Some Applications of Derivatives

- ✓ Mean Value Theorems
- ✓ Taylor's Series
- ➢ Increase and Decrease of a function
- ➢ Extreme Values of a function
- ➢ Convexity & Concavity of a curve

Increase and Decrease of a function

(a) If a differentiable function $f(x)$ is *increasing* ($f(x+\Delta x) > f(x)$ for $\Delta x > 0$) in $[a,b]$ then $f'(x) \geq 0$ in $[a,b]$

(b) If $f(x)$ is continuous in $[a,b]$, differentiable & $f'(x) > 0$ in (a,b) then f is increasing in $[a,b]$

(a) Since $f(x)$ is an increasing function, we have

$$\left.\begin{array}{l} f(x+\Delta x) > f(x) \text{ for } \Delta x > 0 \\ f(x+\Delta x) < f(x) \text{ for } \Delta x < 0 \end{array}\right\} \left.\frac{f(x+\Delta x)-f(x)}{\Delta x} > 0 \right\} \left.\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \geq 0 \right\} f'(x) \geq 0$$

(b) Consider two values x_1 and x_2, $x_1 < x_2$ on the interval $[a,b]$. Using LMVT, we have

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1), \qquad x_1 < \xi < x_2$$

$$f'(\xi) > 0, \Rightarrow f(x_2) > f(x_1) \Rightarrow f \text{ is increasing in } [a,b]$$

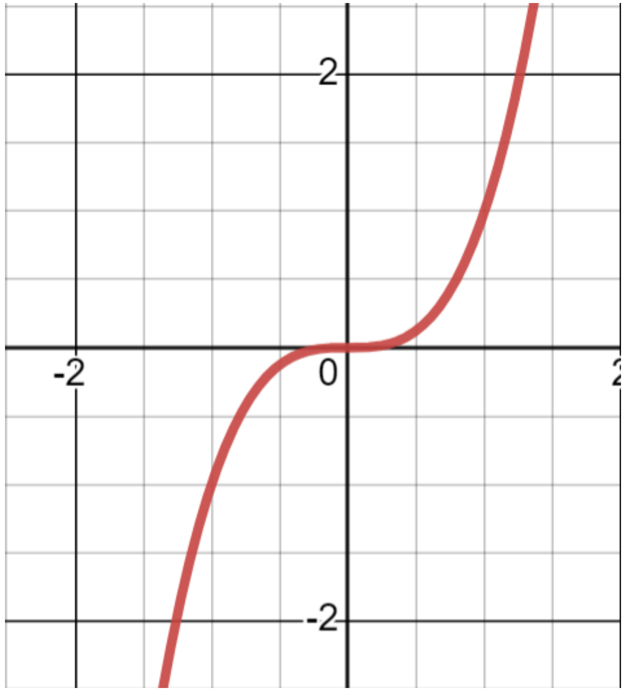
Useful Result: If $f(x) > 0$ (or $f(x) \geq 0$) then $\lim_{x \to x_0} f(x) \geq 0$ $\left(f(x) = e^{-\frac{1}{x^2}};\ \lim_{x\to 0} f(x) = 0 \text{ whereas } f(x) > 0, \forall x \right)$

Increase and Decrease of a function

(a) If a differentiable function $f(x)$ is *increasing* ($f(x + \Delta x) > f(x)$ for $\Delta x > 0$) in $[a, b]$ then $f'(x) \geq 0$ in $[a, b]$

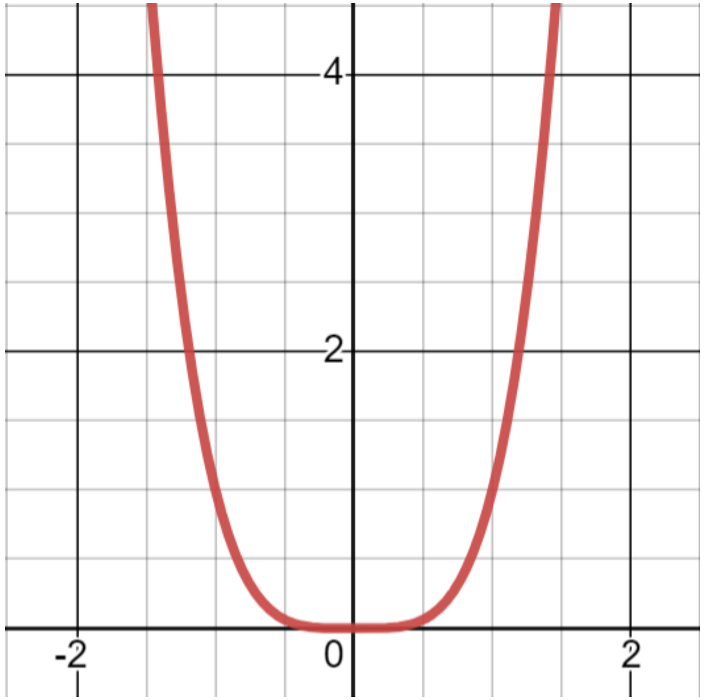
(b) If $f(x)$ is continuous in $[a, b]$, differentiable & $f'(x) > 0$ in (a, b) then f is increasing in $[a, b]$

