devised by Alston Householder for reducing an arbitrary symmetric matrix to a similar tridiagonal matrix. Although there is a clear connection between the problems we are solving in these two sections, Householder's method has such a wide application in areas other than eigenvalue approximation that it deserves special treatment.

·holder (1904–1993) ·n mathematical ·e becoming the ·e Oak Ridge ·oratory in Tennessee ·egan work on ·r systems in the · was when these · developed.

Householder's method is used to find a symmetric tridiagonal matrix B that is similar to a given symmetric matrix A. Theorem 9.16 implies that A is similar to a diagonal matrix D since an orthogonal matrix Q exists with the property that $D = Q^{-1}AQ = Q^tAQ$. Because the matrix Q (and consequently D) is generally difficult to compute, Householder's method offers a compromise. After Householder's method has been implemented, efficient methods such as the QR algorithm can be used for accurate approximation of the eigenvalues of the resulting symmetric tridiagonal matrix.

Householder Transformations

Definition 9.21 Let $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{w}^t\mathbf{w} = 1$. The $n \times n$ matrix

$$P = I - 2\mathbf{w}\mathbf{w}^t$$

is called a **Householder transformation.** ■

Householder transformations are used to selectively zero out blocks of entries in vectors or columns of matrices in a manner that is extremely stable with respect to round-off error. (See [Wil2], pp. 152–162, for further discussion.) Properties of Householder transformations are given in the following theorem.

Theorem 9.22 A Householder transformation, $P = I - 2\mathbf{w}\mathbf{w}^t$, is symmetric and orthogonal, so $P^{-1} = P$.

Proof It follows from

$$(\mathbf{w}\mathbf{w}^t)^t = (\mathbf{w}^t)^t\mathbf{w}^t = \mathbf{w}\mathbf{w}^t$$

that

$$P^t = (I - 2\mathbf{w}\mathbf{w}^t)^t = I - 2\mathbf{w}\mathbf{w}^t = P.$$

Further, $\mathbf{w}^t\mathbf{w} = 1$, so

$$\begin{aligned} PP^t = (I - 2\mathbf{w}\mathbf{w}^t)(I - 2\mathbf{w}\mathbf{w}^t) &= I - 2\mathbf{w}\mathbf{w}^t - 2\mathbf{w}\mathbf{w}^t + 4\mathbf{w}\mathbf{w}^t\mathbf{w}\mathbf{w}^t \\ &= I - 4\mathbf{w}\mathbf{w}^t + 4\mathbf{w}\mathbf{w}^t = I, \end{aligned}$$

and $P^{-1} = P^t = P$. ■

Householder's method begins by determining a transformation $P^{(1)}$ with the property that $A^{(2)} = P^{(1)}AP^{(1)}$ zeros out the entries in the first column of A beginning with the third row, that is, such that

$$a_{j1}^{(2)} = 0, \quad \text{for each } j = 3, 4, \ldots, n. \tag{9.8}$$

By symmetry, we also have $a_{1j}^{(2)} = 0$.

We now choose a vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)^t$ so that $\mathbf{w}^t\mathbf{w} = 1$, Eq. (9.8) holds, and in the matrix

$$A^{(2)} = P^{(1)}AP^{(1)} = (I - 2\mathbf{w}\mathbf{w}^t)A(I - 2\mathbf{w}\mathbf{w}^t),$$

we have $a_{11}^{(2)} = a_{11}$ and $a_{j1}^{(2)} = 0$, for each $j = 3, 4, \ldots, n$. This choice imposes n conditions on the n unknowns $w_1, w_2, \ldots, w_n$.

Setting $w_1 = 0$ ensures that $a_{11}^{(2)} = a_{11}$. We want

$$P^{(1)} = I - 2\mathbf{w}\mathbf{w}^t$$

to satisfy

$$P^{(1)}(a_{11}, a_{21}, a_{31}, \ldots, a_{n1})^t = (a_{11}, \alpha, 0, \ldots, 0)^t, \tag{9.9}$$

where α will be chosen later. To simplify notation, let

$$\hat{\mathbf{w}} = (w_2, w_3, \ldots, w_n)^t \in \mathbb{R}^{n-1}, \quad \hat{\mathbf{y}} = (a_{21}, a_{31}, \ldots, a_{n1})^t \in \mathbb{R}^{n-1},$$

