A New Approach to the Energy Control of Toda Chains

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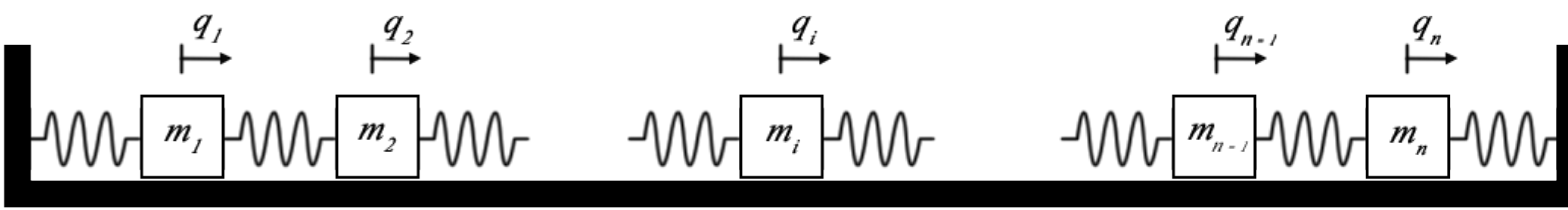
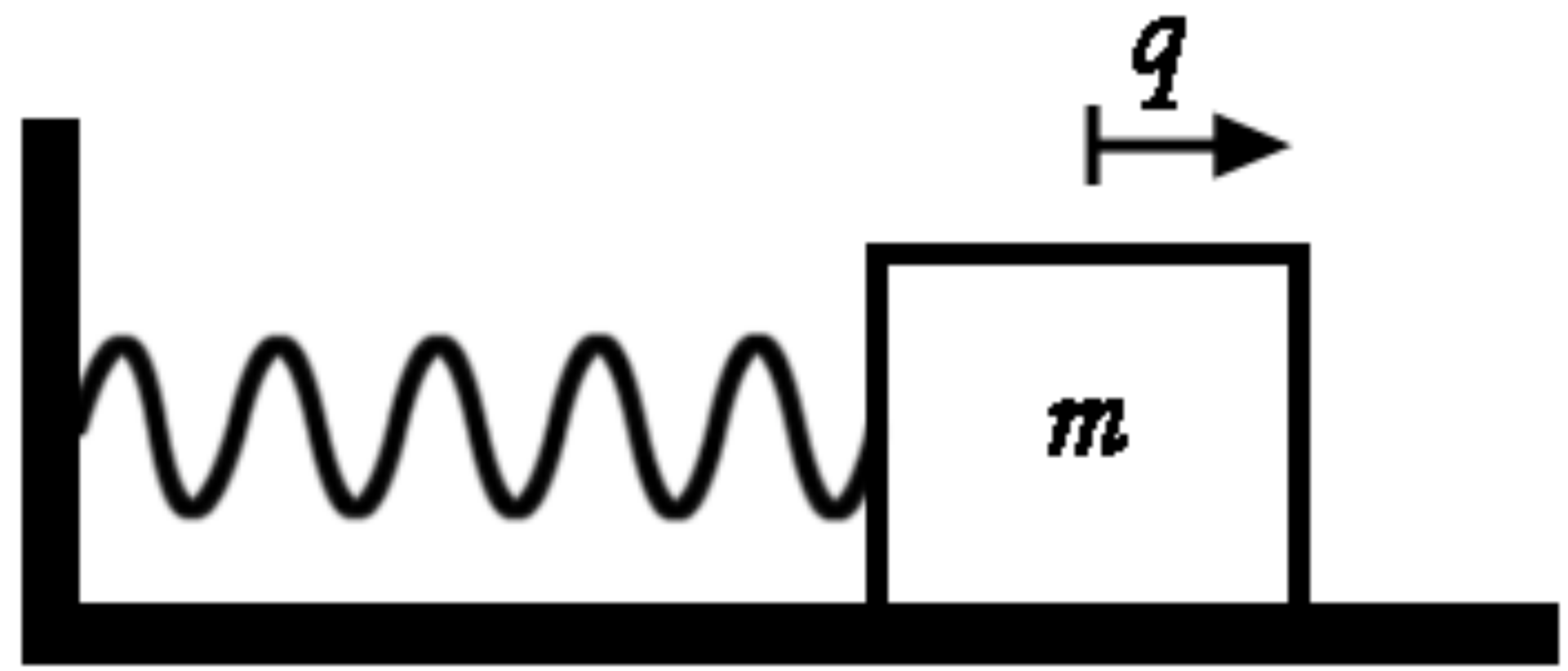
What are Toda Chains ?



Toda Chains

- Named After ‘Marikazu Toda’ (Japanese Physicist) - 1967
- Non-linear One Dimensional Crystal (Lattice)
- Spring-Mass Oscillators with Exponential Spring Constants
- Completely Integrable Hamiltonian System
- Belongs to a class of Non-linear Oscillatory Systems

Physics of the Toda Chain



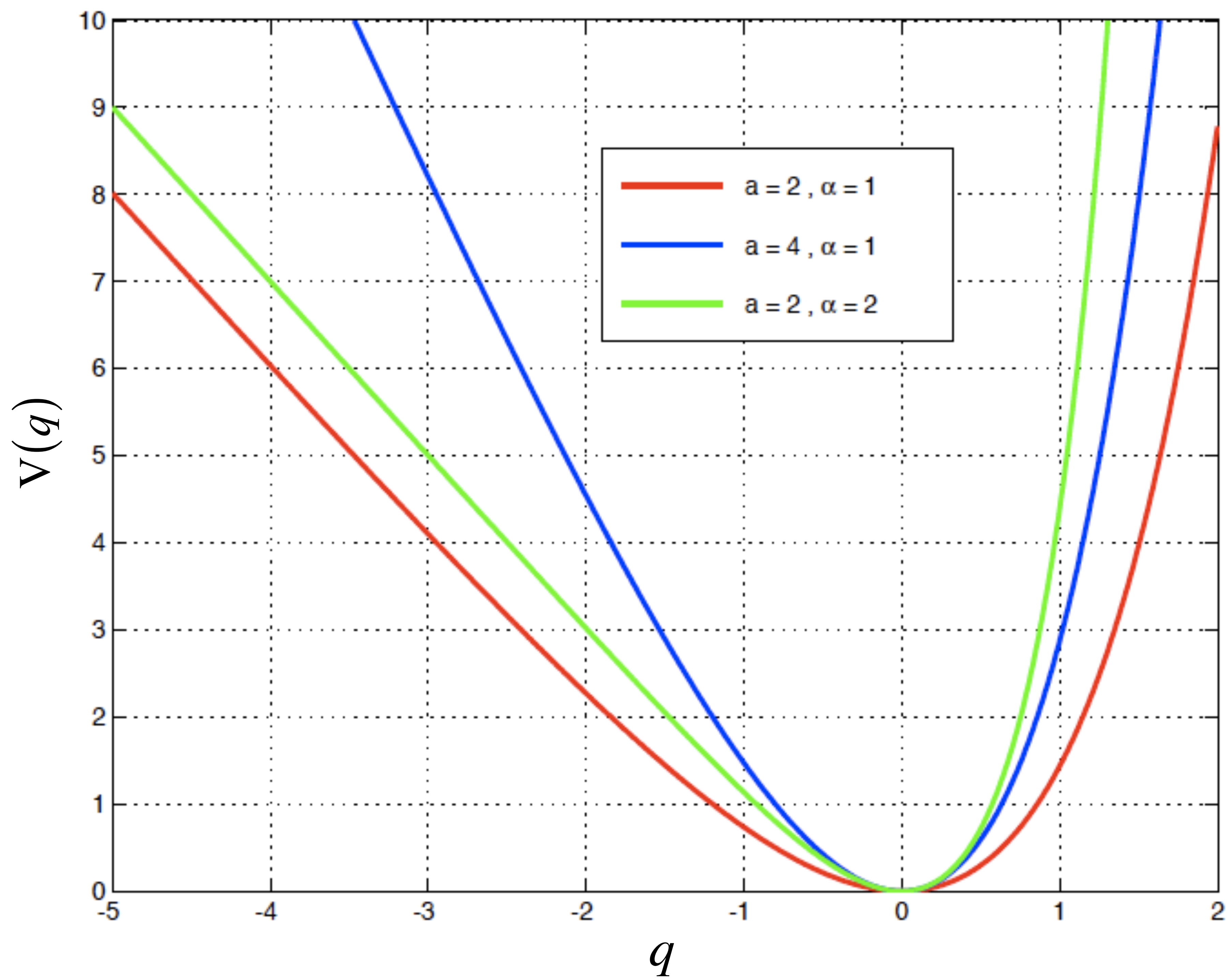
1 DOF Spring-Mass System with Toda Spring Stiffness

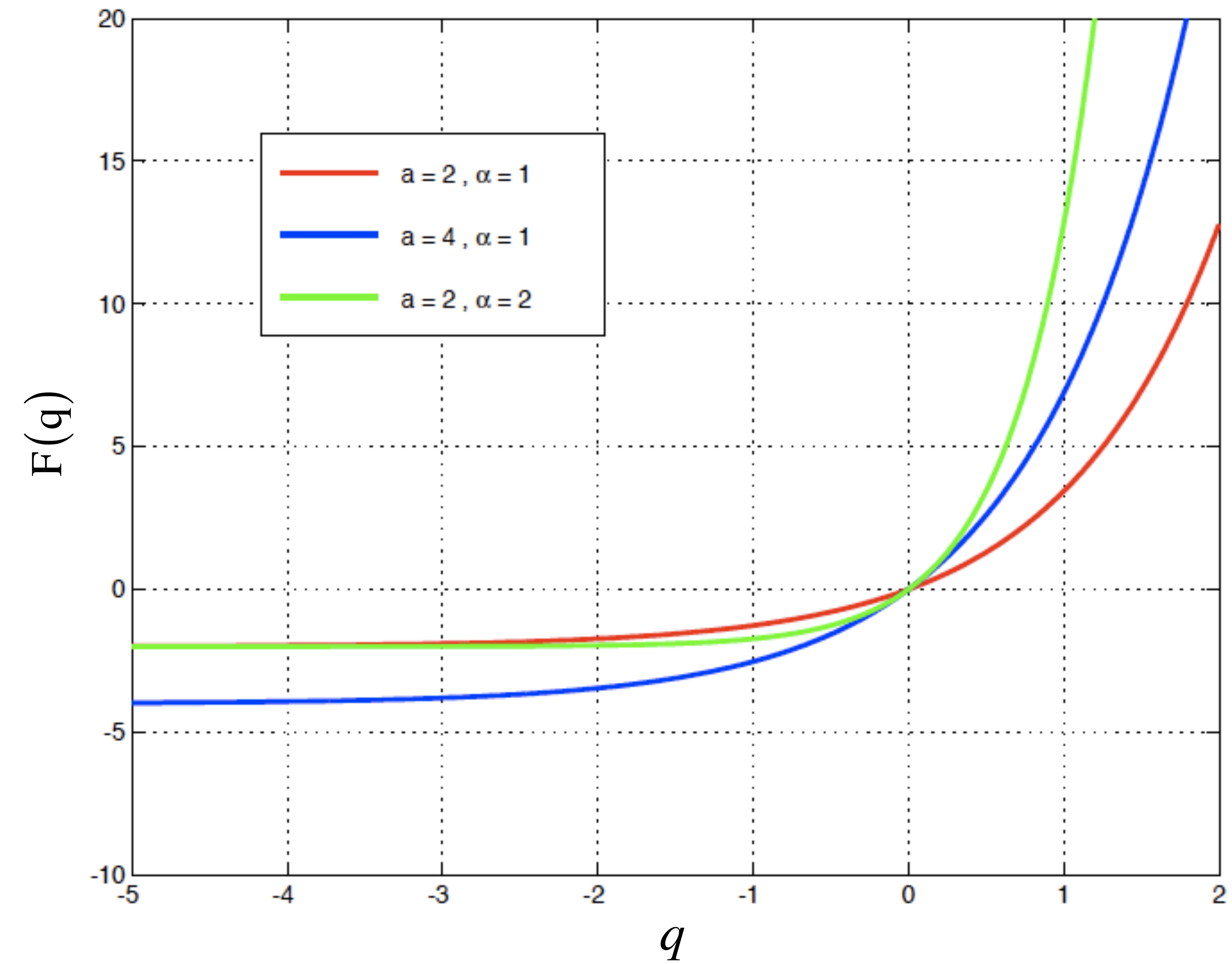
$$V(q) = \frac{a}{\alpha} e^{\alpha q} - aq - \frac{a}{\alpha}$$

$$a > 0, \quad \alpha > 0$$

$$F(q) = -F_{restoring}(q) = \frac{\partial V}{\partial q} = a\left(e^{\alpha q} - 1\right)$$

Physics of the Toda Chain





$$V(q) = \frac{a}{\alpha} e^{\alpha q} - aq - \frac{a}{\alpha}$$

$$a > 0, \quad \alpha > 0$$

$$F(q) = -F_{restoring}(q) = \frac{\partial V}{\partial q} = a\left(e^{\alpha q} - 1\right)$$

Why Toda Chains ?

- Dissimilar Tensile/Compressive Forces
  - Can be used to model physical structures that are made out of flexible cables (e.g. Suspension Bridges)
- 'Finite Dimensional Analog' of the Korteweg-deVries equation that is used to model shallow water waves
- Admits soliton solutions which makes it a remarkable example to analyze for theoretical physicists
- Hamiltonian System - large amount of literature already available for these systems
- Ladder Circuits - electrical counterpart of Toda chains

What is Energy Control ?

$$H\left(q(t),\dot{q}(t)\right) \rightarrow \quad H^*$$

$$as \quad t \quad \rightarrow \quad +\infty$$

Why Energy Control ?

