1. Euler Lagrange Equation

Euler Lagrange Equation is the analog of $f'(x) = 0$ which helped us to find stationary points in a function. In this case it would help us find stationary functions in a functional.

Consider a example where A(x_1, y_1) and B(x_2, y_2) be the two points in a plane and find a function such that the functional is stationary.

$$I[f] = \int_{x_1}^{x_2} F(x, y, \frac{dy}{dx})dx$$

Boundary conditions are : $y(x_1) = y_1, y(x_2) = x_2$

Suppose y(x) be the stationary function it and satisfies the above boundary conditions. This y(x) can be also called the extremal of the functional.

Now we will introduce an arbitrary function such that

$$\eta(x), \eta(x_1) = \eta(x_2) = 0$$

One thing to remember here is that all the functions considered here have continuous second derivative.

Now we will define a function

$$\bar{y}(x) = y(x) + \epsilon\eta(x)$$

where ϵ is an arbitrary parameter. Essentially $\bar{y}$ represents the variation in the extremal y. $\bar{y}$ is also an arbitrary function. Also $\bar{y}$ satisfies the boundary conditions as η vanishes at boundary conditions

$$\bar{y}(x_1) = y(x_1) + \epsilon\eta(x_1) = y(x_1)$$
$$\bar{y}(x_2) = y(x_2) + \epsilon\eta(x_2) = y(x_2)$$

Now because of an arbitrary function η and the parameter ϵ, $\bar{y}$ can be represented as a family of curves. Now we wish to find a particular $\bar{y}$ such that the following functional gets stationary.

$$I[\epsilon] = \int_{x_1}^{x_2} F(x, \bar{y}, \frac{d\bar{y}}{dx})dx$$

A thing to note here is that the quantity I here depends only on the parameter ϵ. That is because x gets integrated and limits are applied and $\bar{y}$ depends upon ϵ. Thus, as I depends on a single variable, to make I stationary we will have to

$$\frac{dI}{d\epsilon} = 0$$

This means that $\bar{y}$(x) would become a stationary function which means $\bar{y}$(x) would become same as y(x). This happens when ϵ becomes zero.

$$\frac{dI}{d\epsilon}_{\epsilon=0} = 0$$

$$\frac{d}{d\epsilon}_{\epsilon=0}\int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')dx = 0$$

But as the $\frac{d}{d\epsilon}$ goes inside the integral it changes to partial derivative as there are two independent parameters x and ϵ.

$$\int_{x_1}^{x_2} \frac{\delta}{\delta\epsilon} F(x, \bar{y}, \bar{y}')dx = 0$$

Now applying the chain rule, we get

$$\int_{x_1}^{x_2} (\frac{\delta F}{\delta x}\frac{\delta x}{\delta\epsilon} + \frac{\delta F}{\delta\bar{y}}\frac{\delta\bar{y}}{\delta\epsilon} + \frac{\delta F}{\delta\bar{y}'}\frac{\delta\bar{y}'}{\delta\epsilon})dx = 0$$

The first term vanishes as x is independent of ϵ.

$$\int_{x_1}^{x_2} (\frac{\delta F}{\delta\bar{y}}\frac{\delta\bar{y}}{\delta\epsilon} + \frac{\delta F}{\delta\bar{y}'}\frac{\delta\bar{y}'}{\delta\epsilon})dx = 0$$

Now we can find that

$$\bar{y}(x) = y(x) + \epsilon\eta(x)$$

$$\bar{y}'(x) = y'(x) + \epsilon\eta'(x)$$

$$\frac{\delta\bar{y}}{\delta\epsilon} = \eta$$

$$\frac{\delta\bar{y}'}{\delta\epsilon} = \eta'$$

So the integral becomes

$$\int_{x_1}^{x_2} (\frac{\delta F}{\delta\bar{y}}\eta + \frac{\delta F}{\delta\bar{y}'}\eta')dx = 0$$

Now integrating the second term by parts

$$\int_{x_1}^{x_2} (\frac{\delta F}{\delta\bar{y}'}\eta')dx$$

$$\frac{\delta F}{\delta \bar{y}'}\int_{x_1}^{x_2}\eta' dx - \int_{x_1}^{x_2}(\int \eta')\frac{d}{dx}(\frac{\delta F}{\delta \bar{y}'})dx$$
$$\frac{\delta F}{\delta \bar{y}'}[\eta]_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\frac{d}{dx}(\frac{\delta F}{\delta \bar{y}'})dx$$

Now the value of η is zero at x_1 and x_2 first term disappears. Now putting the second term in the final integral we get,

$$\int_{x_1}^{x_2}(\frac{\delta F}{\delta \bar{y}}\eta - \eta\frac{d}{dx}\frac{\delta F}{\delta \bar{y}'})dx = 0$$
$$\int_{x_1}^{x_2}(\frac{\delta F}{\delta \bar{y}} - \frac{d}{dx}\frac{\delta F}{\delta \bar{y}'})\eta dx = 0$$

Also we know that ϵ is zero for this condition, so $\bar{y}$=y, and the final integral becomes

$$\int_{x_1}^{x_2}(\frac{\delta F}{\delta y} - \frac{d}{dx}\frac{\delta F}{\delta y'})\eta dx = 0$$

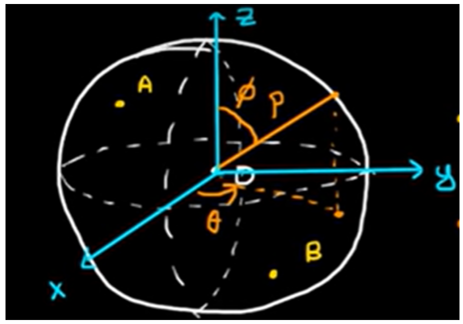
As η is any arbitrary function, to make sure that the integral is zero we will have to make sure that the bracket is zero for any arbitrary η.

$$\frac{\delta F}{\delta y} - \frac{d}{dx}\frac{\delta F}{\delta y'} = 0$$

Thus $\frac{\delta F}{\delta y} - \frac{d}{dx}\frac{\delta F}{\delta y'} = 0$ is the Euler-Lagrange's Equation. If y(x) is an extremum, then it must satisfy this equation. The Euler-Lagrange's equation is a necessary condition for y to be stationary but not necessary. Also this equation gives the functions that make the functional stationary. It doesn't give the nature of the function that is whether its minimum or maximum. We have to determine that by using other methods.