Ex. $f(x) = x^3$



Function is increasing and $f'(x) \geq 0$

Ex. $f(x) = x^4$



For $x > 0$, we have $f' > 0$ and the function increases

For $x < 0$, we have $f' < 0$ and the function decreases

A function $y = f(x)$ has a maximum (or a minimum) at the point $x = x_0$ if at every point in a neighborhood of $x = x_0$, the function assumes a smaller value (or a larger value) than at the point itself.

Such a maximum (or minimum) is called relative or local maximum (or local minimum).

Mathematically, a function $y = f(x)$ has

- a minimum at $x = x_0$ if $f(x_0 + \Delta x) > f(x_0)$, for any sufficiently small Δx (> 0 or < 0)
- a maximum at $x = x_0$ if $f(x_0 + \Delta x) < f(x_0)$, for any sufficiently small Δx (> 0 or < 0)

Maximum and minimum values together are called extreme values.

The smallest and the largest values attained by a function over entire domain including the boundary of the domain are called absolute (or global) minimum and absolute (or global) maximum, respectively.

Necessary condition for the existence of extremum

If at a point $x = x_0$, a differentiable function has a maximum or minimum then $f'(x_0) = 0$

$$f(x_0 + h) = f(x_0) + h\, f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots$$

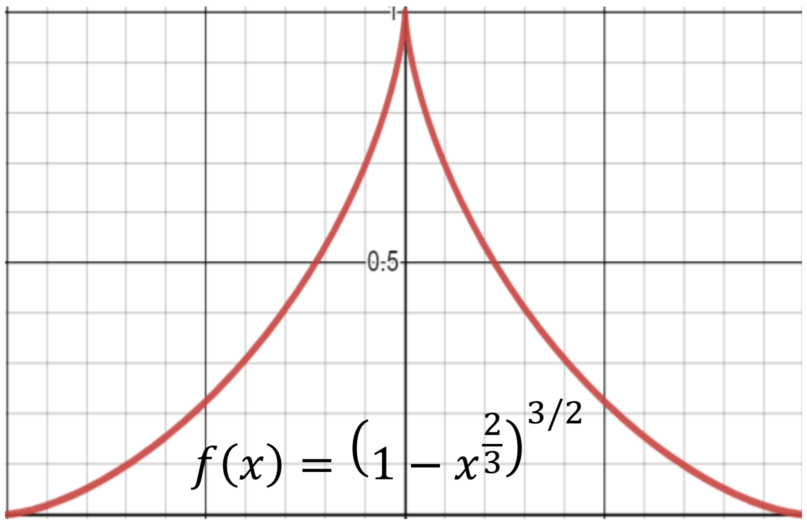
$$\Delta f = f(x_0 + h) - f(x_0) = h\, f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots$$

If $f'(x_0) \neq 0,$ then Δf will be positive for $h > 0$ and Δf will be negative for $h < 0$ and hence x_0 cannot be the point of max/min

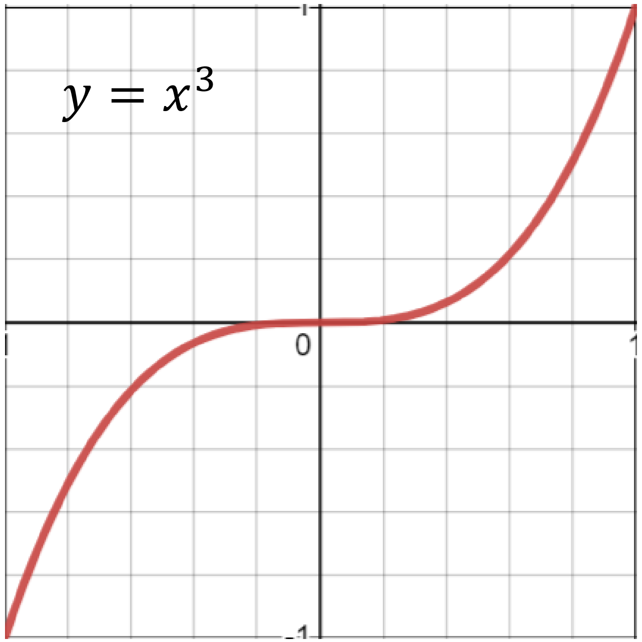
Alternative: Proof of Rolle's Theorem which uses existence of only first derivative

If a function is not differentiable?

