# Multivariate Regression models

Dr. Pritam Anand.
Assistant Professor,
DA-IICT, Gandhinagar.

# Multiple Linear Regression model

| $x_1$ Age | $x_2$ Income ( hundred thousand dollar) | y Balance (thousand dollar) |
|---|---|---|
| 32 | 0.550798 | 5.651202 |
| 22 | 0.708148 | 7.321263 |
| 45 | 0.290905 | 5.167304 |
| 78 | 0.510828 | 5.609367 |
| 54 | 0.892947 | 9.406379 |
| 39 | 0.896293 | 9.379439 |
| 42 | 0.125585 | 2.734997 |
| 51 | 0.207243 | 4.876649 |
| 21 | 0.051467 | 3.584138 |
| 19 | 0.44081 | 5.437239 |

$(x_{i1}, x_{i2}) \rightarrow y$

# Multiple Linear Regression model

$f(X) = \beta_1 x_1 + \beta_2 x_2 + \beta_0$

$\sum_{i=1}^{n} \left(y_i - f(x_i)\right)^2$

$X \rightarrow Y$

$X_1$, $X_2$

| $x_1$ Age | $x_2$ Income ( hundred thousand dollar) | y Balance (thousand dollar) |
|---:|---:|---:|
| 32 | 0.550798 | 5.651202 |
| 22 | 0.708148 | 7.321263 |
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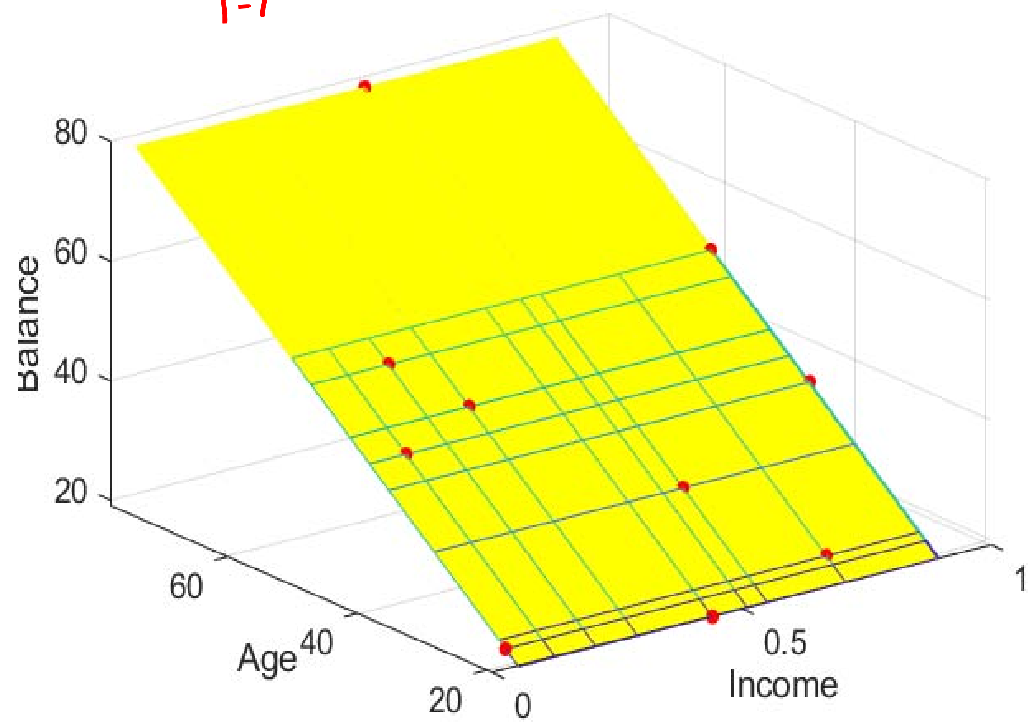


For given Training Set T = { $(x_{11}, x_{12}, y_1), (x_{21}, x_{22}, y_2), \ldots, (x_{n1}, x_{n2}, y_n)$}, we need to solve

$$Min \quad J(\beta_5,\beta_4,\beta_3,\beta_2,\beta_1,\beta_0) = \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}))^2 \quad \ldots..(1)$$

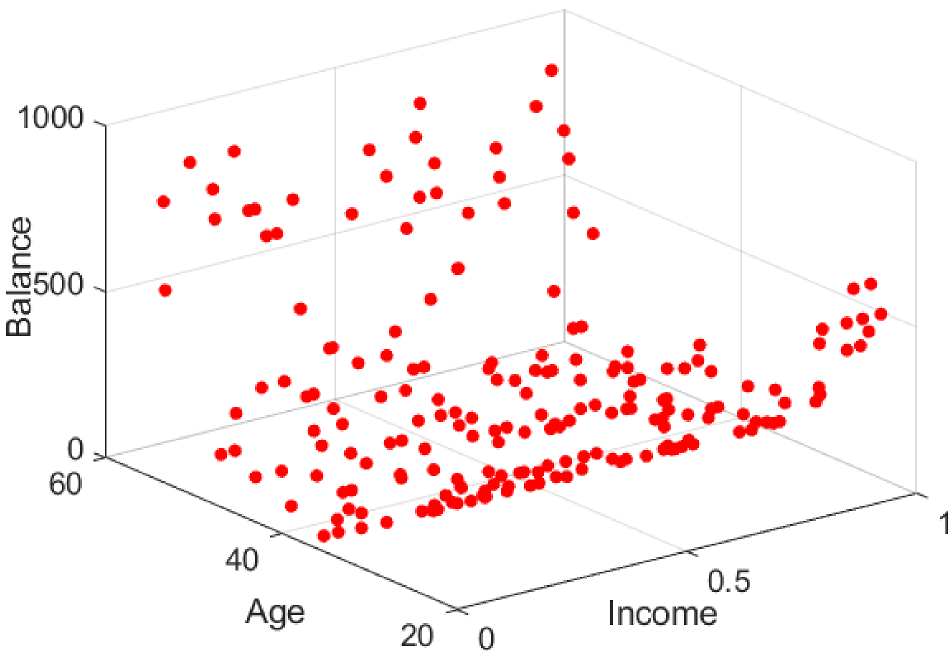
$$\mathbf{u} = \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} \qquad A = \begin{bmatrix} x_{11} & x_{12} & 1 \\ x_{21} & x_{22} & 1 \\ \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_0 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} \beta_1 x_{11} + \beta_2 x_{12} + \beta_0 \\ \beta_1 x_{21} + \beta_2 x_{22} + \cdot \\ \vdots \\ \beta_1 x_{n1} + \beta_2 x_{n2} + \beta_0 \end{bmatrix}$$