2. The Geodesic problem on a sphere

Now we already have the Euler-Lagrange's equation to find the stationary function in the functional. We will use this equation to solve the geodesic problem on a sphere.



Suppose we are given with a sphere of radius R with centre O. There are two points A(x_1, y_1) and B(x_2, y_2) lying on the surface of the sphere. We are asked to find the minimum distance between the points along a path across the sphere. So we would like to minimise

$$L = \int_A^B dS$$

Now using Pythagoras theorem we can replace dS with

$$L = \int_A^B \sqrt{dx^2 + dy^2 + dz^2}$$

But since we are moving along the sphere we have to write it in terms of spherical co-ordinates. Let there be a point P(x,y,z) on the sphere. We can write its co-ordinates as

$$x = R\cos\theta\sin\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\phi$$

where R is the radius of the sphere, θ is the angle P makes with X-axis, and ϕ is the angle P makes with Z-axis.

Now we will find dx,dy and dz in terms of θ, ϕ and R by partial differentiating.

$$dx = \frac{\delta x}{\delta\theta}d\theta + \frac{\delta x}{\delta\phi}d\phi$$
$$dy = \frac{\delta y}{\delta\theta}d\theta + \frac{\delta y}{\delta\phi}d\phi$$

$$dz = \frac{\delta z}{\delta \theta} d\theta + \frac{\delta z}{\delta \phi} d\phi$$

For z, it is independent of θ, so first term vanishes in dz, and after differentiating we get

$$dx = -R\sin\theta\sin\phi d\theta + R\cos\theta\cos\phi d\phi$$
$$dy = R\cos\theta\sin\phi d\theta + R\sin\theta\cos\phi d\phi$$
$$dz = -R\sin\phi d\phi$$

Now squaring and adding dx, dy and dz we get

$$dx^2+dy^2+dz^2 = R^2(\sin^2\theta\sin^2\phi d\theta^2+\cos^2\theta\cos^2\phi d\phi^2+\cos^2\theta\sin^2\phi d\theta^2+\sin^2\theta\cos^2\phi d\phi^2+\sin^2\phi d\phi^2)$$
$$dx^2+dy^2+dz^2 = R^2(d\theta^2\sin^2\phi + d\phi^2\cos^2\phi + \sin^2\phi d\phi^2)$$
$$dx^2+dy^2+dz^2 = R^2(d\phi^2 + d\theta^2\sin^2\phi)$$
$$dx^2+dy^2+dz^2 = R^2 d\phi^2(1+\sin^2\phi(\frac{d\theta}{d\phi})^2)$$

Now putting this in the integral

$$L = \int_{\phi_A}^{\phi_B} R\sqrt{1+\sin^2\phi(\frac{d\theta}{d\phi})^2}\, d\phi$$

Hence, we have obtained the functional of the form

$$L = \int_{\phi_A}^{\phi_B} F(\phi, \theta, \theta')$$

where θ' is $\frac{d\theta}{\phi}$ Now applying Euler-Lagrange's equation we get,

$$\frac{\delta F}{\delta \theta} - \frac{d}{d\phi}\frac{\delta F}{\delta \theta'} = 0$$

Now as F doesn't have any term in θ, first term vanishes and hence we can integrate both sides with respect to ϕ which would give a constant k on the right hand side.

$$\frac{d}{d\phi}\frac{\delta F}{\delta \theta'} = 0$$
$$\frac{\delta F}{\delta \theta'} = k$$

But we can compute $\frac{\delta F}{\delta \theta'}$ which is

$$\frac{\delta F}{\delta \theta'} = \frac{\theta'\sin^2\phi}{\sqrt{\sin^2\phi(\theta')^2+1}}$$

$$\frac{\theta' \sin^2 \phi}{\sqrt{\sin^2 \phi (\theta')^2 + 1}} = k$$
$$\theta'^2 \sin^4 \phi = k^2[\theta'^2 \sin^2 \phi + 1]$$
$$\theta'^2 = \frac{k^2}{\sin^4 \phi - k^2 \sin^2 \phi}$$
$$\theta' = \frac{d\theta}{d\phi} = \frac{k}{\sin \phi \sqrt{\sin^2 \phi - k^2}}$$

Now lets try solving this and obtain the expression in terms of θ and ϕ

$$\theta = \int \frac{k}{\sin \phi \sqrt{\sin^2 \phi - k^2}} d\phi$$

We can conclude that $k < 1$ to make sure that the expression stays real.

To solve this integral we make the substitution

$$u = k \cot \phi$$
$$\sin \phi = \frac{k}{\sqrt{k^2 + u^2}}$$
$$du = -k cosec^2 \phi d\phi$$

Thus integral becomes

$$\theta = -\int \frac{du \sin \phi}{\sqrt{\sin^2 \phi - k^2}}$$
$$\theta = -\int \frac{du}{\sqrt{1 - k^2 - u^2}}$$

As we know $k > 1$, $1 - k^2$ is positive and let it be a. Hence the final integration becomes

$$\theta = \int \frac{du}{\sqrt{a^2 - u^2}}$$
$$\theta = \arccos \frac{u}{a} + \theta_0$$

Now we will back substitute the value of k

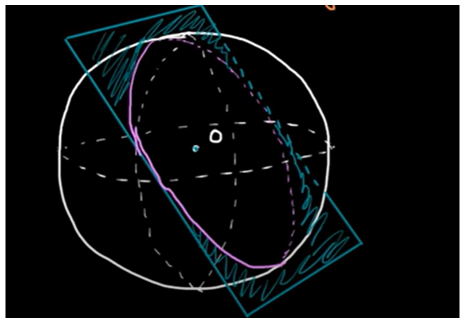
$$\theta = \arccos \frac{k \cot \phi}{a} + \theta_0$$

Let b be $\frac{k}{a}$, then the equation becomes

$$\theta = \arccos b \cot \phi + \theta_0$$

The constants b and θ_0 can be obtained by utilising the initial conditions.