- Classical control theorists dealt with problems of regulation & tracking
- Modern research is being focussed at achieving non-classical control objectives such as -
    - Energy Stabilization
    - Motion Synchronization
- Problems in Vibration Engineering can be reformulated as 'Energy Control' problems

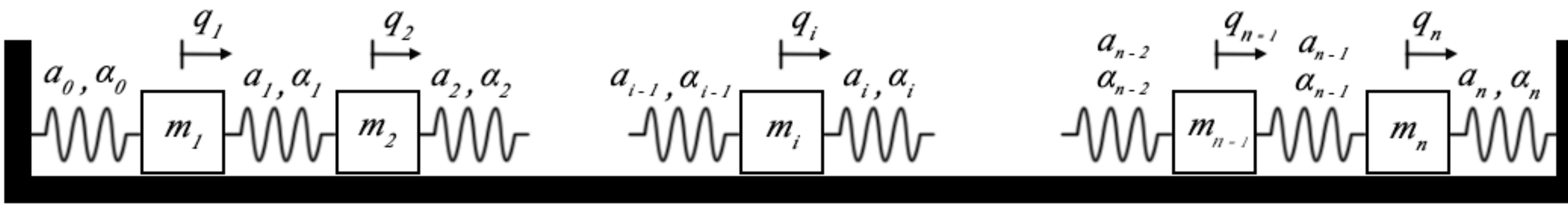
Past Work...

1. Toda M., "Theory of Nonlinear Lattices," Springer-Verlag, New York, 1989.
2. Ford, J., Stoddard, S.D. and Turner, J.S., "On the Integrability of the Toda Chain," Prog. Theor. Phys., 50, 1547-1560, 1973.
3. Henon, M., "Integrals of the Toda Lattice," Phys. Rev. B9, 1921-1925, 1974.
4. Flaschka, H., " The Toda Lattice II. Existence of Integrals" Phys. Rev. B9 1924-1925, 1974.
5. Fradkov, A.L., "Adaptive Control in Complex Systems," Moscow: Nauka, 1990
6. Polushin, I.G., "Energy Control of Toda Lattice," Control of Oscillations and Chaos, Proceedings. 2nd Intl. Conf., 1, pp.26-28, 2000.

Past Work...

- Polushin (2000)
    - Considers a finite dimensional fixed-fixed homogenous Toda chain
    - Employs the speed-gradient control algorithm
    - Discovers that it is possible to control the energy of the entire chain by controlling just a single mass (regardless of the number of masses in the chain)
- Inefficient + Computationally Intensive approach
- “Accurate control of energy waves requires the simultaneous application of more than one control inputs”
- Generalization to multiple control inputs is beyond the scope of the theory presented in Polushin (2000)

In the present paper...



- We consider a finite dimensional non-homogenous Toda chain with
  1] fixed - fixed boundary 2] fixed - free boundary
- Energy control problem is viewed as 'constrained motion problem' which is logically a much simpler approach to the problem
- Fundamental Equation is used to obtain the exact NL control force required to achieve the desired energy stabilization

In the present paper...

- Freedom to choose ONE or MORE control inputs to control the energy waves of the lattice
- General methodology to obtain the control force in closed form when 'k' out of 'n' masses are controlled where $(1 \leq k \leq n)$
- Demonstrate that the closed form control force derived gives us 'Global Asymptotic Convergence' to the energy state H^*
- Numerical Simulations of a Five Mass Toda Chain
  - Fixed - Fixed Ends
  - Fixed - Free Ends

Control applied at various locations

Fundamental Equation

- Unconstrained System

$$M(q,t)\,\ddot{q} = F(q,\dot{q},t)$$

$$q(0) = q_0 \;,\; \dot{q}(0) = \dot{q}_0$$

$$a(q,\dot{q},t) = \left[M(q,t)\right]^{-1} F(q,\dot{q},t)$$

Fundamental Equation

- Constraint Equations - Total of 'm' constraints

$$\varphi_i(q, t) = 0 \qquad \phi_j(q, \dot{q}, t) = 0$$

- Modified Constraint Equations

$$\Lambda_i(q, \dot{q}, \ddot{q}, t) = \ddot{\varphi}_i + c\,\dot{\varphi}_i + k\,\varphi_i = 0$$

$$\Psi_j(q, \dot{q}, \ddot{q}, t) = \dot{\phi}_j + \beta\,\phi_j = 0$$

Fundamental Equation

- Constraint Matrix Representation (General Form)

$$A(q, \dot{q}, t)\,\ddot{q} \;=\; b(q, \dot{q}, t)$$

$A_{m\ x\ n}$ - Constraint matrix of rank 'r'

$b_{m\ x\ 1}$ - Column vector of size 'm'

Fundamental Equation

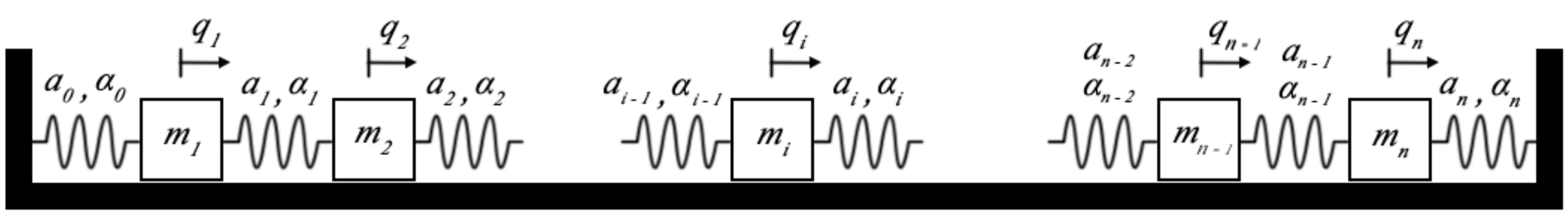
- Constrained System

$$M(q,\,t)\,\ddot{q} \;=\; F(q,\,\dot{q},\,t) \;+\; F^{C}(q,\,\dot{q},\,t)$$

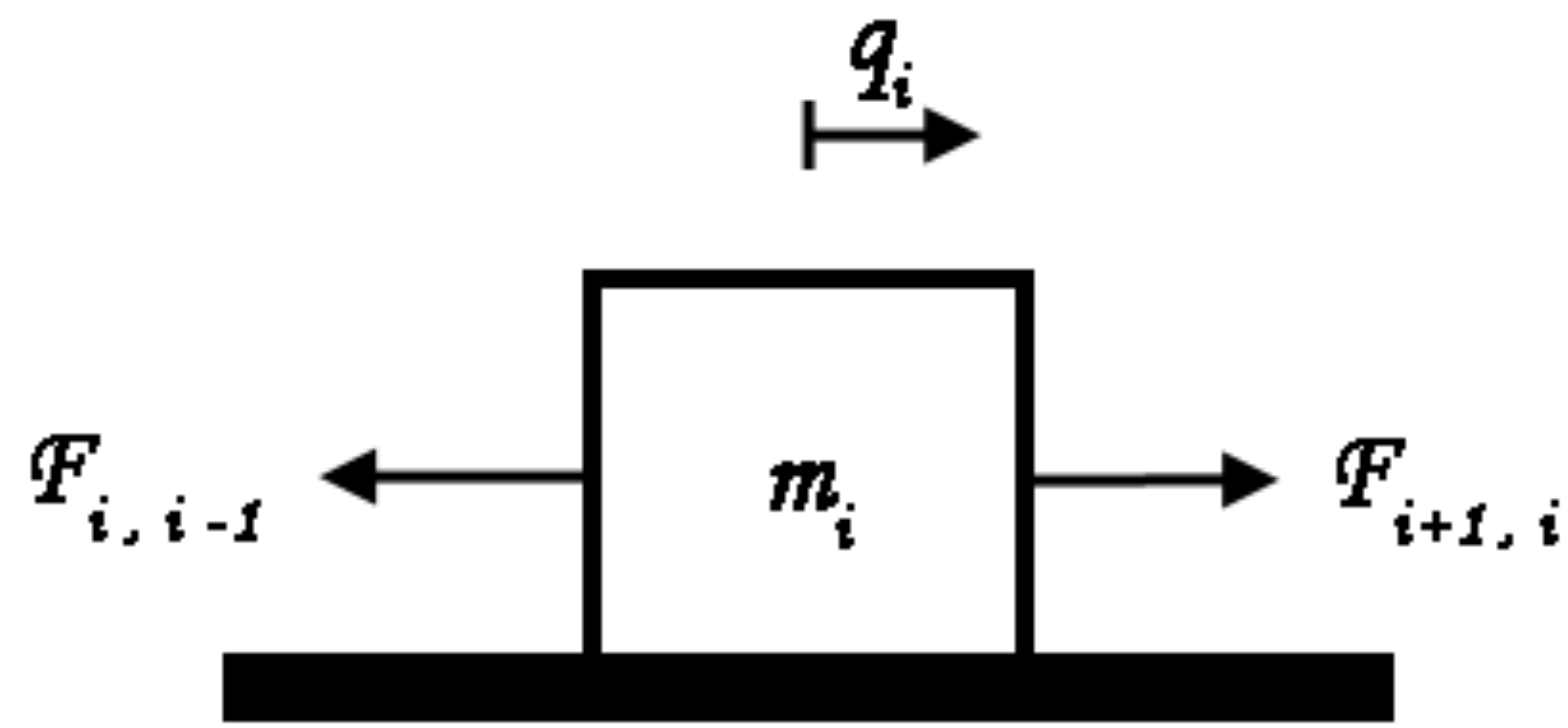
- Constraint Force or Control Force

$$F^{C}(q,\,\dot{q},\,t) \;=\; M^{1/2}\left(AM^{-1/2}\right)^{+}\;(b - Aa)$$

Unconstrained Toda Chain System



Unconstrained Toda Chain System



Forces acting on an individual mass

$$m_i \ddot{q}_i = F_{i+1,\,i} - F_{i,\,i-1}$$

$$\Downarrow$$

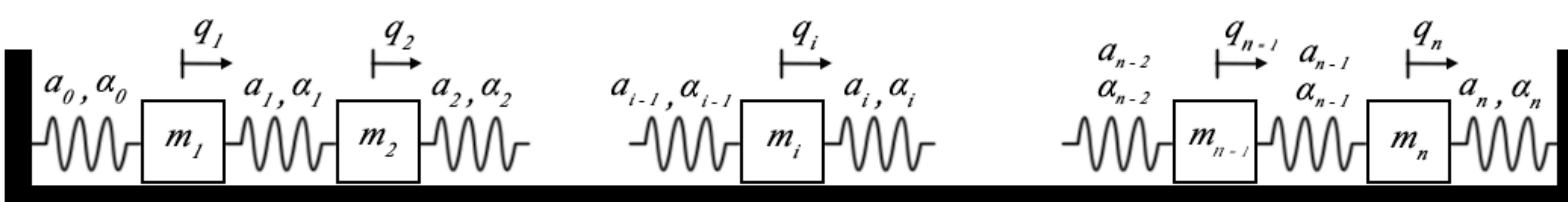
$$m_i \ddot{q}_i = \left[a_i\left(e^{\alpha_i(q_{i+1} - q_i)} - 1\right)\right] - \left[a_{i-1}\left(e^{\alpha_{i-1}(q_i - q_{i-1})} - 1\right)\right]$$

$$M\,\ddot{q} = F$$

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & m_i & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & m_n \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_i \\ \vdots \\ \ddot{q}_n \end{bmatrix} = \begin{bmatrix} a_1\left(e^{\alpha_1(q_2 - q_1)} - 1\right) - a_0\left(e^{\alpha_0(q_1 - q_0)} - 1\right) \\ \vdots \\ a_i\left(e^{\alpha_i(q_{i+1} - q_i)} - 1\right) - a_{i-1}\left(e^{\alpha_{i-1}(q_i - q_{i-1})} - 1\right) \\ \vdots \\ a_n\left(e^{\alpha_n(q_{n+1} - q_n)} - 1\right) - a_{n-1}\left(e^{\alpha_{n-1}(q_n - q_{n-1})} - 1\right) \end{bmatrix}$$

$$\Rightarrow \quad \tilde{a} = M^{-1}F$$

Acceleration of the Unconstrained System

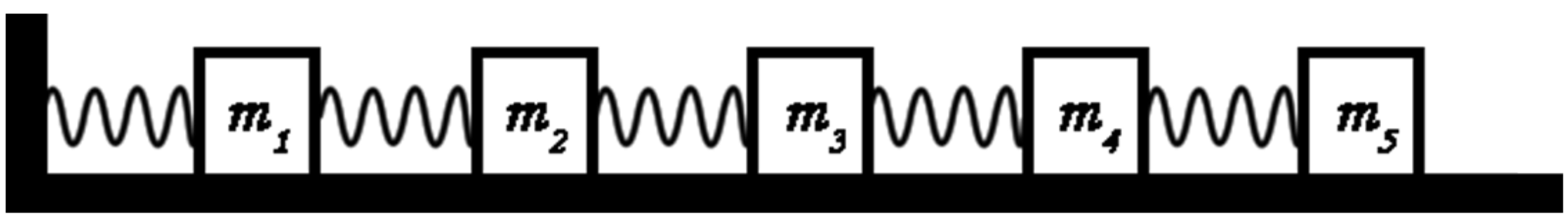


Boundary Conditions

- Fixed - Fixed Boundary Condition

$$q_0 = \dot{q}_0 = 0 \qquad q_{n+1} = \dot{q}_{n+1} = 0$$

- Fixed - Free Boundary Condition



$$q_0 = \dot{q}_0 = 0 \qquad a_n = \alpha_n = 0$$

Constraint Formulation

Essentially two types of constraints act on the Unconstrained System -

- Constraint of 'Energy Stabilization' - 1

- Constraint of 'No Control' on a particular mass - k

A total of '$k + 1$' constraints act on the Unconstrained Toda Chain system

Constraint of Energy Stabilization

- $$\Phi_1(q, \dot{q}, t) = H - H^* = 0$$ Constraint Eqn.

$$= \left[\sum_{i=0}^{n} \left[\left(\frac{1}{2} m_i \dot{q}_i^2 \right) + \left(\frac{a_i}{\alpha_i} e^{\alpha_i (q_{i+1} - q_i)} - a_i (q_{i+1} - q_i) - \frac{a_i}{\alpha_i} \right) \right] - H^* \right]$$

- $$\dot{\Psi}_1 = \dot{\Phi}_1 + \beta \Phi_1 = 0$$ Modified Constraint Eqn.

