Computer Assignment 3: Shock Tube Problem

(Strictly follow instructions for submission as given at the end of the document)

Introduction

The shock-tube problem is an interesting test case as the exact solution is known and any numerical solution can be compared with it for validation. The evaluation of the exact solution has been taught to you in AE 616 Gasdynamics and AE 236 Compressible Fluid Mechanics.

The initial conditions of the shock-tube problem is composed of two uniform states separated by a diaphragm which is usually located at the center as shown in Figure 1. This particular initial value problem is known as Riemann Problem. The initial left and right uniform states are usually introduced by providing the pressure, temperature and velocity. If the viscous effects are negligible along the tube walls and assume that the tube is infinitely long in order to avoid reflections at the tube ends, the exact solution of the Euler equations can be obtained on the basis of a simple wave analysis. When the diaphragm is ruptured, the discontinuity between the two initial states leads into generation of left and right moving waves, which are separated by a contact surface.The resulting wave pattern is composed of a shock wave moving to the right followed by a contact discontinuity, and a rarefaction wave moving to the left whenever initial left state has higher pressure than the initial right state with temperature, velocity being same at the initial state as shown in the Figure 1.

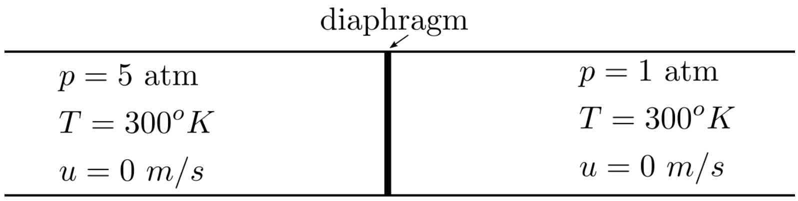

Figure 1: Initial conditions for shock tube problem

Problem Statement

Given the initial conditions as shown in Figure 1, find out the variations of normalized numerical values of pressure, temperature, velocity and Mach number inside the shock tube at 0.75×10^{-3} seconds after the diaphargm is ruptured. Apply Steger and Warming Flux Vector Splitting scheme[1] to obtain the numerical solution. The normalization is to be done based on the maximum numerical value of the flow variable in the computational domain.

Domain Specification and Discretization

Consider N=101 equally spaced grid points in the domain $x = [0, 1]$. You may consider, one additional grid (ghost) points on either sides for applying boundary conditions. Thus, i=0 and i=N+1 will be the ghost points in the left and right boundaries respectively. Thus, you need to consider total 103 grid points. Typical grids to be used is shown in the Figure 2

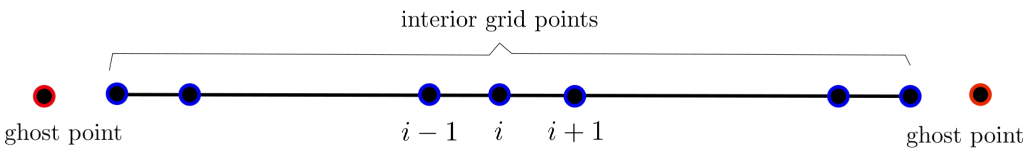

Figure 2: Typical equispaced grid

Numerical Scheme

Steger and Warming Flux Vector Splitting (FVS) method[1] to be used for solving 1D Euler equations to get the solution of shock tube problem is described below.

Governing Equation

Euler equations in conservative form is given by

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{1}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}$$

These equations can be written in a split flux form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^+}{\partial x} + \frac{\partial \mathbf{F}^-}{\partial x} = 0$$

Applying first-order upwind scheme, we get

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F^+}_i^n - \mathbf{F^+}_{i-1}^n\right) - \frac{\Delta t}{\Delta x}\left(\mathbf{F^-}_{i+1}^n - \mathbf{F^-}_i^n\right) \tag{2}$$

Where $\mathbf{F}^+$ and $\mathbf{F}^-$ are determined based on Steger and Warming FVS splitting strategy[1, 2], for which split fluxes are given below. Note, you need to consider all the possible ranges of Mach number in your computer code as specified below.