Now if we look at this equation, we observe that by just looking at the equation, we can't really tell what kind of path we would have to follow to get the minimum distance. We will gain some practical knowledge about this equation with the help of great circle.



A great circle is the intersection of a sphere and a plane passing through the centre of the sphere. Now, we will try obtaining the equation of the great circle.

Assuming that the centre of the sphere is at origin. We know what is the equation of the plane passing through the origin which is

$$Ax + By + Cz = 0$$

Now as this plane passes through the sphere, we will replace x,y and z with spherical co-ordinates.

$$x = R\cos\theta\sin\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\phi$$

The equation thus becomes

$$AR\cos\theta\sin\phi + BR\sin\theta\sin\phi + CR\cos\phi$$
$$A\cos\theta + B\sin\theta = -C\cot\phi$$
$$\sqrt{A^2+B^2}(\frac{A}{\sqrt{A^2+B^2}}\cos\theta + \frac{B}{\sqrt{A^2+B^2}}\sin\theta) = -C\cot\phi$$
$$\sqrt{A^2+B^2}\cos\theta - \theta_0 = -C\cot\phi$$

Here, $\theta_0 = \arctan\frac{B}{A}$, Thus we can obtain theta as,

$$\theta - \theta_0 = \arccos\beta\cot\phi$$

$$\text{where } \beta \text{ is } \frac{-C}{\sqrt{A^2 + B^2}}$$

This equation is same as the path of geodesic that we obtained for a sphere using variational calculus with the help of Euler-Lagrange's Equation. This means that if we want to get the shortest distance between two points on a sphere, we just need to construct a great circle passing through those two points and the segment or arc between those two points of the great circle gives us the geodesic.

3. Beltrami Identity

The Beltrami identity, named after Eugenio Beltrami, is a special case of the Euler-Lagrange equation in the calculus of variations. The Euler-Lagrange equation is used to find functions that extremize action functionals in the form of integrals.

The action functional is defined as:

$$I[u] = \int_a^b L(x, u(x), u'(x))\, dx,$$

where a and b are constants, $u(x)$ represents the unknown function we seek, $u'(x)$ denotes its derivative with respect to x, and $L(x, u(x), u'(x))$ is a function of x, $u(x)$, and $u'(x)$ called the Lagrangian.

If the partial derivative of L with respect to x, denoted as $\frac{\partial L}{\partial x}$, is equal to zero ($\frac{\partial L}{\partial x} = 0$), then the Euler-Lagrange equation reduces to the Beltrami identity:

$$L - u'(x)\frac{\partial L}{\partial u'} = C,$$

where C is a constant.

To derive the Beltrami identity, we start by applying the chain rule to the derivative of L with respect to x:

$$\frac{dL}{dx} = \frac{\partial L}{\partial x}\frac{dx}{dx} + \frac{\partial L}{\partial u}\frac{du}{dx} + \frac{\partial L}{\partial u'}\frac{du'}{dx}.$$

Since we assume that $\frac{\partial L}{\partial x} = 0$, we can simplify the expression to:

$$\frac{dL}{dx} = \frac{\partial L}{\partial u'}\frac{du'}{dx} + \frac{\partial L}{\partial u}\frac{du}{dx}.$$

Using the Euler-Lagrange equation, which states $\frac{\partial L}{\partial u} = \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right)$, we can substitute this expression into the previous one:

$$\frac{dL}{dx} = u'\frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) + u''\frac{\partial L}{\partial u'}.$$

By rearranging terms and applying the product rule, the right side becomes:

$$\frac{dL}{dx} = \frac{d}{dx}\left(u'\frac{\partial L}{\partial u'}\right).$$

Integrating both sides with respect to x and combining the terms, we obtain the Beltrami identity:

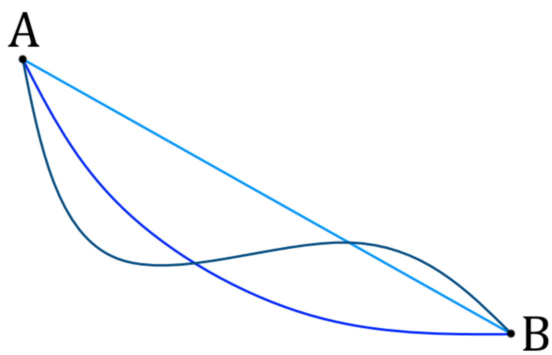
$$L - u' \frac{\partial L}{\partial u'} = C.$$

This equation represents a special case of the Euler-Lagrange equation, and it is known as the Beltrami identity.

4. The Brachistochrone Problem

Problem:

Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time.



Calculus Approach:

let point A is at (0.H) and point B is at (L,0).