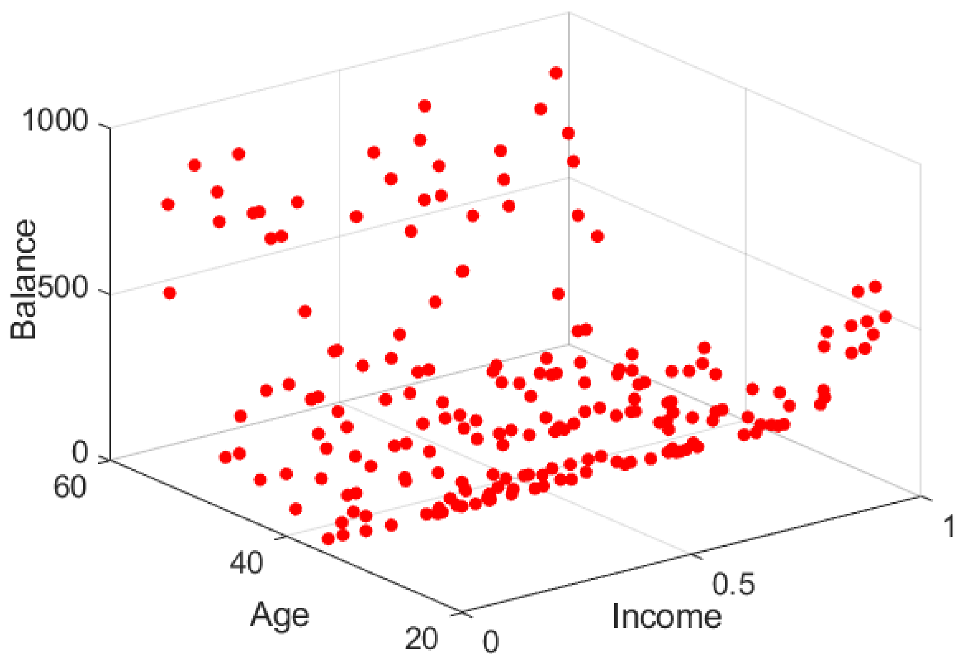
The Least Square problem reduces to

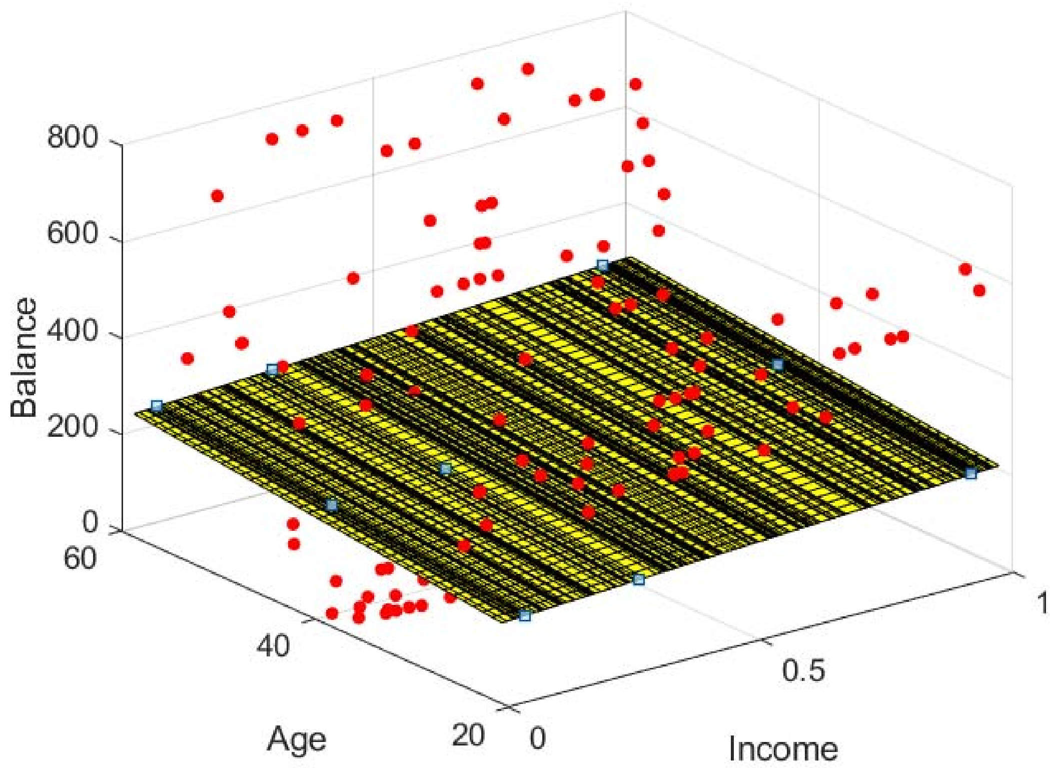
$$Min_{(u)} \quad J(\mathrm{u}) = (Y - Au)^T (Y - Au)$$

$$\boxed{\mathrm{u} = \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = (A^T A)^{-1} A^T Y}$$


Balance
1000
500
0
60
40
20
Age
0
0.5
1
Income

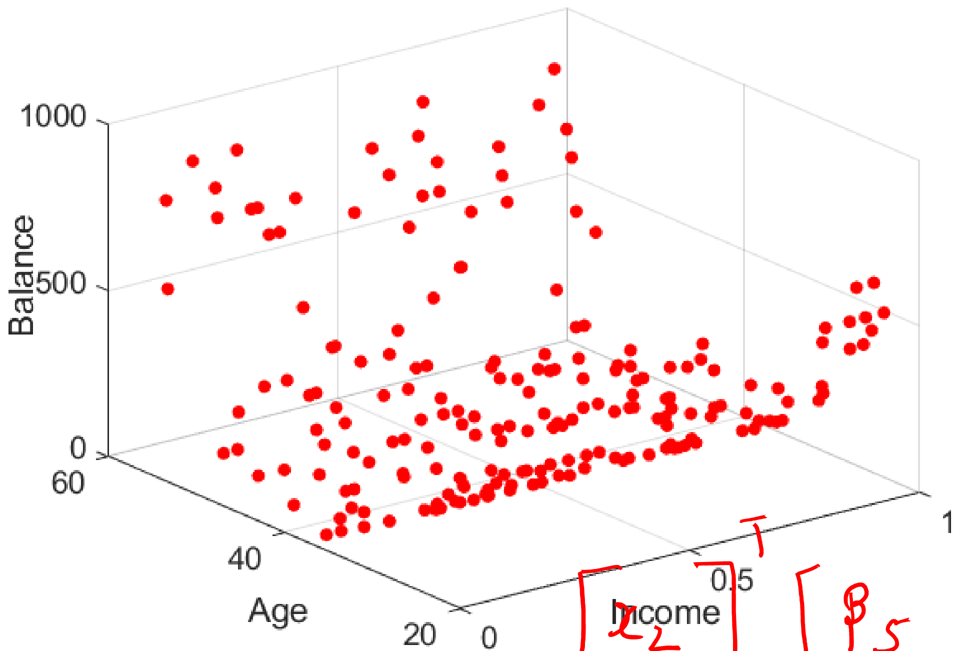
# Quadratic Regression model with two variables

