# Functions of Two Variables

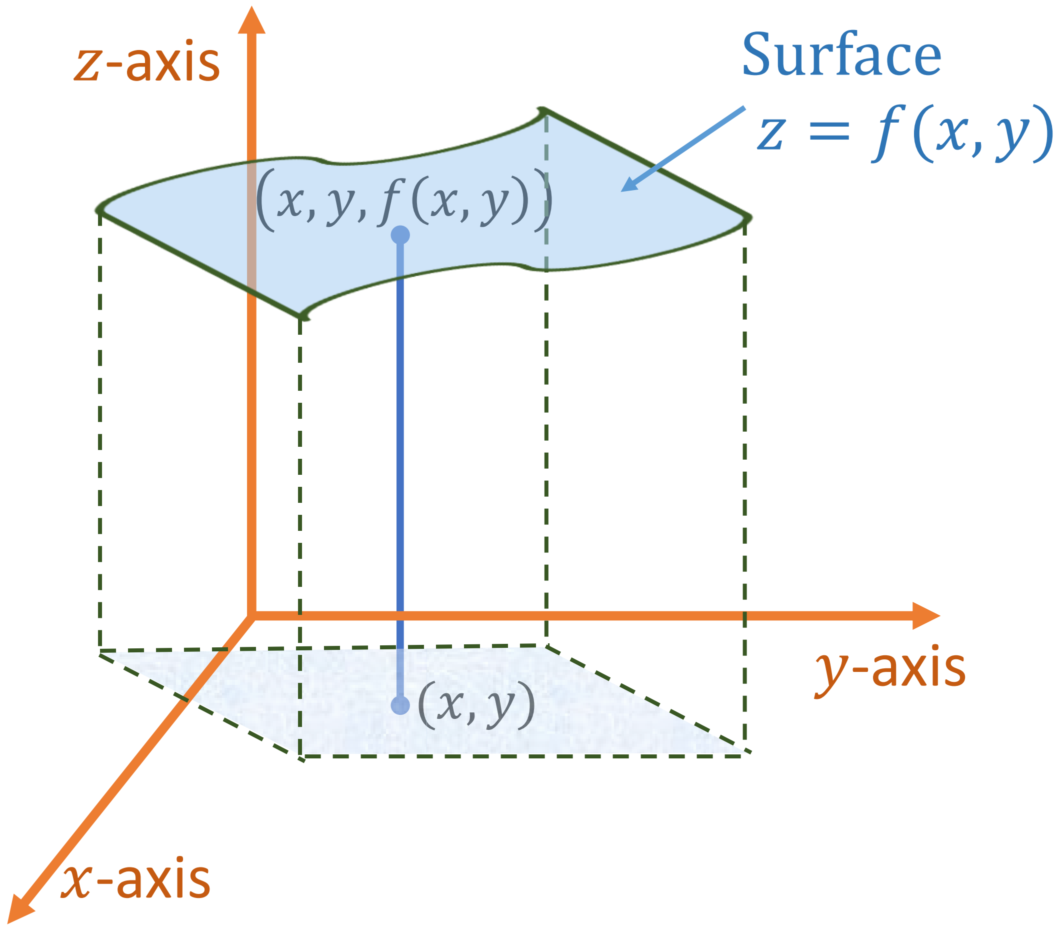
A function $z = f(x, y)$ is a real valued function of two variables $x$ & $y$ if to each point $(x, y)$ of a certain part of $x$-$y$ plane corresponds to a real value $z$ according to some given rule $f(x, y)$.

