with $|u|<1$. So the original P.S converges when $|x^2|<1$ or $|x|<1$. So, $R=1$.

IV. $\sum_{n\geq 0} n^n x^n$. V $\sum_{n\geq 0} \frac{x^n}{n!}$

Def^n : (Analytic function) :

Let $f: I \to R$ be a function and $x_0 \in I$. f is called analytic around x_0 if $\exists$ a $\delta > 0$ $\ni$ for all x with $|x-x_0|<\delta$ –

$$f(x) = \sum_{n\geq 0} a_n (x-x_0)^n ,$$

i.e. f has a power series representation in the neighbourhood of x_0.

PROPERTIES OF POWER SERIES :

Let $F(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ and $\underbrace{\sum_{n=0}^{\infty} b_n (x-x_0)^n}_{= G(x)}$ be two P.S.

I. EQUALITY OF P.S :

iff $a_n = b_n$ $\forall$ $n = 0,1,2,\dots$

II. TERM BY TERM ADDITION :

$$F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$$

III : MULTIPLICATION of POWER SERIES :

$$F(x)\, G(x) = H(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

where $c_n = \sum_{j=1}^{n} a_{n-j} b_j$

IV TERM BY TERM DIFFERENTIATION :

$$\frac{d}{dx} F(x) = F'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$\frac{d}{dx} F'(x) = F''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

ex: 13 I. which of the following represents ~~to~~ P.S. in x)

(a) $1+x^2+x^4+\cdots+x^{2n}+\cdots$ $(x_0=0)$

(b) $1+\sin(x)+(\sin(x))^2+ \quad (\sin x)^n+\cdots$ $(x_0=0)$

(c) $1+x|x|+x^2|x^2|+\cdots+x^n|x^n|+\cdots$ $(x_0=0)$

II. Let $f(x)$ and $g(x)$ be two power series around $x_0=0$, defined by

$$f(x) = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$$

$$g(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$

find R of convergence of $f(x)$ and $g(x)$. Also, for each x in the domain of convergence show that $f'(x)=g(x)$ and $g'(x)=-f(x)$

III. find R, $(x_0=-1)$.

(a) $1+(x+1)+\frac{(x+1)^2}{2!}+\cdots+\frac{(x+1)^n}{n!}+\cdots$

(b) $1+(x+1)+2(x+1)^2+\cdots+n(x+1)^n+\cdots$

SOLUTION IN TERMS OF POWER SERIES

$$y''+a(x)y'+b(x)y=0 \quad -(10)$$

Let a, b be analytic around the point $x_0=0$

let $y=\sum_{k=0}^{\infty} c_k x^k$ be the solⁿ of $-(10)$

eg (14) $y''+y=0$

sol: $a(x)\equiv 0$, $b(x)\equiv 1$ which are analytic around $x_0=0$.

let $y=\sum_{n=0}^{\infty} c_n x^n$ be solⁿ

Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Substituting expn. for y, y', y'' in $y'' + y = 0$

we get $\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$

or $0 = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n$

$= \sum_{n=0}^{\infty} \{ (n+1)(n+2) c_{n+2} + c_n \} x^n$

Here $\forall$ $n = 0, 1, 2, \dots$

$(n+1)(n+2) c_{n+2} + c_n = 0$ or $c_{n+2} = \dfrac{-c_n}{(n+1)(n+2)}$

$\therefore$ we have

$c_2 = \dfrac{-c_0}{2!}$, $c_3 = \dfrac{-c_1}{3!}$

$c_4 = (-1)^2 \dfrac{c_0}{4!}$, $c_5 = (-1)^2 \dfrac{c_1}{5!}$

$\vdots$ $\vdots$

$c_{2n} = (-1)^n \dfrac{c_0}{(2n)!}$, $c_{2n+1} = (-1)^n \dfrac{c_1}{(2n+1)!}$

Here c_0 and c_1 are arbitrary. So

$$y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

or $y = c_0 \cos(x) + c_1 \sin(x)$

for $c_0 = 0, c_1 = 1 \Rightarrow \sin(x)$ is a sol^n

$c_0 = 1, c_1 = 0 \Rightarrow \cos(x)$ "

ex: (15) Solve using P.S.

I. $y' = -y$ $(x_0 = 0)$

II. $y' = 1 + y^2$ $(x_0 = 0)$

III find two L.I. sol^n of

ⓐ $y'' - y = 0$, $(x_0 = 0)$ ⓑ $y'' + 4y = 0$, $(x_0 = 0)$

FROBENIUS THEOREM FOR REGULAR (ORDINARY) POINT:

THEOREM: Let $a(x)$, $b(x)$, $f(x)$ admit a power series representation around a point $x = x_0 \in I$, with non-zero radius of convergence r_1, r_2 and r_3 respectively. Let $R = \min\{r_1, r_2, r_3\}$. Then the equation (1D) has a sol^n y which has a power series representation around x_0 with radius of convergence R.

REMARK: ① Above theorem is true whenever the coeff. of y'' is 1.

② Secondly a point x_0 is called an ORDINARY POINT if $a(x)$, $b(x)$, $f(x)$ admit power series expansion (with non-zero R) around $x = x_0$. x_0 is called SINGULAR POINT if x_0 is not ordinary.

eg: (6) Examine whether x_0 is an ordinary point or a singular point for the following diff. eq.