Now we have to minimize $T = \int_A^B dt$ with respect to $y = f(x)$.

$$dt = \frac{ds}{v(x,y)}$$

ds as we know according to pythagoras theorem is :

$$ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$$

and v(x,y) according to the conservation of energy is:

$$mg(H-y) = \frac{1}{2}mv^2$$

$$v(x,y) = \sqrt{2g(H-y)}$$

Putting this values in the equation we get:

$$T = \int_A^B dt = \int_A^B \frac{ds}{v(x,y)} = \int_A^B \frac{\sqrt{1 + (\frac{dy}{dx})^2} dx}{\sqrt{2g(H-y)}}$$

Here we can use Beltrami's identity as the functional is not dependent on x. So using Beltrami's identity we get:

$$\frac{\sqrt{1+(y')^2} dx}{\sqrt{2g(H-y)}} - \frac{y'^2}{\sqrt{(1+y'^2)(2g(H-y))}} = C$$

Simplifying this equation we get:

$$C\sqrt{(1+(y')^2)(2g(H-y))} = 1$$
$$C^2(1+(y')^2)(2g(H-y)) = 1$$

Taking all the constants on one side and naming them as another constant C1 we get:

$$\frac{dy}{dx} = \sqrt{\frac{C1-(H-y)}{(H-y)}}$$
$$dx = \sqrt{\frac{H-y}{C1-(H-y)}}dy$$

Integrating both side we get:

$$x = \int \sqrt{\frac{H-y}{C1-(H-y)}}dy$$

Using substitution $H-y = C1\sin^2\frac{\theta}{2}$ we have $dy = -C2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$ where C2=2C1.

$$x = \int \sqrt{\frac{C1\sin^2\frac{\theta}{2}}{C1-C1\sin^2\frac{\theta}{2}}}(-C1\sin\frac{\theta}{2}\cos\frac{\theta}{2})d\theta$$

Solving this integral we get:

$$x = \frac{k_1}{2}(\theta-\sin\theta)+k_2$$
$$y = H+\frac{k_1}{2}(1-\cos\theta)$$

Applying the initial conditions we get the values of constants as follows:

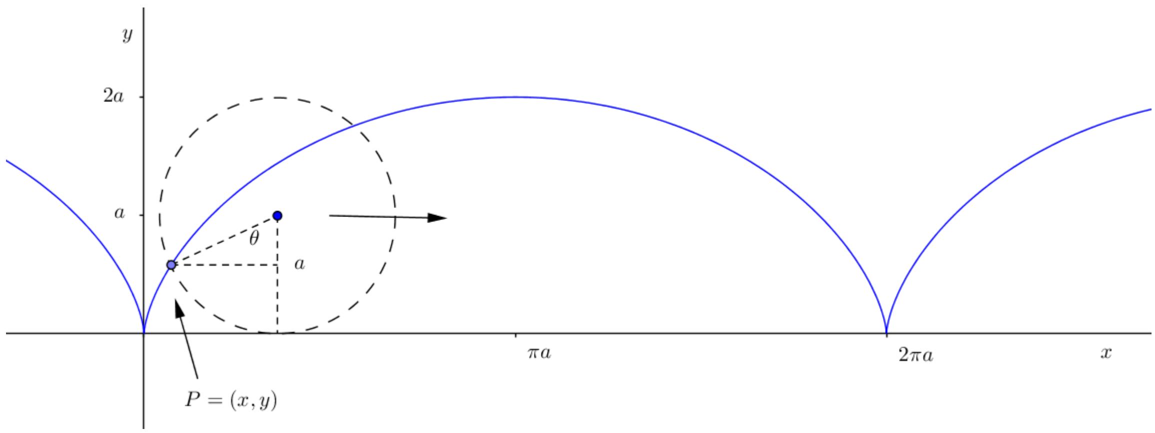
1. At point A:
$$y = H, x = 0 \implies \theta = 0$$
$$x = 0, \theta = 0 \implies k_2 = 0$$

2. At point B:
$$x = L, y = 0$$
$$2L = k_1(\theta-\sin\theta)$$
$$-2H = k_1(1-\cos\theta)$$

Solving these two equations we get the value of k_1 in terms of H and L.

$$x = \frac{k_1}{2}(\theta-\sin\theta)$$
$$y = H+\frac{k_1}{2}(1-\cos\theta)$$

This is parametric equation of cycloid.



The center of circle is at $(a\theta, a)$ so the co-ordinates of point p can be written as follows:

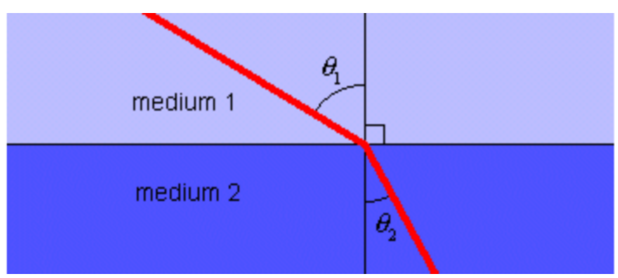
$$x = a\theta - a\sin\theta$$

$$y = a - a\cos\theta$$

The Physical Approach:

The Fermat's Principle: If a beam of light travels from point A to point B, it does so along the fastest path.

Snell's law:



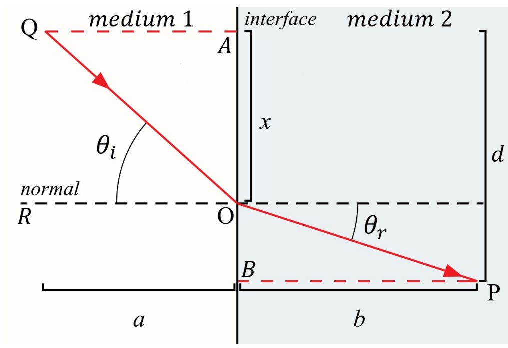
According to Snell's law we have

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

When light travels from one medium to another medium it either slows down or speeds up depending on the relative optical densities of pair of those two mediums.

And according to Fermat's principle light always choose the path that takes minimum time.

Using this two facts we can prove Snell's law.