Domain: The set of points $(x, y)$ in the $x$-$y$ plane for which $f(x, y)$ is defined

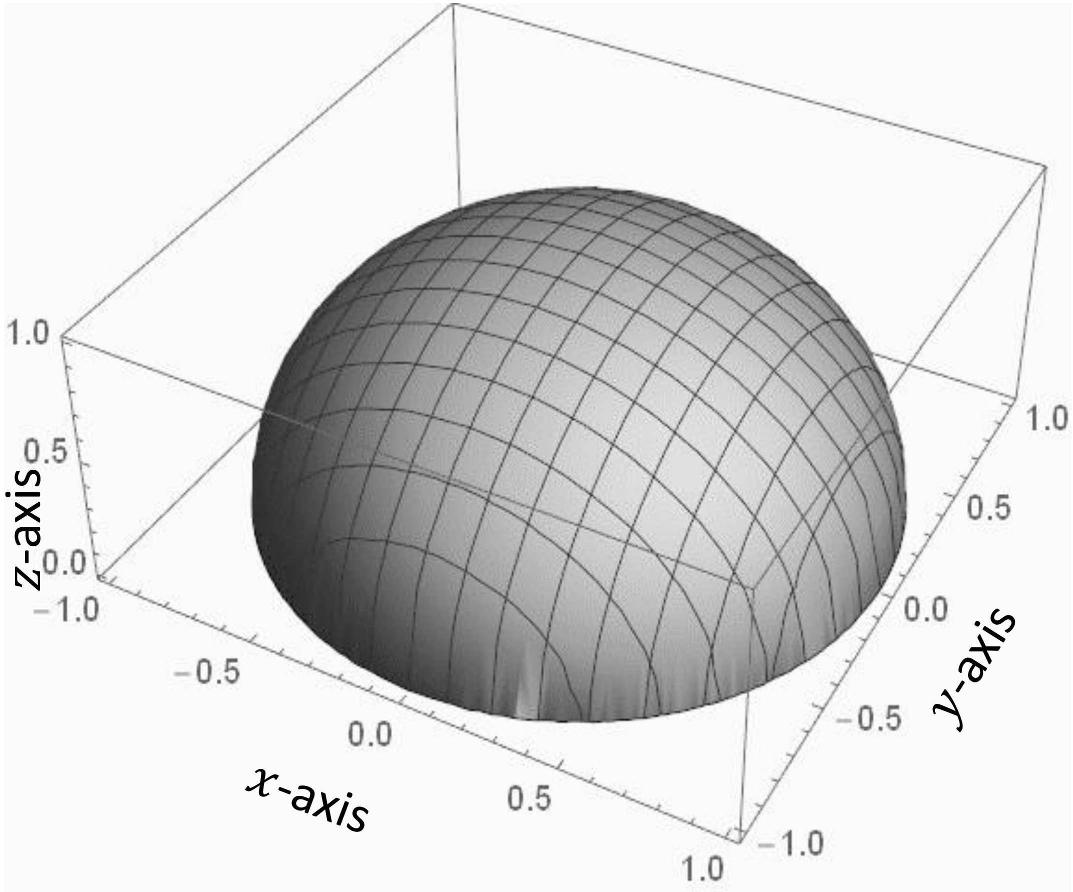
Range: Collection of all possible value of $z$ corresponding to the points $(x, y)$

$x, y \rightarrow$ independent variables

$z \rightarrow$ dependent variable



# Functions of Two Variables




**Example**: $z = \sqrt{1 - x^2 - y^2}$

Since $z$ is real, we must have $(1 - x^2 - y^2) \geq 0$

$\Longrightarrow x^2 + y^2 \leq 1$

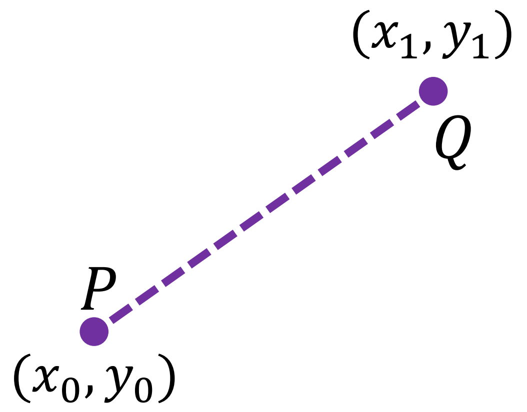
Therefore, Domain:

$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

Range:

