6.5. Matrix factorization.

Q. Can we present $A=LU$, where L is lower triangular and U is upper triangular?

Motivation: Solving $A\vec{x}=\vec{b}$ requires $O(\frac{n^3}{3})$ using Gaussian elimination. However, an upper-triangular system requires only backsubstitution, which takes $O(n^2)$ operations. A similar number of operations is needed to solve a lower-triangular system.

Assume $A=LU$ and one needs to solve $A\vec{x}=\vec{b}$.

Step 1. Let $\vec{y}=U\vec{x}$. Solve $L\vec{y}=\vec{b}$ for $\vec{y}$. This requires $O(n^2)$ operations.

Step 2. Solve $U\vec{x}=\vec{y}$. This requires $O(n^2)$ operations.

Both steps require $O(2n^2)$ operations as opposed to $O(\frac{n^3}{3})$ when A is in a non-factored form.

<u>Determining</u> L and U requires $O(\frac{n^3}{3})$ operations.

Assume L and U can be obtained by Gaussian elimination without row interchanges, which is equivalent to $a_{ii}^{(i)} \neq 0$ $(i=1,\ldots,n)$.

Step 1 of Gaussian elimination

$$(E_j - m_{j,1}E_1) \to (E_j) \quad \text{where } m_{j,1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}} \qquad (6.8).$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Introduce

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ -m_{n1} & 0 & \cdots & 1 \end{bmatrix}$$

Denote $A^{(1)} = A$. Calculate

$$A^{(2)} \equiv M^{(1)} A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ 0 & -m_{21}a_{21}+a_{22} & & -m_{21}a_{n1}+a_{n2} \\ \vdots & & & \\ 0 & -m_{n1}a_{21}+a_{n2} & & -m_{n1}a_{n1}+a_{nn} \end{bmatrix}$$

which is exactly (6.8).

Denote $M^{(1)}\vec{b} = \vec{b}^{(2)}$. Then a solution to $A\vec{x} = \vec{b}$ satisfies

$$A^{(2)}\vec{x} = M^{(1)}A\vec{x} = M^{(1)}\vec{b} = \vec{b}^{(2)}.$$

Step 2. Construct $M^{(2)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & -m_{32} & \ddots & 0 \\ 0 & -m_{n2} & \cdots 0 & 1 \end{bmatrix}$ $\qquad m_{j2} = \dfrac{a_{j2}^{(2)}}{a_{22}^{(2)}}$

$$A^{(3)} \equiv M^{(2)}A^{(2)}; \quad b^{(3)} = M^{(2)}b^{(2)}. \text{ Then}$$

$$A^{(3)}\vec{x} = M^{(2)}A^{(2)}\vec{x} = M^{(2)}\vec{b}^{(2)} = b^{(3)}$$

In general,

$$M^{(k)} = \begin{array}{c|ccccc} & & & k & & \\ \hline & 1 & 0 & 0 & & 0 \\ & & \ddots & \vdots & 0 & \vdots \\ k & & & 1 & \ddots & \\ k+1 & & 0 & -m_{k+1,k} & \ddots & 0 \\ & & & \vdots & 0 & 1 \\ & & & -m_{n,k} & & \end{array}$$

$A^{(k+1)} \vec{x} = \vec{b}^{(k+1)}$, or

$$M^{(k)} \cdots M^{(1)} A x = M^{(k)} \cdots M^{(1)} \vec{b} \qquad (6.9).$$

$A^{(n)}$ will be upper-triangular; $A^{(n)} = U$

$$A^{(n)} = M^{(n-1)} M^{(n-2)} \cdots M^{(1)} A . \qquad (*)$$

$(*)$ forms the upper-triangular portion of $A = LU$.
Now we will identify the complementary L.

