 \Longrightarrow \quad x(3x + 2\lambda) = 0$$

$$F_y = 0 \Longrightarrow \quad y(3 + \lambda) = 0$$

$$F_\lambda = 0 \Longrightarrow \quad x^2 + y^2 = 1$$

Solution of $x(3x+2\lambda)=0$ $\quad y(3+\lambda)=0$ $\quad x^2+y^2=1$

$x=2$ is not possible as it canot satisfy $x^2+y^2=1$

$x(3x+2\lambda)=0$	$y(3+\lambda)=0$	$x^2+y^2=1$
$\Downarrow$	$\Downarrow$	$\Downarrow$
	$y=0\ \&\ \lambda=-3$	
	$\Downarrow$	
$\lambda=\mp\frac{3}{2}$	$y=0$	$x=\pm 1$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$x=0,2$	$\lambda=-3$	$y=\pm 1$

Candidates for extrema $(\pm 1,0)$ $\quad (0,\pm 1)$

Problem - 2: Find maximum/minimum of the function $x^2 - y^2 - 2x$ in the region $x^2 + y^2 \leq 1$

I. Local extrema in the interior $x^2 + y^2 < 1$

Let $f(x, y) = x^2 - y^2 - 2x$

$f_x = 0 \implies x = 1$

Critical Point: $(1,0)$

$f_y = 0 \implies y = 0$

However this point lies on the boundary so no critical point lies in the interior.

II. Local extrema on the boundary $x^2 + y^2 = 1$

Problem $\quad \max/\min \quad x^2 - y^2 - 2x \quad \text{subjet to} \quad x^2 + y^2 = 1$

Auxiliary function for the Lagrange multiplier

$$F(x, y, \lambda) = (x^2 - y^2 - 2x) + \lambda(x^2 + y^2 - 1)$$

Critical Point:

$$F_x = 0 \Longrightarrow \quad x(1 + \lambda) = 1$$

$$F_y = 0 \Longrightarrow \quad y(\lambda - 1) = 0$$

$$F_\lambda = 0 \Longrightarrow \quad x^2 + y^2 = 1$$

II. Local extrema on the boundary $x^2 + y^2 = 1$

$$x(1+\lambda) = 1 \quad\Downarrow\quad \lambda = 0, -2 \quad\Downarrow\quad x = \frac{1}{2}$$

$$y(\lambda - 1) = 0 \quad\Downarrow\quad y = 0 \;\; \& \;\; \lambda = 1 \quad\Downarrow\quad y = 0 \quad\Downarrow\quad \lambda = 1$$

$$x^2 + y^2 = 1 \quad\Downarrow\quad x = \pm 1 \quad\Downarrow\quad y = \pm\frac{\sqrt{3}}{2}$$

Candidates for extrema $(\pm 1, 0)$ $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

III. Function Values:

Candidates for extrema $(\pm 1, 0)$ $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

$$f(x, y) = x^2 - y^2 - 2x$$

Points	$(1, 0)$	$(-1, 0)$	$\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$
Function Value	-1	3	$-\frac{3}{2}$

Maximum value of the function: 3

Minimum value of the function: $-\frac{3}{2}$

Problem - 3: Find maximum and minimum values of the function $f(x,y) = x^2 + y^2$

in the region $(x-2)^2+(y-1)^2 \leq 20$

I. Local extrema in the interior $(x-2)^2+(y-1)^2 < 20$

$$f_x = 0 \implies x = 0$$

$$f_y = 0 \implies y = 0$$

Critical Point: $(0,0)$

II. Local extrema on the boundary $(x-2)^2+(y-1)^2 = 20$

Problem $\quad \max/\min \quad x^2 + y^2 \quad \text{subjet to} \quad (x-2)^2+(y-1)^2 = 20$

Auxiliary function for the Lagrange multiplier

$$F(x,y,\lambda) = (x^2 + y^2) + \lambda\left((x-2)^2 + (y-1)^2 - 20\right)$$

$$F(x,y,\lambda) = (x^2 + y^2) + \lambda\left((x-2)^2 + (y-1)^2 - 20\right)$$

Critical Point:

$$F_x = 0 \Longrightarrow \quad x + \lambda(x-2) = 0 \quad \Longrightarrow \quad (x-2) = -\frac{2}{1+\lambda} \Longrightarrow x = -2, 6$$

$$F_y = 0 \Longrightarrow \quad y + \lambda(y-1) = 0 \quad \Longrightarrow \quad (y-1) = -\frac{1}{1+\lambda} \Longrightarrow y = -1, 3$$

$$F_\lambda = 0 \Longrightarrow \quad (x-2)^2 + (y-1)^2 = 20 \quad \Longrightarrow (1+\lambda) = \pm\frac{1}{2} \Longrightarrow \lambda = -\frac{1}{2}, -\frac{3}{2}$$

Points	$(0,0)$	$(-2,-1)$	$(6,3)$
Function Value	0	5	$\mathbf{45}$

Minimum value : 0 Maximum value: 45

CONCLUSIONS

Method of Lagrange's Multiplier

max/min $u = f(x, y)$

with the constraint $\phi(x, y) = 0$

Auxiliary function $F(x, y, \lambda) = f(x, y) + \lambda\, \phi(x, y)$

Necessary conditions for extrema of F

$$F_x = 0 \Longrightarrow \quad f_x + \lambda \phi_x = 0$$

$$F_y = 0 \Longrightarrow \quad f_y + \lambda \phi_y = 0$$

$$F_\lambda = 0 \Longrightarrow \quad \phi = 0$$