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$$W_4 = \left\{ y : 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = 7 \right\} \subseteq V$$

— Not subspace

Linear span :-

Let V(F) is a vector space and S is a non-empty set of V. Then the linear span of S is the set of all linear combination of finite sets of elements of S such that

$$L(S) = \{ a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n : \alpha_1, \alpha_2, \cdots \alpha_n \in R \}$$

Q. Which of the following are V-spaces

~~$S = \{(1,1), (1,2)\} \subseteq R^2$~~

$\times$	✓	✓	✓	$\times$
R(C) (scalar multipli...)	C(R)	R(R)	R(Q)	Q(R) (scalar multiplication)

Q.

$$S = \{(1,1), (1,2)\} \subseteq R^2$$

let $(x,y) \in R^2$

$a(1,1) + b(1,2) = (x,y)$

$(a+b, a+2b) = (x,y)$

$$\left.\begin{array}{l} a+b = x \\ a+2b = y \end{array}\right\} \quad \left\{ \left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 2 & y \end{array}\right] \right\}$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y-x \end{array}\right]$$

Apply back-subs...

$b = y - x$

$a = x + x - y = 2x - y$

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Remark 1: The set $S = \{0\}$ is linearly dependent in any vector space.

Proof: $c_1 \cdot 0 = 0$, for any value of c_1 other than 0.

Remark 2: The set of two vectors $\{\alpha_1, \alpha_2\}$ is L.D. when one of them is a scalar multiple of the other we say that α_1 and α_2 are collinear

$$\Rightarrow G(x,y) = (x-y)(1,1) + (y-x)(1,2)$$

Linearly Dependent :- (L.D.)

Let $V(F)$ be a vector space over F and $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq V$ then ther elements of V are said to be linearly dependent if $\exists$ some scalars $a_1, a_2, \cdots, a_n \in F$ which all are not zero $\ni$

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$$

$$a_1\alpha_1 + a_2\alpha_2 = 0$$

$$a_1(1,1) + a_2(1,2) = 0$$

$$a_1 + a_2 = 0 \qquad \left\{ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \right\}$$

$$a_1 + 2a_2 = 0$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow a_1 = 0 \ \& \ a_2 = 0$$

$\Rightarrow \alpha_1 \ \& \ \alpha_2$ are linearly independent

Basis:

Let V be a vector space over F and S is a non-empty subset of V. S is said to be a basis of V if

Theorem: Let $S = \{\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n\}$, $n \geq 2$, be a set of vectors in V-space V. Then S is linearly dependent if one of the vector in S can be expressed as a linear combination of the rest.

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① Elements of S are linearly independent

② S span V i.e. $L(S) = V$

eg: $S = \{(1,1), (1,2)\}$ is a basis of R^2

$S_1 = \{(1,0), (0,1)\}$

$S_1 = \{(1,0), (0,1)\}$

$S_2 = \{(1,1)\}$

$S_3 = \{(1,0), (0,1), (1,1)\}$

for S_1: $a_1(1,0) + a_2(0,1) = 0$

$a_1 + 0 = 0$

$0 + a_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1-\cancel{k} & 0 \end{array}\right]$$

~~k~~ ~~$(1-k)a_2 = 0$~~

$\therefore a_2 = 0 \Rightarrow a_1 = 0$

$\therefore$ linearly independent

for S_2: $a_1(1,1) = 0$

$a_1 = 0$ $\therefore$ L.I.

for S_3: $a_1(1,0) + a_2(0,1) + a_3(1,1) = 0$

$\therefore a_1 + a_2 \cdot 0 + a_3 = 0$

$a_1 \cdot 0 + a_2 \cdot 1 + a_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$\therefore a_2 = -a_3$, $a_1 = -a_3$

$\Rightarrow$ if $a_3 = 0 \Rightarrow$ L.I.

else $a_3 \neq 0 \Rightarrow$ L.D.

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Theorem 1> Any subset of L.I. set is also L.I.

