- Orientable Surfaces

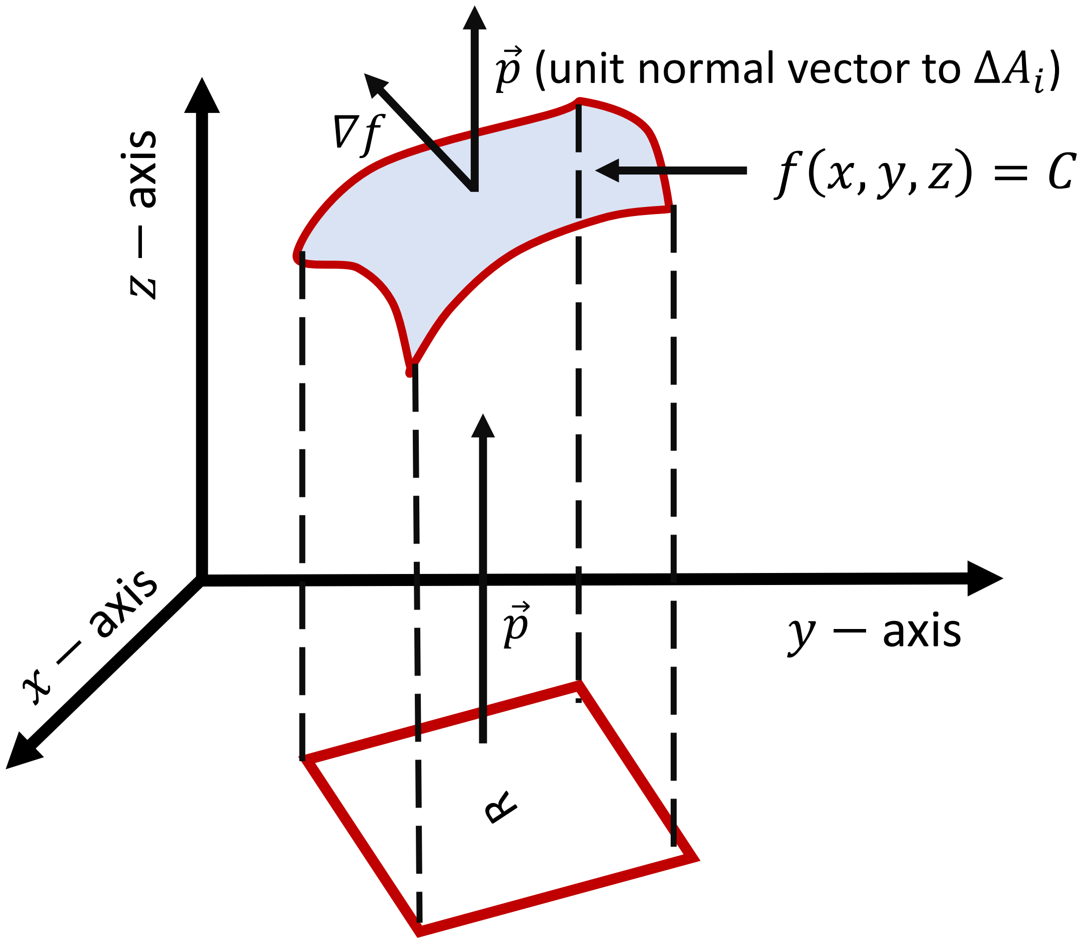
- Flux Integrals

Surface integral of g over S

$$\iint_S g(x,y,z)\, d\sigma = \iint_R g(x,y,z)\, \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA$$

R is the projection of S on on the xy, yz or zx plane

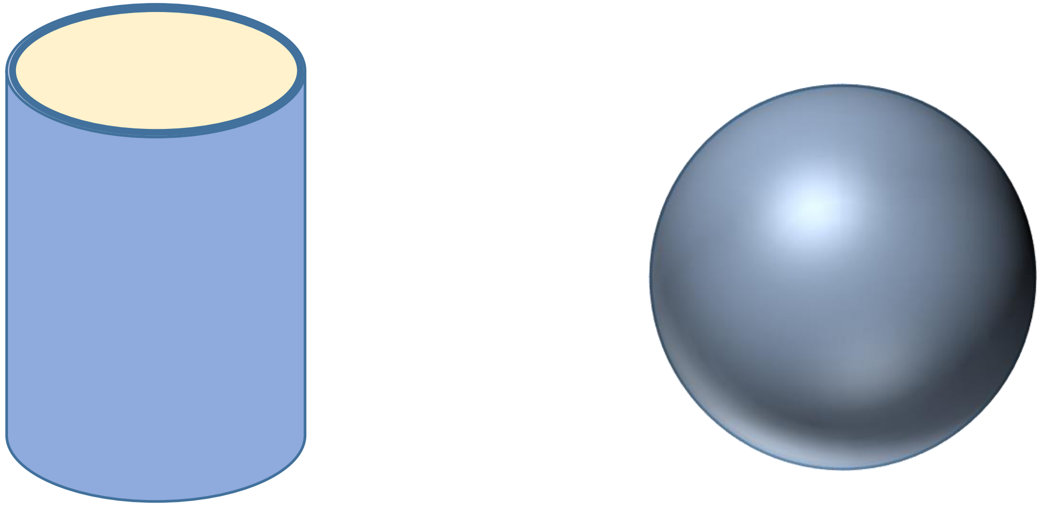
$\vec{p}$ is the unit normal to R and $\nabla f \cdot \vec{p} \neq 0$



Orientable Surface

S is an orientable surface if it has two sides which may be painted in two different colors.

Non-Orientable Surface



Orientable Surfaces

Flux of a vector field $\vec{F}$ through a surface S

The flux of a vector field $\vec{F}$ across an orientable surface S in the direction of $\vec{n}$ (unit normal to S) is given by the integral

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,d\sigma$$

Geometrically, a flux integral is the surface integral over S of the normal component of $\vec{F}$.

If $\vec{F}$ is the continuous velocity field of a fluid and $\rho(x, y, z)$ is the density of the fluid at (x, y, z) then the flux integral

$$\iint_S \rho\ \vec{F}\cdot\vec{n}\ d\sigma$$

represents the mass of the fluid flowing across S per unit of time.

