Review Example:

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 4 \end{array}\right]$$ Augmented matrix

$\downarrow$ Row reduce

pivots

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\downarrow$

$$x_1 + x_3 = 0$$

$$x_2 + x_3 = 0$$

$$0 x_3 = 1 \leftarrow ???$$ No solution!

This exists because:

a) $\det(\underline{A}) = 0 \rightarrow \underline{A}^{-1}$ does not exist

b) $\underline{A}$ has a non-trivial nullspace, where the nullspace of $\underline{A}$ is all vectors $\underline{v}$, such that $\underline{A}\,\underline{v} = \underline{0}$

Here, one finds

$$\underline{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\underline{A}\,\underline{v} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 2+1-3 \\ 1+1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the columns of $\underline{A}$ are not independent!

In addition to nullspace, it is useful to identify the column space of $\underline{A}$.

Altogether, there are four subspaces of a matrix:

(1) Column space

(2) Nullspace

(3) Row space

(4) Left nullspace

Column Space

Recall that a matrix-vector product is simply a linear combination of the matrix columns:

$$\underset{m\times n \text{ matrix}}{\underline{A}}\ \overset{n\times 1 \text{ vector}}{\underline{x}} = \left[\underline{a}_1, \underset{m\times 1 \text{ vectors}}{\underline{a}_2}, \dots, \underline{a}_n\right] \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix} = \underset{m\times 1 \text{ vector}}{\underline{b}}$$

$$\underline{b} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n$$

All possible vectors $\underline{b}$ exist in the column space $C(\underline{A})$ of the matrix

The column space is a subspace of $\mathbb{R}^m$

The column space is very important when solving $\underline{A}\,\underline{x} = \underline{b}$

Theorem: The system $\underline{A}\underline{x} = \underline{b}$ has at least one solution iff $\underline{b}$ is in the column space of $\underline{A}$

Note: At least one solution!

Case 1: $\underline{A}^{-1}$ exists

$\underline{A}\,\underline{x} = \underline{b}$ with $\underline{b}$ in $C(\underline{A})$

then $\underline{x} = \underline{A}^{-1}\underline{b} \Rightarrow$ a solution

$\Rightarrow$ any $\underline{b}$ is a valid right-hand side (rhs)

With $\underline{A} \in M_{nn}$ (square matrix)

$\Rightarrow$ If $\underline{A}^{-1}$ exists, then $\underline{A}$ has n independent columns, which means it spans all of $\mathbb{R}^n$

Example:

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{A}^{-1}$ exists; all 3 columns are linearly independent; columns span all of $\mathbb{R}^3$

Case 2: $\underline{A}^{-1}$ does not exist

then $\underline{b}$ must be in the column space $C(\underline{A})$ to have a solution

Example: $\underline{A}\,\underline{x} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Only two independent columns; $\underline{A}^{-1}$ does not exist

Can one write

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ?$$

Try $x = 1,\ y = 1,\ z = a$

$$(1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + a \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ ok ✓

$\underline{b}$ is in the column space $C(\underline{A})$

and has at least one solution,

actually an infinity of solutions

Example: $\underline{A}\,\underline{x} = \underline{c}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

changed only this element

Can one write

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ?$$

Is $\underline{c}$ in $C(\underline{A})$? No!

No x, y, z to satisfy above equation

$\therefore$ No solution

Nullspace

Another important subspace is the nullspace,

given by $N(\underline{A})$

The nullspace is all vectors $\underline{v}$, such that

$$\underline{A}\,\underline{v} = \underline{0}$$

Is this a vector space?

Let $\underline{v}$ + $\underline{w}$ be in $N(\underline{A})$

$(\underline{A}\,\underline{v} = \underline{0}$ and $\underline{A}\,\underline{w} = \underline{0})$

① $\underline{A}(\underline{v}+\underline{w}) = \underline{A}\,\underline{v} + \underline{A}\,\underline{w} = \underline{0} + \underline{0} = \underline{0}$ ✓

② $\underline{A}(c\underline{v}) = c(\underline{A}\,\underline{v}) = c\underline{0} = \underline{0}$ ✓

Yes, the nullspace is a vector space

The nullspace is always non-empty;

$\underline{0}$ is always in the nullspace

$$\underset{m\times n}{\underline{A}}\;\underset{n\times 1}{\underline{0}} = \underset{m\times 1}{\underline{0}} \qquad\qquad \underline{A}\,\underline{v} = \underline{0}$$

If $\underline{A}^{-1}$ exists, then

$$\underline{A}\,\underline{v} = \underline{0} \Rightarrow \underline{v} = \underline{A}^{-1}\underline{0} = \underline{0}$$

$\Rightarrow$ If $\underline{A}^{-1}$ exists, then the only vector in $N(\underline{A})$ is $\underline{0}$

If the nullspace has any vector in addition to $\underline{0}$, then $\underline{A}^{-1}$ does not exist.