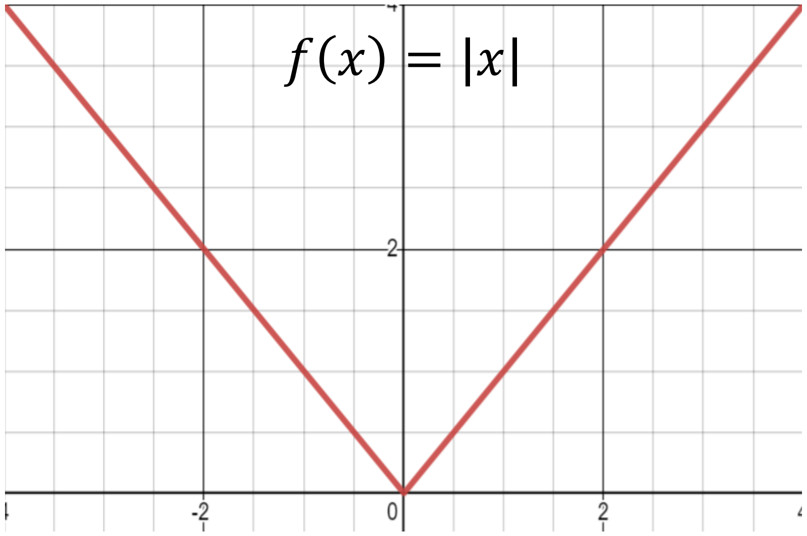
If a function f has extremum at x_0 — Either derivative does not exist at x_0; Or derivative is ZERO at x_0



The point $x = x_0$ is called **critical point** (or stationary point) of $f(x)$ if $f'(x_0) = 0$ OR $f'(x_0)$ does not exist.



A critical point where the function has no minimum or maximum is called a **Inflection point** (*saddle point* for functions of several variables).



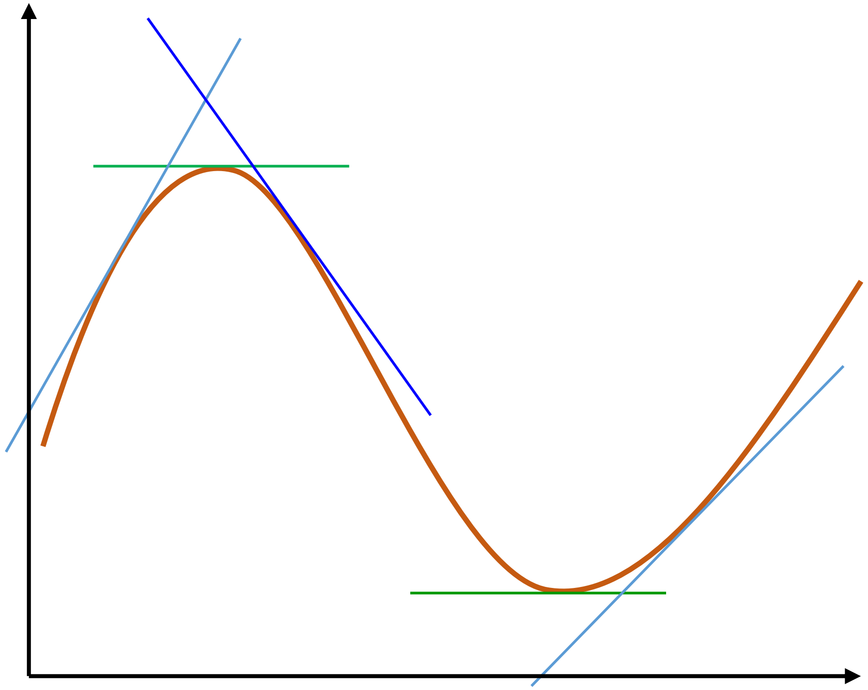
Sufficient condition for the existence of extremum

$$\text{if} \begin{cases} f'(x) > 0 \text{ when } x < x_0 \\ f'(x) < 0 \text{ when } x > x_0 \end{cases}$$

Then function f has maximum at $x = x_0$

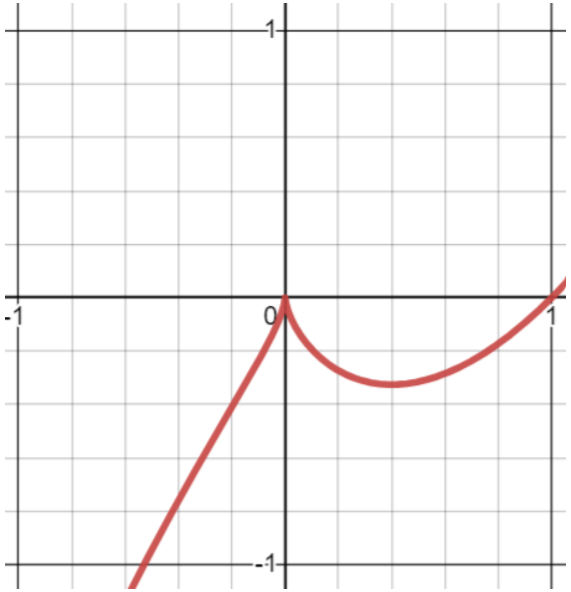
$$\text{if} \begin{cases} f'(x) < 0 \text{ when } x < x_0 \\ f'(x) > 0 \text{ when } x > x_0 \end{cases}$$