- $$\underbrace{\left[m_1 \dot{q}_1 \quad \ldots \quad m_i \dot{q}_i \quad \ldots \quad m_n \dot{q}_n \right]}_{A} \ddot{q}_{n \times 1} =$$ Constraint Matrix Form

$$\underbrace{\left[-\sum_{i=0}^{n} \left[a_i (\dot{q}_{i+1} - \dot{q}_i) \left(e^{\alpha_i (q_{i+1} - q_i)} - 1 \right) \right] - \beta \left(H - H^* \right) \right]}_{b}$$

Constraint of 'No Control'

Total Number of Masses - n
Number of Masses to which control is applied - α
Number of Masses to which control is not applied - k

$\alpha + k = n$

j_υ - denotes any mass m_i picked from amongst 'n' masses to which control is NOT applied

$$j_1 < j_2 < j_3 < \cdots\cdots < j_\upsilon < \cdots < j_k$$

c_γ - denotes any mass m_i picked from amongst the remaining 'n - k' masses to which control IS applied

$$c_1 < c_2 < c_3 < \cdots\cdots < c_\gamma < \cdots < c_\alpha$$

- $\Phi_\upsilon = u_{j_\upsilon} = 0 \qquad \upsilon : 1 \to k$

$$= m_{j_\upsilon} \ddot{q}_{j_\upsilon} - \left[a_{j_\upsilon} \left(e^{\alpha_{j_\upsilon} \left(q_{j_\upsilon + 1} - q_{j_\upsilon} \right)} - 1 \right) - a_{j_\upsilon - 1} \left(e^{\alpha_{j_\upsilon - 1} \left(q_{j_\upsilon} - q_{j_\upsilon - 1} \right)} - 1 \right) \right] = 0$$

Constraint Matrix Representation

$$
\begin{bmatrix}
m_1\dot{q}_1 & \cdots & m_i\dot{q}_i & \cdots & m_n\dot{q}_n \\
0 & m_{j_1} & 0 & \dots & 0 \\
\vdots & & \dots & & \vdots \\
0 & \dots & m_{j_\omega} & \dots & 0 \\
\vdots & & \dots & & \vdots \\
0 & \dots & \dots & m_{j_k} & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\ \vdots \\ \vdots \\ \vdots \\ \ddot{q}_n
\end{bmatrix}
=
\begin{bmatrix}
-\sum_{i=0}^{n}\left[a_i\left(\dot{q}_{i+1} - \dot{q}_i\right)\left(e^{\alpha_i\left(q_{i+1} - q_i\right)} - 1\right)\right] - \beta\left(H - H^*\right) \\
a_{j_1}\left(e^{\alpha_{j_1}\left(q_{j_1+1} - q_{j_1}\right)} - 1\right) - a_{j_1-1}\left(e^{\alpha_{j_1-1}\left(q_{j_1} - q_{j_1-1}\right)} - 1\right) \\
\vdots \\
a_{j_\omega}\left(e^{\alpha_{j_\omega}\left(q_{j_\omega+1} - q_{j_\omega}\right)} - 1\right) - a_{j_\omega-1}\left(e^{\alpha_{j_\omega-1}\left(q_{j_\omega} - q_{j_\omega-1}\right)} - 1\right) \\
\vdots \\
a_{j_k}\left(e^{\alpha_{j_k}\left(q_{j_k+1} - q_{j_k}\right)} - 1\right) - a_{j_k-1}\left(e^{\alpha_{j_k-1}\left(q_{j_k} - q_{j_k-1}\right)} - 1\right)
\end{bmatrix}
$$

$$
A_{(k+1)\,\mathrm{x}\,n} \cdot \ddot{q}_{n\,\mathrm{x}\,1} = b_{(k+1)\,\mathrm{x}\,1}
$$

Derivation of Control Force

- Unconstrained System $\Rightarrow$ M , F , $\tilde{a}$
- Constraint Equations $\Rightarrow$ A , b
- Constraint / Control Force $\Rightarrow$ $F^C = M^{1/2}\left(AM^{-1/2}\right)^+ \left(b - A\tilde{a}\right)$

$$\boxed{F^C = -\frac{\beta\left(H - H^*\right)}{\left(\sum_{\gamma}^{n-k} m_{c_\gamma}\dot{q}_{c_\gamma}^2\right)} \cdot \mu_{n\ x\ n} \cdot \begin{bmatrix} m_1\dot{q}_1 \\ \vdots \\ m_i\dot{q}_i \\ \vdots \\ m_n\dot{q}_n \end{bmatrix}}$$

CLOSED FORM EXPRESSION OF THE EXPLICIT CONTROL FORCE

$$F^C = -\xi\left(H - H^*\right) \bullet \mu_{n\ x\ n} \bullet \begin{bmatrix} m_1\dot{q}_1 \\ \vdots \\ m_i\dot{q}_i \\ \vdots \\ m_n\dot{q}_n \end{bmatrix}$$

- $\beta = \xi\left(\sum_{\gamma}^{n-k} m_{c_\gamma}\dot{q}_{c_\gamma}^2\right)^r$ where $\xi > 0$ and $r \geq 1$
- $\mu_{(j,j)} = \begin{Bmatrix} 1 & , & if & j \in C \\ 0 & , & if & j \notin C \end{Bmatrix} = \begin{bmatrix} 1 & & & & \mathbf{0} \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ \mathbf{0} & & & & 1 \end{bmatrix}$ where $C = \{c_1, c_2, c_3 \ldots c_\alpha\}$

Global Asymptotic Convergence to the Energy State H^*

1-DOF Spring Mass System with Toda Stiffness

1 DOF Spring-Mass System

$$m\,\ddot{q} + a\left(e^{\alpha q} - 1\right) = 0$$

$$q(0) = q_0\,, \quad \dot{q}(0) = \dot{q}_0$$

$$F^C = -\,\xi\cdot\left(H - H^*\right)\cdot\dot{q}$$

Control Force

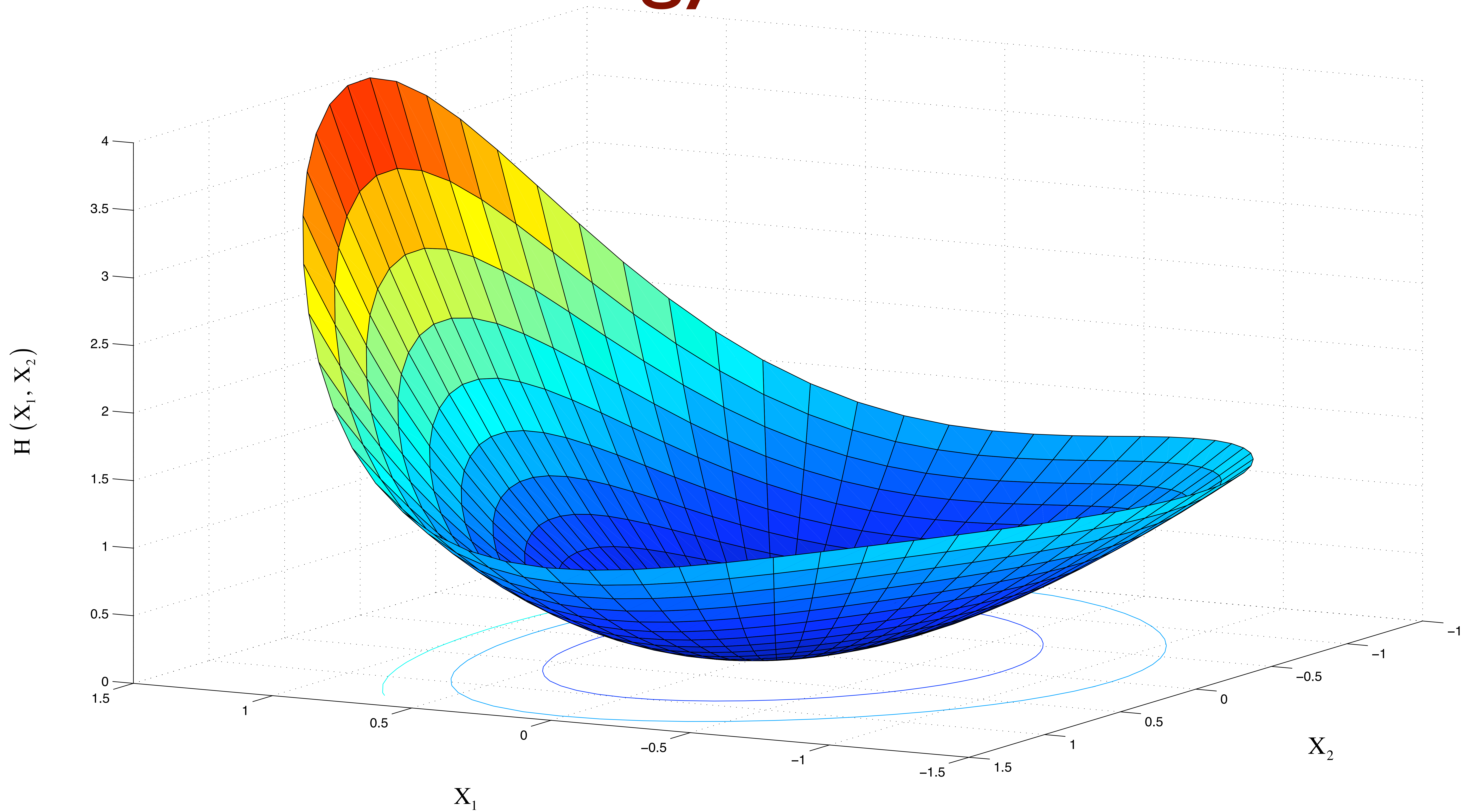
$$m\,\ddot{q} + \xi\cdot\left(H - H^*\right)\cdot\dot{q} + a\left(e^{\alpha q} - 1\right) = 0$$

Equations of Motion of the Constrained System

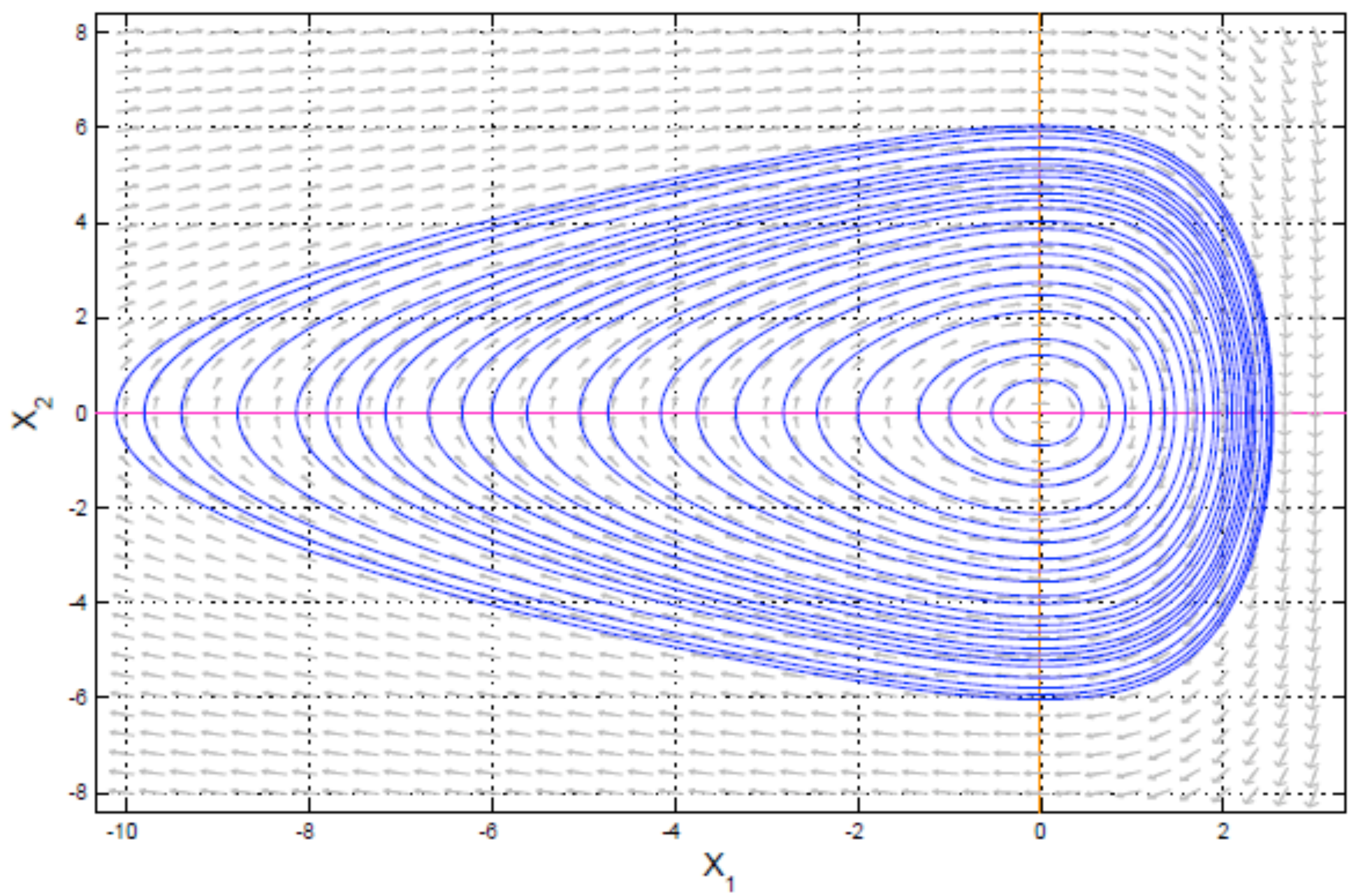
$$H = \frac{1}{2}\,m\,\dot{q}^2 + \frac{a}{\alpha}\,e^{\alpha q} - aq - \frac{a}{\alpha}$$

