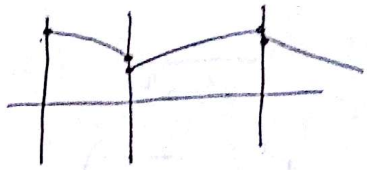
LAPLACE TRANSFORM (L.T.)

Application: L.T. reduces a linear diff. eq. with const. coeff. to an algebraic eq., which can be solved by algebraic methods.

Defn: (Piece-wise continuous function)

1. $f(t)$ is piece wise continuous on a closed interval $[a,b] \subset \mathbb{R}$, if $\exists$ finite no. of points $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, $\ni$ $f(t)$ is continuous in each of intervals and has finite limits.



Defn: Laplace transform

Let $f: [0, \infty) \to \mathbb{R}$ and $s \in \mathbb{R}$. Then $F(s)$, for $s \in \mathbb{R}$ is called Laplace transform, and is defined as

$$L(f(t)) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Remark: ~~Existence~~ Existence of Laplace transform

1. Let $f(t)$ be an exponentially bounded func(t) i.e. $\exists$ $|f(t)| \le Me^{\alpha t}$ $\forall\, t > 0$ and for some real no.s α and M with $M > 0$, then the Laplace transform exists.

2. If $F(s)$ is a Laplace transform of $f(t)$ then

$$\lim_{s \to \infty} F(s) = 0 \quad \text{(always)}$$

If $\lim_{s \to \infty} F(s) = D.N.E$ or $\neq 0$ then it can't be a Laplace transform

Defn: Inverse Laplace transform: $f(t)$ is $L^{-1}(F(s))$

$f(t)$	$L(f(t)) = F(s)$	$f(t)$	$L(f(t)) = F(s)$
1	$\frac{1}{s}$, $s > 0$	t	$\frac{1}{s^2}$, $s > 0$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$	e^{at}	$\frac{1}{s-a}$, $s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$, $s > 0$	$\cos(at)$	$\frac{s}{s^2+a^2}$, $s > 0$

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Properties

Lemma 1. Linearity of Laplace transform

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t))$$

2. $$L^{-1}(aF(s) + bG(s)) = af(t) + bg(t)$$

eg 1) find Laplace transform of $\cosh(at)$

NOTE: $\cosh(at) = \dfrac{e^{at} + e^{-at}}{2}$

sol: $\dfrac{s}{s^2 - a^2}$

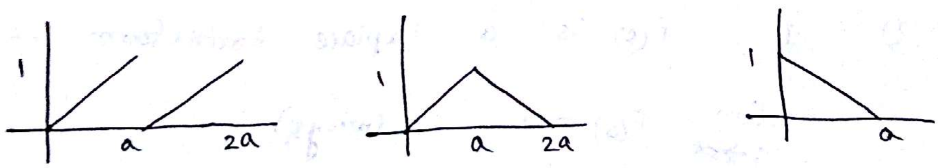
eg: $L^{-1}\left(\dfrac{1}{s(s+1)}\right)$ Ans: $1 - e^{-t}$

Theorem (Scaling by a): Let $f(t)$ be piecewise cont... fun() with $L(f(t)) = F(s)$, then for $a > 0$

$$L(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

ex: (2) a) find L.T. of $t^2 + at + b$, $\cos(\omega t + \theta)$, $\cos^2(t)$, $\sinh^2 t$ where a, b, ω and θ are arbitrary constant.

b) find L.T. of following fun(s)



c) If $L(f(t)) = \dfrac{1}{s^2+1} + \dfrac{1}{2s+1}$, find $f(t)$

Theorem L.T. of Differentiable func(s):

Let $f(t)$, for $t > 0$ be differentiable fun() with derivative $f'(t)$ being continuous. Suppose $\exists$ const(s) M and T $\ni$ $|f(t)| \le Me^{\alpha t}$ $\forall$ $t \geq T$.

If $L(f(t)) = F(s)$ then

$$L(f'(t)) = sF(s) - f(0), \quad \text{for } s > \alpha$$

Corollary: ~~Let~~ Let $L(f(t)) = F(s)$. If $f'(t), f''(t), \dots, f^{(n-1)}t$, $f^{(n)}t$, ~~exist~~ exist and $f^{(n)}t$ is continuous $\forall\, t \geq 0$ then