2> " L.D. " " L.D.

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eg: <u>Vector space</u> : $V(F)$

$$V = \{ (a, b, c) : a, b, c \in \mathbb{R} \}$$

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

$$k(a_1, b_1, c_1) = (ka_1, kb_1, kc_1)$$

* <u>Subspace</u> : $W \subseteq V(F)$

$$W = \{ (a, a, a), a \in R \}$$

$\alpha = (a, a, a)$; $\beta = (b, b, b)$; $k \in R$

$k(a, a, a) + (b, b, b) = (ka+b, ka+b, ka+b)$

* <u>Linear dependence</u> & <u>independence</u>:

$$a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n = 0$$

$S_1 = \{ (1, 1, 1) \}$ — L.I. $[1, 1, 1]$

$S_2 = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ we have no zero row, this collection is LI.

Remark: The independent set of vectors spans the vector space uniquely

Remark: For dependent set of vectors, a vector in the spanning subspace can be expressed in many different ways

Remark: If it is desirable, that a vector can be expressed, as a linear combⁿ of given vectors uniquely, then the vectors in the set must be linearly independent.

Linear span (Linear combination)

$$L(S) = \{ a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n \}$$

$$\alpha_1, \alpha_2, \cdots \alpha_n \in S$$

$$a_1, a_2, \cdots, a_n \in F$$

$$\alpha = \underbrace{a_1 e_1 + a_2 e_2 + a_3 e_3}_{\text{coordinates}}$$

$$(x, y, z) = a_1 (1,0,0) + a_2 (0,1,0) + a_3 (0,0,1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right] \qquad \begin{array}{l} a_1 = x \\ a_2 = y \\ a_3 = z \end{array}$$

S_2 span V

* Coordinates :

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ \| & \| & \| \\ x & y & z \end{array}$$

(x, y, z) - coordinates

* Basis :- $S \subseteq V$

S_1 is basis for W

S_2 is a basis for V

Dimension:

No. of vectors in V

$\dim W = 1$

$\dim V = 3$

If the vectors have equal dimension & their intersection is non-zero then usually the vectors are equal vectors.

Rank of the matrix $\Rightarrow$ no. of non-zero rows in row echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & 10 & 1 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & \boxed{7} & 0 \end{bmatrix} \quad R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow L.D.$$

Rank = 2

$\boxed{R_3 = 3R_1 - R_2}$ $\rightarrow$ linear combination

NOTE: $\rightarrow$ For checking L.I., we can check determinant also:

$\Delta \neq 0$	for L.I.
$\Delta = 0$	for L.D.

$\rightarrow$ Dimension is unique but basis need not be unique at least 1 row zero $\Rightarrow$ dependent

$S_4 = \{(1,2,3), (2,0,1), (3,1,2)\}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$a_1(1,2,3) + a_2(2,0,1) + a_3(3,1,2) = 0$

$a_1 + 2a_2 + 3a_3 = 0$

$2a_1 + a_3 = 0$

$3a_1 + a_2 + 2a_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -5 & 0 \\ 0 & \boxed{-5} & -7 & 0 \end{array}\right] \quad R_3 - \frac{5}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -5 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 \end{bmatrix}$$

$-7 + \frac{25}{4}$

Rank $= 3$

Also $\Rightarrow$ Linearly independent

* Linear span

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 0 & -4 & -5 & y-2x \\ 0 & 0 & -3/4 & \frac{4z-3x-5y}{4} \end{array}\right]$$

$c = \frac{-1}{3}\left(\frac{4z-3x-5y}{1}\right)$

Illly $b = ?$

$c = ?$

eg: $S_5 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$e_1 = [1\ 0\ 0\ 0]$
$e_2 = [0\ 1\ 0\ 0]$
$e_3 = [0\ 0\ 1\ 0]$
$e_4 = [0\ 0\ 0\ 1]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L.I.$$

To check linear span:

$$\begin{bmatrix} x & y \\ w & z \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

** eg: Show that the set $S = \{(1,1), (2,3)\}$ is a basis for R^2.

Hint: To show this, we have to prove the following two:

(i) S is independent

(ii) S spans V.