Then function f has minimum at $x = x_0$



Ex. Find local max/min of the function

$$y = (x-1)\, x^{\frac{2}{3}}$$

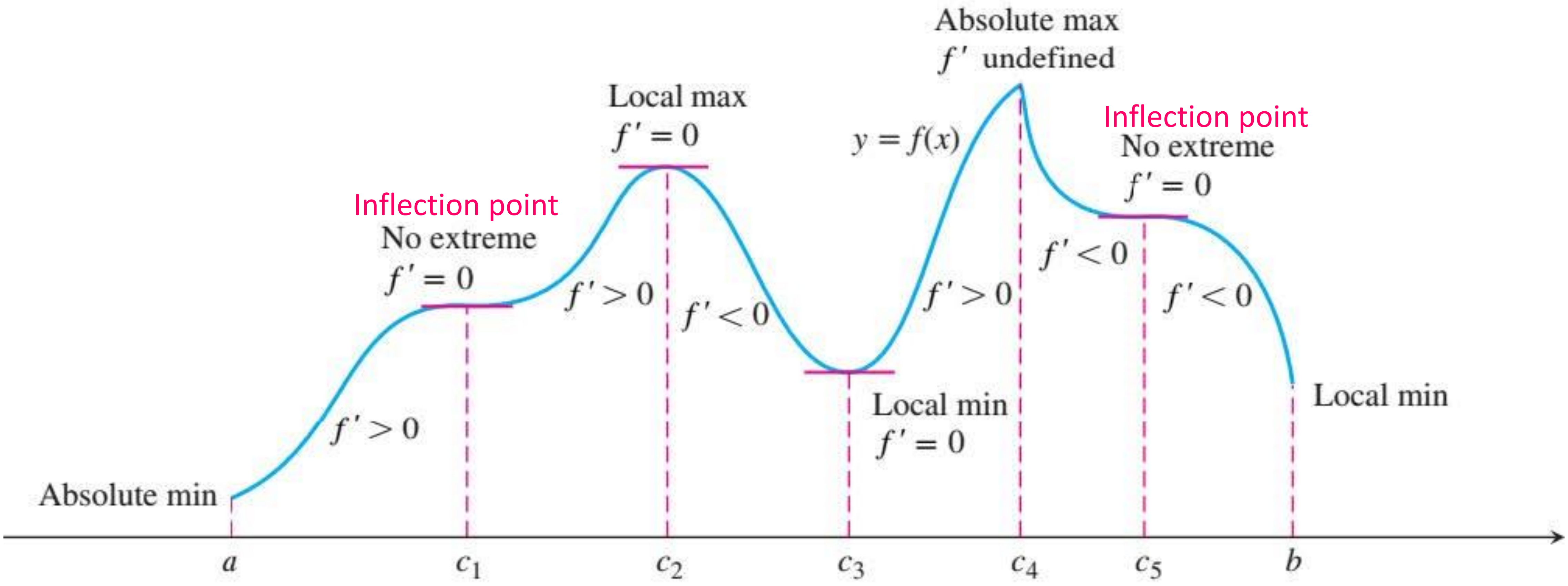


LMVT: $f(x) - f(x_0) = f'(\xi)(x - x_0)$

$$\left.\begin{array}{l} f'(x) > 0 \text{ when } x < x_0 \Rightarrow f'(\xi)(x - x_0) < 0 \Rightarrow f(x) < f(x_0) \\ f'(x) < 0 \text{ when } x > x_0 \Rightarrow f'(\xi)(x - x_0) < 0 \Rightarrow f(x) < f(x_0) \end{array}\right\}$$

Function f has maximum at $x = x_0$

Example:



Absolute max
f' undefined
Local max
$f' = 0$
$y = f(x)$
Inflection point
No extreme
$f' = 0$
Inflection point
No extreme
$f' = 0$
$f' > 0$
$f' < 0$
$f' > 0$
$f' < 0$
$f' < 0$
Local min
$f' = 0$
Local min
$f' > 0$
Absolute min
a
c_1
c_2
c_3
c_4
c_5
b

Let $f'(x_0) = 0$. The function f has maximum at x_0 if $f''(x_0) < 0$ and minimum if $f''(x_0) > 0$

Using Taylor's series: $f(x_0 + h) = f(x_0) + h\, f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots$

$$\Rightarrow \Delta f = f(x_0 + h) - f(x_0) = \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots$$

