SC421: Introduction to Modern Algebra **Assignment 1** **Jan 15, 2025**

DA-IICT, B.Tech, winter 2024

1. In the following determine whether the systems described are groups. If they are not, point out which of the Group axioms fail to hold.
   (a) $G =$ set of all integers, $a * b = a - b$.
   (b) $G =$ set of all positive integers, $a * b = ab$ the usual product of integers.
   (c) $G =$ set of all rational numbers with odd denominators, $a * b = a + b$ the usual addition.
   (d) Let S be a set. Which of the following is a group?
   (i) $\langle \mathcal{P}(S), \cup \rangle$, (ii) $\langle \mathcal{P}(S), \cap \rangle$, (iii) $\langle \mathcal{P}(S), \Delta \rangle$, where $A \Delta B = A \cup B \backslash A \cap B$

2. Let $G = \{\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} | x, y \in \mathbb{R}\}$. Let the binary operation on G be the matrix multiplication.
   (a) Is G closed under multiplcation ?
   (b) Find a matrix $E \in G$ such that $AE = A, \forall A \in G\}$.
   (c) Is $EA = A, \forall A \in G$?
   (d) Is G a group?
   (e) Show that in a group G, $a * e = a, \forall a \in G \quad \implies \quad e * a = a$

3. (a) A permutation is a one-one mapping of a set of n natural numbers onto itself. Justify that the set of all permutations on n objects denoted as S_n forms a group under the operation of composition of permutations.
   (b) Consider the set $\{1, 2, 3\}$. Consider the two permutations $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ of S_3 the permutation group on three elements. Write down all the elements of this group using composites of σ and τ. Work out the group table .

4. (a) Prove that a group of order 4 have to be abelian.
   (b) Write down all possible group table of order 4.

5. Let $G = \{(a, b) | a, b \in \mathbb{Q}\}$ and $G^* = G - \{(0, 0)\}$.
   (a) Let the binary operation in G^* be defined as $(a_1, b_1) \times (a_2, b_2) = (a_1 a_2, b_1 b_2)$. Is G^* along with this binary operation a group?
   (b) Now let the binary operation be defined as $(a_1, b_1) \times (a_2, b_2) = (a_1 a_2 + 2b_1 b_2, a_1 b_2 + b_1 a_2)$. Is G^* along with this binary operation a group?
   (c) Redefine G^* so that G^* along with the binary operation in part **(a)** forms a group.

6. If G is a group of even order , prove that it has an element $a \neq e$ such that $a^2 = e$.

7. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 2 and the determinant is non zero. Prove that G is a group of order 6 under the operation of matrix multiplication.