Examine $[M^{(k)}]^{-1}$:

det

$$
L^{(k)} = \begin{array}{c|c} & \begin{array}{cccccc} & & & k & & \end{array} \\ \hline \begin{array}{c} \\ \\ k \\ \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & & \vdots \\ \vdots & & 0 & 1 & & \\ \vdots & & \vdots & m_{k+1,k} & 1 & 0 \\ 0 & \cdots & 0 & m_{n,k} & \cdots & 1 \end{bmatrix} \end{array}
$$

Compute $C^{(k)} = M^{(k)} \cdot L^{(k)}$:

$$
M^{(k)} \cdot L^{(k)} = \begin{array}{c|c} & \begin{array}{cccc} & k-1 & k & \end{array} \\ \hline \begin{array}{c} \\ k \\ k+1 \\ \end{array} & \begin{bmatrix} 1 & & & \\ & \ddots\ 1 & & O \\ & & 1 & \\ O & & & \ddots\ 1 \end{bmatrix} \end{array}
$$

$$
C^{(k)}_{k+p,j} = -m_{k+p,k} \cdot l_{k,j} + 1 \cdot l_{k+p,j} = \begin{cases} 0, & j < k \\ -m_{k+p,k} \cdot 1 + m_{k+p,k} = 0, & j = k \\ -m_{k+p,k} \cdot \underbrace{l_{k,k+p}}_{0} + 1 \cdot 1 = 1 & j = k+p \\ \quad 0 & j > k+p \end{cases}
$$

Thus, $C^{(k)} = I$ meaning that

$$L^{(k)} = \left[M^{(k)}\right]^{-1}.$$

Introduce

$$L = L^{(1)} L^{(2)} \ldots L^{(n-1)}.$$

Then

$$LU = L^{(1)}L^{(2)}\cdots L^{(n-1)} \cdot M^{(n-1)}M^{(n-2)}\cdots M^{(2)}M^{(1)}A = A.$$

Thm 6.19. If Gaussian elimination can be performed on the linear system $A\vec{x}=\vec{b}$ without row interchanges, then the matrix A can be factored into the product of a lower-triangular matrix L and an upper-triangular matrix U, that is, $A = LU$,

where $m_{ji} = \dfrac{a_{ji}^{(i)}}{a_{ii}^{(i)}}$,

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & & \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & & \\ \vdots & & \ddots & \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

Permutation matrices.

We assumed that $A\vec{x}=\vec{b}$ can be solved by Gaussian elimination without row interchanges. What if the raw interchanges are required? Introduce a class of matrices allowing for rearranging, or permuting rows of a given matrix.

An $n \times n$ permutation matrix $P = \{p_{ij}\}$ is a matrix obtained from I_n, the identity matrix, by rearranging its rows. P has exactly one nonzero entry in each row and in each column, and each nonzero entry is 1.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Multiplying by P has the effect of interchanging the second and third rows (columns if multiplying on the right) of A.

Assume $k_1, \dots, k_n$ is a permutation of the integers $1, \dots, n$ and $P = [p_{ij}]$ where

$$p_{ij} = \begin{cases} 1, & j = k_i \\ 0, & \text{otherwise} \end{cases}$$

Then

(1) PA permutes the rows of A:

$$PA = \begin{bmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 n} \\ \cdot & - & - & - \\ a_{k_n 1} & a_{k_n 2} & \cdots & a_{k_n n} \end{bmatrix}$$

(2) P^{-1} exists and $P^{-1} = P^t$.

There is a rearrangement of the equations in the system $A\vec{x} = \vec{b}$ such that no row interchanges are needed, meaning that there exists a permutation matrix P for which

$$PA\vec{x} = P\vec{b}$$

can be solved without row interchanges. Then

$$PA = LU,$$

as above.

$$A = P^{-1}LU = (P^t L)U.$$

$P^t L$ is not lower triangular unless $P = I$.

Theorem (6.19)

If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, then the matrix A can be factored into the product of a lower-triangular matrix L and an upper-triangular matrix U, that is, $A = LU$, where $m_{ji} = a_{ji}^{(i)}/a_{ii}^{(i)}$,

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & & \\ & & & a_{n-1,n}^{(n-1)} \\ 0 & & 0 & a_{nn}^{(n)} \end{bmatrix}, \text{ and } L = \begin{bmatrix} 1 & 0 & & 0 \\ m_{21} & 1 & & \\ & & & 0 \\ m_{n1} & & m_{n,n-1} & 1 \end{bmatrix}.$$

Permutation matrix

An $n \times n$ **permutation matrix** $P = [p_{ij}]$ is a matrix obtained by rearranging the rows of I_n, the identity matrix. This gives a matrix with precisely one nonzero entry in each row and in each column, and each nonzero entry is a 1.