1. For $M < -1$,

$$\mathbf{F}^+ = 0 \quad \text{and} \quad \mathbf{F}^- = \mathbf{F}$$

2. For $-1 < M \leq 0$,

$$\mathbf{F}^+ = \frac{\rho}{2\gamma}(u+a)\begin{bmatrix} 1 \\ u+a \\ \frac{u^2}{2} + \frac{a^2}{\gamma-1} + au \end{bmatrix}$$

$$\mathbf{F}^- = \frac{\gamma-1}{\gamma}\rho u \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} + \frac{\rho}{2\gamma}(u-a)\begin{bmatrix} 1 \\ u-a \\ \frac{u^2}{2} + \frac{a^2}{\gamma-1} - au \end{bmatrix}$$

3. For $0 < M \leq 1$,

$$\mathbf{F}^{+} = \frac{\gamma - 1}{\gamma}\rho u \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} + \frac{\rho}{2\gamma}(u+a) \begin{bmatrix} 1 \\ u+a \\ \frac{u^2}{2} + \frac{a^2}{\gamma-1} + au \end{bmatrix}$$

$$\mathbf{F}^{-} = \frac{\rho}{2\gamma}(u-a) \begin{bmatrix} 1 \\ u-a \\ \frac{u^2}{2} + \frac{a^2}{\gamma-1} - au \end{bmatrix}$$

4. For $M < -1$,

$$\mathbf{F}^{+} = \mathbf{F} \quad \text{and} \quad \mathbf{F}^{-} = 0$$

Where $a = \sqrt{\gamma RT}$ is the speed of sound in the medium. The ratio of specific heats, $\gamma = 1.4$ for air.

Time Step from Stability Condition

The time step Δt for time-marching is calculated as

$$\Delta t = \nu \frac{\Delta x}{\lambda_{max}}$$

where

$$\lambda_{max} = max\left\{(|u[1]| + a[1]), (|u[2]| + a[2]), \cdots, (|u[N]| + a[N])\right\}$$

and ν is CFL number. For stability $\nu \leq 1$.

Initial Condition

Temperature and velocity at all grid points are set to 300^{o}K and zero respectively. Whereas, pressure is set to 5 atm at the points located in the range $0 \leq x < 0.5$ and 1 atm in the range $0.5 \leq x \leq 1$, because the diaphragm is located at x=0.5. Above mentioned conditions are explained in Figure 1.

Boundary Conditions

Since the waves will not reach the boundaries in the time 0.75×10^{-3} seconds, following boundary conditions can be specified.

At Left Boundary

$$
\begin{aligned}
p[0] &= p[1] \\
T[0] &= T[1] \\
u[0] &= p[1]
\end{aligned}
$$

At Right Boundary

$$
\begin{aligned}
p[N+1] &= p[N] \\
T[N+1] &= T[N] \\
u[N+1] &= p[N]
\end{aligned}
$$

Results to be Obtained

1. Plot all the following quantities at time $t_{end} = 0.75 \times 10^{-3}$ unit in a single graph

   (a) p/p_{max}

   (b) T/T_{max}

   (c) u/u_{max}

   (d) M/M_{max}

   along the shock tube.

2. Those who have credited AE 706/236 should use the exact solution for validation of the numerical results (optional).

Report

Apart from the C-code, output files, you need to submit a brief report (in pdf format) presenting the results and comments.

References

[1] Steger J. and Warming R., *Flux vector splitting of the inviscid gas dynamics equation with applications to the finite difference methods*, Joutnal of Computational Physics, vol 40, pp 263-293, 1981.

[2] Charles Hirsch, *Numerical Computation of Internal and External Flows*, Volume II, Wiley, 1990.

General Instructions for Submission

- **Checklist for submission:**
  - The code (written in C only) with proper inline documentation for each function. (Use meaningful variable names).
  - A "README.txt" file which contains the proper description on how to run the code and get the plots.
  - The plots submitted by you must be reproducible independently by the TA's from your code.
- **Instruction for submission:**
  - Rename your program file as your roll number (example: 184010006.c).
  - Zip the folder (*.zip) which contain the code and plots (See the checklist) and name it as your roll number (example: 194010006.zip).
- **Notes:**
  - Marks will be given only if the program is working and showing correct result. No step marks will be given.
  - The assignment will be evaluated only if the code is written in C language. Otherwise, zero marks will be awarded.

  - Assignment will not be evaluated if "instruction for submission" are not followed properly.

- **Copying program from each other will lead to severe penalty.**

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