Example: To obtain $N(\underline{A})$ Solve $\underline{A}\,\underline{x} = \underline{0}$

Two possibilities: consider $\underline{A} \in M_{33}$

(1)

$$\underline{A} \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(\underline{A}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Only trivial $N(\underline{A}) \Rightarrow \underline{A}^{-1}$ exists

(2)

$$\underline{A} \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ or something similar

Example:

$$\underline{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Big\downarrow \text{rref}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \end{array}\right.$$

$$\begin{array}{c} \downarrow \\ \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right] \\ \uparrow \\ \text{free} \\ \text{column/variable} \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_3 = c$ for $c \in \mathbb{R}$

or simply $x_3 = 1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 1 = 0 \Rightarrow x_1 = 1$$

$$x_2 - 2 = 0 \Rightarrow x_2 = 2$$

$\therefore \underline{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is in nullspace of $\underline{A}$

such that $\underline{A}\,\underline{x} = \underline{0}$

But so is

$2\underline{x} = \begin{bmatrix} 2 \\ 4 \\ \end{bmatrix}, -3x = \begin{bmatrix} -3 \\ -6 \\ \end{bmatrix} \ldots$

$$2\underline{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, -3\underline{x} = \begin{bmatrix} -3 \\ -6 \\ -3 \end{bmatrix}, \ldots$$

Nullspace is not limited to square matrices.

Example:

$$\underline{A} = \begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 10 & 13 \end{bmatrix}$$

Find $N(\underline{A})$: All vectors which span $N(\underline{A})$

$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 10 & 13 \end{bmatrix}$$

$\downarrow$ rref

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

	$\uparrow$ free column (variable)		$\uparrow$ free column (variable)

$$\underline{B} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 2 & 8 & 10 \\ 3 & 10 & 13 \end{bmatrix}$$

rref

$x_1, x_3 \Rightarrow$ "fixed"

$x_2, x_4 \Rightarrow$ "free"

To determine the nullspace, set one free variable to 1, the others to zero & find fixed variables (x_1 & x_3 here)

Set $x_2=1$, $x_4=0$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x_1+3 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x_1=-3 \\ x_3=0 \end{matrix}$$

$$\rightarrow \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ is in } N(\underline{A})$$

Set $x_2=0$, $x_4=1$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x_1+1 \\ x_3+1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x_1=-1 \\ x_3=-1 \end{matrix}$$

$$\begin{bmatrix} -1 \end{bmatrix}$$

$\rightarrow \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ is in $N(\underline{A})$

$$\therefore N(\underline{A}) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Recall, with span, any linear combination of these vectors is in $N(\underline{A})$

Matrix Subspaces: Recap

Let $\underline{A} \in M_{mn}$

m: # rows, n: # columns

① Column Space $C(\underline{A})$ is the subspace of $\mathbb{R}^m$ that is spanned by the columns of $\underline{A}$. (Also called the Range Space)

$$\underset{}{\underline{A}}\ \underset{\mathbb{R}^n}{\underline{x}} = \underset{\mathbb{R}^m}{\underline{b}}$$

$\underline{b}$ is a linear combination of the

$\in \mathbb{R}^n$ $\quad \in \mathbb{R}^m$ $\quad$ combination of the columns of $\underline{A}$

② **Nullspace** $N(\underline{A})$ is the subspace of $\mathbb{R}^n$ that is spanned by all vectors, which are solutions of $\underline{A}\,\underline{x} = \underline{0}$. (Also called the Kernel Space)

All matrices have a nullspace, because $\underset{n \times 1}{\underline{0}}$ is always in $N(\underline{A})$

$$\underset{m \times n}{\underline{A}}\ \underset{n \times 1}{\underline{0}} = \underset{m \times 1}{\underline{0}}$$

③ **Row Space** $C(\underline{A}^T)$ is the subspace of $\mathbb{R}^n$ that is spanned by the rows of $\underline{A}$.