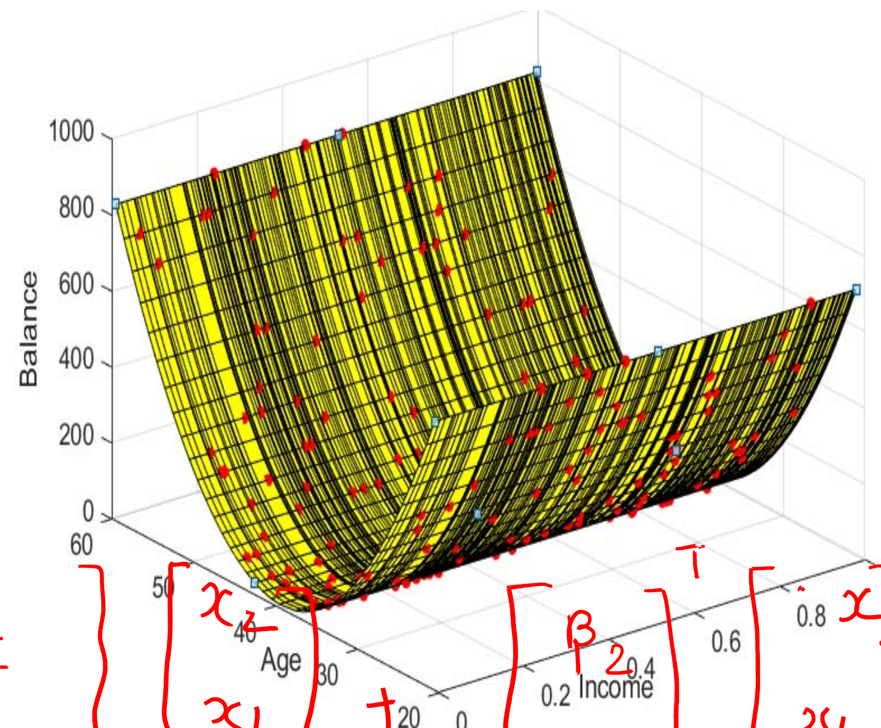


# Quadratic Regression model with two variables

$$f(X) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \underline{\beta_3 x_1 x_2 + \beta_4 x_1^2 + \beta_5 x_2^2}$$

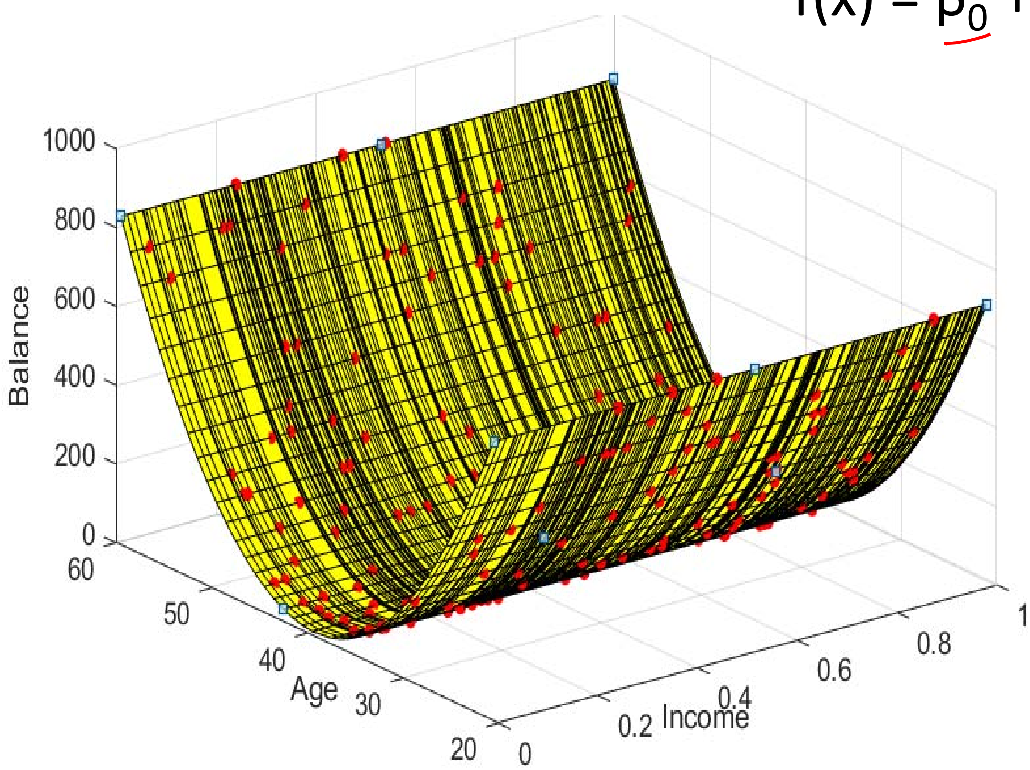



$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}^T \begin{bmatrix} \beta_5 & \frac{\beta_3}{2} \\ \frac{\beta_3}{2} & \beta_4 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix}^T \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \beta_0$$

# Quadratic Regression model with two variables

$$f(x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \beta_4 x_2^2 + \beta_5 x_1^2$$



$$\sum_{i=1}^{n}$$

For given Training Set T = { $\underbrace{(x_{11}, x_{12}}_{X_1}$, ,$y_1$), $\underbrace{(x_{21},x_{22}}_{X_2}$, $y_2$),....., $\underbrace{(x_{n1},x_{n2}}_{X_n}$ $y_n$)}, we need to solve

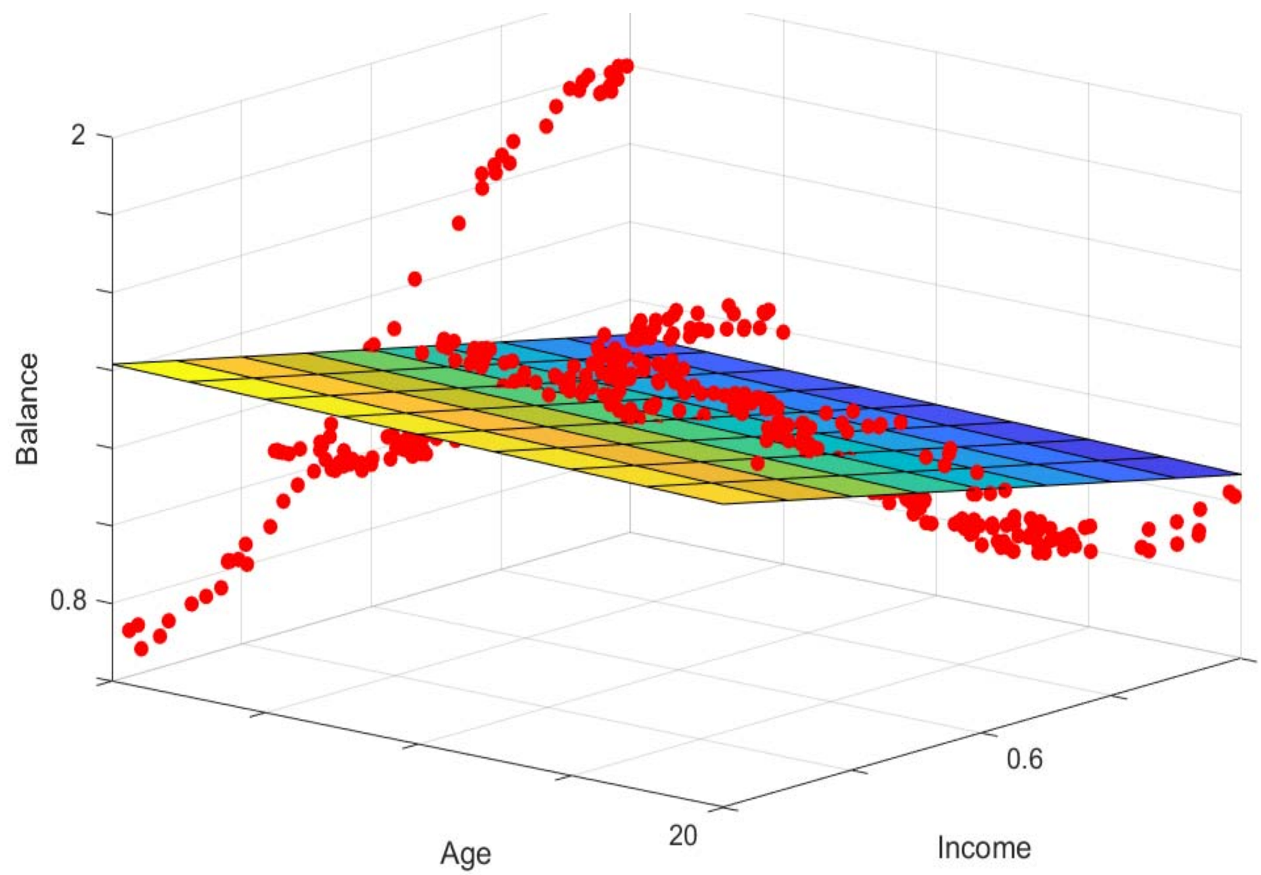
$$Min \quad J(\beta_5,\beta_4,\beta_3,\beta_2,\beta_1,\beta_0) = \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{1i}x_{2i} + \beta_4 x_{2i}^2 + \beta_5 x_{1i}^2)\right)^2 \quad .....(1)$$