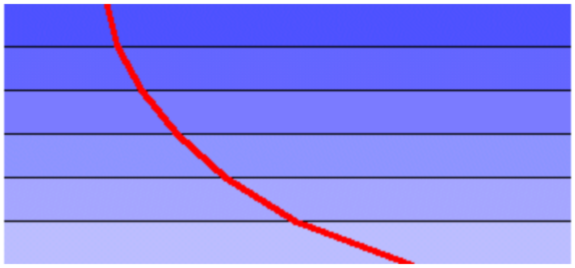


Light will choose such a path that it minimise the time taken from traveling from Q to P.

$$T = t_1 + t_2$$
$$t_1 = \frac{\sqrt{a^2+x^2}}{v_1}$$
$$t_2 = \frac{\sqrt{b^2+(d-x)^2}}{v_2}$$
$$\frac{dT}{dx} = 0$$

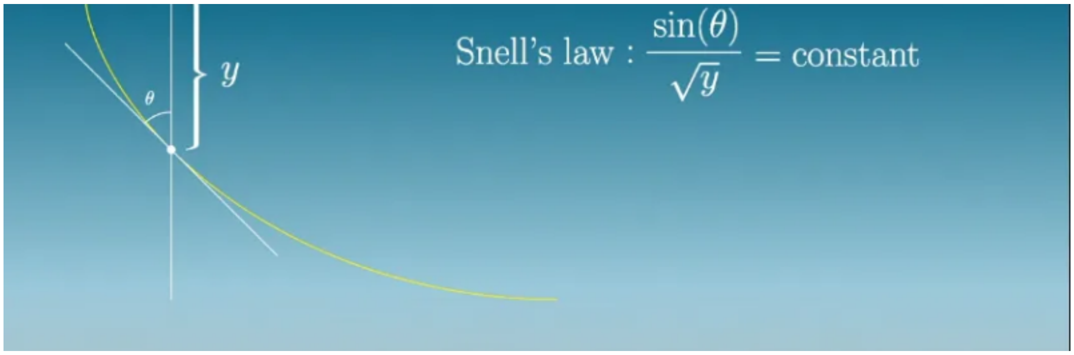
Solving this equation, we get $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$ So fundamentally Snell's law is giving the direction in which the light travel after crossing the junction between the two media such that for any given pair of velocities it takes minimum time.

So using this analogy we can solve our Brachistochrone problem. We will assume that instead of the ball the light is passing through infinitely small such media.



Now here we assume that the velocity of light is inversely proportional to the distance y shown in the diagram.

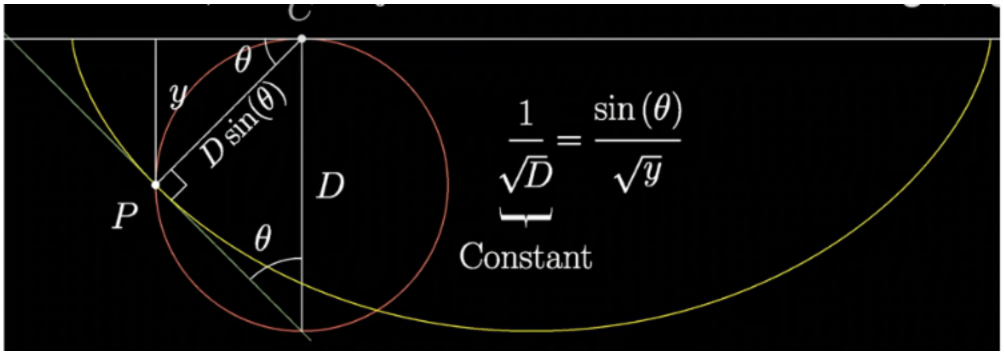
Now on applying Snell's law here we get the direction or the path on which the light moves in order to minimise time with same speed as that of the rolling ball.



So we get

$$\frac{\sin\theta}{y} = constant$$

Now this is the equation of the cycloid. we can prove this by the following construction.



Here the circle moves on the straightline. the curve traced by the point p on the circle is known as cycloid. C is the contact point of that circle with ground. tangent at p will pass through the end point of the diameter.

So from the geometry it is clear that $y = D\sin^2\theta$.

5. Isoperimetric Problem

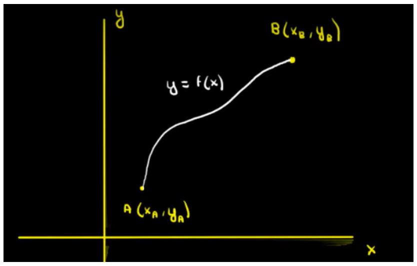
Problem:

Find the curve assumed by wire when it is connected to two points such that area under that curve is maximum.(given length of wire is more than distance between two given points to which wire is connected.)

History of the Problem:

This is also known as Dido's problem and it is the oldest problem in variational calculus. Dido founded the city of Carthage, in Tunisia. According to legend, she arrived at the site with her entourage, a refugee from a power struggle with her brother in Tyre in the Lebanon.She asked the locals for as much land as could be bound by a bull's hide. She cut the hide into a long thin strip and bounded the maximum possible area with this. The maximum possible area bounded by a curve of fixed length is a circle. So the city of Carthage is circular in shape. The Greek mathematician Zenodorus managed to show that the area of a circle is large than the area of any regular polygon with a perimeter of the same length.

Variational Calculus Approach:



We have to find the shape of wire y that maximise the area under the wire. this is a problem of constrained variational calculus problem because we have to maximise $I = \int_{x_A}^{x_B} y dx$ such that $J = \int_{x_A}^{x_B} \sqrt{1 + (y')^2} dx = L$ As usual we let

$$F = I + \lambda J$$

where λ is a lagrange multiplier.