Sol^n: i) $a_1(1,1) + a_2(2,3) = 0$

$\therefore\ a_1 + 2a_2 = 0$

$a_1 + 3a_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow a_2 = 0 \Rightarrow a_1 = 0 \quad \therefore$ L.I.

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Theorem: Any set containing null vector is L.D.

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(ii) $(x,y) = a_1(1,1) + a_2(2,3)$, where $(x,y) \in V$

$\therefore$ $a_1 + 2a_2 = x$

$a_1 + 3a_2 = y$

$$\left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & 3 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & y-x \end{array}\right]$$

$\Rightarrow a_2 = y - x \Rightarrow a_1 = x - 2a_2$

$= 3x - 2y$ (unique sol^n)

$\therefore$ S spans V.

<u>NOTE:</u> We define the zero vector space to have dimension <u>zero</u>.

Inner Product Space :-

Let F be the field of real no.s or that of complex no.s and V be a vector space over F. An Inner Product on V is a function, which assigns to each ~~other~~ order pair of vectors α, β in V a scalar $\langle \alpha, \beta \rangle$ in F in such a way that for $\alpha, \beta, \gamma \in V$ & all scalar $c \in F$.

1) $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$
2) $\langle c\alpha, \beta \rangle = c \langle \alpha, \beta \rangle$

$\Big\}$ $\langle a\alpha + \beta, \gamma \rangle = a \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

Linearity Property

3) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ $\rightarrow$ conjugate symmetry

4) $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ if and only if $\alpha = 0$ - Non-negativity Property

eg: 1) On $V_n(C)$ there is an inner product, which we call the standard inner product.
if $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$
$\gamma = (c_1, c_2, \cdots, c_n)$ $\in V_n(C)$
then we define
$\langle \alpha, \beta \rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots a_n\bar{b}_n$

Linearity :- $\alpha, \beta, \gamma \in V_n(C)$
then $a\alpha + \beta = (aa_1 + b_1, aa_2 + b_2, \cdots, aa_n + b_n)$

$\langle a\alpha + \beta, \gamma \rangle = \langle (aa_1 + b_1, aa_2 + b_2, \cdots, aa_n + b_n), (c_1, c_2, \cdots, c_n) \rangle$
$= (aa_1 + b_1)\bar{c}_1 + (aa_2 + b_2)\bar{c}_2 + \cdots + (aa_n + b_n)\bar{c}_n$

$= (aa_1\bar{c}_1 + aa_2\bar{c}_2 + \cdots + aa_n\bar{c}_n) + (b_1\bar{c}_1 + b_2\bar{c}_2 + \cdots + b_n\bar{c}_n)$

$= a(a_1\bar{c}_1 + \cdots + a_n\bar{c}_n) + (b_1\bar{c}_1 + \cdots + b_n\bar{c}_n)$

$= a\langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle$

<u>Conjugate symmetry:</u>

$\langle\beta,\alpha\rangle = b_1\bar{a}_1 + b_2\bar{a}_2 + \cdots + b_n\bar{a}_n$

$\overline{\langle\beta,\alpha\rangle} = \overline{(b_1\bar{a}_1)} + \overline{(b_2\bar{a}_2)} + \cdots + \overline{(b_n\bar{a}_n)}$

$= \bar{b}_1 a_1 + \bar{b}_2 a_2 + \cdots + \bar{b}_n a_n$

$= a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$

$= \langle\alpha,\beta\rangle$

<u>Non-Negativity Property:-</u>

$\langle\alpha,\alpha\rangle = a_1\bar{a}_1 + a_2\bar{a}_2 + \cdots + a_n\bar{a}_n$