$$u = \begin{bmatrix} \beta_5 \\ \beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} \qquad A = \begin{array}{c} \begin{bmatrix} x_{11}^2 & x_{21}^2 & x_{11}x_{12} & x_{12} & x_{11} & 1 \\ x_{21}^2 & x_{22}^2 & x_{21}x_{22} & x_{22} & x_{21} & 1 \\ & & & & & \\ & & & & & \\ x_{n1}^2 & x_{n2}^2 & x_{n1}x_{n2} & x_{n2} & x_{n1} & 1 \end{bmatrix} \end{array} \begin{array}{l} \beta_5 \\ \beta_4 \\ \\ \\ \\ \end{array} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \\ \\ y_n \end{bmatrix}$$

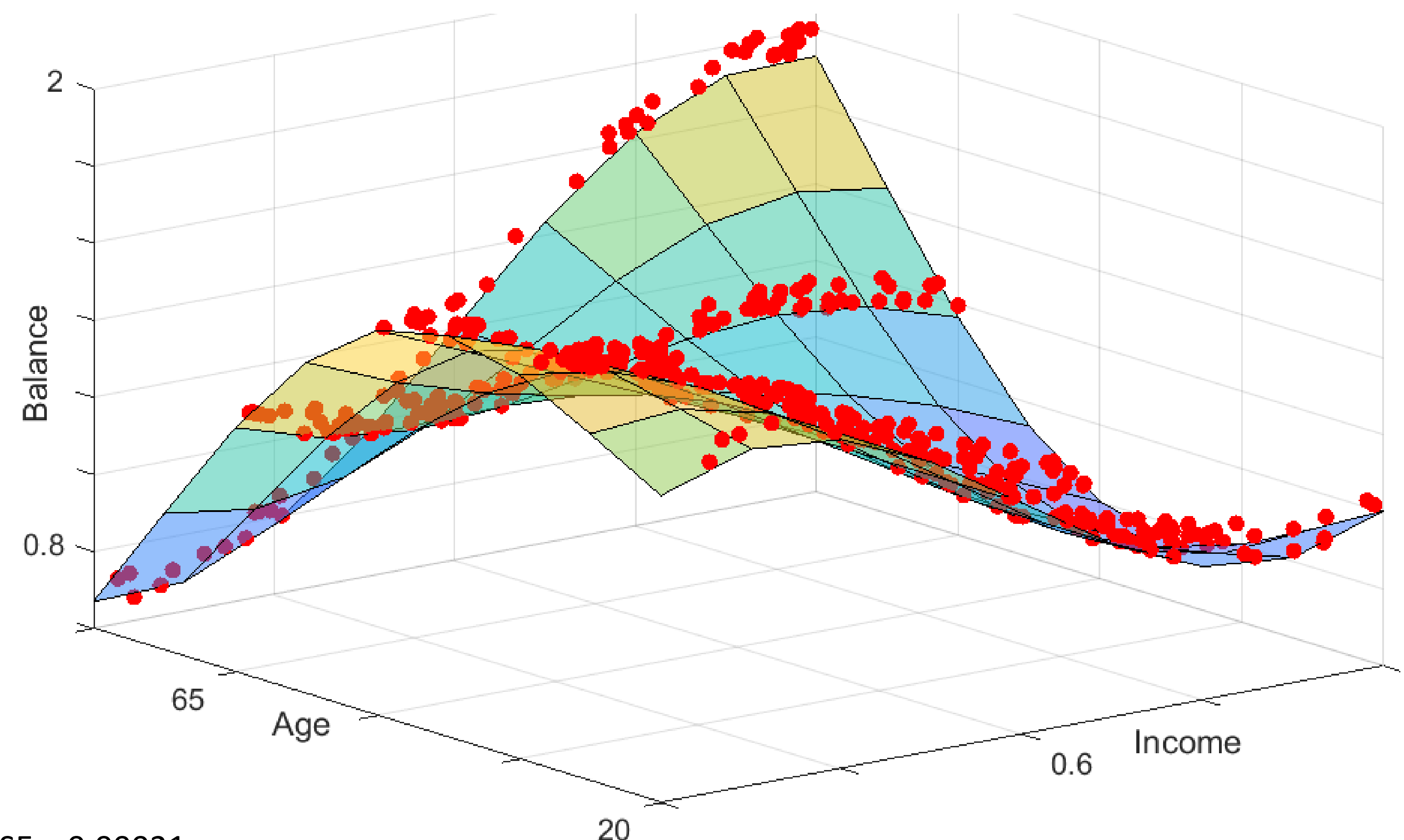
$$u = \begin{bmatrix} \beta_5 \\ \beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = (A^T A)^{-1} \; A^T Y$$