The sign of Δf will depend on the leading term $\frac{h^2}{2} f''(x_0)$; Since $\frac{h^2}{3}$ is positive the sign will depend on $f''(x_0)$

Generalization: Let $f^{(n)}(x_0)$ be the first non-vanishing derivative in Taylor's series expansion. Then

If n is even $\begin{cases} f(x) \text{ has a maximum if } f^{(n)}(x_0) < 0 \\ f(x) \text{ has a minimum if } f^{(n)}(x_0) > 0 \end{cases}$

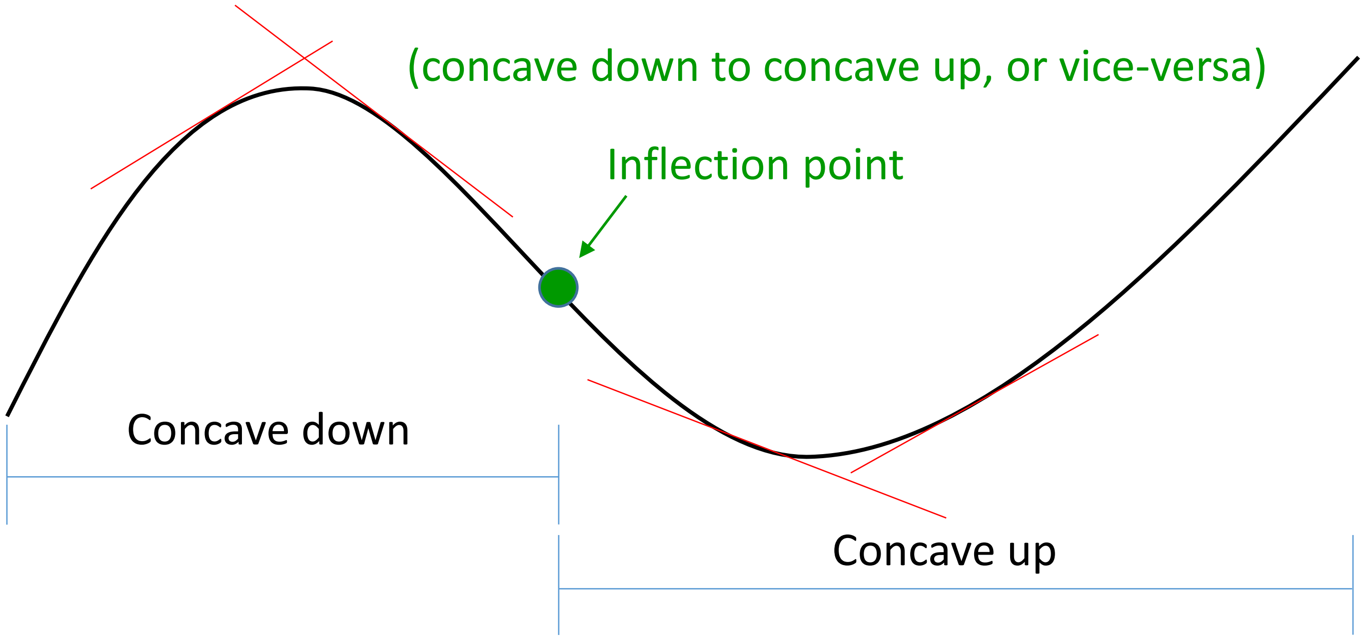
If n is odd $\begin{cases} f(x) \text{ decreases if } f^{(n)}(x_0) < 0 \\ f(x) \text{ increases if } f^{(n)}(x_0) > 0 \end{cases}$

Concave up and Concave down

The bending of a curve is measured by its concavity.