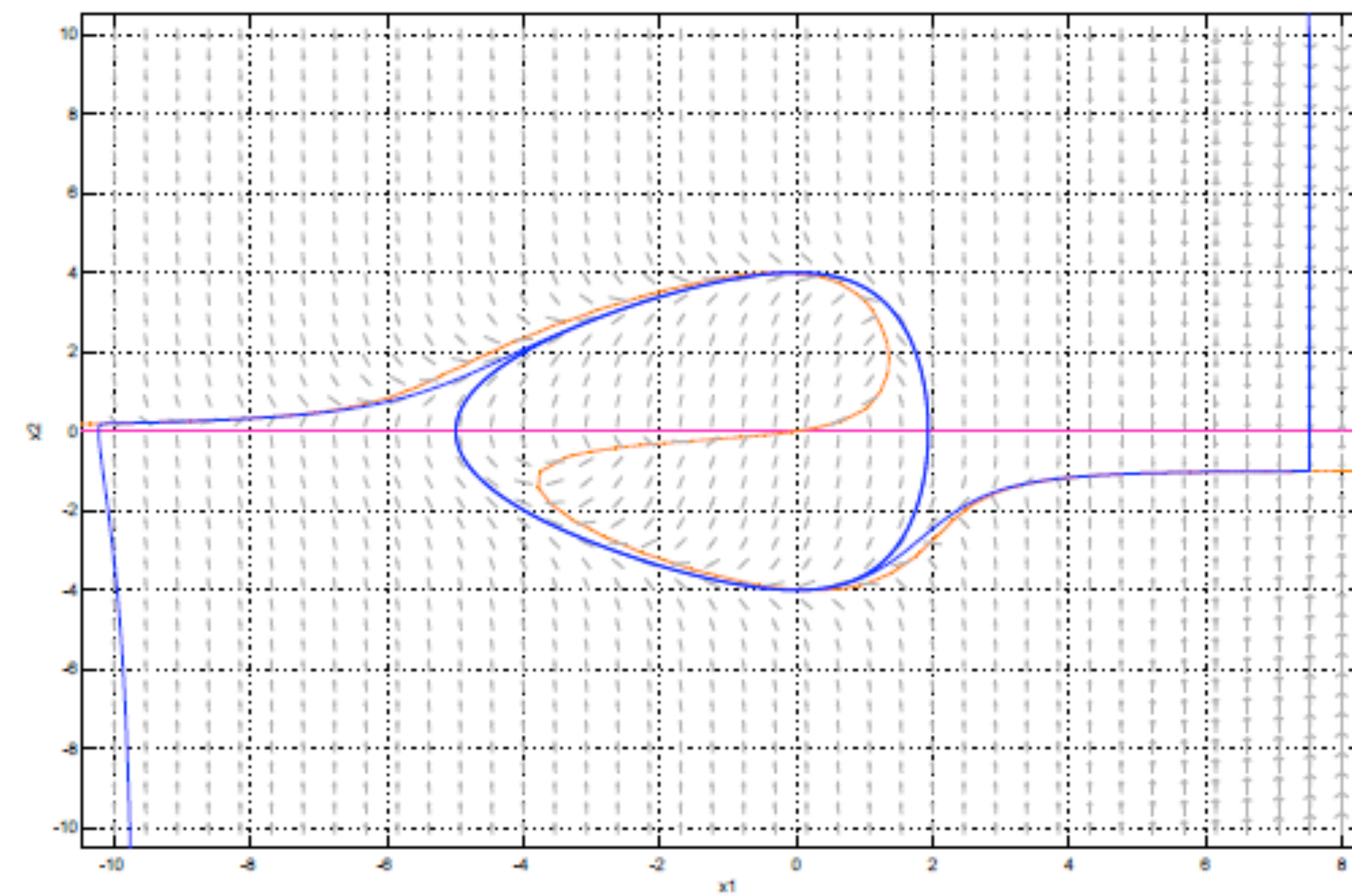
Energy of the System

Energy Surface



1-DOF Spring Mass System with Toda Stiffness





Phase Portrait
of the
Unconstrained System

Phase Portrait
of the
Constrained System

Invariance Principle

Construct an Ω set that is compact and positively invariant

- V is continuously differentiable in Ω
- $\dot{V} \leq 0$ in Ω
- E be the set of all points where $\dot{V} = 0$
- M be the largest invariant set in E

Then, every solution $x(t)$ starting in Ω approaches M as $t \to \infty$

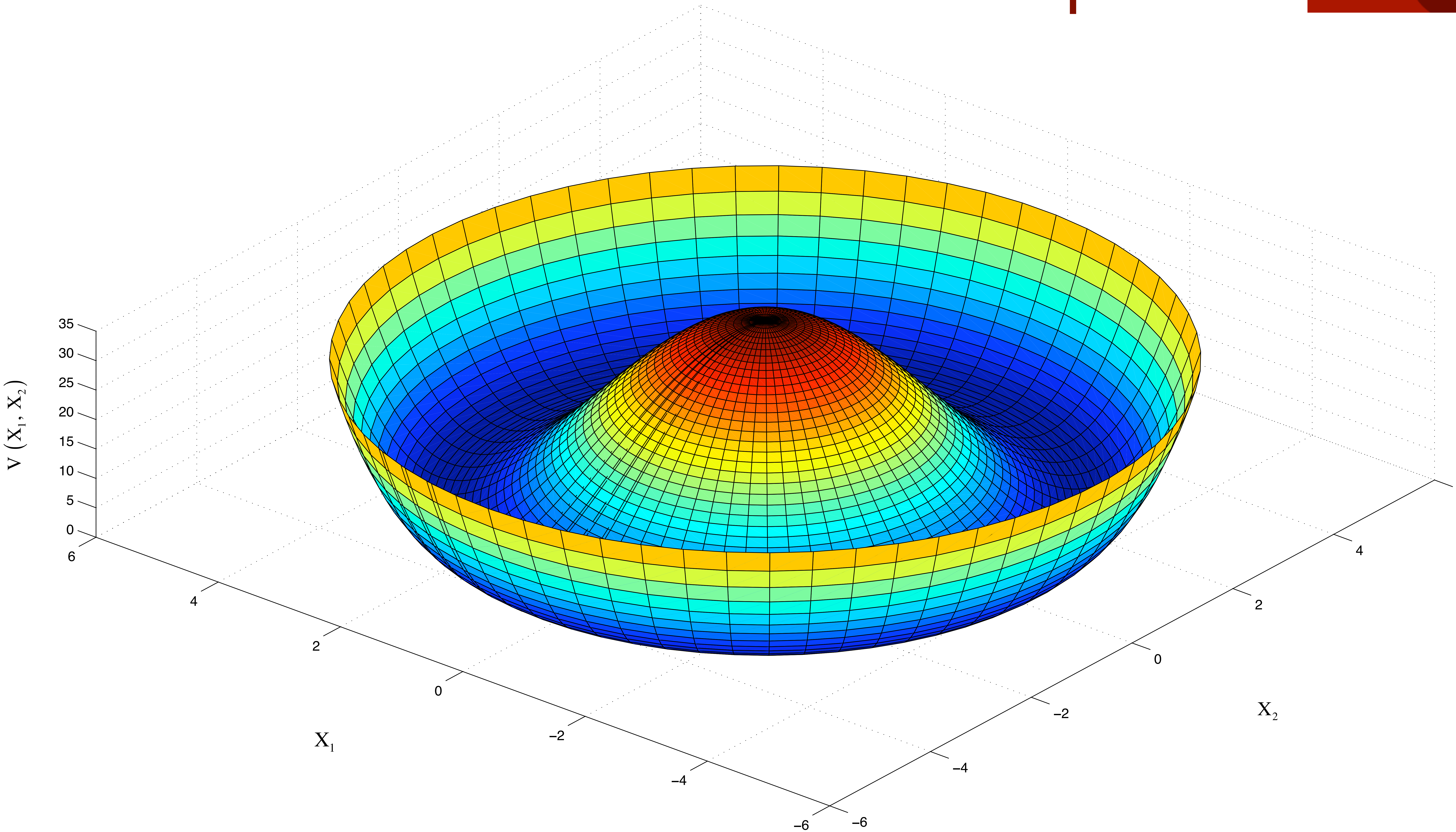
1 DOF System - Invariance Principle

- $V = \frac{1}{2}\left(H - H^*\right)^2$
- $\dot{V} = \left(H - H^*\right) \cdot \frac{dH}{dt} = -\,\xi\,\left(H - H^*\right)^2\,\dot{q}^2$

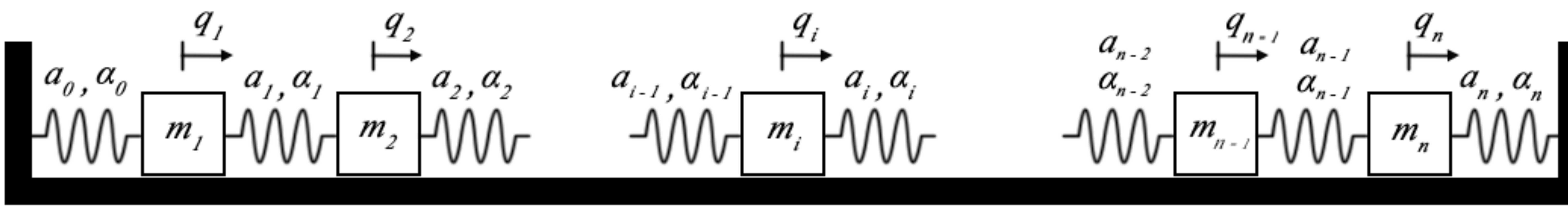
$$\boxed{\begin{array}{lll} \dot{V} = 0 & for & S = \left\{ \begin{array}{c} \dot{q} = 0 \\ H = H^* \end{array} \right\} \\ \dot{V} < 0 & \forall & \Re^2 - S \end{array}}$$

- $L^+ = \left\{ (0,\,0) \;\; ; \;\; (q,\,\dot{q}) \mid H = H^* \right\}$

Global Invariance Principle



‘n’ - DOF Toda Chain



Constrained Equation of Motion

$$m_i \ddot{q}_i = a_i \left[e^{\alpha_i (q_{i+1} - q_i)} - 1 \right] - a_{i-1} \left[e^{\alpha_{i-1} (q_i - q_{i-1})} - 1 \right] - \xi \left(H - H^* \right) \dot{q}_i$$

$$V = \frac{1}{2} \left(H - H^* \right)^2$$

$$\dot{V} = \left(H - H^* \right) \cdot \frac{dH}{dt} = - \sum_{i=0}^{n} \xi \left(H - H^* \right)^2 \dot{q}_{c_\gamma}^2$$

'n' - DOF Toda Chain

$$\dot{V} = 0 \qquad for \qquad S = \left\{ \begin{array}{c} \dot{q}_{c_\gamma} = 0 \\ H = H^* \end{array} \right\}$$

$$\dot{V} < 0 \qquad \forall \qquad \Re^{2n} - S$$

$$m_i \, \ddot{q}_i = a_i \left[e^{\alpha_i (q_{i+1} - q_i)} - 1 \right] - a_{i-1} \left[e^{\alpha_{i-1} (q_i - q_{i-1})} - 1 \right] = 0$$

$$\Rightarrow \quad a_i \left[e^{\alpha_i (q_{i+1} - q_i)} - 1 \right] = a_{i-1} \left[e^{\alpha_{i-1} (q_i - q_{i-1})} - 1 \right]$$

$$\Rightarrow \qquad q_i = 0 \qquad \forall \quad i$$

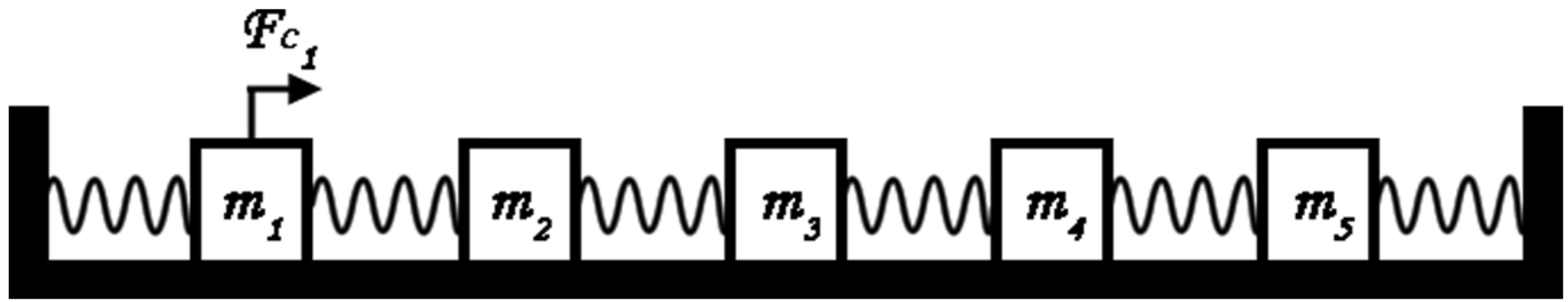
$$L^+ = \left\{ (q, \dot{q}) = (0, 0) \; ; \quad (q, \dot{q}) \mid H = H^* \right\}$$

CLAIM
Energy surface
in
2n - dimensions
is
COMPACT!