The Least Square problem reduces to

$$Min_{(u)} \quad J(u) = \frac{1}{2}(Y - Au)^T (Y - Au)$$




Train RMSE = 0.4621




Train RMSE = 0.00021

$$\begin{bmatrix} x_2^3 \\ x_1^3 \\ x_1 x_2^2 \\ x_1^2 x_2 \\ x_2^2 \\ x_1^2 \\ x_1 x_2 \\ x_2 \\ x_1 \\ 1 \end{bmatrix} \begin{matrix} \to \varphi_9(x) \\ \to \varphi_8(x) \\ \to \varphi_7(x) \\ \\ \\ \\ \\ \\ \\ \to \varphi_0(x) \end{matrix}$$

$$\varphi(x) = \begin{bmatrix} \varphi_9(x) \\ \varphi_8(x) \\ \vdots \\ \varphi_0(x) \end{bmatrix}$$

$$\frac{(m+n)!}{m!\ n!}$$

$$w^T \varphi(x) = ?$$

$$x \in R^2 \qquad (x_1, x_2)$$

$$\varphi_0(x) = 1$$

$$\varphi_1(x) = x_1$$

$$\varphi_2(x) = x_2$$

$$w = \begin{bmatrix} \beta_9 \\ \beta_8 \\ \vdots \\ \beta_0 \end{bmatrix}$$

$$\beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$w^T x \qquad w = \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

$$\min_{w \in \mathbb{R}^{10}} \frac{1}{n} \sum_{i=1}^{n} \left( y_i - w^T \varphi(x_i) \right)^2, \text{ where}$$

$$\varphi(x_i) = \begin{bmatrix} \varphi_9(x_i) \\ \vdots \\ \varphi_0(x_i) \end{bmatrix} = \begin{bmatrix} x_{2i}^3 \\ x_{1i}^3 \\ x_{1i} x_{2i}^2 \\ - \\ = \\ - \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} x_{12}^3 & x_{11}^3 & x_{11} x_{12}^2, & x_{11}^2 x_{12}, & \cdots & , 1 \\ x_{22}^3 & x_{21}^3 & x_{21} x_{22}^2 & x_{21}^2 x_{22} & & 1 \\ - & - & - & - & & - \\ x_{n2}^3 & x_{n1}^3 & x_{n1} x_{n2}^2 & x_{n1}^2 x_{n2}, & \cdots & 1 \end{bmatrix}$$

$$Au = \begin{bmatrix} \beta_9 x_{12}^3 + \beta_8 x_{11}^3 + \beta_7 x_{11} x_{12}^2 + \cdots & \beta_0 \\ \beta_9 x_{22}^3 + \beta_8 x_{21}^3 + \beta_7 x_{21} x_{22}^2 + \cdots & \beta_0 \\ = & \end{bmatrix}$$

$$A = \begin{bmatrix} \varphi_9(x_1), & \varphi_8(x_1) & \cdots & \varphi_0(x_1) \\ \varphi_9(x_2) & \varphi_8(x_2) & & \varphi_0(x_2) \\ & & & \\ \varphi_9(x_n) & \varphi_8(x_n) & & \varphi_0(x_n) \end{bmatrix}$$

$$y = \bar{w} = \begin{bmatrix} \beta_9 \\ - \\ - \\ - \\ \beta_0 \end{bmatrix} = (A^T A)^{-1} A^T y$$

# Multiple Regression model working with k variables

| $x_1$ | $x_2$ | x3 | -- | $x_k$ |
|---|---|---|---|---|
| 32 | 0.550798 | 283 | -- | 2 |
| 22 | 0.708148 | 483 | - | 3 |
| 45 | 0.290905 | 514 | - | 4 |
| 78 | 0.510828 | 681 | - | 3 |
| 54 | 0.892947 | 357 | - | 2 |
| 39 | 0.896293 | 569 | - | 4 |
| 42 | 0.125585 | 259 | - | 2 |
| 51 | 0.207243 | 512 | - | 2 |
| 21 | 0.051467 | 266 | - | 5 |
| 19 | 0.44081 | 491 | - | 3 |

| y Balance (thousand dollar) |
|---|
| 5.651202 |
| 7.321263 |
| 5.167304 |
| 5.609367 |
| 9.406379 |
| 9.379439 |
| 2.734997 |
| 4.876649 |
| 3.584138 |
| 5.437239 |

$\bar{w} \in R^{k+1}$

$w^T \phi(x)$

# Multiple Regression model working with k variables

$\bar{w}^T \bar{x}$

Linear Function:-

$w^T x + \beta_0 \to b$

$$f(x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \ldots + \beta_k x_k$$

$w^T x + b$

$w \in R^k \quad b \in R$

$$W = \begin{bmatrix} \beta_k \\ \beta_{k-1} \\ \vdots \\ \beta_1 \\ \beta_0 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_1 \\ 1 \end{bmatrix}$$

For given Training Set T = { $(x_{11}, x_{12}, \ldots, x_{1k}, y_1), (x_{21}, x_{22}, \ldots, x_{2k}, y_2), \ldots, (x_{n1}, x_{n2}, \ldots, xnk, y_n)$ }, we solve

$$\text{Min } J(\beta_k, \ldots, \beta_4, \beta_3, \beta_2, \beta_1, \beta_0) = \sum_{i=1}^{n} (yi - (\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \ldots + \beta_k x_k))^2$$

$$u = \begin{bmatrix} \beta_k \\ \\ \\ \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} \qquad A = \begin{bmatrix} x_{11} & x_{21} & -- & x_{1k} & 1 \\ x_{21} & x_{22} & -- & x_{2k} & 1 \\ \\ \\ x_{n1} & x_{n2} & -- & x_{nk} & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \\ \\ y_n \end{bmatrix}$$

The Least Square problem reduces to

$$Min_{(u)} \quad J(\text{u}) = \quad (Y - Au)^T \, (Y - Au)$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}$$

$$\underline{\underline{u}} = \begin{bmatrix} \beta_k \\ \\ \\ \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = (A^T A)^{-1} \; A^T Y$$

https://colab.research.google.com/drive/1APfTBXi3U_1ADX1PmLkJ8DyhVCQBegtE?usp=sharing

$\beta_0 \leftarrow \quad \phi_0(x) = 1$

$\beta_1 \leftarrow \quad \phi_1(x) = x_1$

$\phi_2(x) = x_3$

$\vdots$

$\phi_K(x) = x_K$

$\phi_{K+1}(x) = x_1 x_2$

$\phi_{K+2}(x) = x_2 x_3$

$\vdots$

$\beta_m \leftarrow \quad \phi_m(x) = x_K^2$

$$m = \frac{(K+1)(K+2)}{2}$$

$$x \in R^n$$

$$x = \frac{(K+2)!}{K!\,2!}$$

$$= \sum_{i=0}^{m} \beta_i \varphi_i(x) = w^T \varphi(x)$$

$$\Rightarrow w = \begin{bmatrix} \beta_m \\ \beta_{m-1} \\ - \\ \beta_1 \\ \beta_0 \end{bmatrix} \qquad \varphi(x) = \begin{bmatrix} \varphi_m(x) \\ \varphi_{m-1}(x) \\ \\ \varphi_1(x) \\ \varphi_0 \end{bmatrix}$$

$$\min_{w} \sum_{i=1}^{N} \left(y_i - w^T \varphi(x_i)\right)^2$$

$$w = (A^T A)^{-1} A^T y$$

$$f(\cancel{x}) = \beta_0 + \cancel{\beta_1 x_1} + \beta_1 x_1 + \beta_2 x_2$$

$$\min_{\beta_0,\beta_1,\beta_2} \left( \sum_{i=1}^{N} \left( y_i - \{\beta_0 + \beta_1 x_{1i} - \beta_2 x_{2i}\} \right)^2 \right)$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$(y - Au) = \begin{bmatrix} y_1 - \beta_2 x_{11} \\ \quad - \beta_2 x_{12} \\ \quad - \beta_0 \\ \\ \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad A = \begin{bmatrix} x_{11} & x_{12} & 1 \\ x_{21} & x_{22} & 1 \\ \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & 1 \end{bmatrix} \qquad u = \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

$$y_i -$$

$$(y - Au) = \begin{bmatrix} y_1 - \beta_2 x_{11} - \beta_1 x_{12} - \beta_0 \\ \vdots \\ \vdots \end{bmatrix}$$

$$\min_u (y - Au)^T (y - Au) =$$

$$A^T(y - Au) = 0$$

$$\Rightarrow \quad u = (A^T A)^{-1} A^T y$$

$\varphi_2(\cdot$ $\qquad \varphi_1(x)$ $\qquad \varphi_2(x$ $\qquad x$ $\qquad \varphi_4$

$$f(x_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{1i} x_{2i} + \beta_4 x_{1i}^2$$