R$= \{z \in \mathbb{R}, 0 \leq z \leq 1\}$

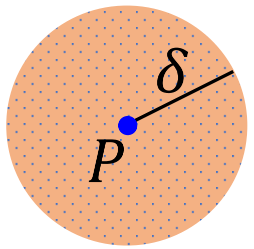
## Distance between the two points

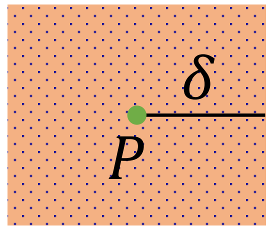



Distance $|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

## Neighborhood of a point $P(x_0, y_0)$

$\delta$-neighborhood of $P$ $(N_\delta(P)\ \ OR\ N(P,\delta))$




$$N_\delta(P) \coloneqq \left\{(x,y): \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\right\}$$

$$N_\delta(P) \coloneqq \{(x,y): x_0 - \delta < x < x_0 + \delta,\ \ y_0 - \delta < y < y_0 + \delta\}$$

## Limit of a Function of One Variable (Recall)

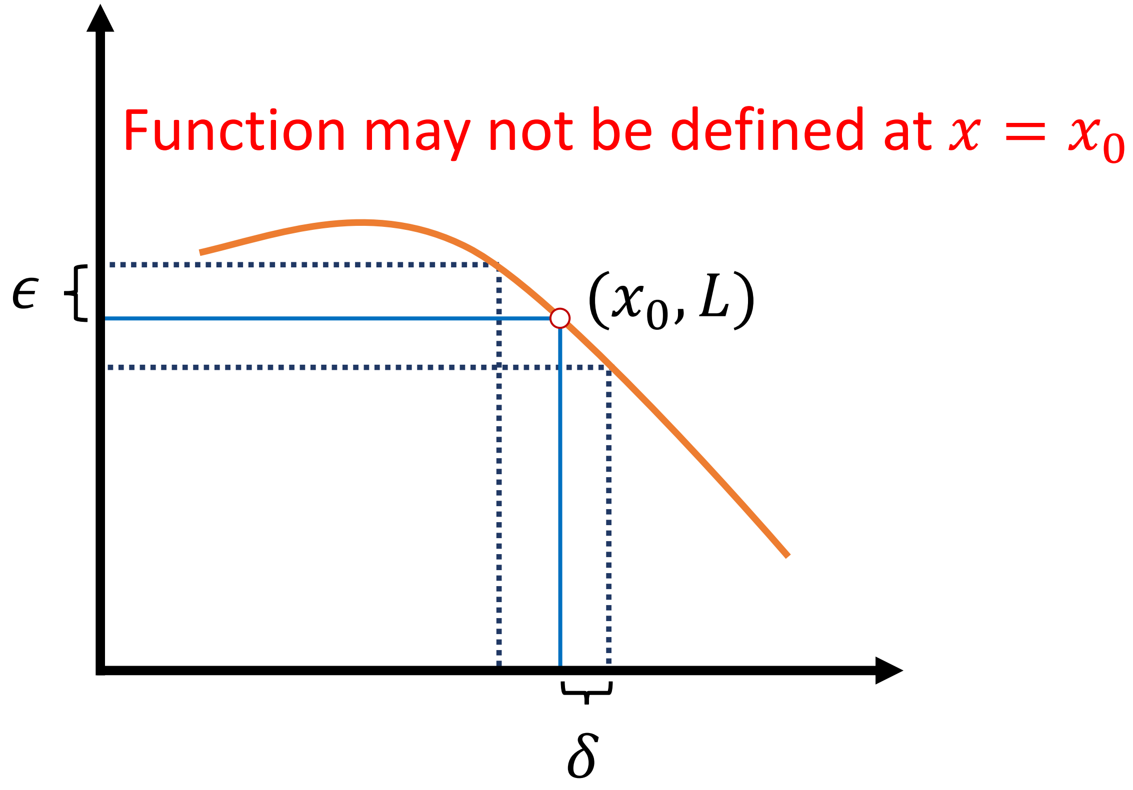
We say $\lim_{x \to x_0} f(x) = L$, if for every $\epsilon > 0$,

there exists $\delta > 0$, such that $\forall\, x$,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