Evaluation of Flux Integral $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$

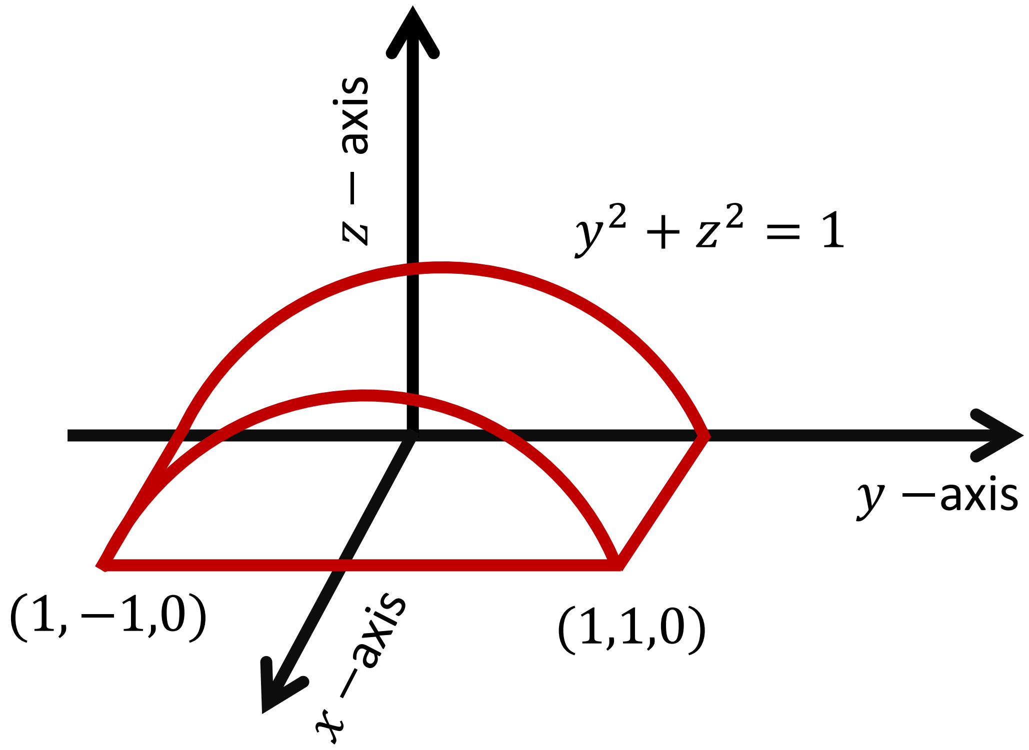
Suppose S is a part of a level surface $f(x, y, z) = C$, then $\vec{n}$ may be taken either of the two unit vectors

$$\vec{n} = \pm \frac{\nabla f}{|\nabla f|}$$

$$\text{Flux} = \pm \iint_R \vec{F} \cdot \left(\frac{\nabla f}{|\nabla f|} \right) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA$$

$$= \pm \iint_R \vec{F} \cdot \frac{\nabla f}{|\nabla f \cdot \vec{p}|}\, dA$$

Problem-1 Find the flux of $\vec{F} = yz\hat{j} + z^2\hat{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z > 0$ by the planes $x = 0$ and $x = 1$.



Solution Surface $f(x, y, z) = C$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{2y\hat{j} + 2z\hat{k}}{\sqrt{4(y^2 + z^2)}} = y\hat{j} + z\hat{k} \qquad \vec{p} = \vec{k}$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA = \frac{|\nabla f|}{|\nabla f \cdot \vec{k}|} dA = \frac{2}{|2z|} dA = \frac{1}{z} dA$$

Also $\vec{F} \cdot \vec{n} = y^2 z + z^3 = z$

Flux through S: $\displaystyle\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iint_{R_{xy}} z \times \frac{1}{z} \, dA = \iint_{R_{xy}} dA = 2$

Problem-2 Evaluate the integral

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma \quad \text{where} \qquad \vec{F} = 6z\,\hat{\imath} + 6\,\hat{\jmath} + 3y\,\hat{k}$$

and S is the portion of the plane $2x + 3y + 4z = 12$ which is in the first octant.

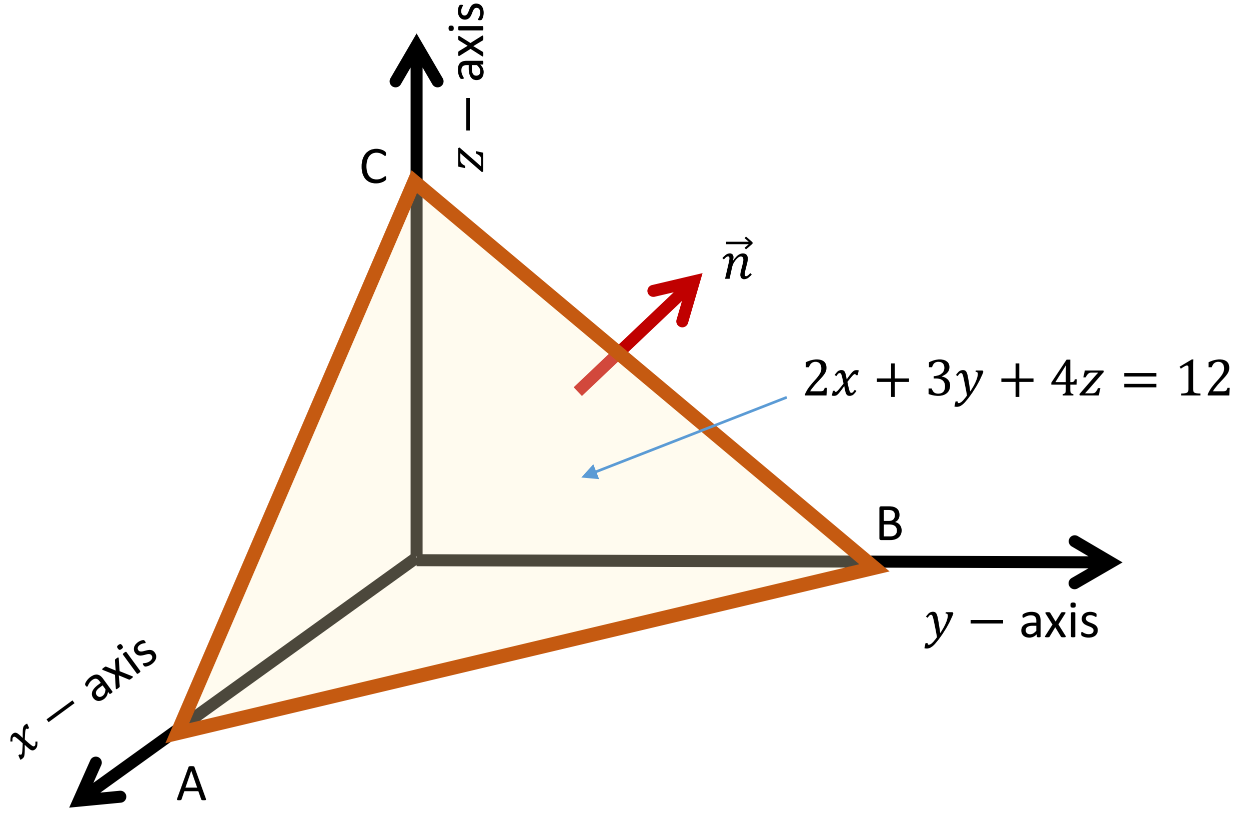
Solution Let $f(x, y, z) = 2x + 3y + 4z \quad \Rightarrow \nabla f = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

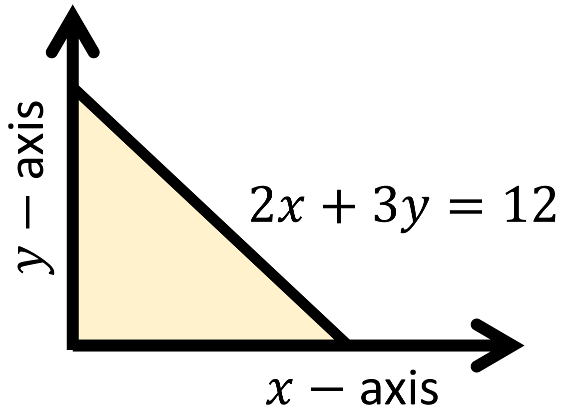
$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{29}}(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) \qquad \vec{F} \cdot \vec{n} = \frac{1}{\sqrt{29}}(12z + 18 + 12y)$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{k}\,|} dA = \frac{\sqrt{29}}{4}\, dA \qquad (\vec{p} = \hat{k})$$

We are projecting of S on the xy plane.

