SC421: Introduction to Modern Algebra **Assignment 2 Jan 31, 2025**

DA-IICT, B.Tech, winter 2024

1. Prove that if H and K are subgroups of a group G then $H \bigcap K$ is also a subgroup of G.
   Is $H \bigcup K$ always a subgroup of G ?
   soln:
   Let $g_1, g_2 \in H \bigcap K$. Then $g_1, g_2 \in H, \implies g_1g_2 \in H$, by closure of H under the group operation.
   Similarly $g_1g_2 \in K$. So $g_1g_2 \in H \bigcap K$.
   Also $\forall g \in H \bigcap K, \ \ g^{-1} \in H$ and $g^{-1} \in K$.
   $\therefore \ \ g^{-1} \in H \bigcap K$.
   So by lemma 3 $H \bigcap K$ is a subgroup of G.

   $H \bigcup K$ is not always a subgroup of G. We consider a case to justify this. Let $h \in H$ but $h \notin K$ and $k \in K$ but $k \notin H$. Then we can't have $hk \in H \bigcup K$.
   Let us assume $hk \in H \bigcup K$. Then $hk \in H$ or $hk \in K$.
   If $hk \in H$ then $h^{-1}hk = k \in H$ since H is closed under the group operation. This is a contradiction to our consideration that $k \notin H$.
   We will arrive at a similar contradiction if we let $hk \in K$.
   $\therefore \ \ H \bigcup K$ is not a subgroup of G in general.

2. Let S be a subset of a group G. Let $N(S) = \{g \in G | gs = sg \ \forall \ s \in S\}$. Prove that $N(S)$ is a subgroup of G.

   soln:
   Consider $g_1, g_2 \in N(S)$. Then $g_1 s = s g_1$ and $g_2 s = s g_2 \ \forall \ s \in S$.
   $\therefore \ \ g_1 g_2 s = g_1 s g_2 = s g_1 g_2 \ \ \forall \ \ s \in S$.
   $\therefore \ \ g_1 g_2 \in N(S)$.
   Also $gs = sg \implies \ \ sg^{-1} = g^{-1}s$ by multiplying g^{-1} from left and from the right.
   $\therefore \ \ \forall \ g \in N(S), g^{-1} \in N(S)$.
   By lemma 3, $N(S)$ is a subgroup of G.

3. Let H and K be subgroups of a group G.
   Define HK as $HK = \{x \in G | x = hk, h \in H, \ k \in K\}$.
   Show that HK is a subgroup of G if and only if $HK = KH$.

   soln:
   if part:
   Suppose $HK = KH$. Let h_1k_1 and $h_2k_2 \in HK$.
   Consider $h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1k_3h_2^{-1}$, where $k_3 = k_1k_2^{-1} \in K$.
   Since $HK = KH, \ k_3h_2^{-1} = h_3k_4 \in HK$.
   Hence $h_1k_1(h_2k_2)^{-1} = h_1h_3k_4 = h_4k_4 \in HK$.
   We have proved earlier that if $a, b \in H \Rightarrow ab^{-1} \in H$ then H is a subgroup of G.

So HK is a subgroup of G.

only if part:
Let it be given that HK is a subgroup. We will first prove the following:
Given a group A let us define a set A^{-1} as $A^{-1} = \{x^{-1} | x \in A\}$. We will show that $A^{-1} = A$.
$x \in A \;\; \Rightarrow \;\; x^{-1} \in A \;\; \Rightarrow \;\; x \in A^{-1} \;\; \Rightarrow \;\; A \subseteq A^{-1}$.
$x \in A^{-1} \;\; \Rightarrow \;\; x^{-1} \in A \;\; \Rightarrow \;\; x \in A \;\; \Rightarrow \;\; A^{-1} \subseteq A$. So we have proved that $A^{-1} = A$.

Since HK is a group this result implies that $(HK)^{-1} = HK$.
$(HK)^{-1} = \{(hk)^{-1} = k^{-1}h^{-1} | h \in H, k \in K\} = K^{-1}H^{-1} = KH$.
Hence $KH = HK$. This completes the proof.

4. Write out all the right and left cosets of H in G where

   (a) $G = \langle a \rangle$ is a cyclic group of order 10 and $H = \langle a^2 \rangle$ is the subgroup of G generated by $\langle a^2 \rangle$.
   soln:
   $o(H) = 5$. H has only two cosets in G. The left and right cosets are same since the group operation is abelian. The cosets are $\{H = \{e, a^2, a^4, a^6, a^8\}, Ha = \{a, a^3, a^5, a^7, a^9\}\}$.

   (b) G as in part **(a)**, $H = \langle a^5 \rangle$ is the subgroup generated by a^5.
   soln
   $H = \{e, a^5\}$. $o(H) = 2 \;\Rightarrow\; [G : H] = 5$.
   The five cosets are $\{H, Ha = \{a, a^6\}, Ha^2 = \{a^2, a^7\}, Ha^3 = \{a^3, a^8\}, Ha^4 = \{a^4, a^9\}\}$.

5. If N is normal in G and $a \in G$ is of order $o(a)$. Prove that the order , m of Na in G/N is a divisor of $o(a)$.
   soln
   Let $o(a) = k$. Since N is a normal subgroup of G, $(Na)^k = N(a^k) = Ne = N$. So $o(Na)$ divides $k = o(a)$.

6. Let $o(a)$ be finite in G. If $b \in C(a)$ in G then show that $o(b) = o(a)$.
   soln
   Let $o(a) = k$ and $o(b) = l$. $b = gag^{-1}$ where $g \in G$. So $b^k = (gag^{-1})^k = g^k a^k g^{-k} = e$.
   This implies l divides k. Similarly k divides l.
   $\therefore \;\; k = l$, i.e, $o(b) = o(a)$.

7. N and M are two normal subgroups of G such that $N \cap M = \langle e \rangle$. Show that $nm = mn, \;\; \forall n \in N \;\text{ and }\; m \in M$
   soln
   Consider $nm \in nM$.
   Since M is a normal subgroup, $nM = Mn \;\; \forall \;\; n \in N$.

So $\exists m' \in M$ such that $nm = m'n$.
Now $nm \in Nm$ and $m'n \in m'N = Nm' \quad \Rightarrow \quad Nm = Nm' \quad \Rightarrow \quad N = Nm'm^{-1}$.
This implies $m'm^{-1} \in N \quad \Rightarrow \quad m'm^{-1} \in N \cap M \quad \Rightarrow \quad m'm^{-1} = e \quad \Rightarrow \quad m' = m$.
So $nm = mn \; \forall \; n \in N$ and $m \in M$

8. If H is a subgroup of G, let $N(H) = \{g \in G | gHg^{-1} = H\}$. Prove

   (a) $N(H)$ is a subgroup of G.

   (b) H is normal in $N(H)$.

   (c) H is normal in G iff $N(H) = G$.