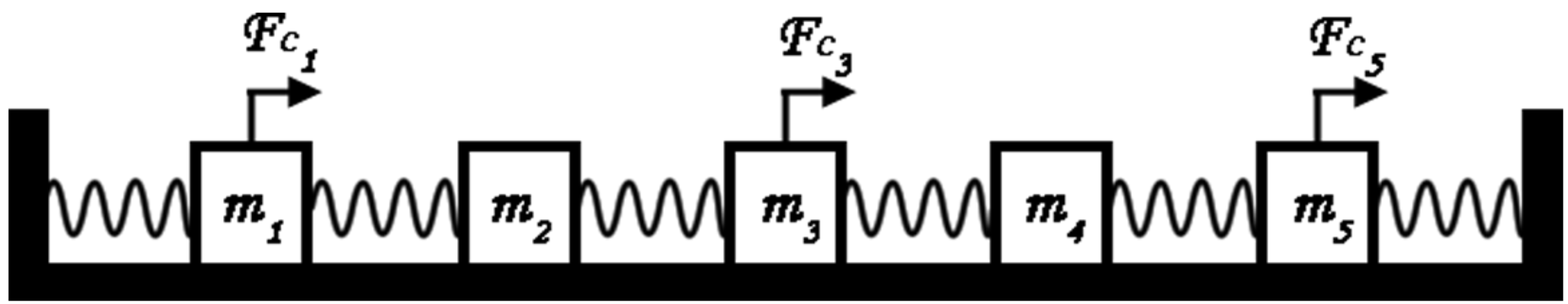
Results & Simulations

Five Mass Toda Chain w/ fixed - fixed ends

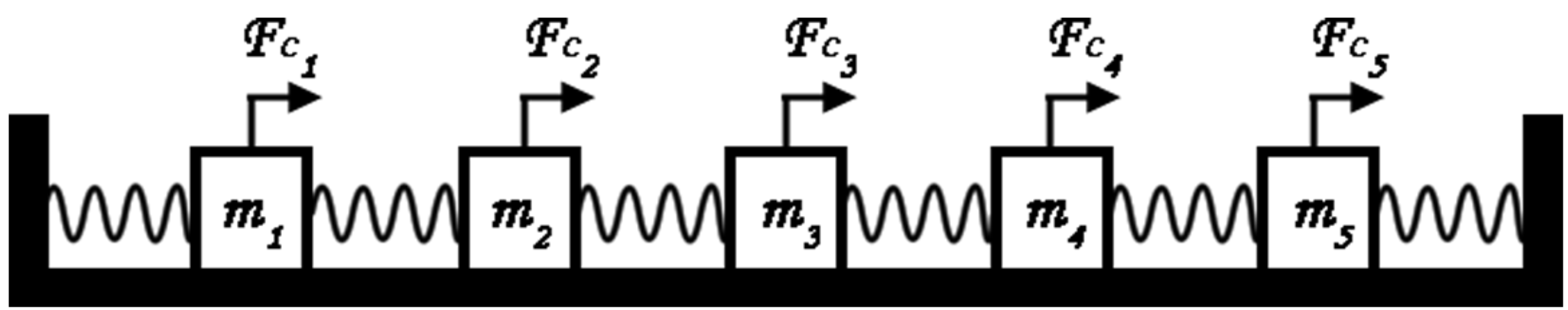
CASE 1



CASE 2



CASE 3



Simulation Parameters

USC Viterbi School of Engineering

Masses

$$m_1 = 1$$
$$m_2 = 2$$
$$m_3 = 3$$
$$m_4 = 2$$
$$m_5 = 1$$

Spring Constants

$$a_0 = 1 \qquad \alpha_0 = 2$$
$$a_1 = 2 \qquad \alpha_1 = 1$$
$$a_2 = 3 \qquad \alpha_2 = 2$$
$$a_3 = 1 \qquad \alpha_3 = 1$$
$$a_4 = 2 \qquad \alpha_4 = 1$$
$$a_5 = 1 \qquad \alpha_5 = 2$$

Desired Energy State

$$H^* = 100$$

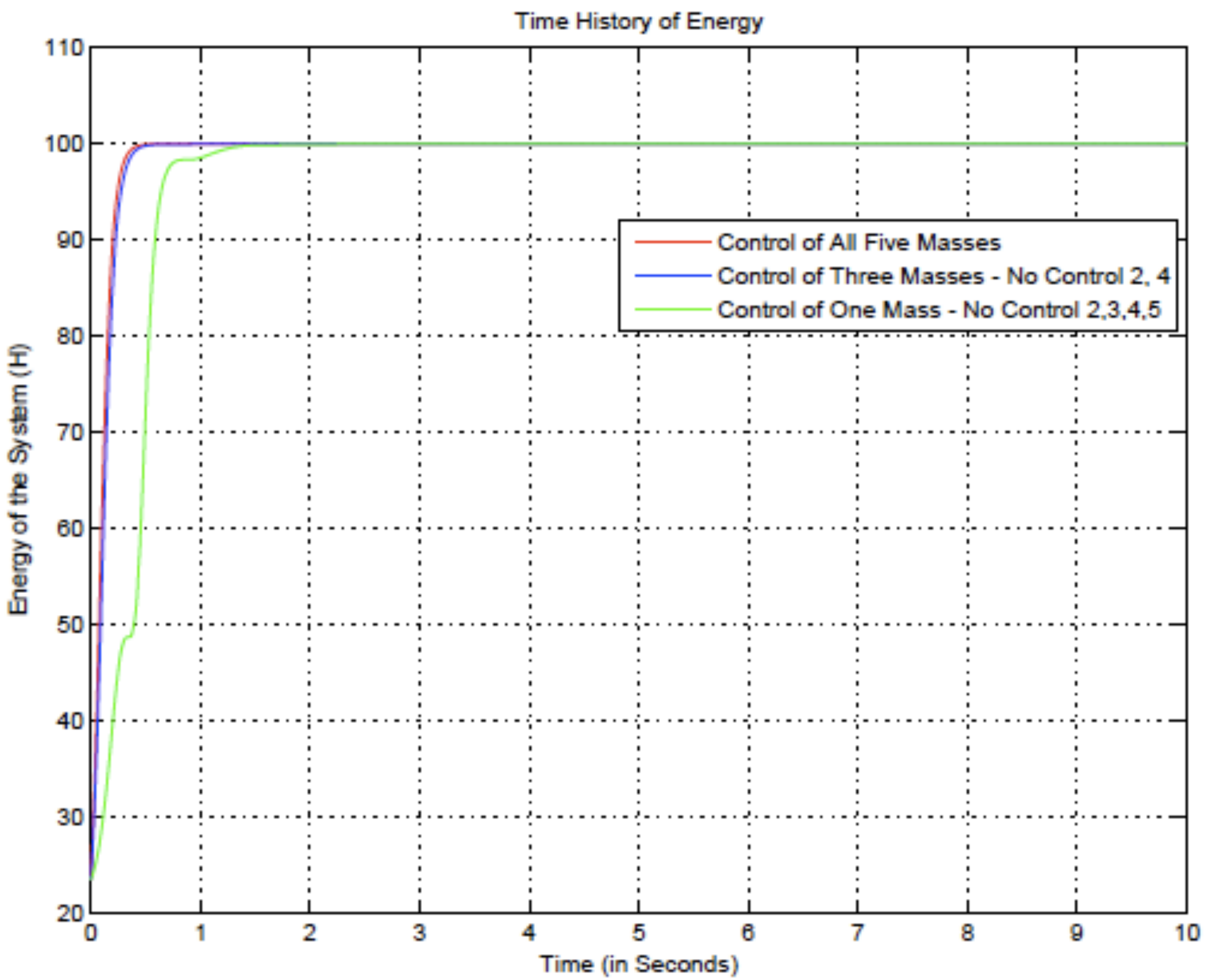
Convergence Parameter

$$\xi = 0.1$$

Initial Conditions

$$q_1(0) = 1 \qquad q_2(0) = 2 \qquad q_3(0) = 1 \qquad q_4(0) = 1 \qquad q_5(0) = 1$$
$$\dot{q}_1(0) = 2 \qquad \dot{q}_2(0) = 1 \qquad \dot{q}_3(0) = 2 \qquad \dot{q}_4(0) = 2 \qquad \dot{q}_5(0) = 3$$