$$L(f^{(n)}t) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

for $n=2$, $L(f^{(2)}t) = s^2 F(s) - s f(0) - f'(0)$

eg ③ I. find $L^{-1}\left(\frac{s}{s^2+1}\right)$

II. find $L(\cos^2(t))$

Lemma Let $f(t)$ be a piecewise continuous func with $L(f(t)) = F(s)$

If $F(s)$ is differentiable, then $L(t f(t)) = -\frac{d}{ds} F(s)$

Equivalently $L^{-1}\left(-\frac{d}{ds} F(s)\right) = t f(t)$

Corollary: Let $L(f(t)) = F(s)$ and $g(t) = \frac{f(t)}{t}$. Then

$$L(g(t)) = G(s) = \int_s^\infty F(p)\, dp$$

eg: ④ 1. Find $L(t \sin(at))$

2. $L^{-1}\left(\frac{4}{(s-1)^3}\right)$

Lemma If $L(f(t)) = F(s)$ then

$$L\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s} \quad \text{OR} \quad L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)\, d\tau$$

eg ⑤ 1. Find $L\left(\int_0^t \sin(az)\, dz\right)$

2. $L\left(\int_0^t \tau^2\, d\tau\right) = ?$

3. $L^{-1}\left(\frac{4}{s(s-1)}\right)$

Lemma (s-shifting) Let $L(f(t)) = F(s)$, then $L(e^{at} f(t)) = F(s-a)$ for $s > a$

eg ⑥ I. $L(e^{at} \sin(bt))$

II. $L^{-1}\left(\frac{s-5}{(s-5)^2+36}\right)$

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Inverse Transform of Rational Functions:

Some methods to calc. $L^{-1}(F(s))$

eg: 1. Denominator of F has distinct REAL ROOTS

if $F(s) = \frac{(s+1)(s+3)}{s(s+2)(s+8)}$ find $f(t)$

Sol^n: $F(s) = \frac{3}{16s} + \frac{1}{12(s+2)} + \frac{35}{48(s+8)}$. Thus,

$$f(t) = \frac{3}{16} + \frac{1}{12}e^{-2t} + \frac{35}{48}e^{-8t}.$$

II. DENOMINATOR OF F HAS DISTINCT COMPLEX ROOTS:

if $F(s) = \frac{4s+3}{s^2+2s+5}$. Find $f(t)$

sol^n: $F(s) = 4 \cdot \frac{(s+1)}{(s+1)^2+2^2} - \frac{1}{2} \cdot \frac{2}{(s+1)^2+2^2}$. thus

$$f(t) = 4e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)$$

III. DENOMINATOR OF F HAS REPEATED REAL ROOTS:

if $F(s) = \frac{3s+4}{(s+1)(s^2+4s+4)}$. find $f(t)$

Sol^n: $F(s) = \frac{3s+4}{(s+1)(s^2+4s+4)} = \frac{3s+4}{(s+1)(s+2)^2} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{(s+2)^2}$

solving for a, b, c we get $F(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{2}{(s+2)^2}$

$$= \frac{1}{s+1} - \frac{1}{s+2} + 2\frac{d}{ds}\left(-\frac{1}{(s+2)}\right)$$

$$\therefore f(t) = e^{-t} - e^{-2t} + 2te^{-2t}$$

Def^n: Unit step function $U_a(t) = \begin{cases} 0, & \text{if } 0 \le t < a \\ 1, & \text{if } t \geq a \end{cases}$

eg: ⑦ $L(U_a(t)) = \int_a^\infty e^{-st}\,dt = \frac{e^{-sa}}{s}$, $s > 0$

Lemma (t-shifting) Let $(f(t)) = F(s)$

$$g(t) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \geq a \end{cases}$$

Then $g_a(t) = U_a(t) f(t-a)$ and $L(g(t)) = e^{-as}F(s)$

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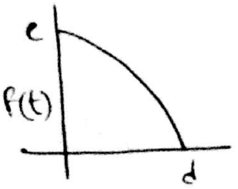

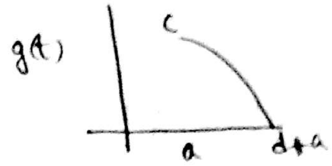

$g(t) = U_a(t) f(t-a)$

eg ⑧ 1. Find $L^{-1}\left(\frac{e^{-ss}}{s^2-4s-5}\right)$

2. Find $L(f(t))$ where $f(t) = \begin{cases} 0 & , t < 2\pi \\ t\cos t & , t \geq 2\pi \end{cases}$

LIMITING THEOREMS

Theorem (First limit theorem)

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s F(s)$$

eg ⑨ 1. For $t \geq 0$, let $Y(s) = L(y(t)) = a(1+s^2)^{-1/2}$. Determine a such that $y(0) = 1$.