$\varphi_1(x)$

$$+ \beta_5 x_{2i}^2$$

$$\begin{bmatrix} x_{11}^2 & x_{21}^2 & x_{11} & x_{21} & x_{11} & x_{21} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1}^2 & x_{n2}^2 & x_{n1} & x_{n2} & x_{n1} & x_{n} & 1 \end{bmatrix}$$

$$f(\cancel{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \beta_4 x_1^2 + \beta_5 x_2^2 + \beta_6 x_1 x_2^2 + \beta_7 x_1^2 x_2 + \beta_8 x_1^3 + \beta_9 x_2^3$$

$$= W^T \phi(x)$$

$$\phi_0(x) = 1$$

$$\phi_1(x) = x_1$$

$$\phi_2(x) = x_2$$

$$\vdots$$

$$\phi_9(x) = x_2^3$$

$$\phi(x) = \begin{bmatrix} \phi_0(x) \\ \phi_2(x) \\ \vdots \\ \phi_9(x) \end{bmatrix}$$

$$W = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_9 \end{bmatrix}$$

$$A = \begin{bmatrix} \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_q(x_1) \\ \phi_0(x_2) & \cdots & & \\ & & & \\ \phi_0(x_N) & \phi_1(x_N) & & \phi_q(x_N) \end{bmatrix}$$

$$f(x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k + \beta_{k+1} x_1 x_2 + \beta_{k+2} x_2 x_3 + \dots$$

$$+ \beta_M x_k^m$$

( k - dim
m - order )

where $\cancel{M} = \cancel{\dfrac{k!\,m!}{(k+r)}}$ $k$

$$M = \frac{(k+m)!}{k!\,m!}$$

$$M = \frac{(k+m)!}{k!\,m!}$$

$\Rightarrow$ Here,

$$\Phi_0(x) = 1$$

$$\Phi_1(x) = x_1$$

$$\Phi_2(x) = x_2$$

$$\vdots$$

$$\Phi_M(x) = x_k^m$$

$$\Phi(x) = \begin{bmatrix} \Phi_M(x) \\ \Phi_{M-1}(x) \\ \vdots \\ \\ \Phi_0(x) \end{bmatrix}$$

$$\omega = \begin{bmatrix} \beta_M \\ \beta_{M-1} \\ \vdots \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

$\Rightarrow$ Now,

$$f(x) = \omega^T \Phi(x)$$

$\Rightarrow$ Error $\Rightarrow$ $J(w)$

~~min w~~

$$\min_{w} \sum_{i=1}^{N} \left( y_i - w^T \Phi(x_i) \right)^2 + \frac{\lambda}{2} w^T w$$

$$A = \begin{bmatrix} \Phi_m(x_1) & \Phi_{m-1}(x_1) & \cdots & \Phi_0(x_1) \\ \Phi_m(x_2) & \Phi_{m-1}(x_2) & \cdots & \Phi_0(x_2) \\ \vdots & & \ddots & \vdots \\ \Phi_m(x_N) & \Phi_{m-1}(x_N) & \cdots & \Phi_0(x_N) \end{bmatrix}$$

$\Rightarrow$ $w = (A^T A)^{-1} A^T y$ $\Rightarrow$ $(A^T A + \lambda I)^{-1} A^T y$

$$(A^T A + \lambda I)^{-1} A^T y$$

$$\min_{w} J(w) = \frac{1}{2}\sum_{i=1}^{N}\left(y_i - w^T\varphi(x_i)\right)^2 + \frac{\lambda}{2} w^T w$$

$w \in R^m$

$$\nabla_w J(w) = \boxed{\lambda w + \sum_{i=1}^{N} -\left(y_i - w^T\varphi(x_i)\right)\varphi(x_i)}$$

$$w^{(0)} \in R^m$$

$$w^{(1)} = w^{(0)} - \eta_0\left(\lambda w^{(0)} + \sum_{i=1}^{N} -\left(y_i - w^{(0)T}\varphi(x_i)\right)\varphi(x_i)\right)$$

$$w^{(2)} = w^{(1)} - \eta_0\left(\lambda w^{(1)} + \sum_{i=1}^{N} -\left(y_i - w^{(0)T}\varphi(x_i)\right)\varphi(x_i)\right)$$

## Gradient Descent Algo

Intialize $w^0 \in R^m$

Repeat

$$w^{(k+1)} = w^{(k)} - \eta_k \left( -\sum_{i=1}^{N} \left( y_i - w^{(k)^T} \phi(x_i) \right) \phi(x_i) + \lambda w^{(k)} \right)$$

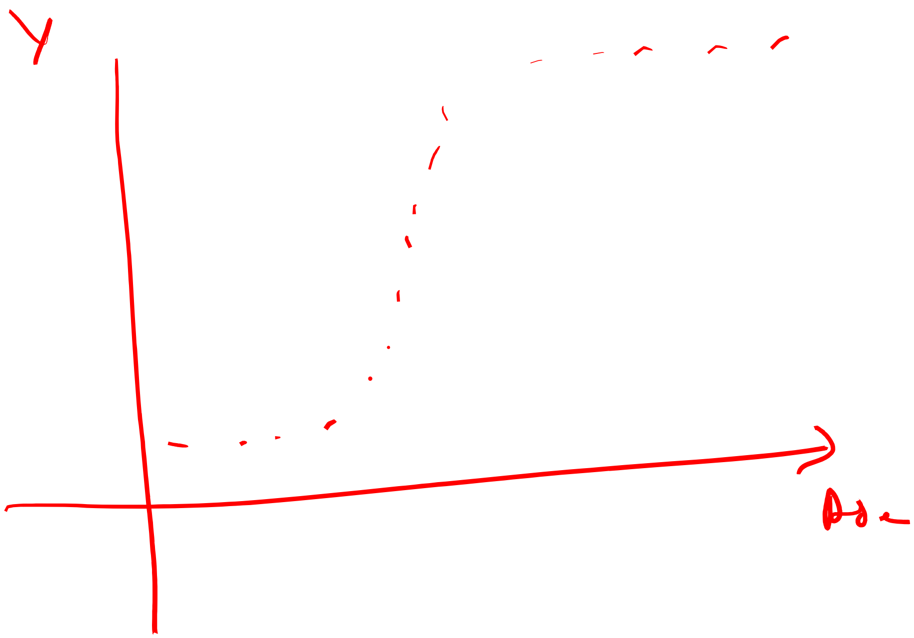
Until $\left\| -\sum_{i=1}^{N} \left( y_i - w^{(k)^T} \phi(x_i) \right) \phi(x_i) + \lambda w^{(k)} \right\| \leq \epsilon_{tol}$