The projection R is bounded by x-axis, y-axis and $2x + 3y = 12$





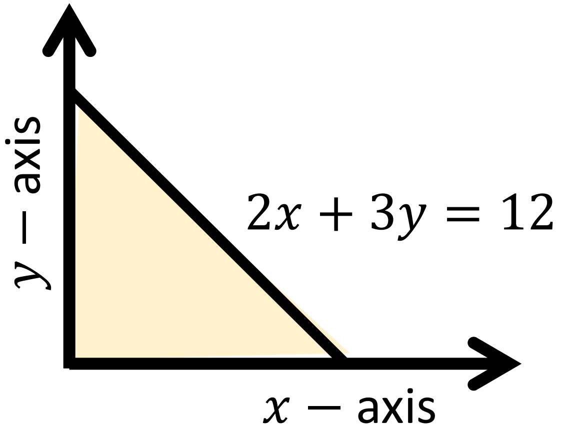
Note that $\vec{F}\cdot\vec{n} = \dfrac{1}{\sqrt{29}}(12z + 18 + 12y)$

Also given surface $2x + 3y + 4z = 12$

$$d\sigma = \frac{\sqrt{29}}{4}\, dA$$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_R \frac{1}{\sqrt{29}}(3(12 - 2x - 3y) + 18 + 12y)\left(\frac{\sqrt{29}}{4}\right) dA$$

$$= \frac{1}{4}\iint_R (54 - 6x + 3y)\, dA$$



$$= \frac{1}{4}\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (54 - 6x + 3y)\, dy\, dx$$

$$= 138$$

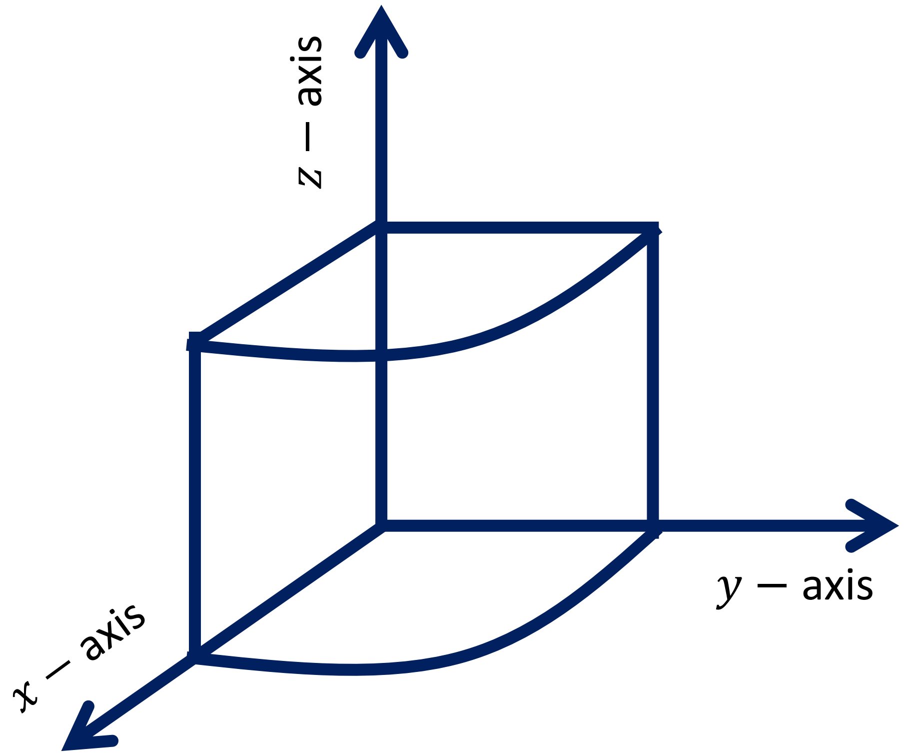
Problem-3 Evaluate the surface integral $\iint\limits_S \vec{F}\cdot\vec{n}\,d\sigma$ where $\vec{F} = z^2\,\hat{\imath} + xy\,\hat{\jmath} - y^2\,\hat{k}$ and S is the portion of the surface of the cylinder $x^2 + y^2 = 36$, $0 \le z \le 4$ included in the first octant.

Solution Let $f(x,y,z) = x^2 + y^2 - 36$

$$\Rightarrow \nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath} \Rightarrow |\nabla f| = \sqrt{4\times 36} = 12$$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{12}(2x\,\hat{\imath} + 2y\,\hat{\jmath})$$

$$= \frac{1}{6}(x\,\hat{\imath} + y\,\hat{\jmath})$$



$$d\sigma = \frac{|\nabla f|}{|\nabla f . \vec{p}|} dA \qquad \vec{p} = i \text{ (if projection is on } yz \text{ plane)}$$

$$\nabla f = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$$

$$|\nabla f| = 12$$

$$d\sigma = \frac{12}{|2x|} dA = \frac{6}{x} dA$$

$$\vec{F} = z^2\,\hat{\imath} + xy\,\hat{\jmath} - y^2\,\hat{k}$$

$$\vec{n} = \frac{1}{6}(x\,\hat{\imath} + y\,\hat{\jmath})$$

Therefore $\displaystyle \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{R_{yz}} \frac{1}{6}(xz^2 + xy^2)\frac{6}{x}\, dA$

$$= \int_{z=0}^{4} \int_{y=0}^{6} (y^2 + z^2)\, dydz = \int_0^4 \left[\frac{y^3}{3} + z^2 y\right]_0^6 dz$$

$$= \int_0^4 (72 + 6z^2) dz = 72 \times 4 + \frac{6}{3} \times 64 = 416$$

KEY TAKEAWAY

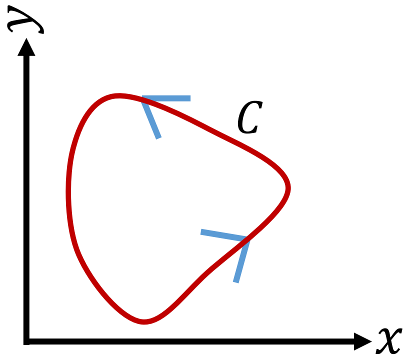
Surface Integrals

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_R \vec{F} \cdot \left(\frac{\nabla f}{|\nabla f|} \right) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA$$

$$= \iint_R \vec{F} \cdot \frac{\nabla f}{|\nabla f \cdot \vec{p}|}\, dA$$

- **Stokes' Theorem (Generalization of Green's Theorem)**

Green's Theorem (Recall):



Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C

Let $\vec{F} = F_1(x, y)\,\hat{\imath} + F_2(x, y)\hat{\jmath}$ be smooth vector field (F_1 & F_2 are C^1 functions) on both R and C, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

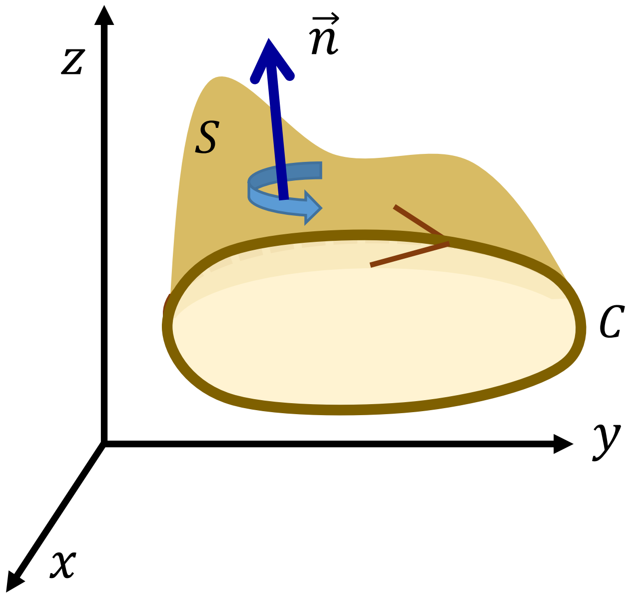
Note that the above can also be written as

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl } \vec{F}) \cdot \hat{k} \; dA$$

Stokes’ Theorem

Green’s theorem in the plane $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k}\ dxdy$

Let C be a closed curve in 3-D space which forms the boundary of a surface S whose unit normal vector is $\vec{n}$



Then for a continuously differentiable vector field $\vec{F}$, we have

$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\ ds$ where the direction of the line integral around C and the normal $\vec{n}$ are oriented in a right-handed sense

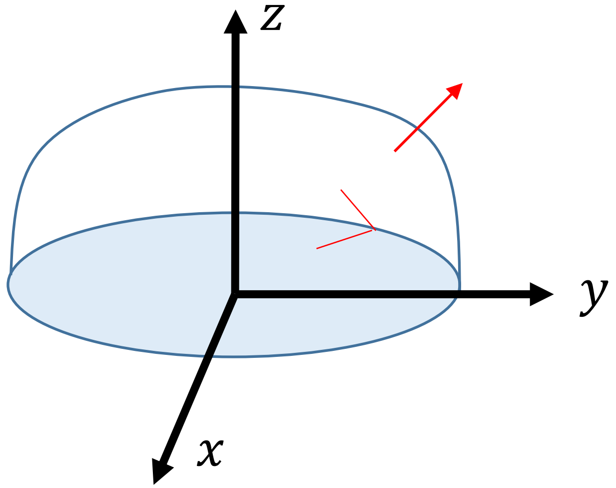
If $\nabla \times \vec{F} = 0$ ($\vec{F}$ is irrotational, or $\vec{F}$ is conservative) then, Stokes’ theorem tells us that

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

Problem-1 Verify Stokes' theorem for the hemisphere $S: x^2 + y^2 + z^2 = 9$, $z \geq 0$, its boundary $C: x^2 + y^2 = 9$, $z = 0$ and the field $\vec{F} = y\hat{\imath} - x\hat{\jmath}$

Stokes' theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\ ds$

Solution: Parametric equation of the curve



$$\vec{r}(\theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath}, \qquad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \frac{d\vec{r}}{d\theta} = -3\sin\theta\,\hat{\imath} + 3\cos\theta\,\hat{\jmath}$$

$$\vec{F} = 3\sin\theta\,\hat{\imath} - 3\cos\theta\ \hat{\jmath}$$

$$\vec{F} \cdot \frac{d\vec{r}}{d\theta} = -9\sin^2\theta - 9\cos^2\theta\ = -9$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} \vec{F} \cdot \frac{d\vec{r}}{d\theta} d\theta$$

$$= \int_0^{2\pi} -9\ d\theta\ = -18\pi$$

$$S: x^2 + y^2 + z^2 = 9$$

$$f = x^2 + y^2 + z^2$$

$$\vec{F} = y\hat{\imath} - x\hat{\jmath}$$

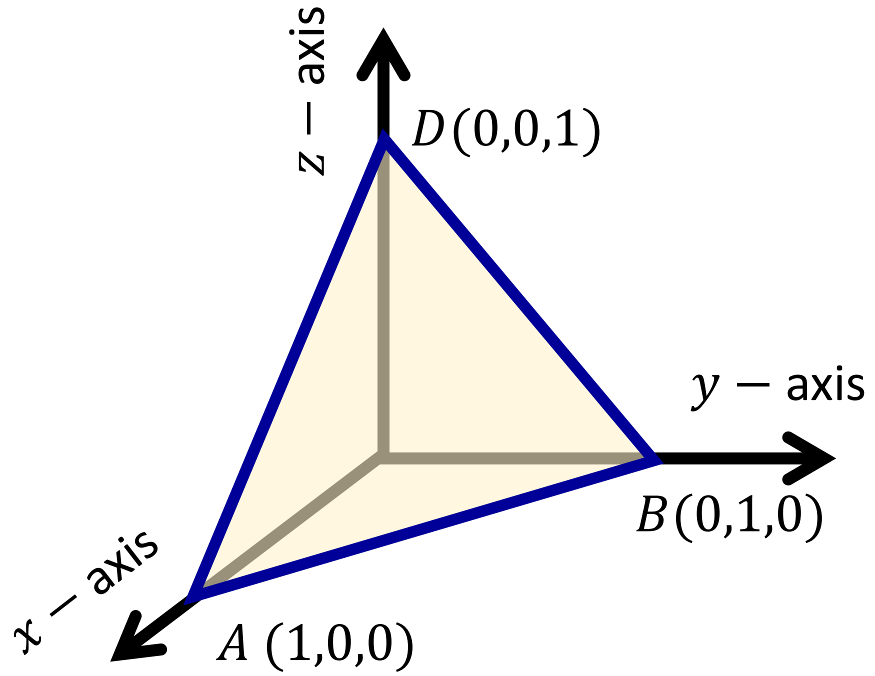
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \hat{\imath}\,(0) + \hat{\jmath}\,(0) + \hat{k}(-1-1) = -2\hat{k}$$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}}{\sqrt{4 \times 9}} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{3}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds = \iint_{x^2+y^2 \le 9} -\frac{2z}{3} \frac{|\nabla(x^2 + y^2 + z^2)|}{|\nabla(x^2 + y^2 + z^2) \cdot \hat{k}|} dxdy$$

$$= \iint_{x^2+y^2 \le 9} -\frac{2z}{3} \frac{6}{2z} dxdy = -2 \iint_{x^2+y^2 \le 9} dxdy = -18\pi$$

Problem-2 Verify Stokes' theorem for the function $\vec{F} = x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}$ over the plane surface $x + y + z = 1$ lying in the first quadrant.



