1 Normal Subgroups

Consider $S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$.
For the subgroup $H = \{e, \tau\}$ the left cosets are

$$H, \ \ \sigma H = \{\sigma, \sigma\tau\}, \sigma^2 H = \{\sigma^2\tau\}$$

The right cosets are

$$H, \ \ H\sigma = \{\sigma, \sigma^2\tau\}, H\sigma^2 = \{\sigma^2, \sigma\tau\}$$

The collection of left cosets and the collection of right cosets are different.

Now consuder the subgroup $N = \{e, \sigma, \sigma^2\}$.
The collection of left cosets is

$$N, \tau N = \{\tau, \sigma^2\tau, \sigma\tau\}$$

The collection of right cosets is

$$N, N\tau = \{\tau, \sigma\tau, \sigma^2\tau\}$$

We see that the collection of left cosets is the same as the collection of the right cosets.

If N is a subgroup of G then aNa^{-1} is also a subgroup of $G, \ \forall \ a \in G$ (verify).
It may so happen that $aNa^{-1} = N$.
$\Rightarrow \ \ aN = Na \ \forall \ a \in G$.
This is the case with the subgroup N of S_3.
Such subgroups are called Normal subgroups.

- *Def.***Normal Subgroup**
  A subgroup N of a group G is called a normal subgroup if $aNa^{-1} \subseteq N, \ \forall \ a \in G$.
- For any group G the trivial subgroups $\{e\}$ and G are normal subgroups of G.
- All subgroups of an abellain group are normal since $aNa^{-1} = N$ always.

- *Lemma* 7:
  N is a normal subgroup of G if and only if $aNa^{-1} = N, \ \forall \ a \in G$.

Proof:
if part:

Consider $a \in G$.
$aNa^{-1} = N \ \ \Rightarrow \ \ aNa^{-1} \subseteq N$. So N is normal.

only if part:

N is normal.
Consider $a \in G$.

$aNa^{-1} \subseteq N \;\; \Rightarrow \;\; N \subseteq a^{-1}Na.$

Also $a^{-1}Na \subseteq N$ since $a^{-1} \in G$.

So we conclude $aNa^{-1} = N$.

- *Lemma* 8:
  A subgroup N of a group G is normal if and only if every left coset of N is also a right coset.

Proof:
only if part:

Let N be normal .
$\therefore aNa^{-1} = N, \;\forall a \in G \;\; \Rightarrow \;\; aN = Na \;\forall a \in G.$

So every left coset of N is also a right coset.

if part:

Every left coset is equal to a right coset.

So let $aN = Nb,$ for $a, b \in G$.

$a \in aN, \Rightarrow a \in Nb$

Also $a \in Na \;\; \Rightarrow \;\; Nb = Na$

$\therefore \; aN = Na \Rightarrow aNa^{-1} = N.$

So N is a normal subgroup of G.

- If H and K are two subgroups of G, what about the subset HK of G?
- We can show that if $HK = KH$ then HK is also a subgroup of G.
- If one of these subgroups, say, K is normal then in view of lemma 8 $HK = KH, \;\Rightarrow\;\; HK$ is also a subgroup.
- Is HK normal?

  If both H and K are normal then HK is normal.

Structure in the set of cosets

2 Normal Subgroups contd...

Let N be a subgroup of G. Consider the collection of right cosets of N in G i.e $\{Na|a \in G\}$
When we multiply(set multiplication) two such right cosets do we get a right coset?
This happens only if N is a normal subgroup.

- *Lemma*: 9
  N is a normal subgroup of G if and only if the product of two right(left) cosets of N is also a right(left) coset of N in G.

  proof:
  only if part

  Let N be a normal subgroup of G.

  Consider two right cosets Na and Nb of N in G.
  Consider the product $NaNb$. We have

$$NaNb = N(aN)b = N(Na)b = (NN)(ab) = N(ab)$$

which is a right coset of N in G.

if part:

Let $NaNb = Nc$, a right coset.