(a) $(x-1)\, y'' + \sin(y) = 0$, $x_0 = 0$

(b) $y'' + \dfrac{\sin(x)}{x-1}\, y = 0$, $x_0 = 0$

(c) Find two linearly independent sol^{n}s of

(d) $(1-x^2)\, y'' - 2xy' + n(n+1)\, y = 0$, $x_0 = 0$, n is real const.

II. Show that the following equations admit power series solution around given x_0. Also, find the power series solutions if it exists.

(a) $y'' + y = 0$, $x_0 = 0$

(b) $xy'' + y = 0$, $x_0 = 0$

(c) $y'' + 9y = 0$, $x_0 = 0$

Legendre Equations and Legendre Polynomials.

Def^n: The eq. $(1-x^2)y'' - 2xy' + p(p+1)y = 0$, $-1 < x < 1$

(11) where $p \in \mathbb{R}$ is called LEGENDRE EQUATION of order p.

$$y'' - \underbrace{\frac{2x}{(1-x^2)}}\, y' + \underbrace{\frac{p(p+1)}{(1-x^2)}}\, y = 0$$

analytic around $x_0 = 0$

$$y_1 = 1 - \frac{p(p+1)}{2!}x^2 + \dots + (-1)^m \frac{(p-2m+2)\cdots(p+2m-1)}{(2m)!}x^{2m} + \dots$$

$$y_2 = x - \frac{(p-1)(p+2)}{3!}x^3 + \dots + (-1)^m \frac{(p-2m+1)\cdots(p+2m)}{(2m+1)!}x^{2m+1} + \dots$$

Remark: y_1 and y_2 are two L.I. sol^{n}s of (11)
It now follows that general sol^n of (11)

$y = c_1y_1 + c_2y_2$ where $c_1, c_2 \in \mathbb{R}$

Legendre Polynomials:

Def^n: A polynomial $P_n(x)$ of (11) is called a LEGENDRE POLYNOMIAL whenever $P_n(1) = 1$

PROPOSITION: Let $p = n$ be a non-negative even integer. Then any polynomial ~~solu~~ sol^n y of (11) which has only even powers of x is a multiple of $P_n(x)$.
Illly if $p = n$ is a non-negative odd integer, then any polynomial sol^n y of (11) which has only odd powers of x is a multiple of $P_n(x)$.

Theorem: (RODRIGUE'S FORMULA): The Legendre Polynomials $P_n(x)$ for $n = 1, 2, \dots$ are given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

eg: (11) I. When $n=0$, $P_0(x)=1$

II. When $n=1$, $P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$

III. When $n=2$, $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\{12x^2-4\}$

$$= \frac{3}{2}x^2 - \frac{1}{2}$$

NOTE: RODRIGUE'S FORMULA IS useful in the computation of $P_n(x)$ for "small" values of n.

THEOREM Let $P_n(x)$ denote, the lengendre Polynomial of degree n. Then $\int_{-1}^{1} P_n(x)\cdot P_m(x)\,dx = 0$; if $m \neq n$

THEOREM: For $n = 0, 1, 2, \ldots$

$$\int_{-1}^{1}\cancel{P_n^2(x)} \qquad \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

THEOREM: Let $f(x)$ be a real valued continuous fun() defined in $[-1,1]$. Then

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad , \quad x \in [-1,1]$$

where $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx$

Legendre polynomials can also be generated by a suitable function. To do that use the following theorem:

Theorem: Let $P_n(x)$ be the Legendre polynomial of degree n. Then

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n \quad , \quad t \neq 1 \qquad (12)$$

the fun() $h(t) = \frac{1}{\sqrt{1-2xt+t^2}}$ admits a power series expansion in t (for small t) and the coeff. of t^n in $P_n(x)$. The fun() $h(t)$ is called the GENERATING FUNCTION for the legendre polynomials.

eg: (18) I. By using the Rodrigue's formula, find $P_0(x)$, $P_1(x)$ and $P_2(x)$

II. Using Generating fun() - (12)

(a) find $P_0(x)$, $P_1(x)$, $P_2(x)$

(b) to show that $P_n(x)$ is an odd fun() whenever n is odd and is an even fun() whenever n is even.

eg (19) I. find a polynomial solution $y(x)$ of $(1-x^2)y'' - 2xy' + 20y = 0$, ∋ $y(1) = 10$

II. Prove the following

(a) $\int_{-1}^{1} P_m(x)\,dx = 0 \quad \forall$ positive int. $m \geq 1$

(b) $\int_{-1}^{1} x^{2n+1} P_{2m}(x)\,dx = 0$ whenever m and n are positive integers with $m \neq n$

(c) $\int_{-1}^{1} x^m P_n(x)\,dx = 0$ whenever m and n are +ve integers with $m < n$

III. Show that $P_n'(1) = \frac{n(n+1)}{2}$ and $P_n'(-1) = (-1)^{n-1}\frac{n(n+1)}{2}$

IV. Establish the following recurrence relations

(a) $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$

(b) $(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$