Solution Stokes' theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\ ds$

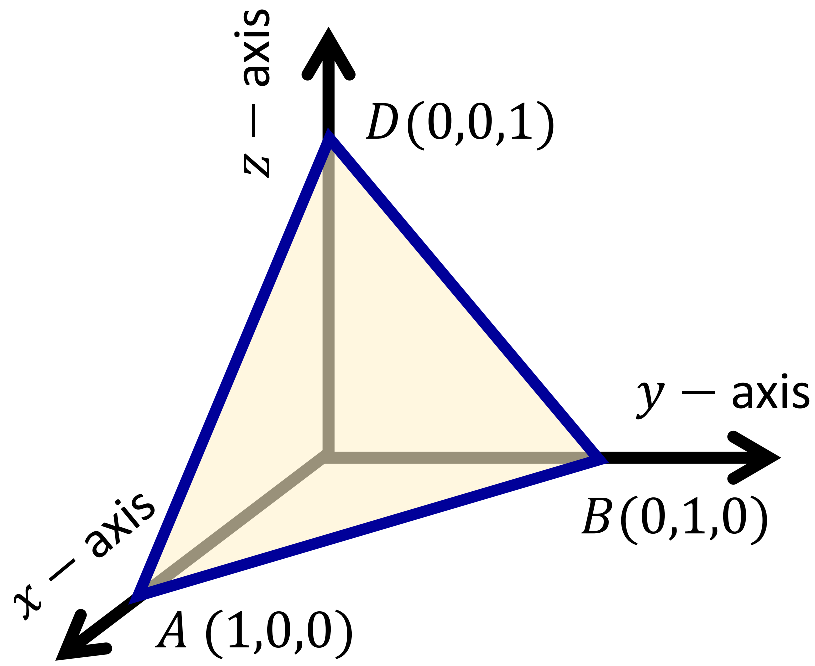
S: triangle ABD $\qquad$ C: lines AB, BD and DA

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}) \cdot (\hat{\imath}\, dx + \hat{\jmath}\ dy + \hat{k}\, dz)$$

$$= \int_{AB} x\, dx + z^2 dy + y^2 dz + \int_{BD} xdx + z^2 dy + y^2 dz + \int_{DA} xdx + z^2 dy + y^2 dz$$

Equating to the line AB: $\dfrac{x-1}{0-1}=\dfrac{y-0}{1-0}=\dfrac{z-0}{0-0}=t$

$$x=1-t \qquad y=t \qquad z=0$$



$$\int_{AB} xdx+z^2dy+y^2dz=\int_{t=0}^{1}(1-t)(-dt)=\left[\frac{(1-t)^2}{2}\right]_0^1=-\frac{1}{2}$$

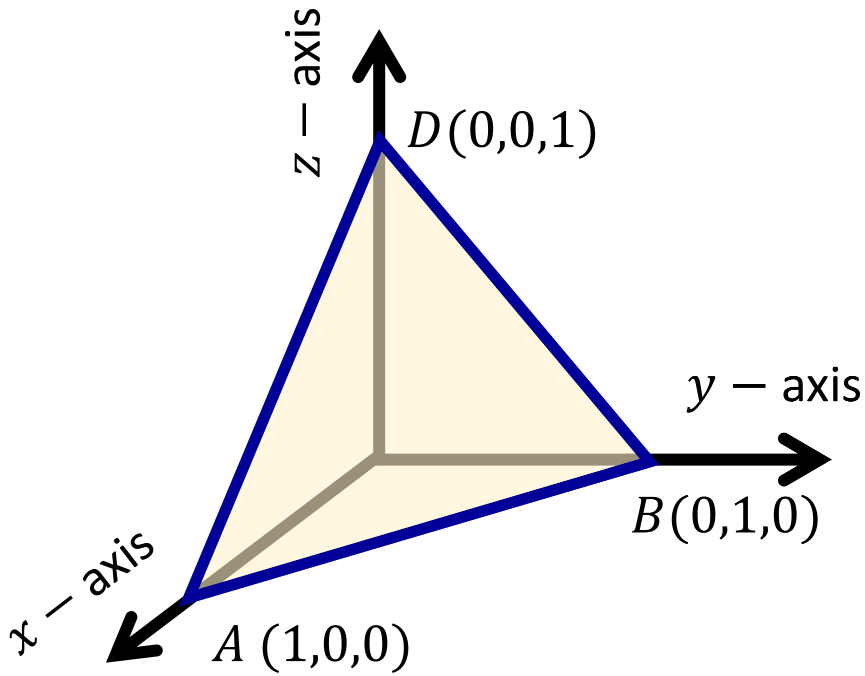
Equating to the line *BD*: $\dfrac{x-0}{0-0}=\dfrac{y-1}{0-1}=\dfrac{z-0}{1-0}=t \quad x=0 \quad y=1-t \quad z=t$

$$\int_{BD} xdx+z^2dy+y^2dz=\int_{t=0}^{1}t^2(-dt)+(1-t)^2\,dt=\int_{t=0}^{1}(1-2t)\,dt=0$$