Time Taken For Convergence



Case 1 - Control of One Mass

$$t_{99.995} = 3.68$$

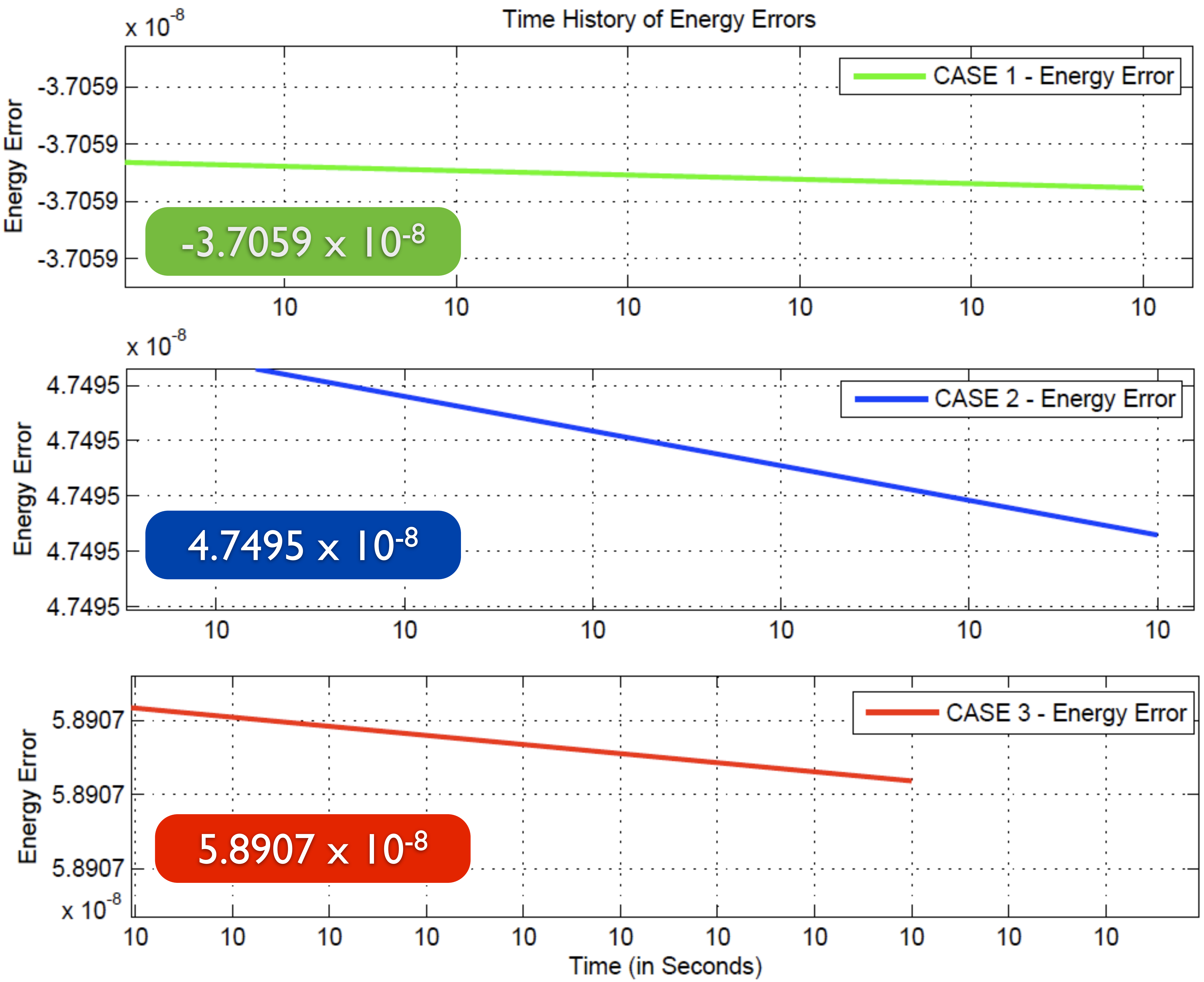
Case 2 - Control of Three Masses

$$t_{99.995} = 1.35$$

Case 3 - Control of Five Masses

$$t_{99.995} = 0.652$$

Energy Errors



Energy Error

$$e = H - H^*$$

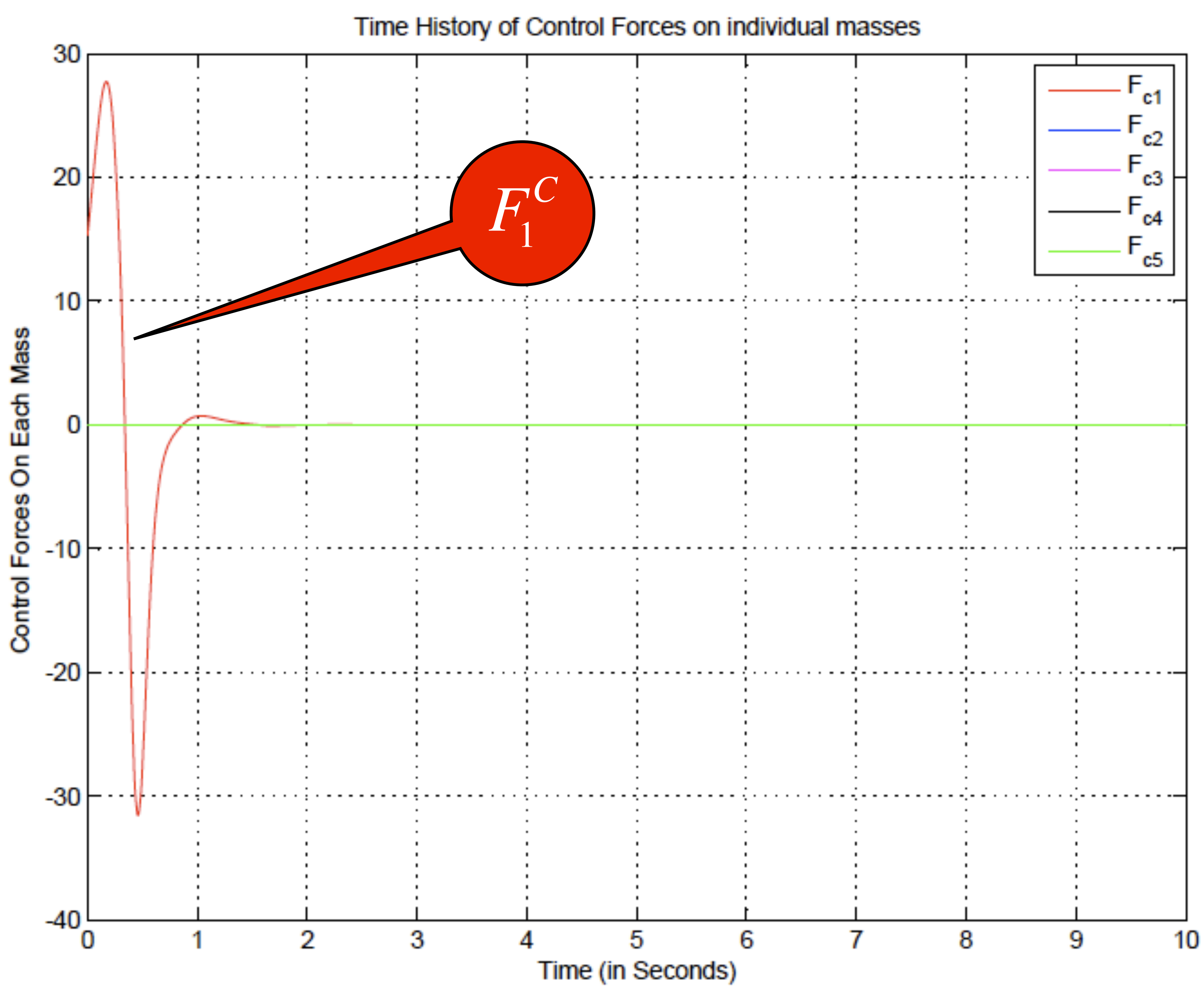
is of the order of 10^{-8}

for all three simulation cases

$$Rel.\,Error\ Tolerance = 10^{-8}$$

$$Abs.\,Error\ Tolerance = 10^{-12}$$

Case 1 - Control Forces



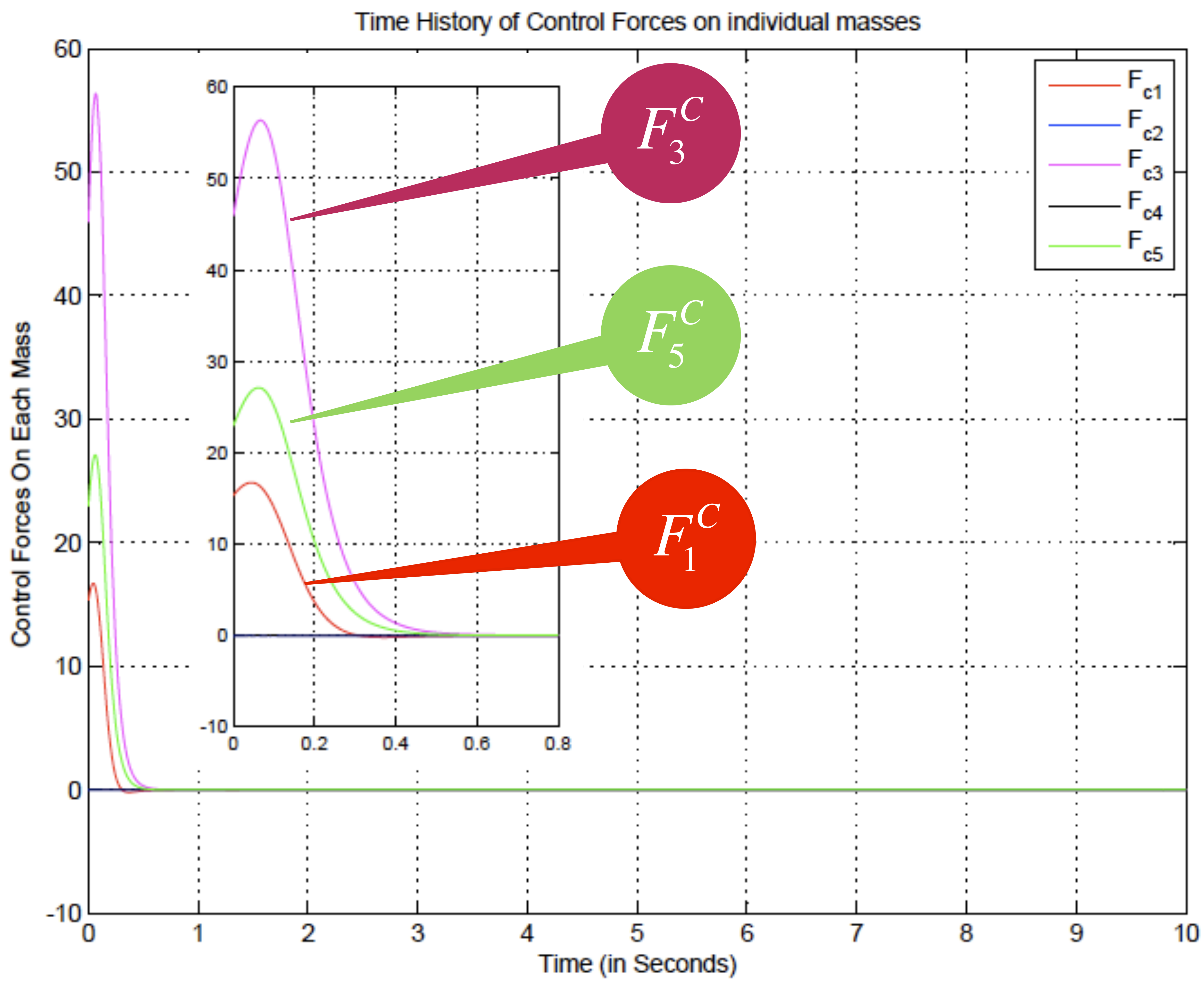
Control of One Mass

Control is applied
only to the first mass (red)

No Control
is applied
to masses 2, 3, 4 and 5

Therefore, control force on
these masses is zero for all time

Case 2 - Control Forces



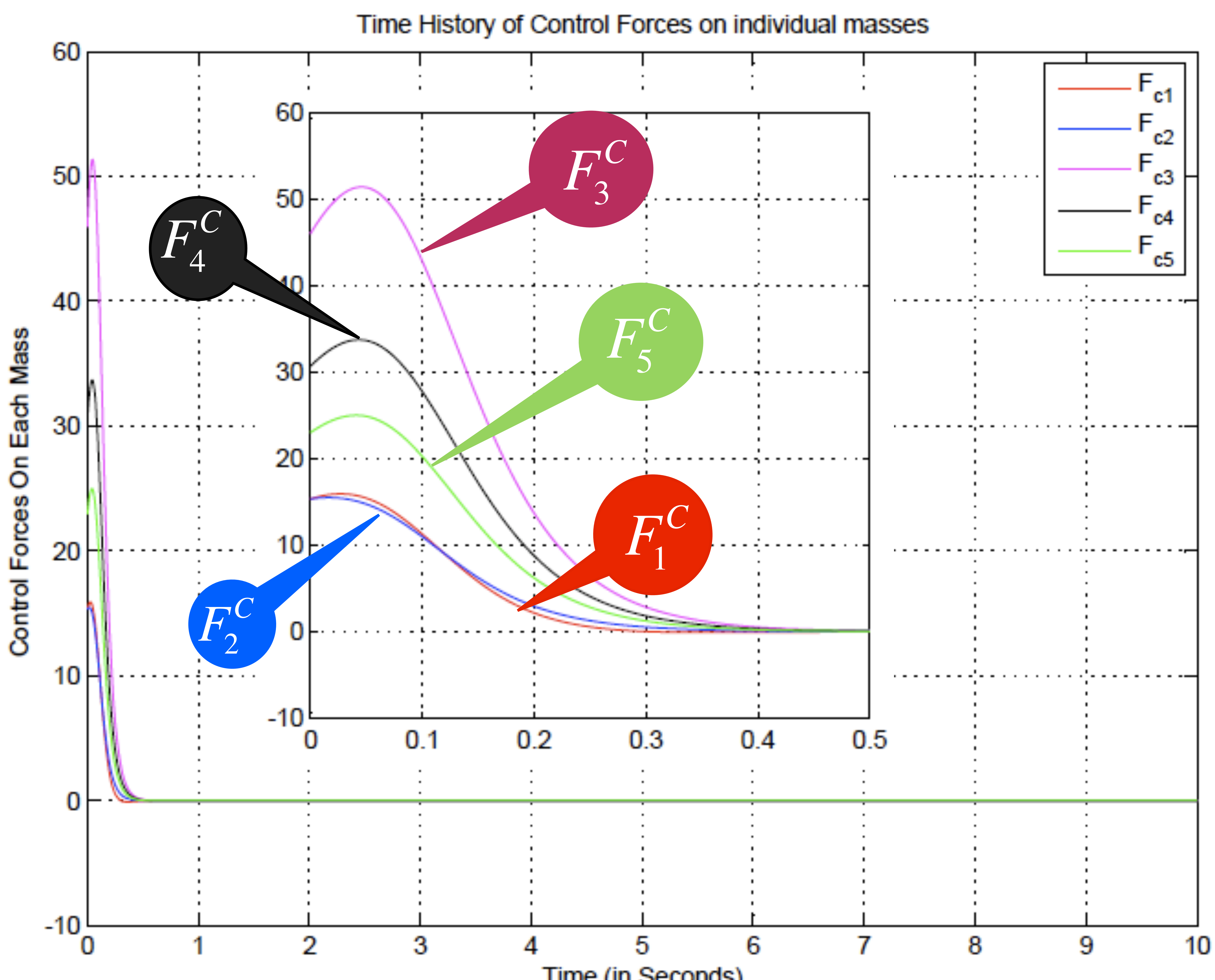
Control of Three Masses

Control is applied to the first mass (red), third mass (pink), and fifth mass (green)

No Control is applied to masses 2 and 4

Therefore, control force on these masses is zero for all time

Case 3 - Control Forces

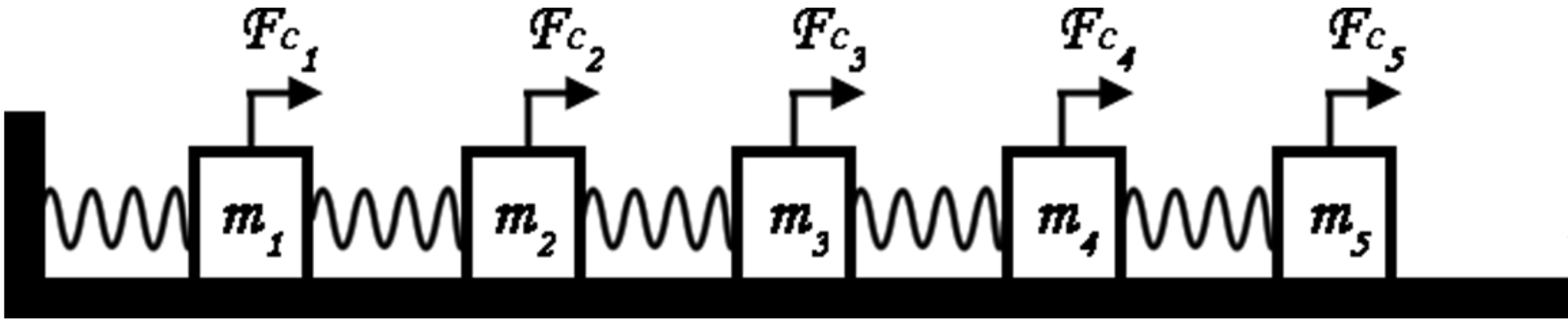


Control of All Five Masses

Control is applied to the

first mass (red),
second mass (blue),
third mass (pink),
fourth mass (black),
and fifth mass (green)

Five Mass Toda Chain w/ fixed - free ends



Control Applied to All Five Masses

$$q_0 = \dot{q}_0 = 0 \quad ; \quad a_5 = \alpha_5 = 0$$

$$t_{99.995} = 0.635$$

Conclusions

- Energy Control Problem in Toda Lattices has been approached from a new perspective
- Use of multiple control inputs to achieve energy stabilization for both fixed - fixed and fixed - free Toda Chains
- General methodology to obtain the control force in closed form for the energy stabilization of a finite dimensional non-homogenous Toda chain when 'k' out of 'n' masses are controlled
- Use of 'Invariance Principle' to prove that the closed form control force gives us global asymptotic convergence to the desired energy state
- Numerical simulations demonstrating perfect error convergence for the case of a five mass Toda chain w/ appropriate boundary conditions.

Thank You :)

Any Q's ?

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Completely Integrable

A Hamiltonian system defined by -

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}$$

$$\{\ \} - Poisson\ Brackets$$

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

is said to be completely integrable if and only if there exist precisely 'n' functionally independent integrals of motion $I_i(q,p);\ i = 1,2,\ldots,n$ such that -

$$\dot{I}_i = \{I_i, H\} = 0$$

$$\{I_i, I_j\} = 0$$

A system is said to be integrable if the first relation holds and a system is said to be completely integrable if both the relations hold. Due to existence of 'n' DOF and 'n' integrals, it is possible to perform a canonical transformation from Hamiltonian variables to Action-Angle variables. For a completely integrable system in action-angle variables, time evolution reduces to a linear flow on an n-dimensional torus parameterized by the angle variable.

KdV Equation

$$u_t(x,t) + 6\,u(x,t)\,u_x(x,t) + u_{xxx}(x,t) = 0$$

- Derived by Korteweg & deVries (1895) to describe weakly nonlinear shallow water waves
- Existence of solitary wave solutions - have behavior similar to the superposition principle although wave solutions are highly nonlinear
- Exhibits Galilean Invariance
- Lax showed 'isospectral integrability condition'

Solitary Waves

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Scott Russell spent some time making practical and theoretical investigations of solitary waves. He built wave tanks at his home and noticed some key properties:

- The waves are stable, and can travel over very large distances (normal waves would tend to either flatten out, or steepen and topple over)
- The speed depends on the size of the wave, and its width on the depth of water.
- Unlike normal waves they will never merge—so a small wave is overtaken by a large one, rather than the two combining.
- If a wave is too big for the depth of water, it splits into two, one big and one small.