$(|z|^2 = z\bar{z})$ $= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 > 0$

if $\langle\alpha,\alpha\rangle = 0$

then $|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 = 0$

only when $a_1 = a_2 = \cdots = a_n = 0$

$\therefore \alpha = 0$

* Euclidean space (R) & Unitary space (C)
(over real no's) (over complex no's)

eg: $V(R)$, $\alpha = (a_1, a_2) \in V$, $\beta = (b_1, b_2) \in V$, $\gamma = (c_1, c_2) \in V$

$\langle\alpha,\beta\rangle = a_1b_1 + 4a_2b_2$

Linearity ✓, Symmetry ✓

Inner Product Space

V(F) is a vector space
then A function $\langle , \rangle$ is said to be inner product of

1) $\langle a\alpha+\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$, $\alpha, \beta, \gamma \in V$
$a \in F$

2) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ where ' $\overline{}$ ' is complex conjugate

3) $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ if $\alpha = 0$

Positive Definite Property:

$V \times V \rightarrow F$

eg: V(R) be a vector space of all continuous real valued function on the interval [0,1] if $f, g \in V$ defined $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$

Show that it is an inner product on V.

① Linearity:

$\left.\begin{array}{l} f(t) \in V \\ g(t) \in V \\ h(t) \in V \end{array}\right\} a \in F$

$\langle af(t) + g(t), h(t) \rangle = \int_0^1 (a f(t) + g(t))\, h(t)\, dt$

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$$= \int_0^1 [a\,f(t)\,h(t) + g(t)\,h(t)]\,dt$$

$$= \int_0^1 a\,f(t)\,h(t)\,dt + \int_0^1 g(t)\cdot h(t)\,dt$$

$$= a\int_0^1 f(t)\cdot h(t)\,dt + \int_0^1 g(t)\,h(t)\,dt = a\langle f,h\rangle + \langle g,h\rangle$$

② $\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt$

$$= \int_0^1 g(t)\,f(t)\,dt$$

$$= \langle g,f\rangle \quad \text{(symmetry)}$$

③ $\langle f,f\rangle = \int_0^1 [f(t)^2]\,dt \geq 0$

eg: $f(t) = \cancel{t+}\ t-1$

$g(t) = t+2$

$$\langle f,g\rangle = \int_0^1 (t-1)(t+2)\,dt$$

$$= \frac{1}{3} + \frac{1}{2} - 2 = \frac{5-12}{6} = \cancel{\frac{-11}{6}}\ \frac{-7}{6}$$ //

<u>Norm of a vector</u> :-

Let V be an inner product space. If $\alpha \in V$ then norm ~~of~~ (or length of vector) is defined as $\|\alpha\| = \sqrt{\langle \alpha,\alpha\rangle}$

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eg: $\|f(f)\| = \frac{1}{\sqrt{3}}$ where $f = (t-1)$

Normalization of vector:

$$\hat{\alpha} = \frac{1}{\|\alpha\|}\alpha$$

$$\hat{f} = \frac{1}{\|f\|} f = \frac{\sqrt{3}}{\cancel{\sqrt{3}}} f - \frac{\sqrt{3}}{\cancel{\sqrt{3}}} \qquad \|\hat{f}\| = \sqrt{\langle \hat{f}, \hat{f}\rangle}$$

Q.2 $V_2(R) = R^2 = \{(a_1, a_2),\ a_1, a_2 \in R\}$

$\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$

I) $\langle \alpha, \beta \rangle = a_1^2 + a_1 b_1 + a_2 b_2 + b_1^2$ (Not linear $\Rightarrow$ Not Inner prod)

II) $\langle \alpha, \beta \rangle = a_1 a_2 + 2b_1 b_2$ (Not symmetric $\Rightarrow$ Not I.P.)

III) $\langle \alpha, \beta \rangle = a_1 b_1 + 2a_2 b_2 + 5$ (Not linear $\Rightarrow$ Not I.P.)