Equating to the line DA: $\frac{x-0}{1-0}=\frac{y-0}{0-0}=\frac{z-1}{0-1}=t$

$$x=t \qquad y=0 \qquad z=1-t$$

$$\int_{DA} xdx+z^2dy+y^2dz=\int_{t=0}^{1} t\ dt=\frac{1}{2}$$

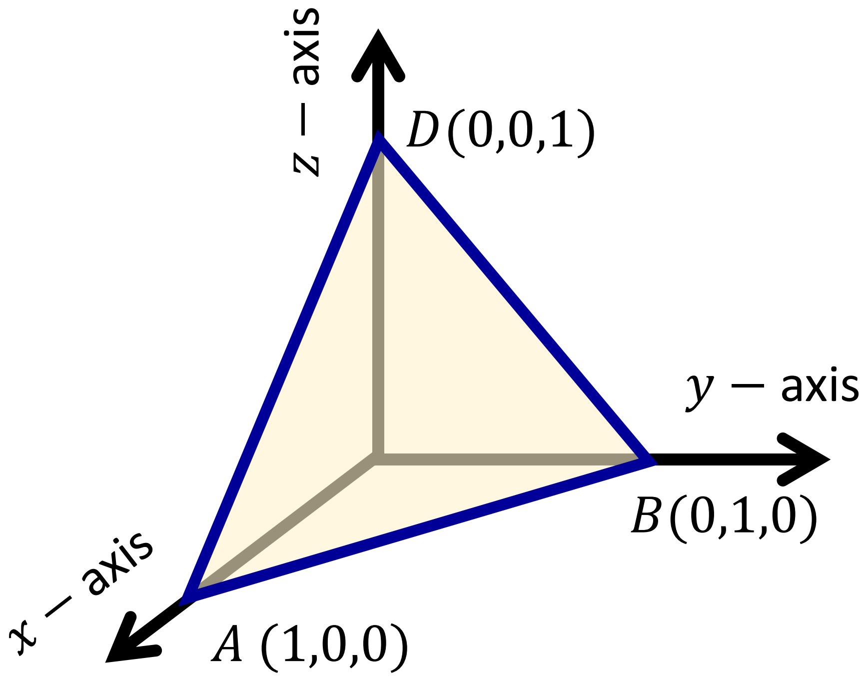


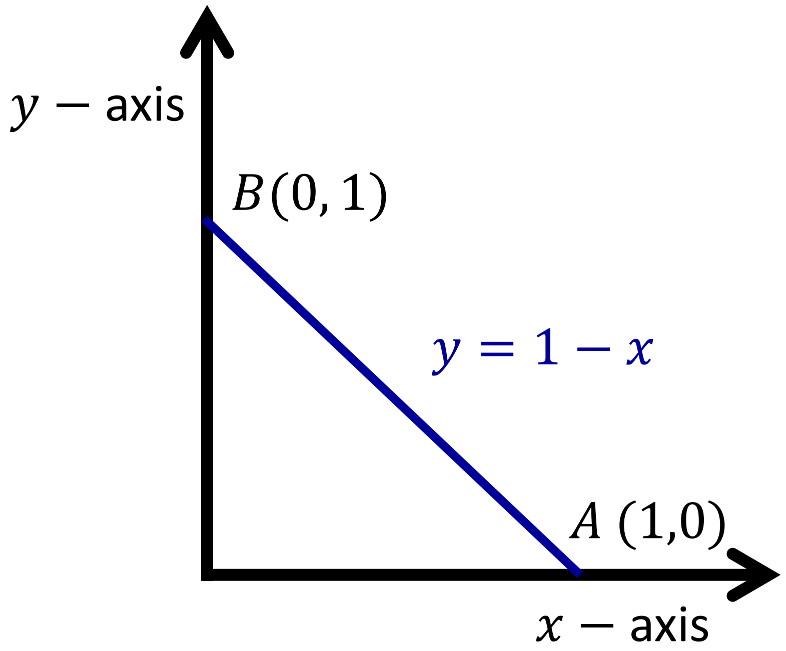
We have $\oint_{AB} \vec{F}\cdot d\vec{r}=-\frac{1}{2}$ $\qquad \oint_{BD} \vec{F}\cdot d\vec{r}=0$ $\qquad \oint_{DA} \vec{F}\cdot d\vec{r}=\frac{1}{2}$

$$\Rightarrow \oint_{C} \vec{F}\cdot d\vec{r}=0$$

Projecting S on the x-y plane, let R be its projection.

R is bounded by the x-axis, y-axis and straight line AB.





Given surface $f = x + y + z = 1 \Rightarrow \nabla f = \hat{\imath} + \hat{\jmath} + \hat{k}$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}}$$

$$\frac{|\nabla f|}{|\nabla f \cdot \hat{k}|} = \frac{\sqrt{3}}{|1|} = \sqrt{3}$$

$$\text{curl}\,\vec{F}\cdot\vec{n} = (2(y-z)\,\hat{\imath})\cdot\left(\frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}(y-z) = \frac{2}{\sqrt{3}}(2y+x-1)$$

$$\vec{F} = x\hat{\imath} + z^2\hat{\jmath} + y^2\hat{k}$$

$$S:\ x+y+z=1$$

$$\vec{n} = \frac{\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}$$

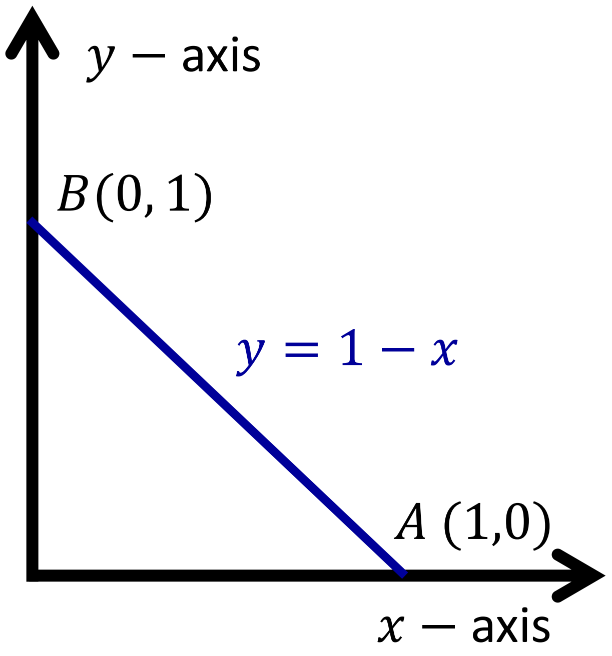
$$\frac{|\nabla f|}{|\nabla f\cdot\hat{k}|} = \sqrt{3}$$

$$\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds = \iint_{R_{xy}} \frac{2}{\sqrt{3}}(2y+x-1)\sqrt{3}\ dxdy$$

$$= 2\int_0^1\int_0^{1-x}(2y+x-1)\,dy\,dx$$

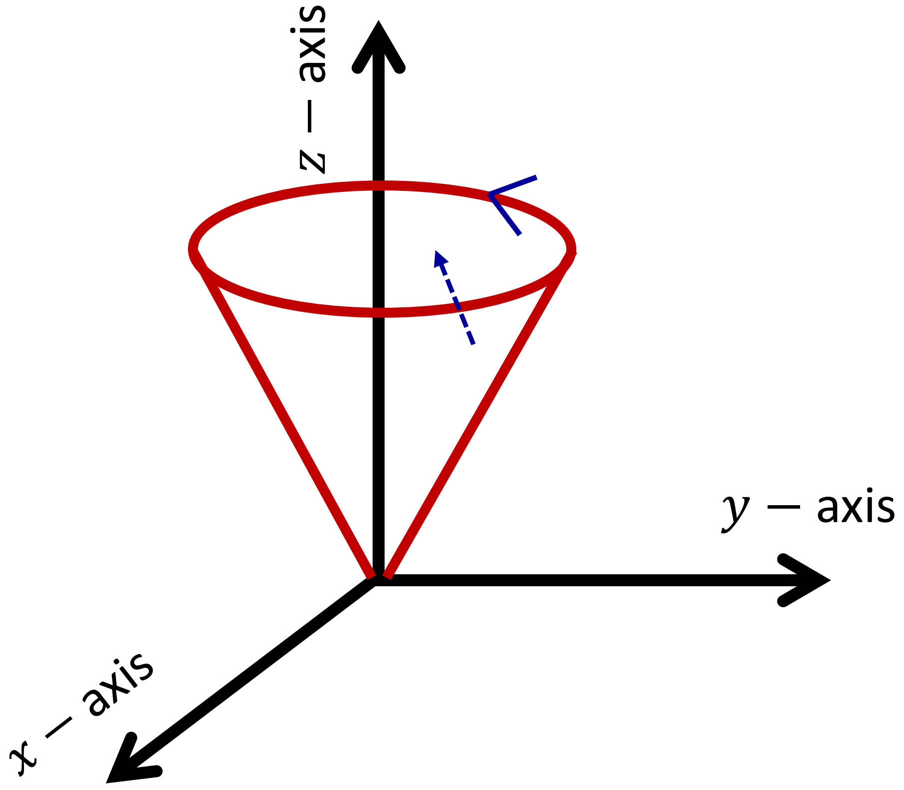
$$= 2\int_0^1 (1-x)^2+(x-1)(1-x)\ dx$$

$$= 0$$



Problem: Let $\vec{F} = -y\hat{\imath} + x\,\hat{\jmath} - xyz\,\hat{k}$ and let S be the part of cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \leq 9$.

Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ or $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$ whichever appears easier. Here $\vec{n}$ is the inner normal vector.



$C: x^2 + y^2 = 9 \;\&\; z = 3$

$x = 3\cos t\,, \qquad y = 3\sin t\,, \qquad z = 3$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C -y\,dx + x\,dy - xyz\,dz$$

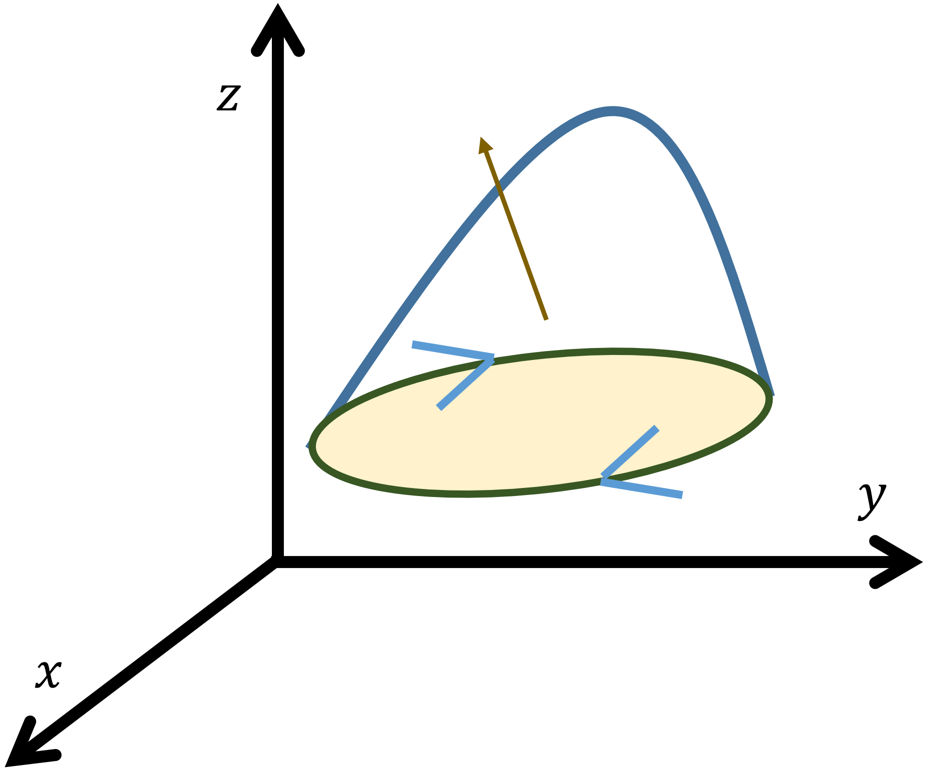
$$= \oint_0^{2\pi} (3\sin t)\,(3\sin t)\,dt + 3\cos t\,(3\cos \mathrm{t})\,\mathrm{dt}$$

$$= 9\oint_0^{2\pi} \mathrm{dt} = 18\pi$$

KEY TAKEAWAY

Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ ds$$



- **Divergence Theorem (**volume integrals ↔ surface integrals**)**

Recall Green′s Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{k} \;\, dA$$

Its generalization in space $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \;\, ds$ Stokes’ Theorem

The Divergence Theorem (Generalization of Green's Theorem)

Green's Theorem: $\oint_C \vec{F} \cdot \hat{n} \; ds = \iint_D \nabla \cdot \vec{F} \, dA$

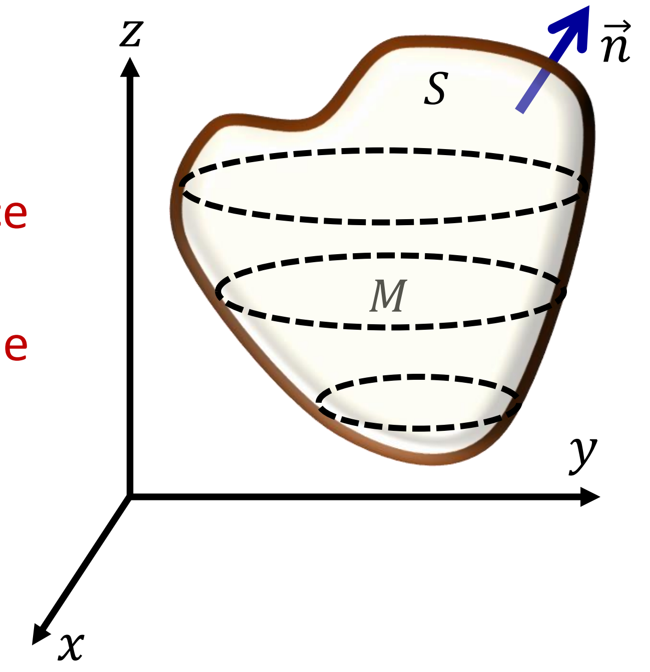
Replace the closed curve C $\rightarrow$ a closed surface S in 3D

Replace the bounding domain D $\rightarrow$ the bounding volume M

The vector field $\vec{F}(x, y)$ $\rightarrow$ The vector field $\vec{F}(x, y, z)$

$$\iint_S \vec{F} \cdot \; \hat{n} \, d\sigma = \iiint_M \nabla \cdot \vec{F} \, dV$$

The Divergence Theorem



The flux of a vector field $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ across a closed oriented surface S in the direction of the surface's outward unit normal field $\hat{n}$ equals the integral of $\nabla \cdot \vec{F}$ over the region M enclosed by the surface

$$\iint\limits_{S} \vec{F} \cdot \hat{n}\, d\sigma = \iiint\limits_{M} \nabla \cdot \vec{F}\, dV$$

Intuitively, it states that sum of all sources minus the sum of all sinks gives the net flow of a region.