④ **Left Nullspace** $N(\underline{A}^T)$ is the subspace of $\mathbb{R}^m$ that is spanned by all vectors, which are solutions of $\underline{A}^T \underline{x} = \underline{0}$

Note that

$$\underline{A}^T \underline{x} = \underline{0} \Rightarrow (\underline{A}^T \underline{x})^T = \underline{0}^T$$

$$\underline{x}^T \underline{A} = \underline{0}^T$$

Hence, left nullspace

Summary

Let $\underline{A} \in M_{mn}$

Matrix Subspace	Notation	Subspace of
Column Space	$C(\underline{A})$	$\mathbb{R}^m$
Nullspace	$N(\underline{A})$	$\mathbb{R}^n$
Row Space	$C(\underline{A}^T)$	$\mathbb{R}^n$
Left Nullspace	$N(\underline{A}^T)$	$\mathbb{R}^m$

Orthogonal Complements:

- Column space & left nullspace in $\mathbb{R}^m$

$$C(\underline{A}) \quad \& \quad N(\underline{A}^T)$$

$$\text{dim:} \quad r \quad + \quad (m-r) \quad = \quad m$$

$$\text{dim:} \quad r \quad + \quad (m-r) \quad = \quad m$$

- Row space & nullspace in $\mathbb{R}^n$

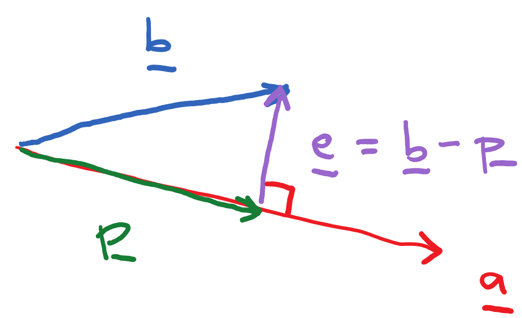
$$C(\underline{A}^T) \quad \& \quad N(\underline{A})$$

$$\text{dim:} \quad r \quad + \quad (n-r) \quad = \quad n$$

Projections

First consider projections onto a vector (line), then generalize onto a subspace.

A vector projection is the determination of which part of one vector lies on another vector



$$\underline{p} + \underline{e} = \underline{b}$$
$$\underline{e} = \underline{b} - \underline{p}$$

$\underline{b}$: Generic vector in $\mathbb{R}^n$

$\underline{a}$: Another vector in $\mathbb{R}^n$

$\underline{p}$: Projection of $\underline{b}$ onto $\underline{a}$

$\underline{e}$: "Error" vector indicating how far $\underline{b}$ is from $\underline{a}$ $\sim$ $\underline{e} \perp \underline{a}$

To find $\underline{p}$, first define it as

$$\underline{p} = \hat{x}\,\underline{a}$$

where $\hat{x}$ is the relative distance along $\underline{a}$

Then, the error

$$\underline{e} = \underline{b} - \underline{p} = \underline{b} - \hat{x}\,\underline{a}$$

but

$$\underline{e} \perp \underline{a} \Rightarrow \underline{a} \cdot \underline{e} = 0 \Rightarrow \underline{a}^T \underline{e} = 0$$

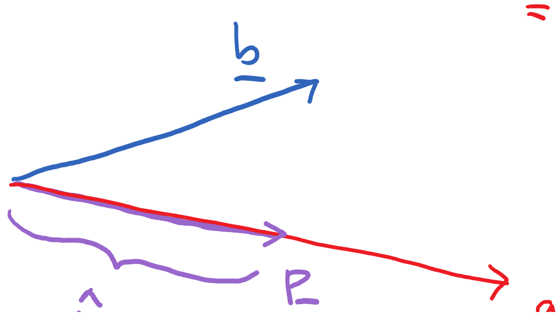
$$\underline{a}^T \underline{e} = \underline{a}^T (\underline{b} - \underline{p}) = \underline{a}^T (\underline{b} - \hat{x}\,\underline{a}) = 0$$

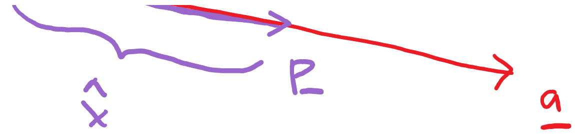
$$= \underline{a}^T \underline{b} - \underbrace{\hat{x}}_{\text{scalar}}\, \underline{a}^T \underline{a} = 0$$

$$\Rightarrow \hat{x} = \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \quad \text{scalars}$$

$$\therefore \underline{p} = \hat{x}\,\underline{a} = \underbrace{\left(\frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \right)}_{\text{scalar}} \underline{a} = \underline{a} \left(\frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \right)$$

$$= \underbrace{\left(\frac{\underline{a}\,\underline{a}^T}{\underline{a}^T \underline{a}} \right)}_{\text{scalar? Matrix!}} \underline{b}$$





Scalar? Matrix!

Shortcoming: $\underline{p}$ is only for a specific $\underline{b}$

How can this be generalized?