Speed - Gradient Control

- Control of Oscillations in Lossless Nonlinear Systems (AS Shiriaev)

$$u = -\Psi\left[\nabla_u \dot{Q}(x)\right]$$

where Q is the goal function and $\Psi(z)^T z > 0 \;\; for \;\; z \neq 0$

- $L_g V$ type Speed Gradient Algorithm Control Law -

$$u_1 = -\phi\left(\left(H - H^*\right) p_1\right)$$

where ϕ is a smooth function which satisfies $\phi(0) = 0$ and $y \cdot \phi(y) > 0 \;\forall\; y \neq 0$

Invariance Principle

Let Ω be a compact set (Ω is a subset of D where $D \subset \Re^{n}$) that is positively invariant i.e. it has the property that every solution $x(t)$ of the system $\dot{x} = f(x)$ which starts in Ω remains in Ω for all future time. Let V : D → R be a continuously differentiable function such that $\dot{V} \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V} = 0$. Let M be the largest invariant set in E. Then, every solution $x(t)$ starting in Ω approaches M as $t \to \infty$. Here, M is the union of all invariant sets within E.

For some $c > 0$, the compact set Ω is defined as $\Omega = \left\{ x \in \Re^{n} \mid V \leq c \right\}$

Lemma - If a solution $x(t)$ of the system $\dot{x} = f(x)$ is bounded and belongs to D for all $t \geq 0$, then its positive limit set L^{+} is non-empty, compact and invariant. Moreover, $x(t)$ approaches L^{+} as $t \to \infty$

Positive Limit Set - A point p is said to be a positive limit point of $x(t)$ if there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the positive limit set of $x(t)$.

Positively Invariant Set - A set M is said to be a positively invariant set if $x(0) \in M \Rightarrow x(t) \in M$ for all $t \geq 0$.

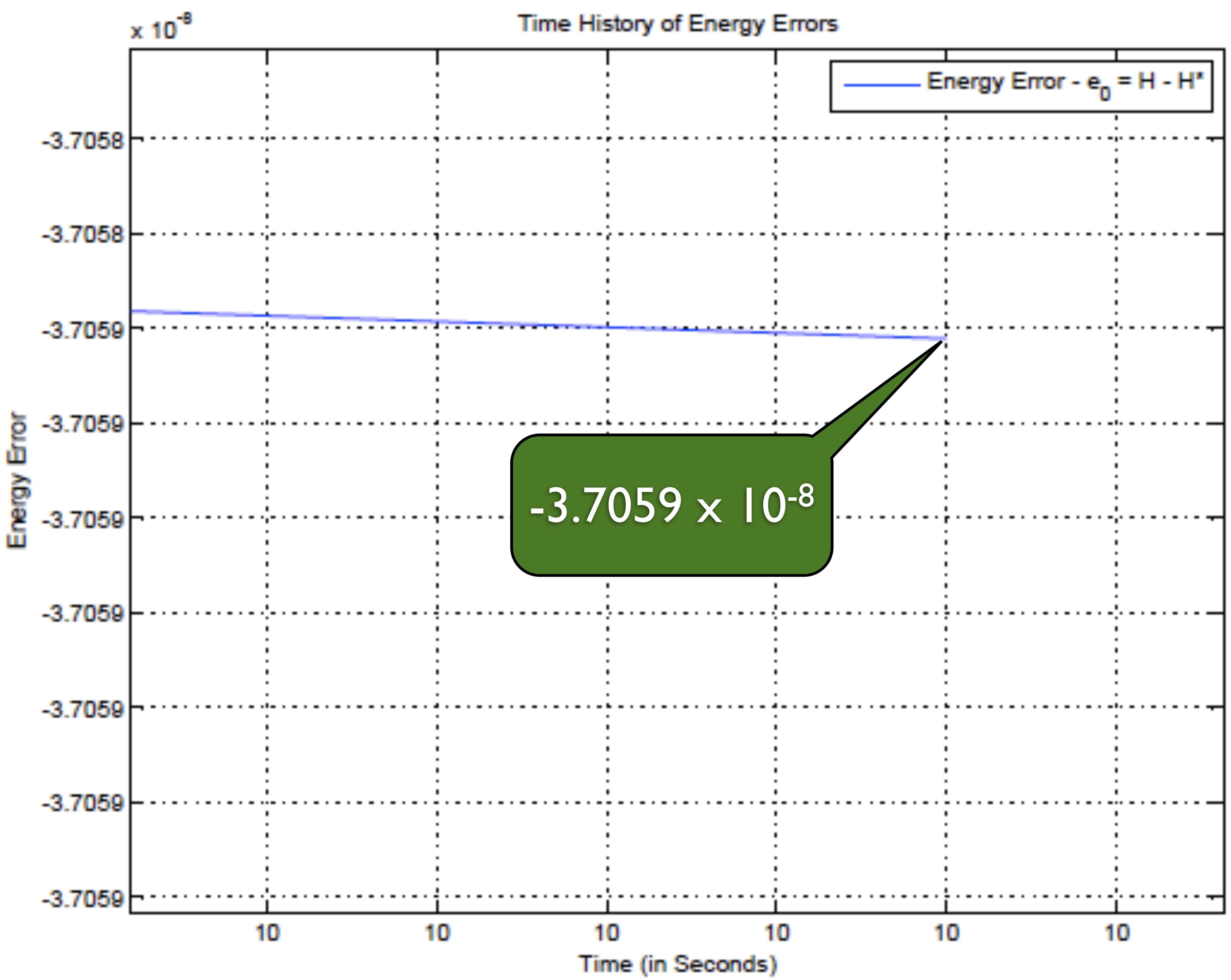
Error Tolerances

RelTol — This tolerance is a measure of the error relative to the size of each solution component. Roughly, it controls the number of correct digits in all solution components, except those smaller than thresholds AbsTol(i).

AbsTol — AbsTol(i) is a threshold below which the value of the ith solution component is unimportant. The absolute error tolerances determine the accuracy when the solution approaches zero.

If AbsTol is a vector, the length of AbsTol must be the same as the length of the solution vector y. If AbsTol is a scalar, the value applies to all components of y.