Problem-1 Verify Divergence theorem for the field $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ over the sphere $x^2 + y^2 + z^2 = 9$

Solution: $\vec{n} = \dfrac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{3} \Rightarrow \vec{F} \cdot \vec{n} = \dfrac{1}{3}(x^2 + y^2 + z^2) = 3$

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_S 3\, d\sigma = 3 \iint_S d\sigma = 3\,(4\pi\, 3^2) = 108\,\pi$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\Longrightarrow \iiint_D \vec{\nabla} \cdot \vec{F}\, dV = \iiint 3\, dV = 3 \times \frac{4}{3}\pi\, 3^3 = 108\,\pi$$

Problem-2 Find the flux of $\vec{F} = xy\,\hat{\imath} + yz\hat{\jmath} + xz\,\hat{k}$ outward through the surface of the cube from the first octant by the planes $x = 2$, $y = 2$ and $z = 2$.

Solution: $\nabla \cdot \vec{F} = y + z + x$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \vec{\nabla} \cdot \vec{F}\, dV$$

Divergence Theorem

$$= \int_0^2 \int_0^2 \int_0^2 (x + y + z)\, dxdydz$$

$$= 24$$

Problem-3 If V is the volume enclosed by a closed surface S and $\vec{F} = 3x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$ show that

$$\iint_S \vec{F}\cdot\vec{n}\,ds = 6V$$

Solution: $$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z) = 6$$

By Gauss Divergence theorem: $$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV$$

$$= 6\iiint_D dV = 6V$$

Problem-4 Evaluate $\iint\limits_S \left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + 2\hat{k}\right)\cdot \hat{n}\, d\sigma$ where S denotes the surface of the cube bounded by the planes $x = 0, x = 3, y = 0, y = 3, z = 0, z = 3$

Solution: $\nabla \cdot \vec{F} = 3x^2 - 2x^2 - 0 \;= x^2$

By Gauss Divergence theorem:

$$\iint\limits_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint\limits_D \nabla\cdot\vec{F}\;\, dV \;= \iiint\limits_D x^2\, dxdydz$$

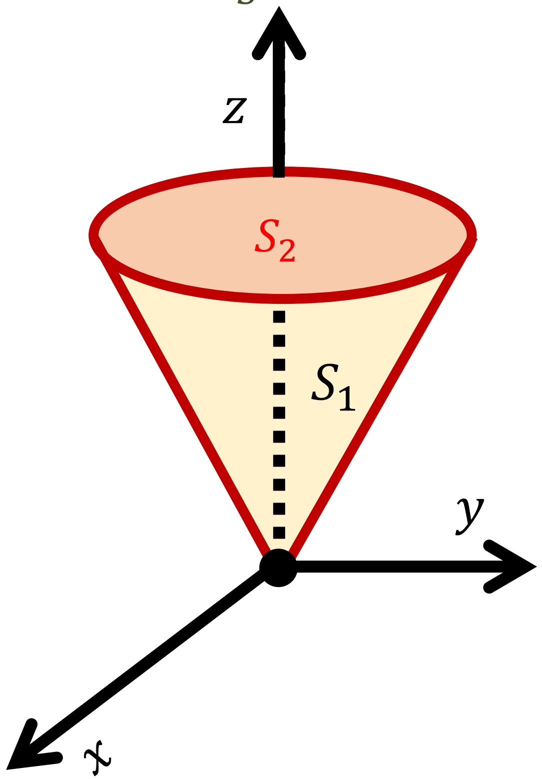
$$= \int_0^3\int_0^3\int_0^3 x^2\, dxdydz \;= 81$$

Problem-5 Let S be given by the cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \leq 1$ together with the disk $x^2 + y^2 \leq 1, z = 1$. For $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, verify the divergence theorem.

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_M \nabla \cdot \vec{F}\, dV$$

Solution Let S_1: $z = \sqrt{x^2 + y^2}, \quad x^2 + y^2 \leq 1$

Let S_2: $x^2 + y^2 \leq 1, \qquad z = 1$



Surface Integral: $\displaystyle \iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_{S_1} \vec{F} \cdot \hat{n}\, d\sigma + \iint_{S_2} \vec{F} \cdot \hat{n}\, d\sigma$

For S_1: $\hat{n} = \dfrac{2x\,\hat{\imath} + 2y\,\hat{\jmath} - 2z\,\hat{k}}{2\sqrt{x^2 + y^2 + z^2}} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath} - z\,\hat{k}}{\sqrt{2}\, z}$ $\qquad \vec{F} \cdot \hat{n} = 0$

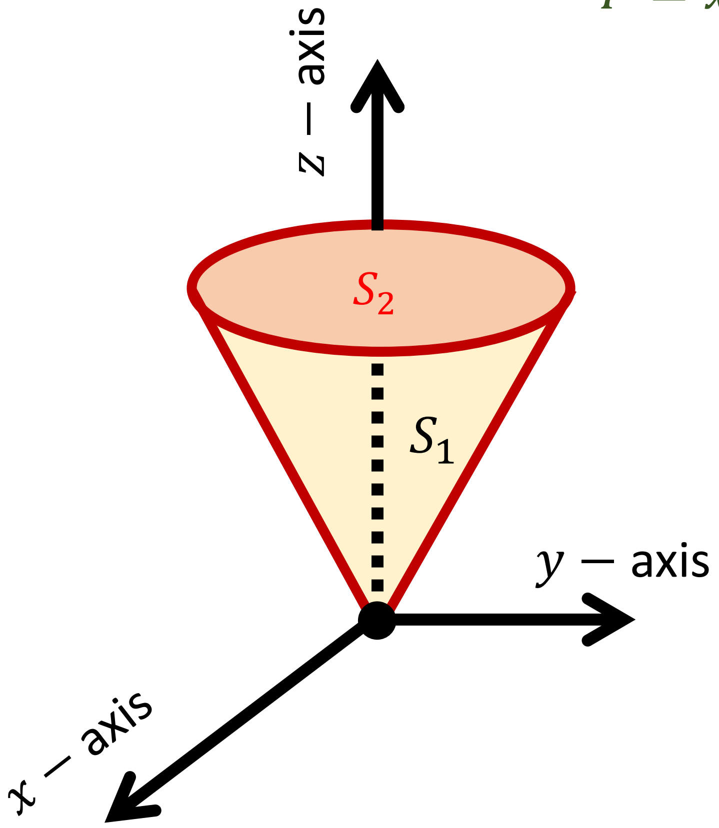
For S_2: $\hat{n} = k$ $\vec{F} \cdot \hat{n} = z$

S_2: $x^2 + y^2 \leq 1,$ $z = 1$

$\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_{S_1} \vec{F} \cdot \hat{n}\, d\sigma + \iint_{S_2} \vec{F} \cdot \hat{n}\, d\sigma$$

$$= \iint_{S_2} d\sigma = \pi$$



Volume Integral $\iiint_M \nabla \cdot \vec{F}\, dV = 3 \iiint_M dV = 3 \times \pi(1)^2 \frac{1}{3} = \pi$

Volume of a cone of height h and radius $r = \pi r^2 \frac{h}{3}$

KEY TAKEAWAY

The Divergence Theorem:

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_M \nabla \cdot \vec{F}\, dV$$