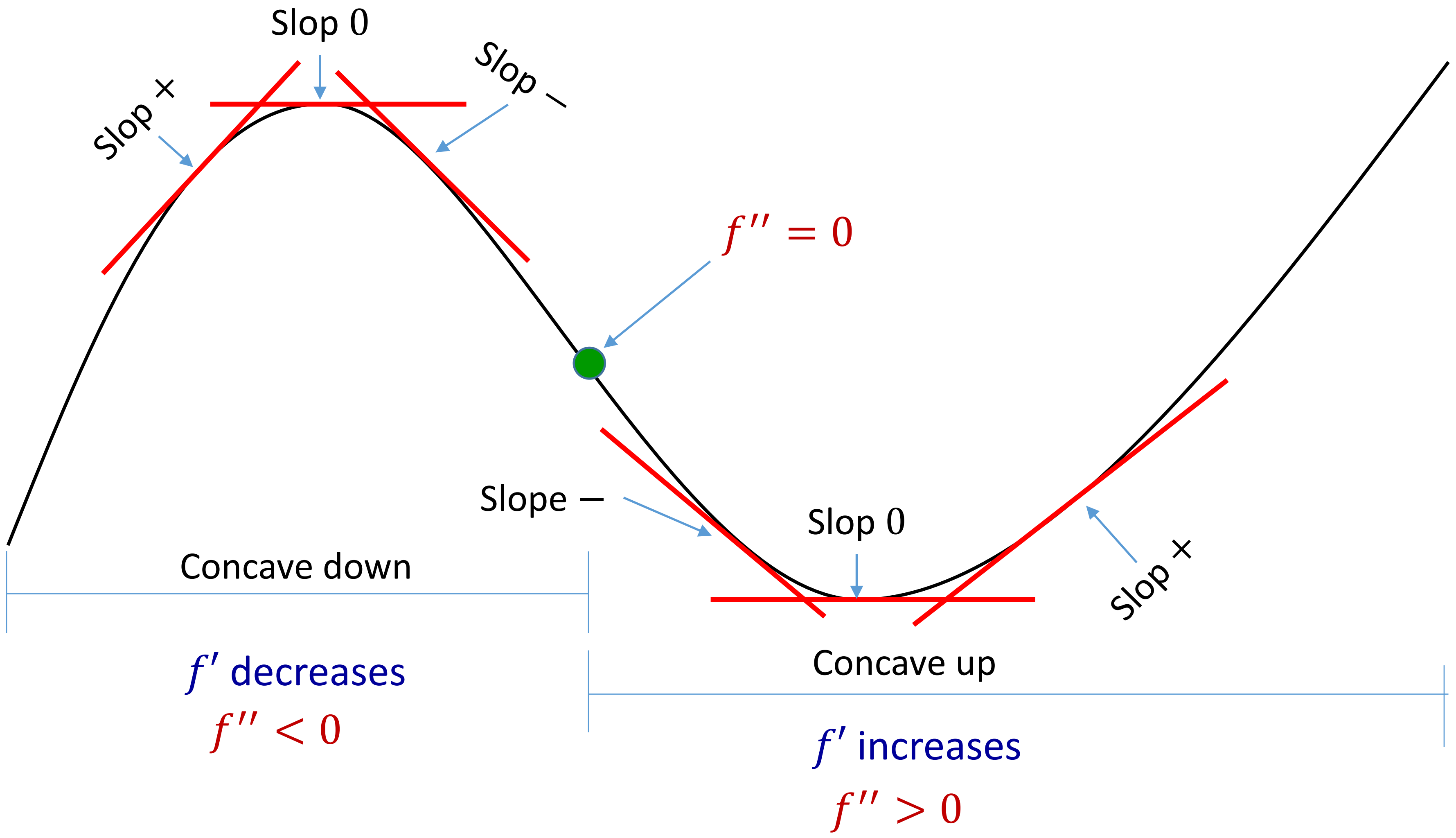
- If the graph of a function f (a curve) lies **above all of its tangent lines** on an interval I, then we say the function (the curve) is **concave up** on I.

- If the graph of a function f (a curve) lies **below all of its tangent lines** on an interval I, then we say the function (the curve) is **concave down** on I.



