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→ Diagonal entries of a skew-symmetric matrix is 0.

→ Periodic matrix : $A^{k+1} = A$ ($A \to$ square matrix)

$\Rightarrow k$ is period

→ Nilpotent matrix : $A^p = 0$

$p \to$ index of nilpotent matrix

→ Orthogonal matrix : $A \cdot A^T = I$

Row-reduction and Echelon form:

Elementary row operations on $m \times n$ matrix A

1) Multiplication of one row of A by a non-zero scalar c.

2) Replacement of i^{th} row of A by an i^{th} row plus c times j^{th} row where c is any scalar.

3) Interchange of 2 rows of A.

eg: $x_1 + 2x_2 = 1$

$2x_1 + x_2 = 3$

Def^n: Row equivalent: If A and B are $m \times n$ matrices over the field F, we say that B is row-equivalent to A if B can be obtained from A by performing a finite seq. of elementary row operations.

eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix}$ are row - equiv.

Echelon form OR Row Echelon form :-

1) All non-zero rows are above any row of all zeroes.

eg:
$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \rightarrow \text{not Echelon form}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Echelon form}$$

2) Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3) All entries in a column below a leading entry are zero.

eg:
$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \quad R_4 \leftrightarrow R_1$$

Pivot

$$
\begin{array}{l}
\text{Pivot} \leftarrow \\
\\
\text{eliminate} \\
\\
\end{array}
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ \boxed{-1} & -2 & -1 & 3 & 1 \\ \boxed{-2} & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}
\begin{array}{l} \\ \\ R_2 + R_1 \\ R_3 + 2R_1 \end{array}
$$

Pivot

$$
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & \boxed{5} & 10 & -15 & -15 \\ 0 & \boxed{-3} & -6 & 4 & 9 \end{bmatrix}
\begin{array}{l} \\ \\ R_3 - \frac{5}{2} R_2 \\ R_4 + \frac{3}{2} R_2 \end{array}
$$

eliminate

$$
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}
\quad R_3 \leftrightarrow R_4
$$

$$
\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\quad \text{Pivot entries}
$$

$\uparrow \quad \uparrow \quad \uparrow \rightarrow$ Pivot columns

__Reduced Echelon form__ OR __Row - reduced Echelon form :-__

1) Make all leading entries 1
2) Other than the leading entries, the other entries of the particular column is 0.

eg: take last one. In that the final matrix is

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{R_2}{2}, \frac{R_3}{-5}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & -3 & 3 & 5 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 + 3R_3, R_1 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduce Echelon form:

If Echelon form satisfies following 2 $cond^n(s)$

4) leading entry in each non zero ~~column~~ row is one.

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5) Each leading 1 is the only non-zero entry in its column.

———×———×———×———

Linear system of eq. in n unknowns $x_1, x_2, \dots, x_n$ is of the form:-

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

The system is called linear because each var. x_i, $i = 1$ to n appears in the first power only

$a_{11}, a_{12}, \dots$ are called the coefficient of the system

$b_1, b_2, \dots, b_m$ are given constant

→ system of linear eq's are called <u>Homogeneous</u> if all $b_1, b_2, \dots, b_n$ are 0.

eg:
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_2 + x_3 = 0$$
$$x_1 + 5x_3 = 0$$

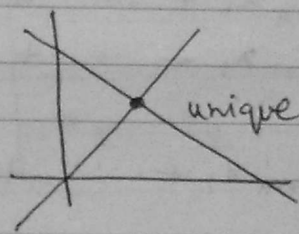
Teacher's Signature

Non-homogeneous linear system:

If at least one of b_j is non-zero

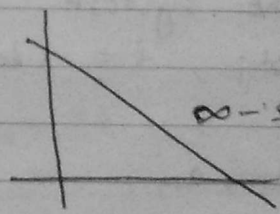
$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_1 \quad + x_3 = 1 \longrightarrow \text{non-zero}$$
$$5x_1 + x_2 + x_3 = 0$$

eg: i) $x_1 + x_2 = 2$
$x_1 - x_2 = 0$



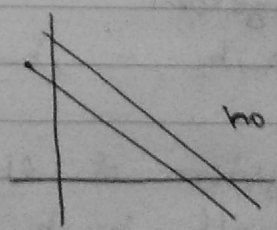
unique sol^n.

ii) $x_1 + x_2 = 2$
$2x_1 + 2x_2 = 4$



∞ - sol^n.

iii) $x_1 + x_2 = 1$
$x_1 + x_2 = 2$



no sol^n.

No. of sol^n(s) can be :-

1) Unique
2) exactly two sol^n
3) no sol^n
4) infinite sol^n.