In other words,

If we can make the difference $|f(x) - L|$ as small as we like by considering a small enough neighborhood around $x_0$, then we say that $\lim_{x \to x_0} f(x) = L$



## Limit of Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables defined in a domain $D$. Let $P(x_0, y_0)$ be a point in $D$. If for a given real number $\epsilon > 0$, we can find a real number $\delta > 0$ such that for every point $(x, y)$ in the $\delta$-neighborhood of $P(x_0, y_0)$ satisfies $|f(x, y) - L| < \epsilon$, i.e.,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

(function may not be defined at $(x_0, y_0)$)

Then the real number $L$ is called the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$

Symbolically, $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$

**Problem - 1** $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) = 0$

For $(x,y) \neq (0,0)$, consider

$$\left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) - 0\right| = (x^2+y^2)\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq (x^2+y^2) < \delta^2 \leq \epsilon$$

Neighborhood of $(0,0)$: $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

For given $\epsilon$ if we choose $\delta^2 \leq \epsilon$, then $\left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) - 0\right| < \epsilon$

**Problem - 2** $\lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x+y}\right) = 0$

$(|x|-|y|)^2 \geq 0 \quad \Rightarrow x^2+y^2-2|xy| \geq 0$

$\Rightarrow 2|xy| \leq x^2+y^2 \Rightarrow (|x|+|y|)^2 \leq 2\,(x^2+y^2)$

$\Rightarrow (|x|+|y|) \leq \sqrt{2}\sqrt{(x^2+y^2)}$

For $(x, y) \neq (0,0)$, consider

$$\left|(x+y)\sin\left(\frac{1}{x+y}\right) - 0\right| \leq |(x+y)| \leq |x|+|y| \leq \sqrt{2}\sqrt{x^2+y^2} < \sqrt{2}\,\delta \leq \epsilon$$

Neighborhood of (0,0): $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

Choose $\delta \leq \frac{\epsilon}{\sqrt{2}}$, then $|f(x,y) - f(0,0)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