and $\hat{P}$ be the $(n-1) \times (n-1)$ Householder transformation

$$\hat{P} = I_{n-1} - 2\hat{\mathbf{w}}\hat{\mathbf{w}}^t.$$

Eq. (9.9) then becomes

$$P^{(1)} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix} = \left[\begin{array}{c:c} 1 & 0 \cdots\cdots 0 \\ \hdashline 0 & \\ \vdots & \hat{P} \\ 0 & \end{array}\right] \cdot \begin{bmatrix} a_{11} \\ \hdashline \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} a_{11} \\ \hdashline \hat{P}\hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} a_{11} \\ \hdashline \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$p^{(1)}$ with $w_1 = 0$

with

$$\hat{P}\hat{\mathbf{y}} = (I_{n-1} - 2\hat{\mathbf{w}}\hat{\mathbf{w}}^t)\hat{\mathbf{y}} = \hat{\mathbf{y}} - 2(\hat{\mathbf{w}}^t\hat{\mathbf{y}})\hat{\mathbf{w}} = (\alpha, 0, \ldots, 0)^t. \tag{9.10}$$

Let $r = \hat{\mathbf{w}}^t\hat{\mathbf{y}}$. Then

$$(\alpha, 0, \ldots, 0)^t = (a_{21} - 2rw_2, a_{31} - 2rw_3, \ldots, a_{n1} - 2rw_n)^t,$$

and we can determine all of the w_i once we know α and r. Equating components gives

$$\alpha = a_{21} - 2rw_2$$

and

$$0 = a_{j1} - 2rw_j, \quad \text{for each } j = 3, \ldots, n.$$

Thus,

$$2rw_2 = a_{21} - \alpha \tag{9.11}$$

and

$$2rw_j = a_{j1}, \quad \text{for each } j = 3, \ldots, n. \tag{9.12}$$

Squaring both sides of each of the equations and adding the corresponding terms gives

$$4r^2 \sum_{j=2}^{n} w_j^2 = (a_{21} - \alpha)^2 + \sum_{j=3}^{n} a_{j1}^2.$$

Since $\mathbf{w}^t\mathbf{w} = 1$ and $w_1 = 0$, we have $\sum_{j=2}^{n} w_j^2 = 1$, and

$$4r^2 = \sum_{j=2}^{n} a_{j1}^2 - 2\alpha a_{21} + \alpha^2. \tag{9.13}$$

Equation (9.10) and the fact that P is orthogonal imply that

$$\alpha^2 = (\alpha, 0, \ldots, 0)(\alpha, 0, \ldots, 0)' = (\hat{P}\hat{\mathbf{y}})'\hat{P}\hat{\mathbf{y}} = \hat{\mathbf{y}}'\hat{P}'\hat{P}\hat{\mathbf{y}} = \hat{\mathbf{y}}'\hat{\mathbf{y}}.$$

Thus,

$$\alpha^2 = \sum_{j=2}^{n} a_{j1}^2,$$

which when substituted into Eq. (9.13) gives

$$2r^2 = \sum_{j=2}^{n} a_{j1}^2 - \alpha a_{21}.$$

To ensure that $2r^2 = 0$ only if $a_{21} = a_{31} = \cdots = a_{n1} = 0$, we choose

$$\alpha = -\operatorname{sgn}(a_{21})\left(\sum_{j=2}^{n} a_{j1}^2\right)^{1/2},$$

which implies that

$$2r^2 = \sum_{j=2}^{n} a_{j1}^2 + |a_{21}|\left(\sum_{j=2}^{n} a_{j1}^2\right)^{1/2}.$$

With this choice of α and $2r^2$, we solve Eqs. (9.11) and (9.12) to obtain

$$w_2 = \frac{a_{21} - \alpha}{2r} \quad \text{and} \quad w_j = \frac{a_{j1}}{2r}, \quad \text{for each } j = 3, \ldots, n.$$

To summarize the choice of $P^{(1)}$, we have

$$\alpha = -\operatorname{sgn}(a_{21})\left(\sum_{j=2}^{n} a_{j1}^2\right)^{1/2},$$

$$r = \left(\frac{1}{2}\alpha^2 - \frac{1}{2}a_{21}\alpha\right)^{1/2},$$

$$w_1 = 0,$$

$$w_2 = \frac{a_{21} - \alpha}{2r},$$

and

$$w_j = \frac{a_{j1}}{2r}, \quad \text{for each } j = 3, \ldots, n.$$

With this choice,

$$A^{(2)} = P^{(1)}AP^{(1)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & 0 & \cdots & 0 \\ a_{21}^{(2)} & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix}.$$