$$F = \int_{x_A}^{x_B} (y + \lambda\sqrt{1 + (y')^2}) dx$$

Applying euler-lagrange equation to the function F we get;

$$\frac{\partial(y + \lambda\sqrt{1 + (y')^2})}{\partial y} - \frac{d[\frac{\partial(y+\lambda\sqrt{1+(y')^2})}{\partial y'}]}{dx} = 0$$

$$1 - \frac{d[\frac{\lambda y'}{\sqrt{1+(y')^2}}]}{dx} = 0$$

$$x + C1 = \frac{\lambda y'}{\sqrt{1+(y')^2}}$$

Simplifying this equation we get

$$\frac{dy}{dx} = \frac{x + C1}{\sqrt{\lambda^2 - (x + C1)^2}}$$

Integrating both side we get

$$y = \int \frac{x + C1}{\sqrt{\lambda^2 - (x + C1)^2}} dx$$

Taking substitution $u = x + C1$ so we have $dx = du$

$$y = \int \frac{u}{\sqrt{\lambda^2 - u^2}} du$$

Now take another substitution as $v = \lambda^2 - u^2$ so we have $dv = -2udu$ putting this into the equation we get:

$$y = -\frac{1}{2} \int \frac{dv}{\sqrt{v}} = -\sqrt{v} - C2$$

Simplifying this equation we have

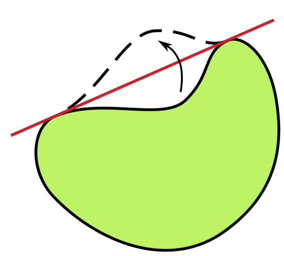
$$(x + c1)^2 + (y + C2)^2 = \lambda^2$$

Which is the equation of a circle. Use the constraints given in the problem to find value of C1,C2 and λ.

Physical Approach:

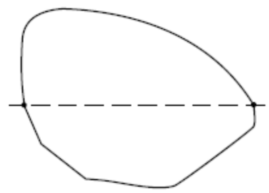
Given a string of length L. We have to create a shape of maximum area that can be formed using that string.

1. The shape can't have a dent.

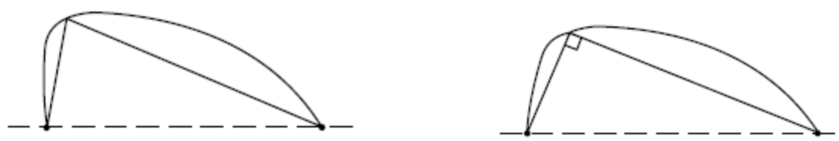


If a region is not convex, a "dent" in its boundary can be "flipped" to increase the area of the region while keeping the perimeter unchanged.

2. The shape must be symmetric about line passing through any two points which divide the perimeter of the string in two equal part. Other wise if one part has lesser area than other than we can replicate the shape of the part having more area to the shape having lesser area in order to maximise the total area. So the shape becomes symmetric.



3. Consider one of these halves. Suppose it is not a semicircle. Then there will be some point on the boundary where lines drawn from the points on the symmetry line meet at an angle that is not a right angle.



Let's assume that the arms of the triangle formed have length a and b. And let the angle between these arms is θ. Then the area A of this triangle can be written as:

$$A = \frac{1}{2}ab\sin\theta$$

If we vary the angle θ then the area of the two segments of the circle remain constant, only area of the triangle changes which is also the change in area of the overall shape. So in order to maximise the area of the shape we have to maximise the area of the triangle.

So the area is maximum only when the $\sin\theta = 1$ that is possible only when the $\theta = \frac{\pi}{2}$. And since we have choosen an arbitrary point this angle must be 90 degrees at all the points on that curve. So this is possible only when that curve is semi-circle.

This proof is given by Jacob Steiner.

Hence it is proved that the string have to form circle in order to maximise the area enclosed by it.

6. The Catenary Problem

Problem :

In the catenary problem, the primary objective is to determine the precise curve that the cable or chain assumes when suspended between two fixed points, such as the end-points of a suspension bridge or the supports of a power line. The challenge lies in accurately describing the shape of the curve, which is essential for structural engineering and design.

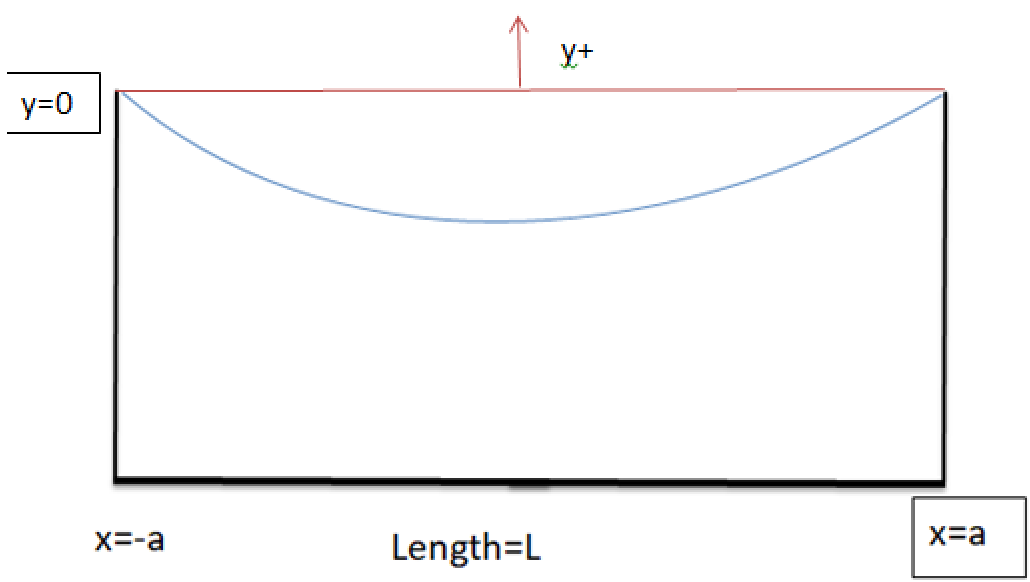
History of the Problem :

The catenary problem is a fascinating mathematical conundrum that revolves around the shape formed by a hanging flexible cable or chain under its own weight. The word "catenary" originates from the Latin word "catena", meaning chain.

The catenary problem has historical significance, with its origins dating back to the 17th century. Renowned mathematicians and physicists, including Galileo Galilei and Johann Bernoulli, made significant contributions to the understanding of catenary curves. Their work laid the foundation for subsequent advancements in mathematics and engineering principles.

In conclusion, the catenary problem delves into the captivating study of the curve formed by a hanging flexible cable or chain under gravity's influence.