System of linear equations can be written in matrix form as : $AX = B$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \rightarrow \text{coeff. matrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

<u>Augmented matrix</u> $[A : B]$

$$A/B = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

eg 1: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Sol: Pivot

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ \boxed{1} & -1 & 0 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

eliminate

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right]$$

transforming gives $\left.\begin{array}{r} x_1 + x_2 = 2 \\ -2x_2 = -2 \end{array}\right]$ can be easily solved

eg2) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Sol: $[A|B] =$ pivot $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ \boxed{2} & 2 & 4 \end{array}\right]$ $R_2 \leftarrow R_2 - 2R_1$

eliminate

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

transformation gives $x_1 + x_2 = 2$

$\therefore$ ∞ - sol^n.

eg3) $[A|B] =$ pivot $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ \boxed{1} & 1 & 2 \end{array}\right]$ $R_2 \leftarrow R_2 - R_1$

Sol: $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$

transforming we get $0 = 1$

$\therefore$ no sol^n.

NOTE: In row reduced Echelon form the no: of non-zero rows = rank of matrix

unique sol^n - $\rho[A] = \rho[A/b] = n$

Infinite - $\rho[A] = \rho[A/b] < n$

no sol^n - $\rho[A] \neq \rho[A/b]$

where ρ denotes rank

Gauss Elimination method:

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 + x_2 + 3x_3 = 1$$
$$x_1 + x_2 + x_3 = 2$$
$$x_2 + 2x_3 = 0$$

$[A|b] \neq$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

pivot

$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ \boxed{2} & 1 & 3 & 1 \\ \boxed{1} & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{array}\right]$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$

eliminate

pivot

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 2 & 0 \end{array}\right]$ $R_4 \leftarrow R_4 + R_2$

eliminate

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -3 \end{array}\right]$ $R_3 \leftrightarrow R_4$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Now backsubstitution eg: $3x_3 = -3 \Rightarrow x_3 = -1$

$-x_2 + x_3 = -3 \Rightarrow x_2 = 2$

$x_1 + x_2 + x_3 = 2 \Rightarrow x_1 = 1$

eg: $x + y + z = 2$

$2x + y + -2z = 1$

$3x + 2y - z = k$

Find the value of k such that the given system of equations has ∞ly many sol^n.

sol: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 3 & 2 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 1 \\ k \end{bmatrix}$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ \boxed{2} & 1 & -2 & 1 \\ \boxed{3} & 2 & -1 & k \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow 2R_1 \\ R_3 \leftarrow 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -4 & -3 \\ 0 & \boxed{-1} & -4 & k-6 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -4 & -3 \\ 0 & 0 & 0 & k-3 \end{array}\right]$$

for ∞ly many sol^n to exist

$$0 = k - 3$$

$$\Rightarrow \boxed{k = 3}$$

for $k \neq 3$ $\exists$ no solution.

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Existence and Uniqueness Theorem:

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column i.e. iff an echelon form of the augmented matrix has no row of the form $[0 \cdots 0 \; b]$ with b nonzero

System of linear Equations:

Homogeneous : $AX=0$

Non-homo... : $AX=b$

- Consistent
  - unique soln.
  - infinitely many solⁿ
- Inconsistent
  - No solution

eg: $x_1+x_2=1$, $x_1+2x_2=0$ (unique soln.)

$x_1+x_2=-2$, $-2x_1+2x_2=4$ (infinitely many solⁿ)

eg: $x_1+x_2=3$, $x_1+x_2=1$ (No solution)

Row operations:

- Row Echelon form (Gauss Elimination method)
- Row-reduced Echelon form (Gauss Jordan Elimination method)

eg 1: $x_1+x_2=1$

$x_1+2x_2=0$

sol: $$[A|b] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

$$= \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

Get x_1, x_2 using back-substitution

upto here it's Gauss Elimination method

we make upper Δ'r matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

Upto here it's Gauss-Jordan elimination method.

$\therefore$ $x_1 = 2$ //

$x_2 = -1$ //

<u>Set</u>: A well-defined collection of objects

binary operation

$S \times S \to S$

$a, b \in S$

$a+b = c$

P1) <u>Closure law</u>:

$+ : N \times N \to N$

$2, 3 \in N$ $(N, +)$

$2+3 = 5$

eg: $(N, -)$ is not ~~to~~ closured as $1-2 = -1 \notin N$

$(I, -)$ is closured

$(N, *)$ is defined under closure law

P2) <u>Associative law</u>: let $a, b, c \in S$

$a + (b+c) = (a+b) + c$

eg i) $(N, +)$ ✓

ii) $(I, -)$ ✗ eg: $1-3-4$