Having found $P^{(1)}$ and computed $A^{(2)}$, the process is repeated for $k = 2, 3, \ldots, n-2$
follows:

$$
\begin{aligned}
\alpha &= -\operatorname{sgn}(a_{k+1,k}^{(k)}) \left(\sum_{j=k+1}^{n} (a_{jk}^{(k)})^2 \right)^{1/2}, \\
r &= \left(\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha a_{k+1,k}^{(k)} \right)^{1/2}, \\
w_1^{(k)} &= w_2^{(k)} = \cdots = w_k^{(k)} = 0, \\
w_{k+1}^{(k)} &= \frac{a_{k+1,k}^{(k)} - \alpha}{2r}, \\
w_j^{(k)} &= \frac{a_{jk}^{(k)}}{2r}, \quad \text{for each} \quad j = k+2, k+3, \ldots, n, \\
P^{(k)} &= I - 2\mathbf{w}^{(k)} \cdot (\mathbf{w}^{(k)})^t,
\end{aligned}
$$

and

$$
A^{(k+1)} = P^{(k)} A^{(k)} P^{(k)},
$$

where

$$
A^{(k+1)} = \begin{bmatrix}
a_{11}^{(k+1)} & a_{12}^{(k+1)} & 0 & \cdots & \cdots & \cdots & 0 \\
a_{21}^{(k+1)} & \ddots & \ddots & \ddots & & & \vdots \\
0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\
\vdots & \ddots & a_{k+1,k}^{(k+1)} & a_{k+1,k+1}^{(k+1)} & a_{k+1,k+2}^{(k+1)} & \cdots & a_{k+1,n}^{(k+1)} \\
\vdots & & 0 & \vdots & \ddots & & \vdots \\
\vdots & & \vdots & \vdots & & \ddots & \vdots \\
0 & \cdots & 0 & a_{n,k+1}^{(k+1)} & \cdots & \cdots & a_{nn}^{(k+1)}
\end{bmatrix}.
$$

Continuing in this manner, the tridiagonal and symmetric matrix $A^{(n-1)}$
where

$$
A^{(n-1)} = P^{(n-2)} P^{(n-3)} \cdots P^{(1)} A P^{(1)} \cdots P^{(n-3)} P^{(n-2)}.
$$

Example 1 Apply Householder transformations to the symmetric 4×4 matrix

$$
A = \begin{bmatrix}
4 & 1 & -2 & 2 \\
1 & 2 & 0 & 1 \\
-2 & 0 & 3 & -2 \\
2 & 1 & -2 & -1
\end{bmatrix}
$$

to produce a symmetric tridiagonal matrix that is similar to A.

Solution For the first application of a Householder transformation,

$$\alpha = -(1)\left(\sum_{j=2}^{4} a_{j1}^2\right)^{1/2} = -3, \; r = \left(\frac{1}{2}(-3)^2 - \frac{1}{2}(1)(-3)\right)^{1/2} = \sqrt{6},$$

$$\mathbf{w} = \left(0, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}\right),$$

$$P^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{\sqrt{6}}{6}\right)^2 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \cdot (0, 2, -1, 1)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

and

$$A^{(2)} = \begin{bmatrix} 4 & -3 & 0 & 0 \\ -3 & \frac{10}{3} & 1 & \frac{4}{3} \\ 0 & 1 & \frac{5}{3} & -\frac{4}{3} \\ 0 & \frac{4}{3} & -\frac{4}{3} & -1 \end{bmatrix}.$$

Continuing to the second iteration,

$$\alpha = -\frac{5}{3}, \quad r = \frac{2\sqrt{5}}{3}, \quad \mathbf{w} = \left(0, 0, 2\sqrt{5}, \frac{\sqrt{5}}{5}\right)^t,$$

$$P^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{3}{5} & -\frac{4}{5} \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix},$$

and the symmetric tridiagonal matrix is

$$A^{(3)} = \begin{bmatrix} 4 & -3 & 0 & 0 \\ -3 & \frac{10}{3} & -\frac{5}{3} & 0 \\ 0 & -\frac{5}{3} & -\frac{33}{25} & \frac{68}{75} \\ 0 & 0 & \frac{68}{75} & \frac{149}{75} \end{bmatrix}.$$ ■