**Problem - 3** $\lim_{(x,y)\to(0,0)} \left( \frac{xy}{\sqrt{x^2+y^2}} \right) = 0$

$(|x| - |y|)^2 \geq 0 \Rightarrow x^2 + y^2 - 2|xy| \geq 0$

$\Rightarrow |xy| \leq \frac{(x^2+y^2)}{2} \leq (x^2+y^2)$

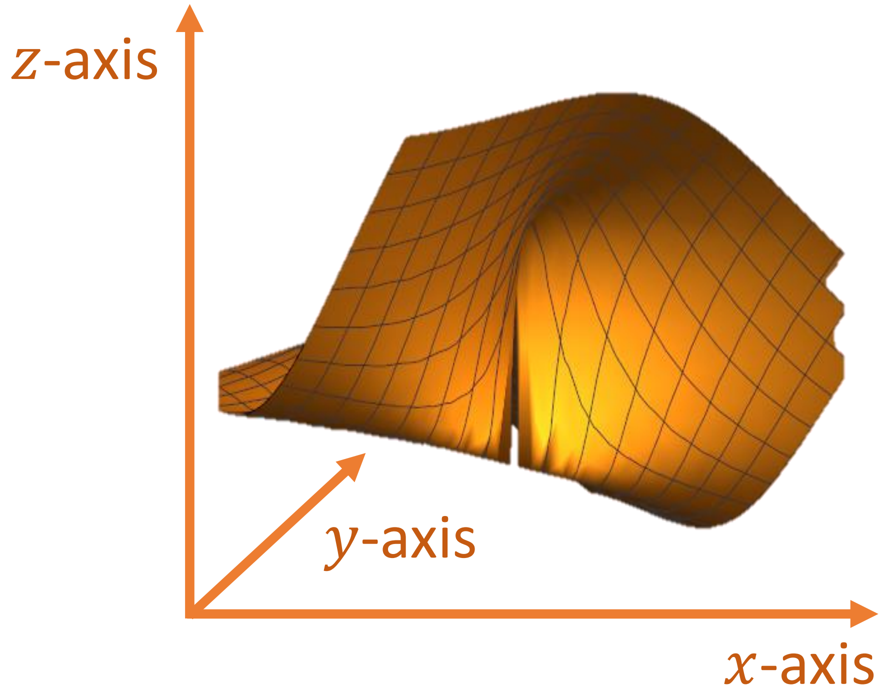
For $(x, y) \neq (0,0)$, consider

$$\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2} < \delta \leq \epsilon$$

Neighborhood of $(0,0)$: $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$

For given $\epsilon$ if we choose $\delta \leq \epsilon$, then $\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| < \epsilon$

# KEY TAKEAWAY




- Functions of Two Variables

$$Z = f(x, y)$$

- Definition of limit ($\epsilon - \delta$)

We need to have some idea about the limit $L$ and then it may be used to verify that $L$ is the limit

## Limit (Previous Lecture)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If for a given real number $\epsilon > 0$, we can find a real number $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Note: $\epsilon - \delta$ approach is useful for verifying that the given number $L$ is the limit

## Working with Limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1 \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$$

$$\lim_{(x,y)\to(x_0,y_0)} [k\, f(x,y)] = k\, L_1$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\, g(x,y)] = L_1\, L_2$$

$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L_1}{L_2} \qquad \text{Provided } \mathrm{L}_2 \neq 0$$

## Working with Limits (generalization)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = \infty.$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = \infty \qquad \lim_{(x,y)\to(x_0,y_0)} [f(x,y) + g(x,y)] = \infty$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = -\infty.$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = -\infty$$

## Working with Limits (generalization)

$$\lim_{(x,y)\to (x_0,y_0)} f(x,y) = \infty \quad \text{and} \quad \lim_{(x,y)\to (x_0,y_0)} g(x,y) = L \text{ (finite real number).}$$

$$\lim_{(x,y)\to (x_0,y_0)} [f(x,y) \pm g(x,y)] = \infty$$

$$\lim_{(x,y)\to (x_0,y_0)} f(x,y) = \infty \quad \text{and} \quad \lim_{(x,y)\to (x_0,y_0)} g(x,y) = L\ (> 0).$$

$$\lim_{(x,y)\to (x_0,y_0)} [f(x,y)g(x,y)] = \infty$$

## Working with Limits (generalization)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L\ (< 0).$$

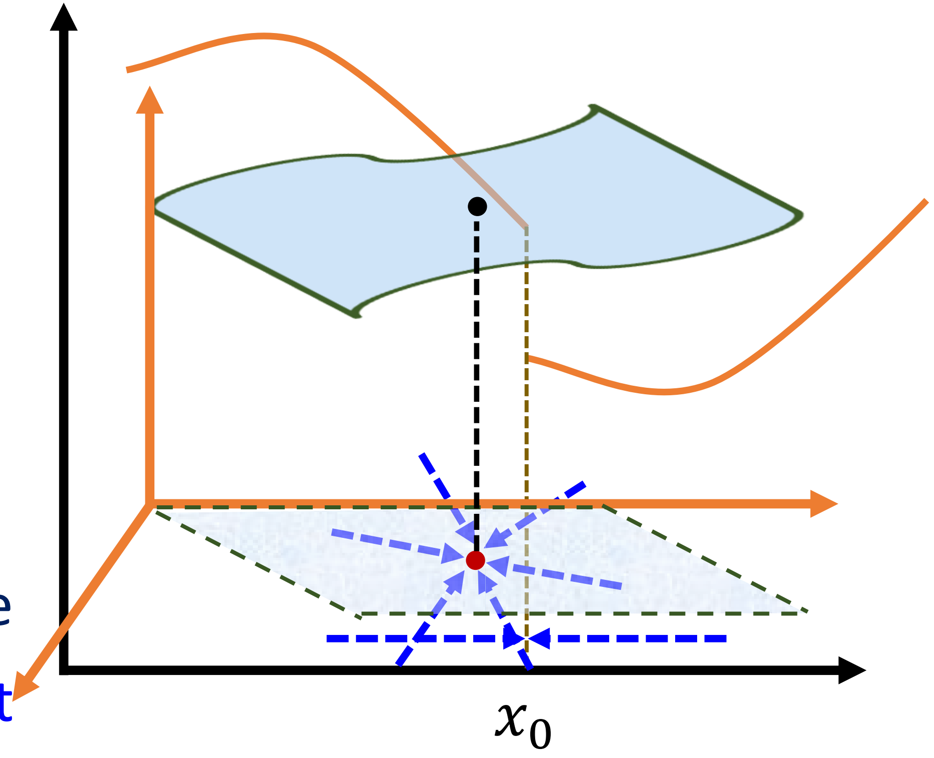
$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = -\infty$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \infty \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L\,.$$