$a \in Na, \ \ b \in Nb \ \ \Rightarrow \ \ ab \in NaNb$

$\Rightarrow \ \ ab \in Nc, \ \ \Rightarrow \ \ Nc = Nab$

$\therefore NaNb = Nab$
If $b = a^{-1}$ then $NaNa^{-1} = Ne = N$

Let $x \in aNa^{-1}$.

Then $Nx \subseteq N$.

Since $x \in Nx \ \ \Rightarrow \ \ x \in N$.
$\therefore aNa^{-1} \subseteq N \ \ \Rightarrow \ \ N$ is a normal subgroup of G.
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- For a normal subgroup N since $Na = aN$ we simply say cosets of N and not right or left cosets of N.

- Lemma 9 says that the collection of all cosets of a normal subgroup N of G is closed under the set multiplication.
  We denote this collection as G/N.

- *Proposition* 6:
  If N is a normal subgroup of G then G/N forms a group under the multiplication of cosets. This group is called the quotient group or the factor group of G by N.

- $o(G/N) = [G : N] = o(G)/o(N)$ if G is finite.

Eg.

$$G = \langle \mathbb{R}, + \rangle, \;\; N = \langle \mathbb{Z}, + \rangle$$

$$G/N = \mathbb{R}/\mathbb{Z} = \{\mathbb{Z}x | 0 \leq x < 1\}$$

One of the elements(a coset) of G/N is

$\mathbb{Z}(0.36) = \{.... - 2.64, -1.64, -0.64, 0.36, 1.36, 2.36, ...\}$

$\mathbb{Z}(0.36) \cdot \mathbb{Z}(0.75) = \mathbb{Z}(0.11)$

Identity element in G/N is $\mathbb{Z}$.
$(\mathbb{Z}x)^{-1} = \mathbb{Z}(1 - x)$

- *Def:*
  An element $b \in G$ is said to be a conjugate of $a \in G$ if $\exists \;\, x \in G$ such that $xax^{-1} = b$.

- If a is conjugate to b then we write $a \sim b$.

- Conjugacy is an equivalence relation on G (exercise).

- The group G gets partitioned into conjugacy classes. Let $C(a)$ denote the conjugacy class of a.
  Then $C(a) = \{x \in G | x \sim a\}$.

- Unlike the cosets the conjugacy classes are not all of the same size.
  for eg. $C(e) = \{e\}$. But the conjugacy class of other elements may contain more than one element.
  If G is an abellian group then $C(a) = \{a\}, \; \forall \;\, a \in G$.

- We denote $|C(a)| = c_a$.

Does $C(a)$ form a subgroup of G? (answer yourself)

- *Def:* Normalizer
  Let $a \in G$. The normalizer of a in G, denoted as $N(a)$ is a subset of G consisting of elements that commute with a.

$$N(a) = \{x \in G | xa = ax\}$$

- $N(a)$ is a subgroup of G.
  Proof is an exercise.

- $N(e) = G$.

- *Proposition* 7:
  If G is a finite group then

$$|C(a)| = [G : N(a)] = i_G(N(a))$$

Proof:
Let $gN(a)$ be a left coset of $N(a)$ in G.
Consider $x \in gN(a)$. Then $x = gh$, where $h \in N(a)$.

$\Rightarrow \quad xax^{-1} = g(hah^{-1})g^{-1} = gag^{-1}$
$\Rightarrow \quad$ all elements in $gN(a)$ yield the same element gag^{-1} in $C(a)$.

Consider $g_1, g_2 \in G$ such that
$g_1ag_1^{-1} = g_2ag_2^{-1} \quad \Rightarrow \quad g_2^{-1}g_1ag_1^{-1}g_2 = a$
$\Rightarrow \quad g_2^{-1}g_1 \in N(a) \quad \Rightarrow \quad g_1 \in g_2N(a)$
$\Rightarrow g_1, g_2$ belongs to the same coset of $N(a)$ in G.

........contd.
.........contd.
So elements from two distinct cosets cannot yield the same element in $C(a)$.

Hence there are as many elements in $C(a)$ as there are cosets of $N(a)$.
This proves the result.
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- Every normal subgroup is a union of certain conjugacy classes of G including $C(e) = \{e\}$.