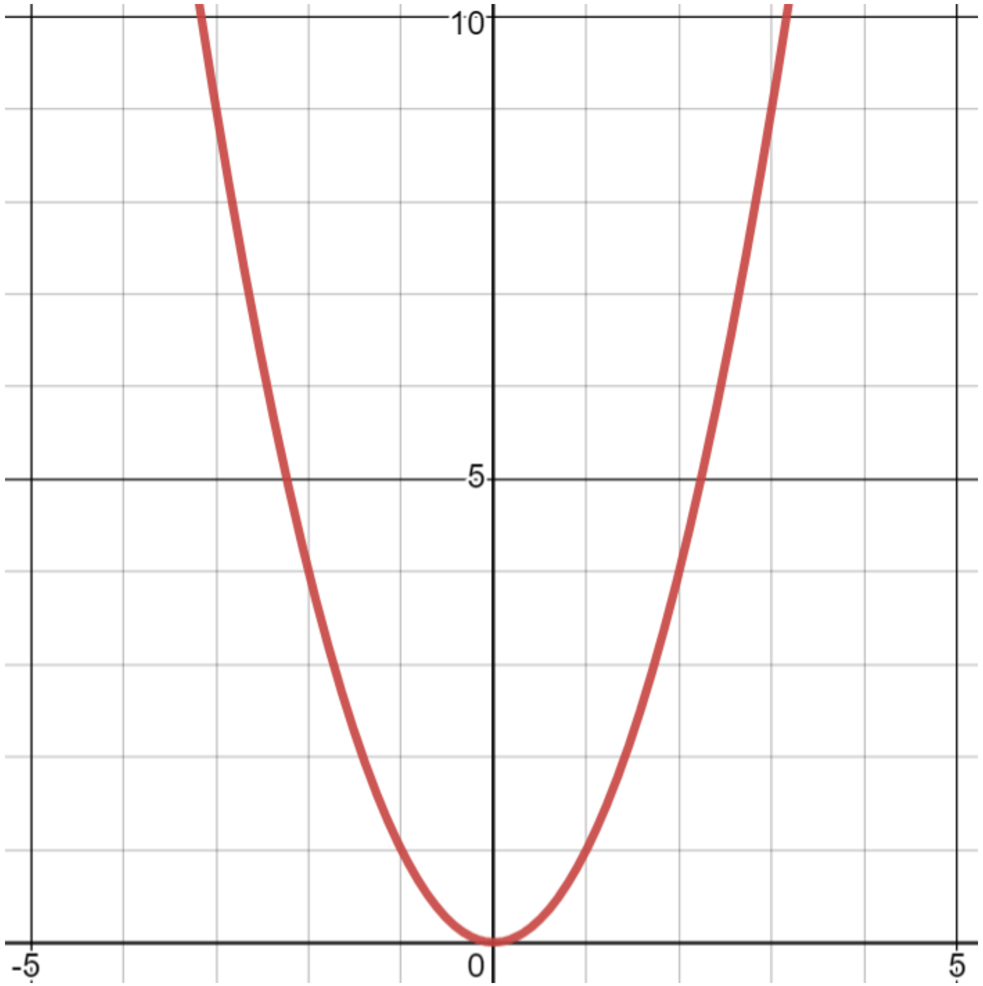
Concave up and Concave down

Application of 1st and 2nd order derivatives for identifying behaviour of a curve/graph of a function



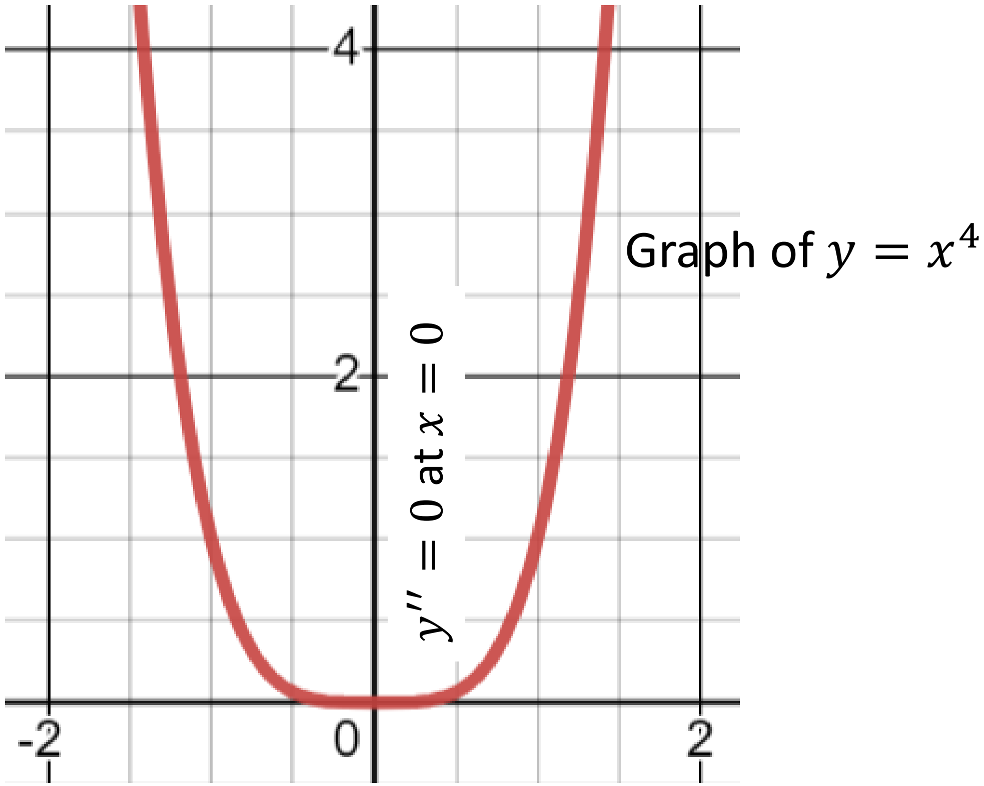
Concave up and Concave down

Example: The curve $y = x^2$ is **concave up** on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive.