IV) $\langle \alpha, \beta \rangle = 2a_1 b_1 + 3a_2 b_2$
(linear, symmetric $\Rightarrow$ I.P.)

(V) $\langle \alpha, \beta \rangle = 2a_1 b_1 - 3a_2 b_2$
(Linear, Symmetric)

sol I) $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$, $\gamma = (c_1, c_2)$

$a\alpha + \beta = a(a_1, a_2) + (b_1, b_2) = (aa_1 + b_1, aa_2 + b_2)$

$\therefore \quad \langle a\alpha+\beta, \gamma\rangle = \langle (aa_1+b_1, aa_2+b_2), (c_1, c_2)\rangle$

$$= (aa_1+b_1)^2 + (aa_1+b_1)c_1 + (aa_2+b_2)c_2 + c_1^2$$

$$= a^2a_1^2 + b_1^2 + 2aa_1b_1 + aa_1c_1 + b_1c_1 + aa_2c_2 + b_2c_2 + c_1^2 \quad -① $$

RHS: $\quad a\langle\alpha,\gamma\rangle = a(a_1^2 + a_1c_1 + a_2c_2 + c_1^2) \quad -②$

$$\langle\beta,\gamma\rangle = b_1^2 + b_1c_1 + b_2c_2 + c_1^2 \quad -③$$

① $\neq$ ② + ③ Hence not linear. Not inner pro. ✓

Sol' II) $\langle\alpha,\beta\rangle = a_1a_2 + 2b_1b_2$

$\langle\beta,\alpha\rangle = b_1b_2 + 2a_1a_2$

$\because$ not symmetric. Hence not inner product. ✓

Sol III) $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$, $\gamma = (c_1, c_2)$

NOTE: Generally $\langle \alpha, \beta \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

eg $\underset{\alpha}{(1,0)}, \underset{\beta}{(0,1)} \in \mathbb{R}^2$

$$\langle \alpha, \beta \rangle = \langle (1,0), (0,1) \rangle$$
$$= 0 \quad ✓$$

Orthogonal Vector : let $V(F)$ be inner product space and $\alpha, \beta \in V$, then α and β are said to be orthogonal if

$$\langle \alpha, \beta \rangle = 0 \quad , \alpha \neq \beta$$

eg: $S_1 = \{(0,1), (1,0), (0,0)\}$ — orthogonal set ✓

$S_2 = \{(0,1), (1,0), \cancel{(0,0)\}}\ (1,1)\}$ — not " ✓

eg: $\alpha = (1,0), \beta = (0,1)$

find $\|\alpha\|$ under $\langle \alpha, \beta \rangle = 2a_1b_1 + 3a_2b_2$

sol: $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle} = \sqrt{2+0} = \sqrt{2}$ ✓

eg: $\alpha = (1,0), \beta = (0,1)$

find $\|\alpha\|$ under $\langle \alpha, \beta \rangle = a_1b_1 + a_2b_2$

sol: $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle} = \sqrt{1 \cdot 1 + 0 \cdot 0} = 1$ ✓

Inner product space:

1) $\langle a\alpha+\beta, \gamma\rangle = a\langle \alpha, \gamma\rangle + \langle \beta, \gamma\rangle$

2) $\langle \alpha, \beta\rangle = \overline{\langle \beta, \alpha\rangle}$ where $\alpha, \beta, \gamma \in V$

& $a \in F$

3) $\langle \alpha, \alpha\rangle \geq 0$, if $\langle \alpha, \alpha\rangle = 0 \Rightarrow \alpha = 0$

Norm of a vector:

$\alpha \in V(F)$

then $\|\alpha\| = \sqrt{\langle \alpha, \alpha\rangle}$

Normalizing a vector:

$$\hat{\alpha} = \frac{1}{\|\alpha\|}\alpha$$

Note that length of a normalizable vector is one. How??