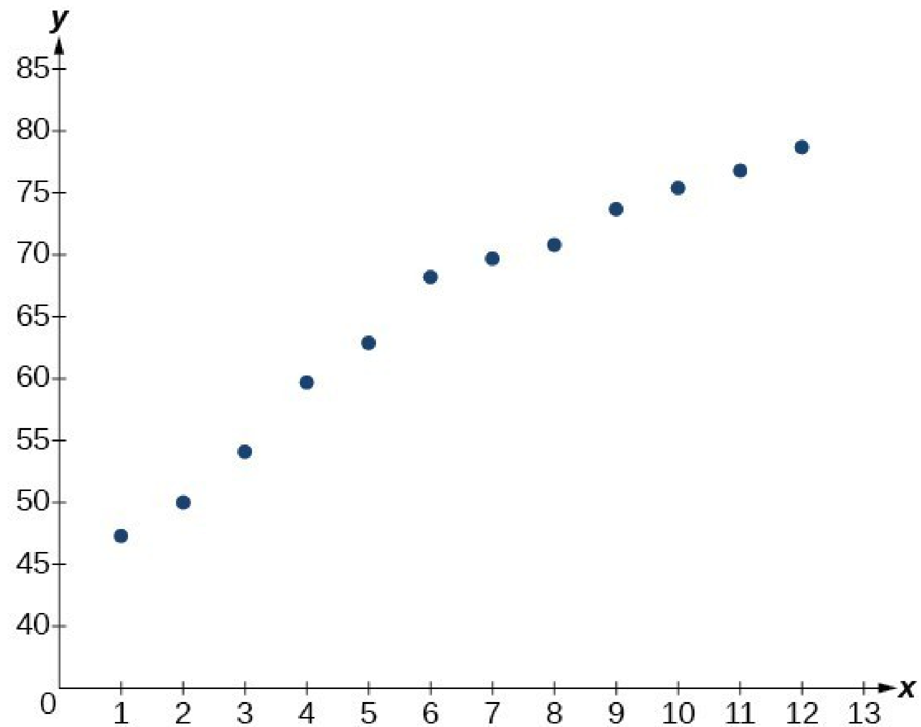
$$\cancel{\eta_k = (0.1)^{0.99}}$$

$$\eta_k = \frac{\eta_{k-1}}{c}$$

$$\eta_k = (\eta_{k-1})^{0.99}$$

# Relationship in data



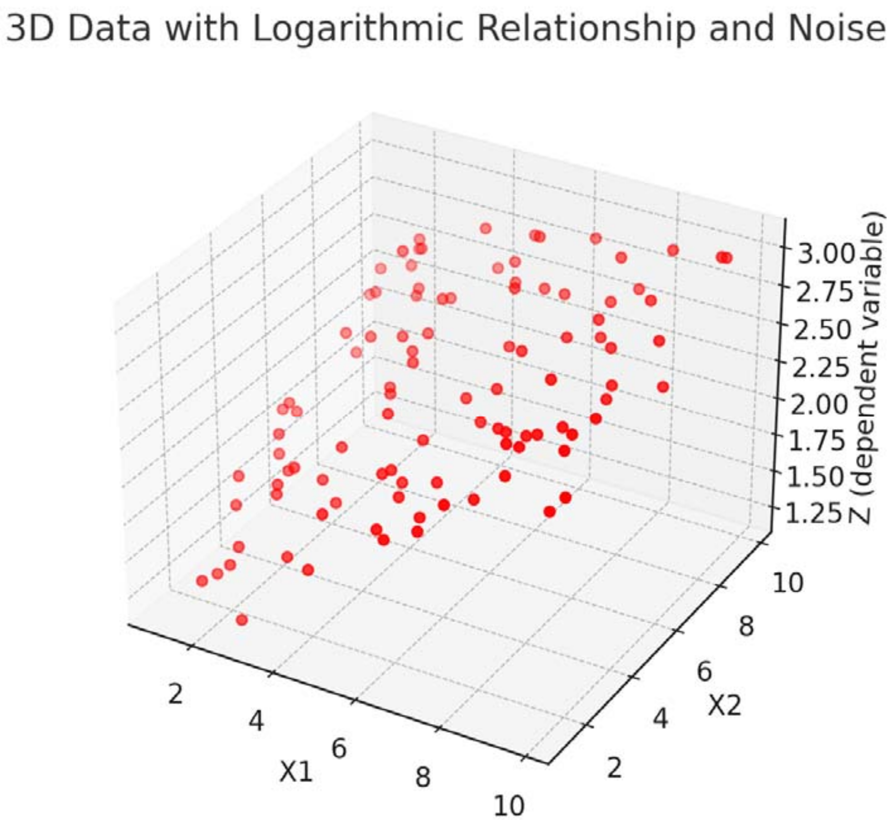
Intialize $w^0 \in R^m$

Repeat

Pick a subset $B$ from Training set $T$ with cardinality $K$

$$w^{(k+1)} = w^k - \eta_k \left( \sum_{i \in B} - \varphi(x_i)\left(y_i - w^{(k)T}\varphi(x_i)\right) + \lambda w^{(k)} \right)$$