Calculus Approach:



Mathematically, the catenary curve is described by a hyperbolic cosine function. Some assumptions that we take before approaching the solution of the problem are as follows:-

1) the string has a uniform string constant of linear density
2) we assume that the end of the string hang from the same level
3) we will assume the direction of positive y upwards.

Now our goal is to derive the equation of catenary given the physical parameters.

As we know that the length of the string is fixed. we can calculate the arc length of the string by using:

$$J = \int_S dS = \int_{-a}^{a} \sqrt{1 + \frac{dy}{dx}^2}\, dx \tag{1}$$

Now to derive the main equation we need to understand what does catenary refers to. For that let assume we throw a ball from some height above the ground.We observe that the ball moves toward the ground and becomes stationary. Thus we can say that at some given height y the ball had potential energy mgy. When released the ball will fall to the ground where it potential energy is zero. Therfore we can conclude that in absence of external forces physical system prefer a lower potential energy. Thus we can say that the shape of our catenary is such that the total gravitational energy U_g of the string is minimized.

Consider a small segment dS of the string:

$$dU_g = (\mu dS)gy \tag{2}$$

So to obtain the total potential energy of the string we will integrate:

$$dU_g = \int_{-a}^{a} \mu gy \sqrt{1 + \frac{dy}{dx}^2}\, dx \tag{3}$$

Now we need minimize U_g given the constraint that the length of the string is constant or is equal to L.

To solve a constraint variation problem we need to construct a new function

$$K = U_g + \lambda J = \int_{-a}^{a} \left[\mu gy \sqrt{1 + \frac{dy}{dx}^2} + \lambda \sqrt{1 + \frac{dy}{dx}^2} \right] dx \tag{4}$$

We will now apply the Euler-lagrange equation which is:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{5}$$

As the whole function F is independent of x we can implement a simpler variant of the 5 which is called as the Beitrami Identity:

$$F - y'\frac{\partial F}{\partial y} = C1 \tag{6}$$

To obtain $\frac{\partial F}{\partial y'}$ we need to diffrentiate F wrt y'

$$\frac{\partial}{\partial y'}\left(\sqrt{1+(y')^2}\right) = \frac{1}{2\sqrt{1+(y')^2}}\frac{\partial}{\partial y'}\left(1+(y')^2\right) = \frac{y'}{\sqrt{1+(y')^2}} \tag{7}$$

By substituting the value of $\frac{\partial F}{\partial y'}$ from 7 into 6 we get

$$\mu g y\sqrt{1+(y')^2} + \lambda\sqrt{1+(y')^2} - y'\left[\mu g y\frac{y'}{\sqrt{1+(y')^2}} + \lambda\frac{y'}{\sqrt{1+(y')^2}}\right] = C1 \tag{8}$$

Further simplification will give us

$$\frac{\mu g y + \lambda}{\sqrt{1+(y')^2}} = C1$$

$$\frac{(\mu g y + \lambda)^2}{1+(y')^2} = C1^2$$

$$(\mu g y + \lambda)^2 = C1^2(1+(y')^2)$$

$$y' = \sqrt{\frac{(\mu g y + \lambda)^2}{C1^2} - 1} \tag{9}$$

$$\int \frac{dy}{\sqrt{\frac{(\mu g y+\lambda)^2}{C1^2} - 1}} = \int dx = x + C2$$

For our convenience we assume $\cosh u = \dfrac{\mu g y + \lambda}{C1}$

$$\int \frac{dy}{\sqrt{(cosh\ u)^2 - 1}} = x + C1$$

By differentiating cosh u wrt y we get

$$dy = \frac{C1sinh\ u}{\mu g} du \tag{10}$$

By substituting the value of du from 10 into 9 we get

$$\begin{aligned} \int \frac{C1du}{\mu g} &= x + c2 \\ \frac{C1u}{\mu g} &= x + C2 \\ \frac{C1}{\mu g} cosh^{-1}\left(\frac{\mu g y + \lambda}{C1}\right) &= x + c2 \\ \frac{\mu g y + \lambda}{C1} = cosh\left[\frac{\mu g}{C1}(x + C2)\right] \end{aligned} \tag{11}$$

By isolating y we the equation of catenary

$$y = \frac{C1}{\mu g} cosh\left[\frac{\mu g}{C1}(x + C2)\right] - \frac{\lambda}{\mu g} \tag{12}$$

But here the equation has three unknown constant C1,C2,λ.therfore we need 3 equation for these three constants.

We can get two of them just from the boundary condition which is when x=-a,a the value of y=0. Thus on substituting the value we the following equations

$$0 = \frac{C1}{\mu g} cosh\left[\frac{\mu g}{C1}(a + C2)\right] - \frac{\lambda}{\mu g} = \frac{C1}{\mu g} cosh\left[\frac{\mu g}{C1}(-a + C2)\right] - \frac{\lambda}{\mu g} \tag{13}$$

Now cosh u=cosh(-u). As both the cosh are equal C2 must be zero thus now we have

$$\begin{aligned} \frac{C1}{\mu g} cosh\left[\frac{\mu g}{C1}(a)\right] &= \frac{\lambda}{\mu g}\ and \\ \frac{C1}{\mu g} cosh\left[\frac{\mu g}{C1}(-a)\right] &= \frac{\lambda}{\mu g} \\ \lambda &= C1cosh\left[\frac{\mu g}{C1}(a)\right] \end{aligned} \tag{14}$$

$$y = \frac{C1}{\mu g}\left[cosh\left(\frac{\mu g x}{C1}\right) - cosh\left(\frac{\mu g a}{C1}\right)\right]] \tag{15}$$