$$\|\hat{\alpha}\| = \sqrt{\langle \hat{\alpha}, \hat{\alpha}\rangle} = \sqrt{\left\langle \frac{1}{\|\alpha\|}\alpha, \frac{1}{\|\alpha\|}\alpha\right\rangle}$$

#

Orthogonal vector:

let $\alpha, \beta \in V(F)$ we say that α is orthogonal to β if $\langle \alpha, \beta\rangle = 0$

Orthonormal vector:

if $\alpha_i, \alpha_j \in V(F)$

and

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

eg: $\alpha = (1, 2, -4)$, $\beta = (2, 3, 2)$, $\in V_3(R) = R^3$

under the standard inner product

$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

1) Find $\langle \alpha, \beta \rangle$ 2) $\|\alpha\|$ 3) $\|\beta\|$

Sol^n: 1) $\langle \alpha, \beta \rangle = \langle (1, 2, -4), (2, 3, 2) \rangle$

$= 2 + 6 - 8 = 0$ ✓ $\therefore \alpha, \beta$ are orthogonal

2) $\|(1, 2, -4)\| = \|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$

$= \sqrt{\langle (1, 2, -4), (1, 2, -4) \rangle}$

$= \sqrt{1 + 4 + 16} = \sqrt{21}$ ✓

3) $\|\beta\| = \sqrt{\langle \beta, \beta \rangle} = \sqrt{\langle (2, 3, 2), (2, 3, 2) \rangle}$

$= \sqrt{4 + 9 + 4}$

$= \sqrt{17}$ ✓

Cauchy - Schwarz Inequality :

$\|\alpha\| \times \|\beta\| \geq |\langle \alpha, \beta \rangle|$

For any u and v in a real inner product space V,

$|\langle \alpha, \beta \rangle| \leq \|u\| \times \|v\|$

Q: set of all

$V = \{$ all continuous real valued function defined on $(-\pi, \pi)\}$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \cdot g(t) \, dt \quad , \quad f, g, \in V$$

$f(t) = \sin(t)$, $g(t) = \cos(t)$

Are $\sin(t)$, $\cos(t)$ orthogonal ?

sol: $$\langle f, g \rangle = \int_{-\pi}^{\pi} \sin(t) \cos(t) \, dt$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \sin(2t) \, dt$$

$$= -\frac{[\cos(2t)]_{-\pi}^{\pi}}{4}$$

$$= \frac{-1}{4}[1-1] = 0$$

$\therefore$ f, g are orthogonal. ✓

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$\{(1,1), (0,1)\}$

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Gram-Schmidt orthogonalization process :-

Suppose $\{\alpha_1, \dots, \alpha_n\}$ is a basis of an inner product space $V(F)$; one can use this basis to contract an orthogonal basis :

$\{w_1, \dots, w_n\}$ of V as

set

$$w_1 = \alpha_1$$

$$w_2 = \alpha_2 - \frac{\langle \alpha_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = \alpha_3 - \frac{\langle \alpha_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle \alpha_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

...

...

$$w_n = \alpha_n - \frac{\langle \alpha_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle \alpha_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \cdots$$

$$\cdots - \frac{\langle \alpha_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$$

eg: $\{(1,1), (0,1)\}$ $\quad \langle v_1, v_2 \rangle$

$$w_1 = \alpha_1 = (1,1)$$

$$w_2 = (0,1) - \frac{\langle (0,1), (1,1) \rangle}{\langle (1,1), (1,1) \rangle} \cdot (1,1)$$

$$= (0,1) - \frac{1}{2}(1,1)$$

$$w_2 = \left(-\frac{1}{2}, +\frac{1}{2}\right)$$

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eg: $V = P_2(t)$ (polynomial of degree 2)

$S = \{1, t, t^2\}$

Now

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt \quad , \ f, g \in V(F) = P_2(t)$$