end

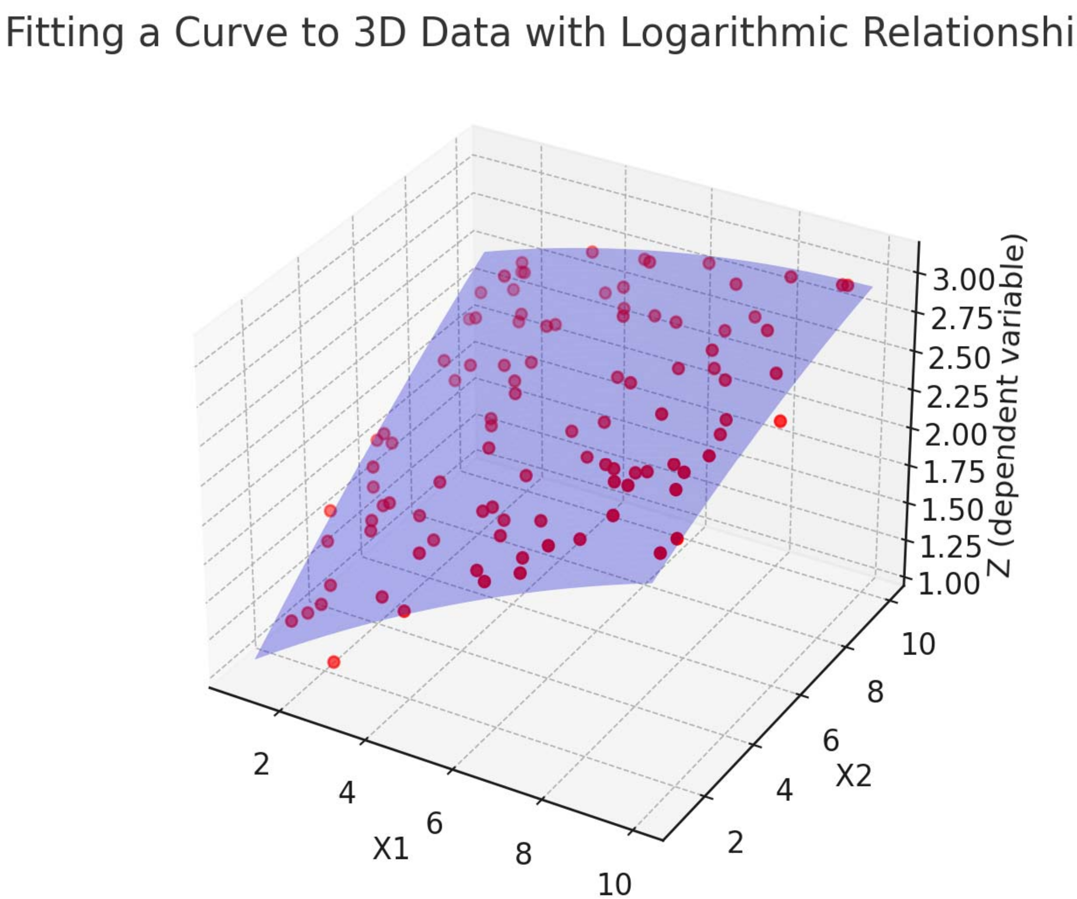


3D Data with Logarithmic Relationship and Noise
Z (dependent variable)
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
X1
2
4
6
8
10
X2
2
4
6
8
10

$\varphi_1(x) = x_1$ ✓

$\varphi_2(x) = x_2$ ✓

$\varphi_6(x) = x_1^3 x_2$ ✓



$$\varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_m(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_1^k \end{bmatrix}$$

$$\begin{bmatrix} \exp -\frac{1}{2s} \\ \\ \\ \end{bmatrix}$$

## Radial Basis Function (RBF)

- We already have learned about polynomial basis functions
- Another class are radial basis functions (RBF). Typical representatives are Gaussian basis functions

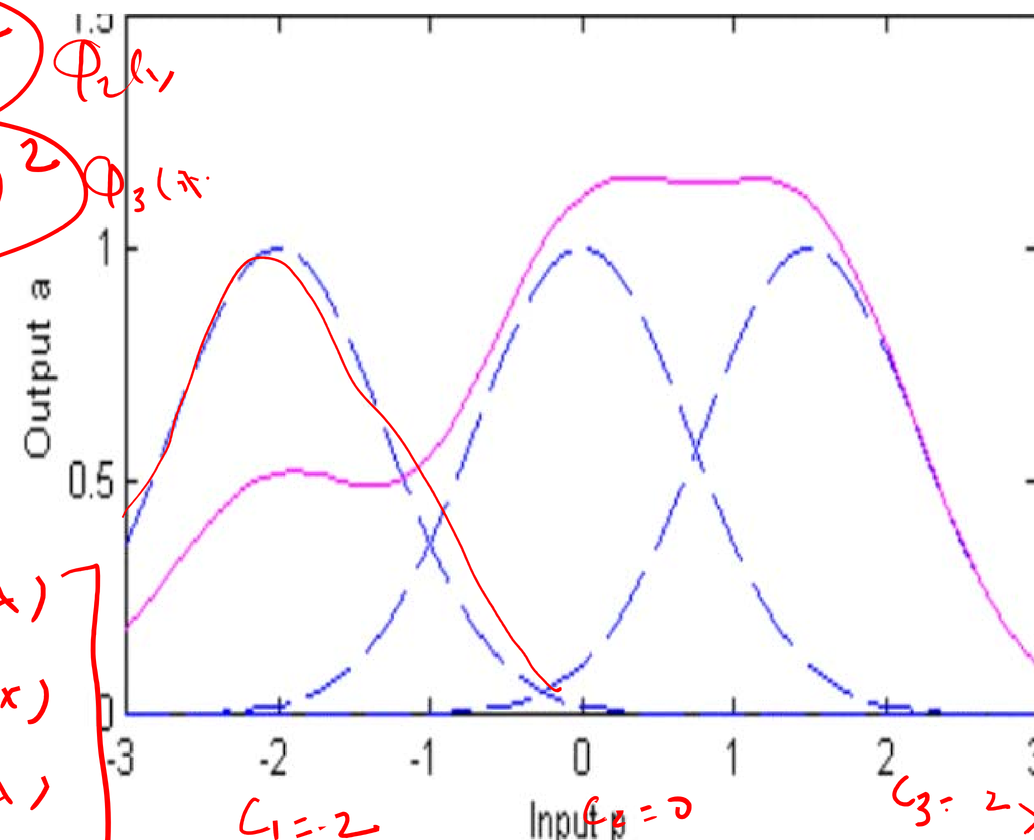
$$\phi_j(\mathbf{x}) = \exp\left(-\frac{1}{2s_j^2}\underline{\|\mathbf{x} - \mathbf{c}_j\|}^2\right)$$

$$f(x) = \underbrace{\beta_0}_{\varphi_0(x)} + \beta_1 \underbrace{\exp\left(-\frac{1}{2s_1}\|x - c_1\|^2\right)}_{\varphi_1(x)}$$

$$+ \beta_2 \underbrace{\exp -\frac{1}{2s_2}\|x - c_2\|^2}_{\varphi_2(x)}$$

$$+ \beta_3 \underbrace{\exp -\frac{1}{2s_3}\|x - c_3\|^2}_{\varphi_3(x)}$$

Three RBFs (blue) form $f(x)$ (pink)




$c_1 = -2$, $s_1 = 2$

$c_2 = 0$, $s_1 = 4$

$c_3 = 2$, $s_1 = 2$

$$f(x) = w^T \varphi(x)$$

$$w = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \quad \varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \\ \varphi_4(x) \end{bmatrix}$$

$$\min_{w} \frac{1}{2} \sum_{i=1}^{N} \left( y_i - w^T \varphi(x_i) \right)^2 + \frac{\lambda}{2} w^T w$$

$$A = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \varphi_3(x_1) & \varphi_4(x_1) \\ \cdots & \cdots & \cdots & \\ \varphi_1(x_N) & \varphi_2(x_N) & \varphi_3(x_N) & \cdots \end{bmatrix}$$

Testing set: $\{ (x_1, y_1) \ldots (x_k, y_k) \}$

$f \rightarrow$

$$RMSE = \sqrt{\frac{1}{k} \sum_{i=1}^{k} \left( y_i - f(x_i) \right)^2}$$

$$\frac{SSE}{SST} = \frac{\sum_{i=1}^{k} \left( y_i - f(x_i) \right)^2}{\sum_{i=1}^{k} \left( y_i - \bar{y} \right)^2}$$

$NMSE =$

$$MAE = \frac{1}{k} \sum_{i=1}^{k} \left| y_i - f(x_i) \right|$$