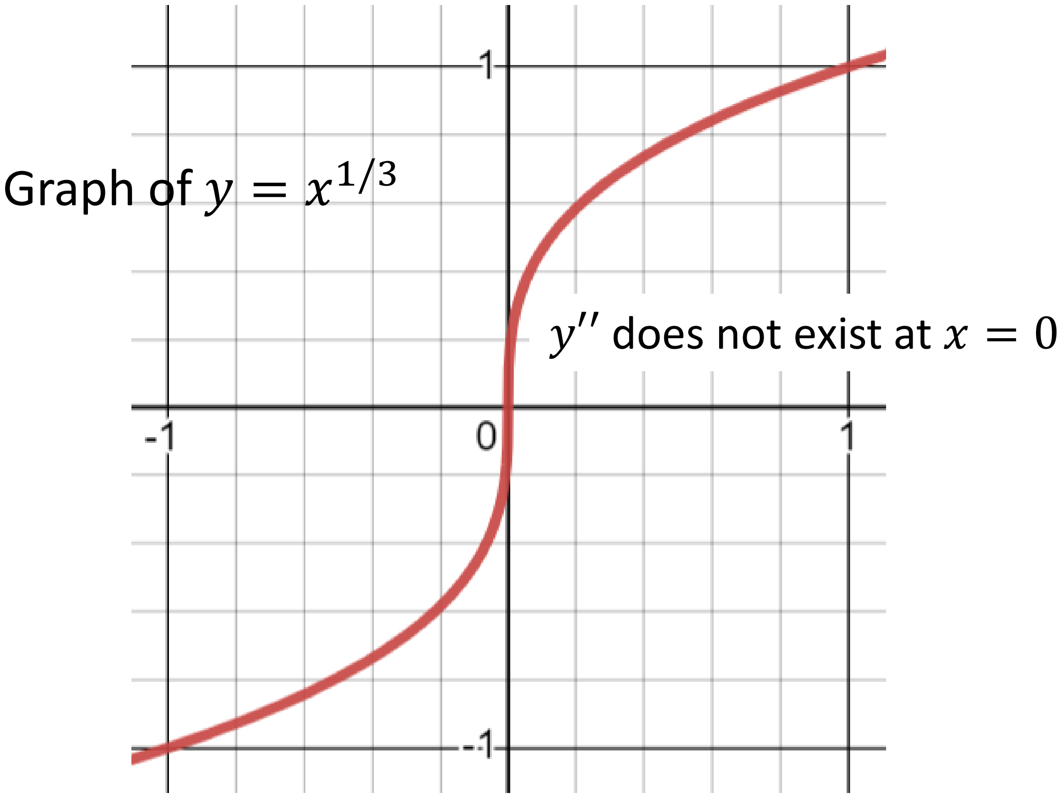


Necessary condition for a point of inflection

Note: An Inflection Point may not exist where $y'' = 0$.



Note: An Inflection Point may occur where y'' does not exist.



The condition $f''(x_0) = 0$ is not sufficient to conclude that x_0 is a point of inflection for f.

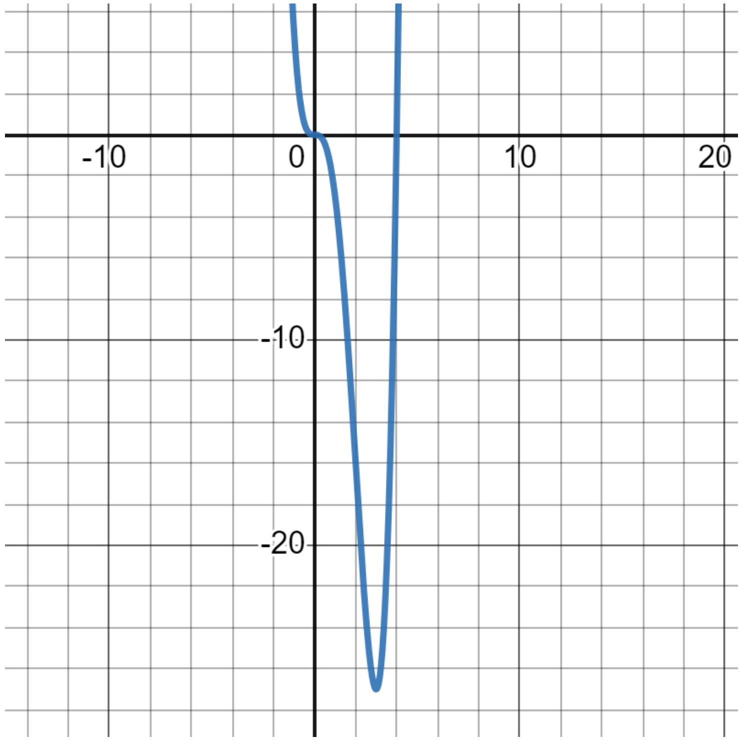
If x_0 is a point of inflection for f, then either $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

Suppose f is thrice differentiable at x_0. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is point of inflection for f.

Note: The condition $f'''(x_0) \neq 0$ is not necessary for x_0 to be a point of inflection for f.

Example: Let $f(x) = x^5$. Then 0 is a point of inflection for f, but $f'''(0) = 0$.

Example: Identify points of inflection for $f(x) = x^4 - 4x^3$



$f' = 4x^3 - 12x^2$

$f'' = 12x^2 - 24x$

$f'' = 0 \Rightarrow x = 0, 2$

$f''' = 24x - 24$

$f''' \neq 0$ for $x = 0$ & 2

$\Rightarrow$ 0 and 2 are points of inection for f.