Sol^n: Set $w_1 = 1$ $\quad [\alpha_1 = 1]$

$[\alpha_2 = t]$

$$w_2 = \alpha_2 - \frac{\langle \alpha_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= t - \frac{\int_{-1}^{1} t\, dt}{2} \cdot 1$$

$$= t - 0$$

$$= t$$

$$w_3 = \alpha_3 - \frac{\langle \alpha_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle \alpha_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= t^2 - \frac{\int_{-1}^{1} t^2 \cdot 1\, dt}{2} \cdot 1 - \frac{\int_{-1}^{1} t^2 \cdot t\, dt}{\int_{-1}^{1} t^2\, dt} \cdot t$$

$$= t^2 - \frac{1}{6}\left[t^3\right]_{-1}^{1} - 0$$

$$= t^2 - \frac{1}{3}$$

Nullity → dimension of null space

Null space:

Let $A = (a_{ij})_{n \times n}$ be a $n \times n$ matrix

The Null space of the matrix A, denoted by $N(A)$ is the set of all n-dimensional column vectors x such that $Ax = 0$

$N(A)$ is the subspace of $\mathbb{R}^n$.

The dimension of $N(A)$ is said to be nullity of A.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$Ax = 0 \Rightarrow \left.\begin{array}{l} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{array}\right.$$

$x_3 = 0,\ x_4 = 1,\ x_2 = -3,\ x_1 = +2;\quad (+2, -3, 0, 1) = v_1$

$x_3 = 1,\ x_4 = 0,\ x_2 = -2,\ x_1 = +1;\quad (+1, -2, 1, 0) = v_2$

$\boxed{\rho(A) + n(A) = n}$ where $n \to$ no. of columns in A

Q: For the given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 2 & 0 & 3 & 4 & 0 \\ 4 & 4 & 9 & 4 & 8 \end{bmatrix}$$

Q/ Find
1) Row space
2) Column space
3) Null space
4) Rank of A
5) Nullity of A

Sol 1) First get row echelon form:

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & -4 & -3 & 4 & -8 \\ 0 & -4 & -3 & 4 & -8 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & -4 & -3 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns   non-pivot columns

Clearly x_1, x_2 are <u>pivot variables</u>

x_3, x_4, x_5 are <u>free variables</u>

Row space = span { pivot rows }
= span { (1,2,3,0,4), (0,−4,−3,4,−8) }

NOTE: If any row can be written in the linear combination of other rows then while writing Echelon form ~~#~~ it will become 0.

Basis of row space $A = \{(1,2,3,0,4), (0,-4,-3,4,-8)\}$

2) Column space of A = span {pivot columns}

$= \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-4\\0\end{pmatrix}\right\}$

Basis of column space $= \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-4\\0\end{pmatrix}\right\}$

3) Null space (zero space) :-

$x_1 + 2x_2 + 3x_3 + 4x_5 = 0$

$-4x_2 - 3x_3 + 4x_4 - 8x_5 = 0$

$x_1 = -2x_2 - 3x_3 - 4x_5 \quad - (1)$

$x_2 = \frac{1}{4}(3x_3 - 4x_4 + 8x_5)$

take $x_3 = 1,\ x_4 = 0,\ x_5 = 0 \quad \therefore x_2 = \frac{3}{4},\ x_1 = \frac{-9}{2}$

$\hookrightarrow \left(\frac{-9}{2}, \frac{3}{4}, 1, 0, 0\right)$

$x_3 = 0,\ x_4 = 1,\ x_5 = 0 \Rightarrow (2, -1, 0, 1, 0)$

$x_3 = 0,\ x_4 = 0,\ x_5 = 1 \Rightarrow (8, 2, 0, 0, 1)$

Basis for null space of A $= \left\{\begin{bmatrix}-9/2\\3/4\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}8\\2\\0\\0\\1\end{bmatrix}\right\}$

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4) rank of A = dimension of column space = 2

5) Nullity of A = dimension of null space = 3

Rank Theorem :

Rank of A + Nullity of A = no. of columns

eg: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_2 = 0 \\ 2x_2 = 0 \end{matrix}$$