$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{g(x,y)}{f(x,y)}\right] = 0$$

## Evaluation of Limit

Remark: Note that $(x, y) \to (x_0, y_0)$ in the two dimensional plane, there are infinite number of paths joining $(x, y)$ to $(x_0, y_0)$.




Since the limit, if exists, is unique, the limit should be the same along all the paths. Thus, the limit cannot be obtained by approaching the point $P$ along a particular path and finding the limit of $f(x, y)$.

If the limit is dependent on a path, then the limit does not exist.

**Example 1:** $\lim_{(x,y)\to(0,0)} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(3x+6y)}$

Set $(x+2y)=t$

$$\lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(3x+6y)} = \lim_{t\to 0} \frac{\sin^{-1}(t)}{\tan^{-1}(3t)}$$

Using L'Hospital's rule $$= \lim_{t\to 0} \frac{\frac{1}{\sqrt{1-t^2}}}{\left(\frac{3}{1+9t^2}\right)} = \frac{1}{3}$$

**Example 2:** $\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$

Fix $y = 1$ and approach along $x$ to 0

$$\lim_{x\to 0-0} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

$$\lim_{x\to 0+0} \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

The limit depends on path and hence it does not exist.

**Example 3:**

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \frac{x^2y}{x^4+y^2} \qquad \lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^2}} \frac{x^2y}{x^4+y^2}$$

Along $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0$$

Along $y = x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \frac{1}{2}$$

Limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist in this case!

**Example 4:** $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

Along $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \frac{m}{1 + m^2}$$

The limit depends on path and hence it does not exist.

**Example 5:** $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ **Alternative Approach**

Change of coordinate system from Cartesian to Polar

$$x = r\cos\theta \qquad \& \qquad y = r\sin\theta$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \cos\theta\sin\theta$$

The limit depends on the angle $\theta$ and hence it does not exist.

**Example 6:** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2 + y^2}$

Change of coordinate system from Cartesian to Polar

$$x = r\cos\theta \qquad \& \qquad y = r\sin\theta$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0 \qquad \text{No dependency on } \theta$$

Hence the limit exists in this case.

## Remark:

*Changing to polar coordinate (subst. $x = r\cos\theta\,, y = r\sin\theta$) and investigating the limit of the resulting expression as $r \to 0$ is often very useful.*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = 0$$

Limit is 0

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

Limit does not exist

## Remark:

*Changing to polar coordinate (subst. $x = r\cos\theta$ , $y = r\sin\theta$) does not always help and the transformation may tempt us to false conclusion.*

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}$$

If we fix $\theta$, then $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0$

Taking the path $r\sin\theta = r^2\cos^2\theta$ $(y = x^2)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta\, r^2\cos^2\theta}{r^4\cos^4\theta + r^4\cos^4\theta} = \frac{1}{2}$$

Limit does not exist

# KEY TAKEAWAY

## LIMIT

Changing to polar coordinate is often useful for evaluation of limit

*Needs to be careful while changing to polar coordinate!*