2. If $F(s) = \frac{(s+1)(s+3)}{s(s+2)(s+8)}$ find $f(0^+)$.

Theorem (Second limit theorem)

Suppose $L(f(t))$ exists. Then $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$

eg: ⑩ If $F(s) = \frac{2(s+3)}{s(s+2)(s+8)}$, find $\lim_{t \to \infty} f(t)$

Def^n. Convolution of functions: let $f(t)$ and $g(t)$ be two smooth functions. The convolution func() $f*g$ is a func() defined as

$$(f*g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

①. $(f*g)(t) = (g*f)(t)$

②. If $f(t) = \cos(t)$ then $(f*f)(t) = \frac{t\cos(t) + \sin(t)}{2}$

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APPLICATION TO DIFFERENTIAL EQUATIONS:

Q. $af''(t) + bf'(t) + cf(t) = g(t)$ with $f(0) = f_0$, $f'(0) = f_1$

A: Let $L(f(t)) = G(s)$. Then

$$G(s) = a(s^2 F(s) - sf(0) - f'(0)) + b(sF(s) - f(0)) + cF(s)$$

$$G(s) = F(s)(as^2 + bs + c) - f(0)(as + b) - af'(0)$$

$$G(s) = F(s)(as^2+bs+c) - f_0(as+b) - af_1$$

$$F(s) = \underbrace{\frac{G(s)}{as^2+bs+c}}_{\text{non-homo... part}} + \underbrace{\frac{f_0(as+b)}{as^2+bs+c} + \frac{af_1}{as^2+bs+c}}_{\text{initial condn..}}$$

eg (11) 1. Solve the IVP

$$y'' - 4y' - 5y = f(t) = \begin{cases} t & \text{if } 0 \le t < 5 \\ t+5 & \text{if } t \ge 5 \end{cases}$$

with $y(0) = 1$, $y'(0) = 4$

sol: $f(t) = t + U_5(t)$. Thus $L(f(t)) = \frac{1}{s^2} + \frac{e^{-5s}}{s}$

~~The~~ Taking L.T. of above eq. we get

$$(s^2 Y(s) - sy(0) - y'(0)) - 4(sY(s) - y(0)) - 5Y(s) = L(f(t))$$

$$= \frac{1}{s^2} + \frac{e^{-5s}}{s}$$

which gives
$$Y(s) = \frac{s}{(s+1)(s-5)} + \frac{e^{-5s}}{s(s+1)(s-5)} + \frac{1}{s^2(s+1)(s-5)}$$

$$= \frac{1}{6}\left[\frac{5}{s-5} + \frac{1}{s+1}\right] + \frac{e^{-5s}}{30}\left[\frac{-6}{s} + \frac{5}{s+1} + \frac{1}{s-5}\right]$$

$$+ \frac{1}{150}\left[-\frac{30}{s^2} + \frac{24}{s} - \frac{25}{s+1} + \frac{1}{s-5}\right]$$

$$y(t) = \frac{5e^{5t}}{6} + \frac{e^{-t}}{6} + U_5(t)\left[-\frac{1}{5} + \frac{e^{-(t-5)}}{6} + \frac{e^{5(t-5)}}{30}\right]$$

$$+ \frac{1}{150}\left[-30t + 24 - 25e^{-t} + e^{5t}\right]$$

Remark Let $y(t)$ be soln of $ay'' + by' + cy = f(t)$. Then $y(t)$ and $y'(t)$ are always continuous functions of time.

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eg: (12) 1. $t y''(t) + y'(t) + t y(t) = 0$, with $y(0) = 1$ and $y'(0) = 0$.
Find $L(y(t))$

2. $y(t) = \int_0^t f(\tau) g(t-\tau) d\tau$ is a solution of

$y''(t) + a y'(t) + b y(t) = f(t)$ with $y(0) = y'(0) = 0$

where $L[g(t)] = \frac{1}{s^2 + as + b}$

3. Show that $\frac{1}{a}\int_0^t f(\tau) \sin(a(t-\tau)) d\tau$ is a sol^n of

$y''(t) + a^2 y(t) = f(t)$, with $y(0) = y'(0) = 0$

4. Solve the IVP: $y'(t) = \int_0^t y(\tau) d\tau + t - 4\sin(t)$; $y(0) = 1$

* Transform of Unit Impulse function :

eg (13) Find L.T. of

$$\delta_h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{h} & 0 \le t < h \\ 0 & t > h \end{cases}$$

Hint $\delta_h(t) = \frac{1}{h}(U_0(t) - U_h(t))$

Unit Impulse func() or Dirac-Delta func()

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

OR $\delta(t) = \lim_{h\to 0} \delta_h(t) = \lim_{h \to 0} \frac{1}{h}(U_0(t) - U_h(t))$

NOTE: $\int_0^\infty \delta_h(t) dt = 1$, $\forall h$

NOTE: $L(\delta_h(t)) = \left(\frac{1-e^{-hs}}{hs}\right)$

$L(\delta(t)) = \lim_{h\to 0} \frac{1-e^{-hs}}{hs} = \lim_{h\to 0} \frac{s e^{-hs}}{s} = 1$

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