NOTE: Any nonsingular matrix A can be factored into $A = P^t LU$.

Example.

#1(a). Solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\underset{L}{\parallel} \qquad \underset{U}{\parallel} \qquad \underset{\vec{x}}{\parallel}$$

Let $U\vec{x} = \vec{y}$

① $L\vec{y} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Forward-substitution:

$y_1 = 2$

$2y_1 + y_2 = -1$

$4 + y_2 = -1$

$y_2 = -5$

$-y_1 + y_3 = 1$

$-2 + y_3 = 1$

$y_3 = 3$

$$\vec{y} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

(2) $\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$

$3x_3 = 3$

$x_3 = 1$

$-2x_2 + 1 = -5$

$x_2 = \frac{-6}{-2} = 3$

$2x_1 + 9 - 1 = 2$

$2x_1 = 2 - 8 = -6$

$x_1 = -3$

$\vec{x} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$.

#3(a)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{E_2 - 2E_1 \to E_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{E_2 \leftrightarrow E_3}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = LU$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 4 & 0 \end{bmatrix}$$

$$m_{31} = 2$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A=(P^tL)U=\begin{bmatrix}1&0&0\\2&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&-1\\0&1&-1\\0&0&2\end{bmatrix}=\begin{bmatrix}1&2&-1\\2&4&0\\0&1&-1\end{bmatrix}$$

$$P^tL=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&0&1\\0&1&0\end{bmatrix}$$

Algorithm 6.4: LU FACTORIZATION

To factor the $n \times n$ matrix $A = [a_{ij}]$ into the product of the lower-triangular matrix $L = [l_{ij}]$ and the upper-triangular matrix $U = [u_{ij}]$; that is, $A = LU$, where the main diagonal of either L or U consists of all ones:

INPUT dimension n; the entries a_{ij}, $1 \le i, j \le n$ of A; the diagonal $l_{11} = \cdots = l_{nn} = 1$ of L or the diagonal $u_{11} = \cdots = u_{nn} = 1$ of U.

OUTPUT the entries l_{ij}, $1 \le j \le i$, $1 \le i \le n$ of L and the entries, u_{ij}, $i \le j \le n$, $1 \le i \le n$ of U.

Step 1 Select l_{11} and u_{11} satisfying $l_{11}u_{11} = a_{11}$.
If $l_{11}u_{11} = 0$ then OUTPUT ('Factorization impossible');
STOP.

Step 2 For $j = 2, \ldots, n$ set $u_{1j} = a_{1j}/l_{11}$; (*First row of U.*)
$l_{j1} = a_{j1}/u_{11}$. (*First column of L.*)

Algorithm 6.4: LU FACTORIZATION

Step 3 For $i = 2, \ldots, n-1$ do Steps 4 and 5.

Step 4 Select l_{ii} and u_{ii} satisfying $l_{ii}u_{ii} = a_{ii} - \sum_{k=1}^{i-1} l_{ik}u_{ki}$.

If $l_{ii}u_{ii} = 0$ then OUTPUT ('Factorization impossible');

STOP.

Step 5 For $j = i+1, \ldots, n$

set $u_{ij} = \frac{1}{l_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}\right]$; (*ith row of U.*)

$l_{ji} = \frac{1}{u_{ii}}\left[a_{ji} - \sum_{k=1}^{i-1} l_{jk}u_{ki}\right]$. (*ith column of L.*)

Step 6 Select l_{nn} and u_{nn} satisfying $l_{nn}u_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk}u_{kn}$.

(*Note: If $l_{nn}u_{nn} = 0$, then $A = LU$ but A is singular.*)

Step 7 OUTPUT (l_{ij} for $j = 1, \ldots, i$ and $i = 1, \ldots, n$);

OUTPUT (u_{ij} for $j = i, \ldots, n$ and $i = 1, \ldots, n$);

STOP.