Rank of $A = 2$

Nullity $= 0$

Eigen Value := A scalar λ is an eigen value of a $n \times n$ matrix A iff λ satisfies the characteristic equation

$$\boxed{\det(A - \lambda I) = 0}$$

eg: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1, 1$$

Eigen vector : For an $n \times n$ matrix A, a non-zero vector X such that $AX = \lambda X$, where λ is an eigenvalue

or

Eigen vector is a non-trivial solution X such that $AX = \lambda X$ ~~or~~

or

Then we say X is an eigenvector corresponding to eigenvalue λ.

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

For $\lambda = 1$

$$AX = \lambda X$$

$$AX = X$$

$$AX - IX = 0$$

$$[A - I]X = 0$$

$$\left[\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right]\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left.\begin{aligned} x\cdot 0 + y &= 0\\ 0\cdot x + 0\cdot y &= 0\end{aligned}\right\}$$

$x = 1,\ y = 0$ eigen vector $\begin{bmatrix}1\\0\end{bmatrix}$

$= \{(1,0), (2,0), \dots\} = \{(k,0) : k \in R\} \subseteq R^2$

$B = \{(1,0)\}$ $\quad a(k,0) = (ak,0)$

eigen space $= \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

eg: $A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

sol: $|A - \lambda I| = \begin{vmatrix}1-\lambda & 0\\0 & 1-\lambda\end{vmatrix} = (1-\lambda)^2 = 0$

eigen value $= 1, 1$

Eigen vector: $[A - I]X = 0$

$$\left[\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right]X = 0$$

$$\Rightarrow \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

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$$\left.\begin{array}{l} 0x + 0y = 0 \\ 0x + 0y = 0 \end{array}\right\}$$

Eigen vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Eigen space = span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

eg: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Sol: $|A - \lambda I| = \left| \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(3-\lambda) - 2 = 0$

$\Rightarrow 3 - 4\lambda + \lambda^2 - 2 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 1 = 0$

$\Rightarrow \lambda = 2 \pm \sqrt{3}$

NOTE: sum of eigen values = trace (A)
product " = $|A|$

eg: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Find eigen val(s) and corresponding eigen vectors

Sol^n: $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$

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$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 1 \\ 1 & \lambda & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)(\lambda^2-\lambda) + 1(\lambda+\lambda) = 0$

$\Rightarrow \quad -\lambda(\lambda-1)^2 + 2\lambda = 0$

$\Rightarrow \quad \lambda = 0 \quad$ or $\quad 2-(\lambda-1)^2 = 0$

or $\quad (\lambda-1)^2 = 2$

or $\quad \lambda = 1 \pm \sqrt{2}$

$\lambda = 0, 0, 3$

Eigenvector for $\lambda = 3$

$[A - 3I]x = 0$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x + y + z &= 0 \\ x - 2y + z &= 0 \\ x + y - 2z &= 0 \end{aligned}\right\}$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$-\cancel{3x}+\cancel{3z}=0$

$-2x+y+z=0$

$-\frac{3}{\cancel{2}}y+\frac{3}{\cancel{2}}z=0 \Rightarrow y=z$

$z=1, y=1, x=1$ i.e. $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Now for $\lambda=0$

$[A-0I]x=0$

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore x+y+z=0$

let $y=1, z=0, x=-1$

$y=0, z=1, x=-1$

$$\therefore \left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$

NOTE: No. of eigen vectors $\leq$ no. of eigen val.(s).

eg: $A=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

Sol^n: $|A-\lambda I| = (1-\lambda)^3=0$

$\lambda=1,1,1 \rightarrow$ eigen value

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eigen vector : (1, 0, 0)

Q7: If 7 is eigen value of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ then calculate eigen vector.

sol: $[A - \lambda I] X = 0$

$$\Rightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -6 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-6x + 6y = 0$

$x = y$

eigen vector : (1, 1)