Case 1 - Energy Error



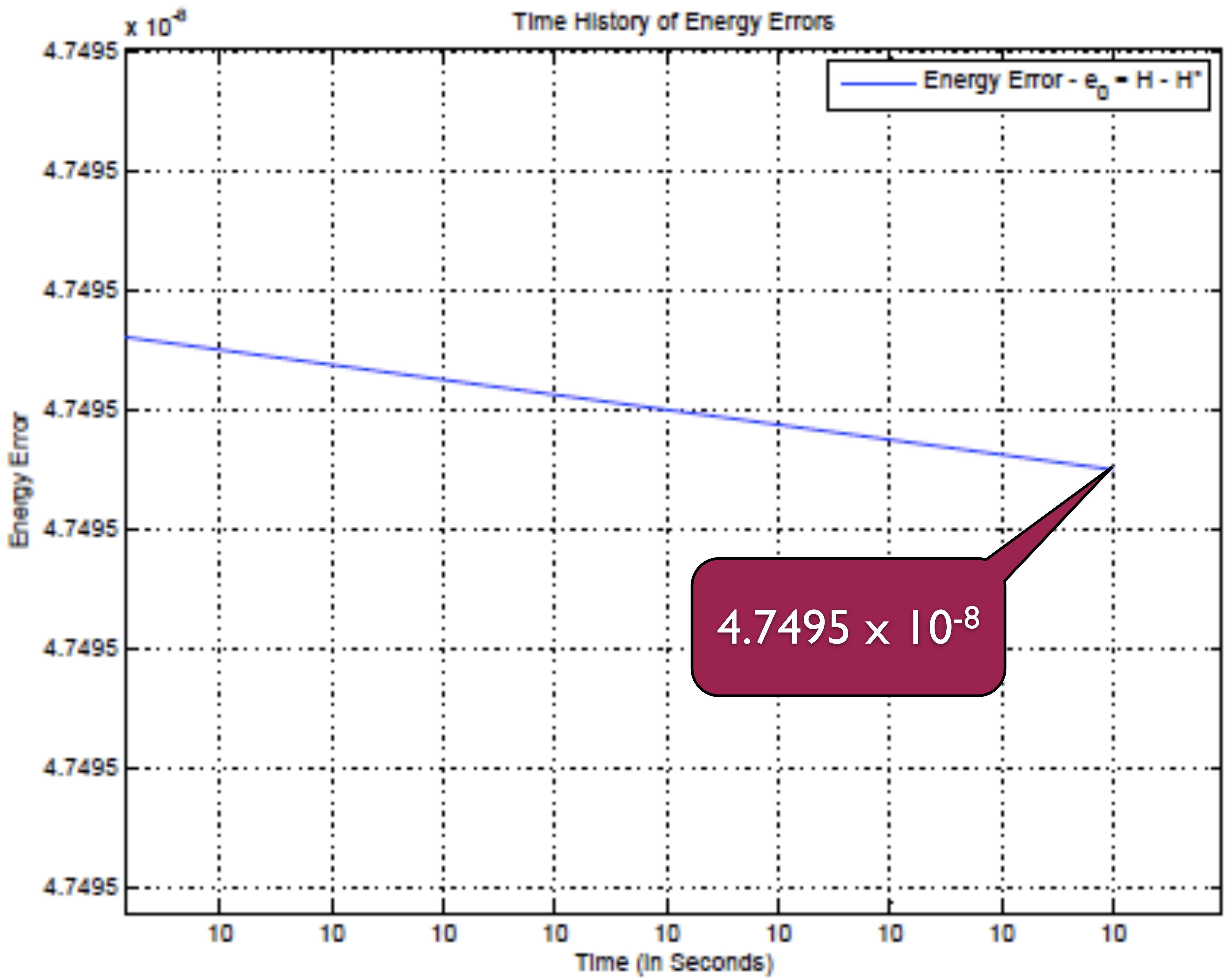
Control of One Mass

Energy Error

$$e = H - H^*$$

is of the order of 10^{-8}

Case 2 - Energy Error



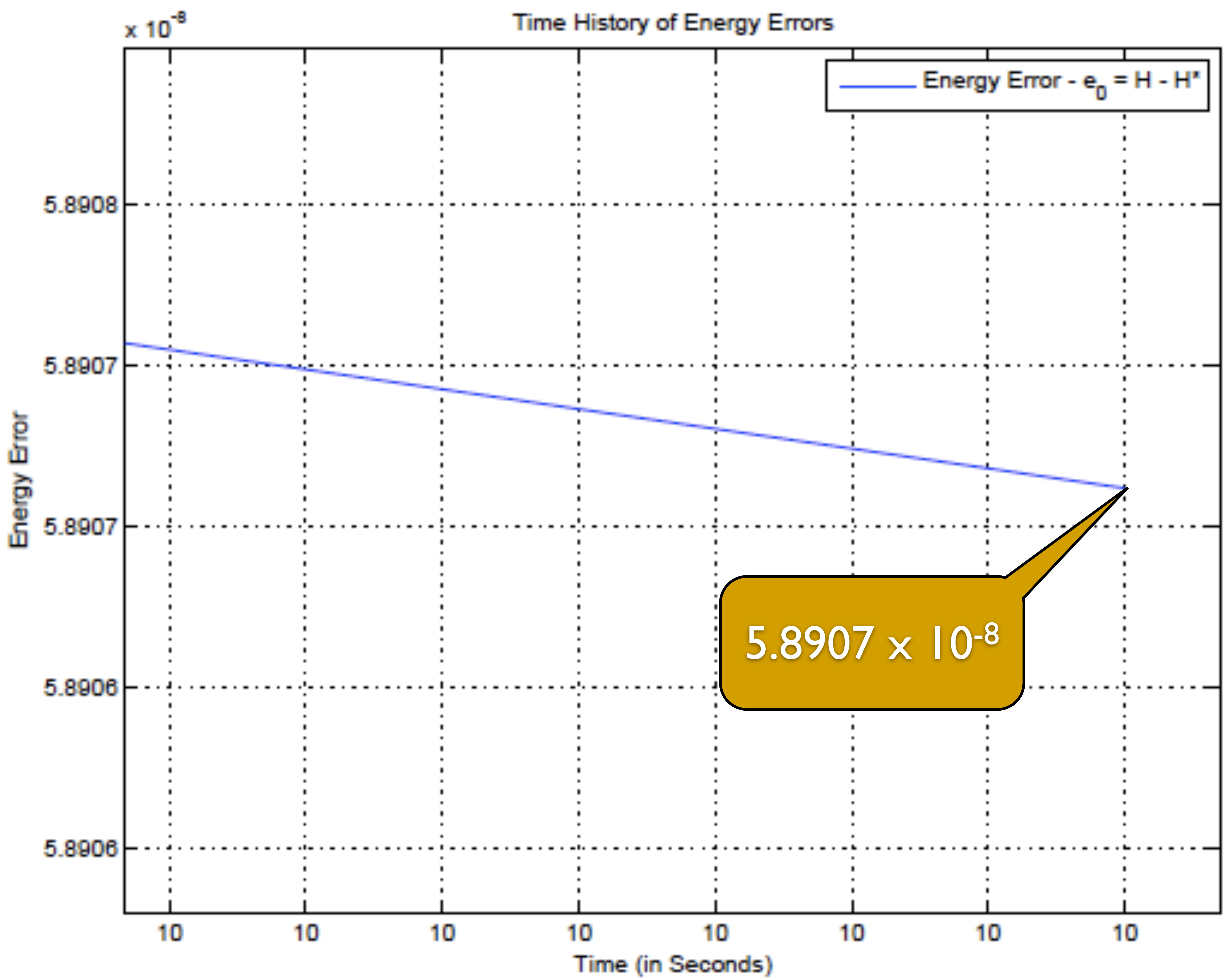
Control of Three Masses

$$Energy\ Error$$

$$e = H - H^*$$

$$is\ of\ the\ order\ of\ 10^{-8}$$

Case 3 - Energy Error



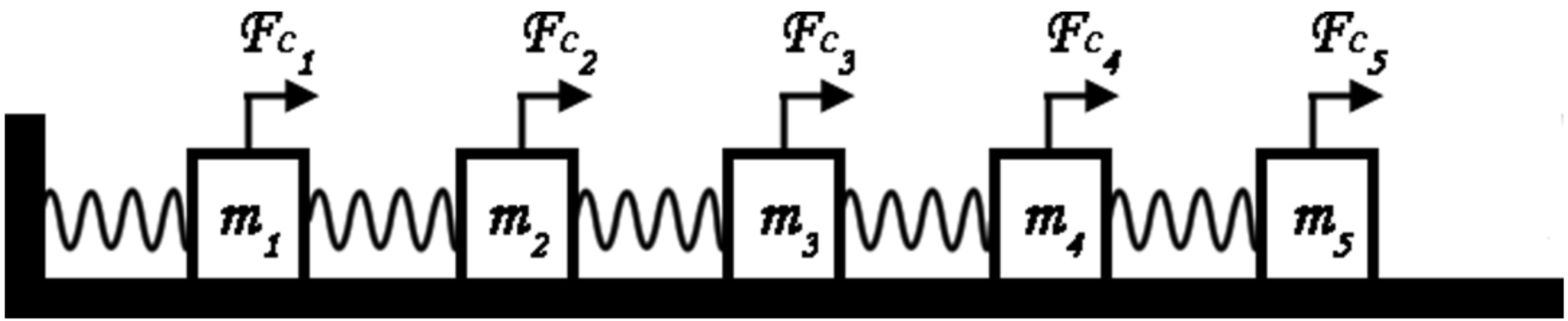
Control of All Five Masses

Energy Error

$$e = H - H^*$$

is of the order of 10^{-8}

Five Mass Toda Chain w/ fixed - free ends

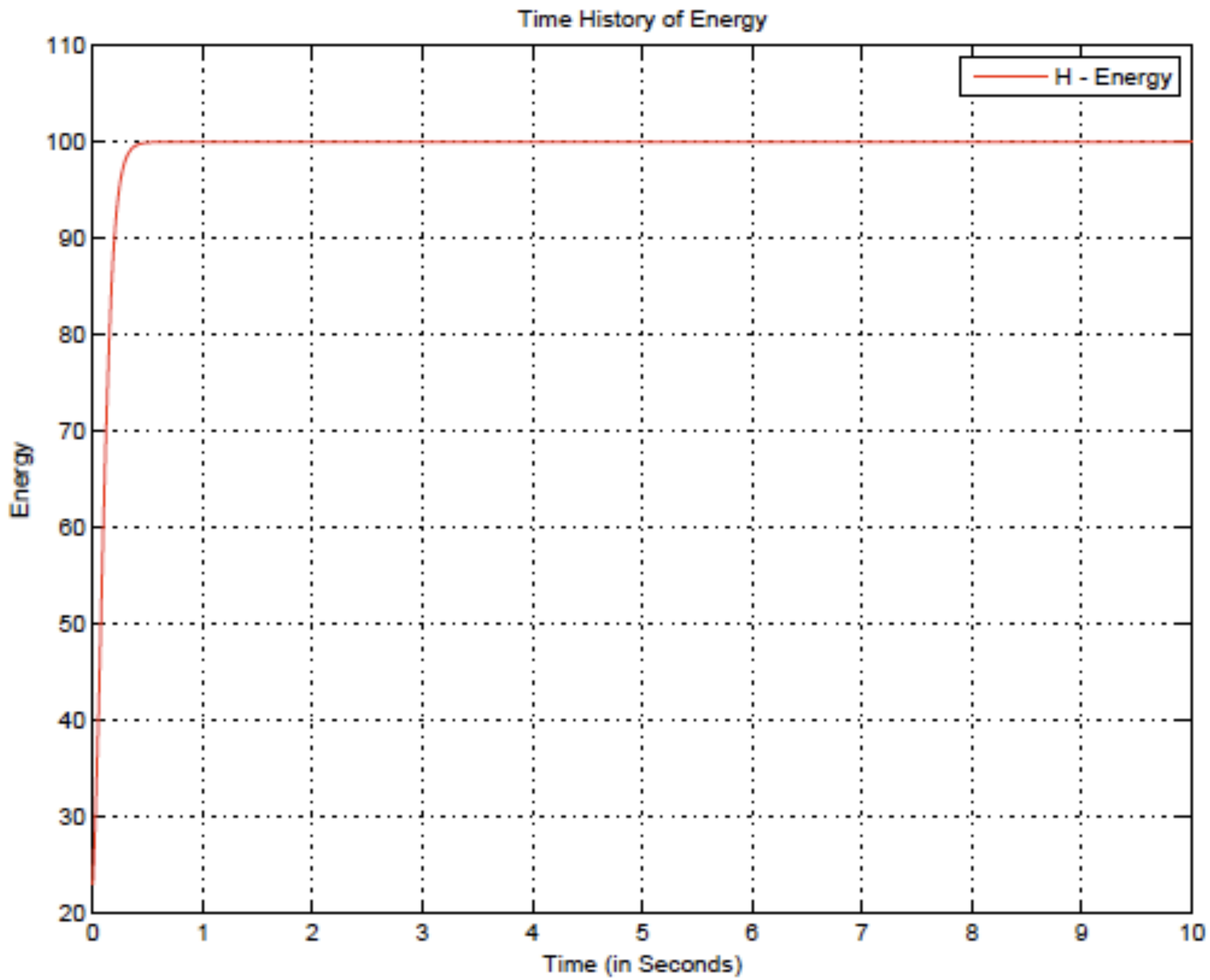


Control Applied to All Five Masses

$$q_0 = \dot{q}_0 = 0 \quad ; \quad a_5 = \alpha_5 = 0$$

$$t_{99.995} = 0.635$$

Time Taken For Convergence



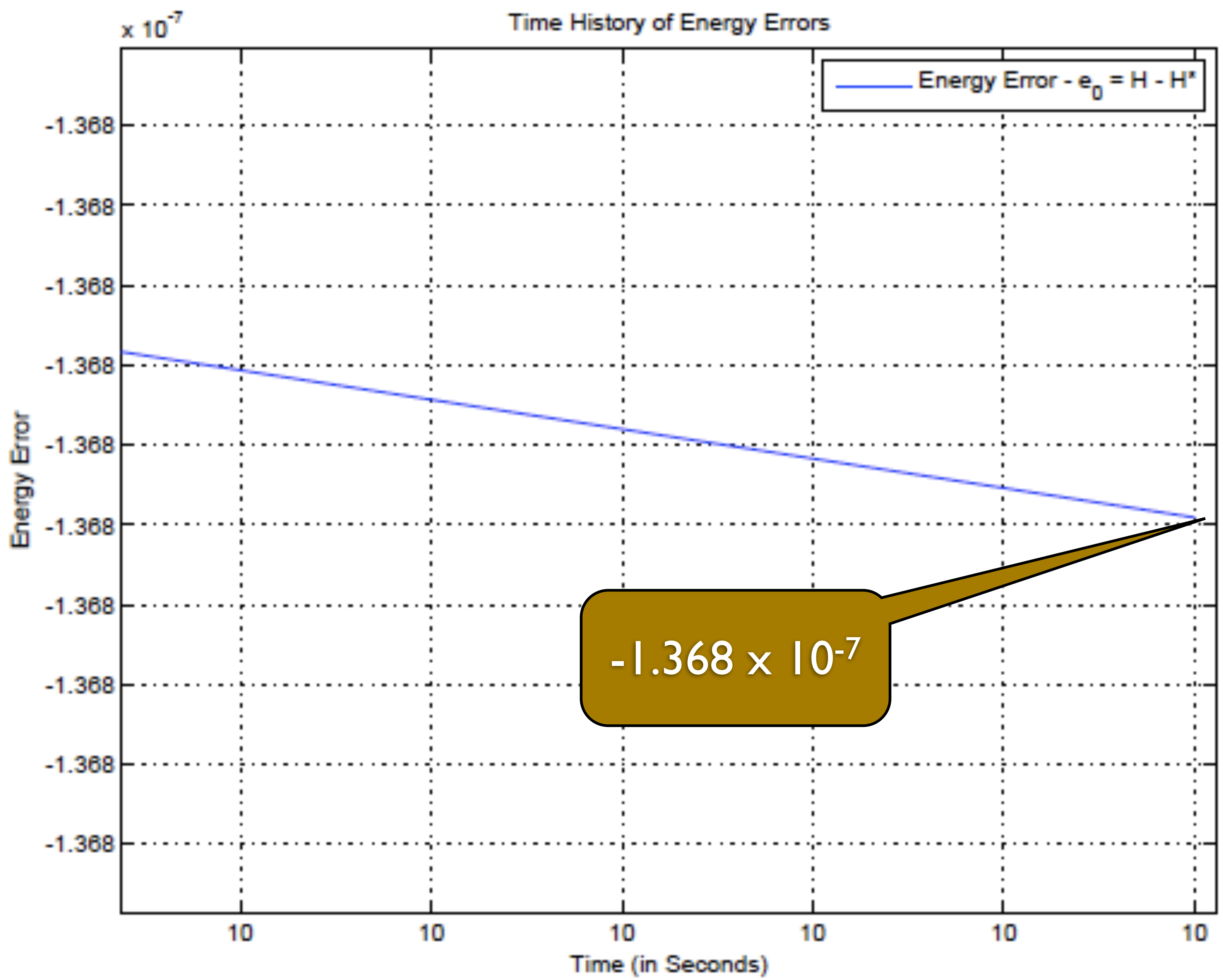
Control of
All Five Masses

$$t_{99.995} = 0.635$$

Lesser time
to converge
than the equivalent
Fixed - Fixed Toda Chain

Fixed - Free Boundary

Energy Error



Control of All Five Masses

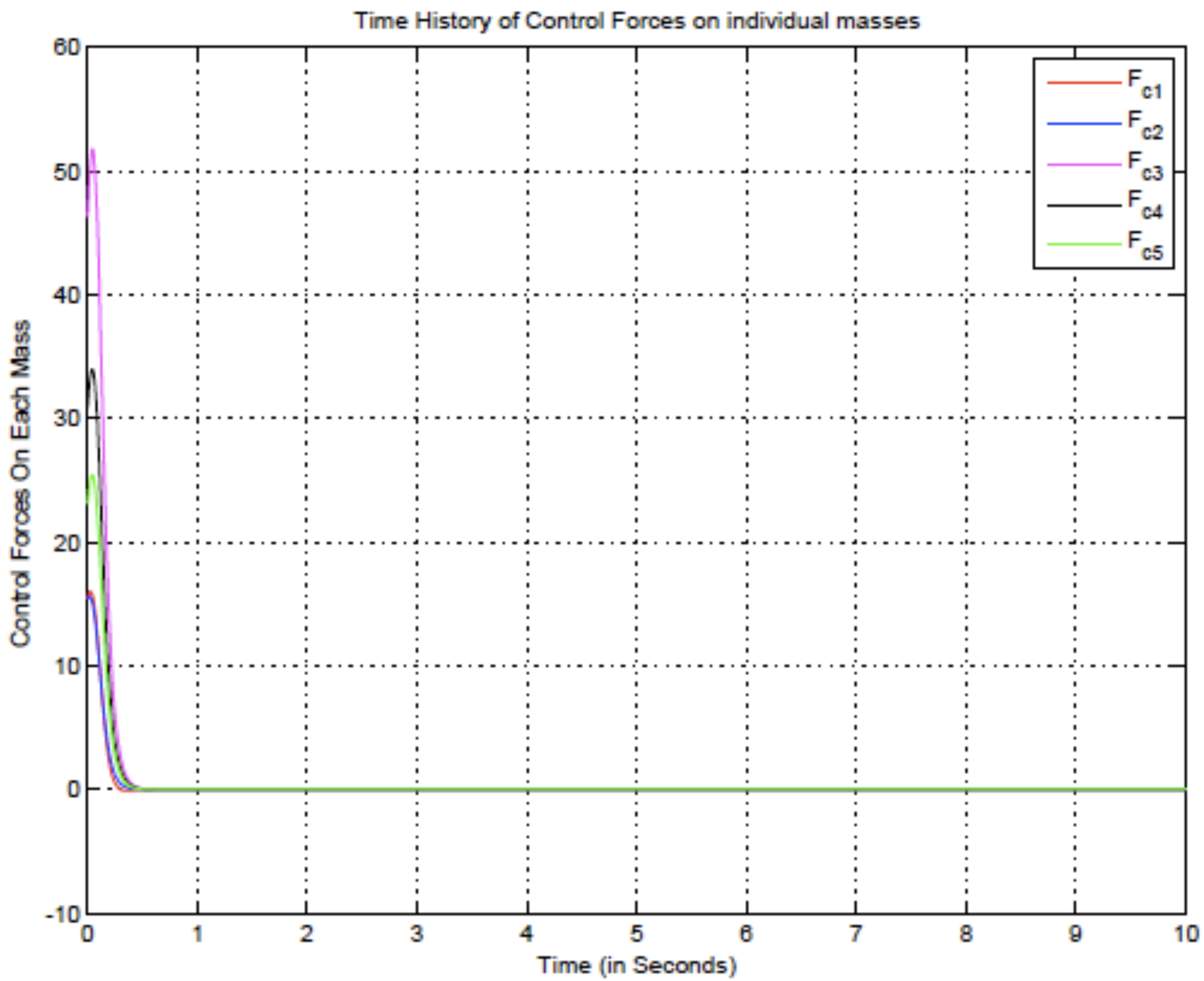
Energy Error

$$e = H - H^*$$

is of the order of 10^{-7}

Fixed - Free Boundary

Control Forces



Control of All Five Masses

Control
is applied to the

first mass (red),
second mass (blue),
third mass (pink),
fourth mass (black),
and fifth mass (green)

Fixed - Free Boundary

Energy Surface