We now differentiate y wrt x

$$\frac{dy}{dx} = sinh\left(\frac{\mu g x}{C1}\right) \tag{16}$$

and substitute it in equation of L and use the identity $sinh^2 x = cosh^2 x$-1

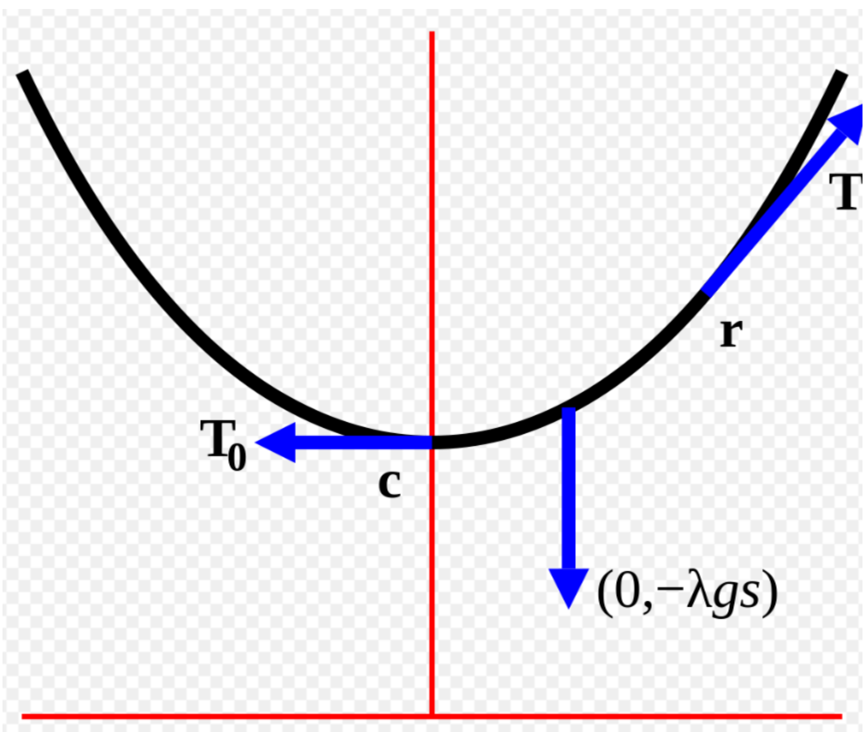
$$L = \int_{-a}^{a} cosh\left(\frac{\mu g x}{C1}\right)\, dx = \left[\frac{C1}{\mu g} sinh\left(\frac{\mu g x}{C1}\right)\right]_{-a}^{a} = \frac{C1}{\mu g}\left[sinh\left(\frac{\mu g a}{C1}\right) - sinh\left(\frac{\mu g(-a)}{C1}\right)\right] \tag{17}$$

sinh(-a)=-sinh(a)

$$L = \frac{2C1}{\mu g} sinh\left(\frac{\mu g a}{C1}\right) \tag{18}$$

Thus by hit trial one can try substituting the value of C1 in L.

Derivation of the Catenary Equations:



The tension at a point on the catenary is denoted by T, and the angle between the tangent at that point and the horizontal axis is φ. We can express the equilibrium condition of the chain using the following equations:

$$T\cos\varphi = T_0$$
$$T\sin\varphi = \lambda g s$$

Dividing the second equation by the first equation, we obtain:

$$\tan\varphi = \frac{\lambda g s}{T_0}$$

We define a parameter a as the length of chain whose weight is equal in magnitude to the tension at point c. Therefore, we have:

$$a = \frac{T_0}{\lambda g}$$

Using the equation $\frac{dy}{dx} = \tan\varphi$, we have:

$$\frac{dy}{dx} = \frac{s}{a}$$

To derive the equations for the catenary curve, we can use the formula for arc length:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \frac{\sqrt{a^2 + s^2}}{a} dx$$

Simplifying, we find:

$$\frac{dx}{ds} = \frac{a}{\sqrt{a^2 + s^2}}$$

Integrating the above equation, we obtain:

$$x = a \sinh\left(\frac{s}{a}\right) + \alpha$$

Similarly, integrating $\frac{dy}{ds} = \frac{s}{\sqrt{a^2+s^2}}$, we have:

$$y = \sqrt{a^2 + s^2} + \beta$$

By shifting the position of the coordinate axes, we can set $\alpha = \beta = 0$. Hence, the equations become:

$$x = a \sinh\left(\frac{s}{a}\right)$$
$$y = \sqrt{a^2 + s^2}$$

By eliminating s, we arrive at the catenary equation:

$$y = a \cosh\left(\frac{x}{a}\right)$$

where a is the constant defined as $a = \frac{T_0}{\lambda g}$.

7. Second Variation

To derive the second variation of a functional using only the Euler-Lagrange equation, we consider a functional of the form:

$$J[u] = \int_a^b F(x, u(x), u'(x))\, dx,$$

where $u(x)$ is the unknown function we seek, and $F(x, u(x), u'(x))$ is a given function. The second variation of $J[u]$ is defined as:

$$\delta^2 J = \left.\frac{d^2 J}{d\epsilon^2}\right|_{\epsilon=0},$$

where ϵ represents a small perturbation to the function $u(x)$. To calculate the second variation, we consider the perturbed function $u(x) + \epsilon v(x)$, where $v(x)$ is another function that satisfies certain boundary conditions.

Using the Euler-Lagrange equation, the first variation of the functional $J[u + \epsilon v]$ is given by:

$$\delta J = \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \left(\frac{\partial F}{\partial u} v(x) + \frac{\partial F}{\partial u'} v'(x) \right) dx.$$

To obtain the second variation, we differentiate the first variation with respect to ϵ and evaluate it at $\epsilon = 0$:

$$\delta^2 J = \left.\frac{d^2 J}{d\epsilon^2}\right|_{\epsilon=0} = \frac{d}{d\epsilon} \left(\int_a^b \left(\frac{\partial F}{\partial u} v(x) + \frac{\partial F}{\partial u'} v'(x) \right) dx \right) \Bigg|_{\epsilon=0}.$$

Now, using the product rule and the Euler-Lagrange equation $\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = \frac{\partial F}{\partial u}$, we can simplify the expression:

$$\delta^2 J = \int_a^b \left(\frac{\partial^2 F}{\partial u^2} v(x)^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} v(x) v'(x) + \frac{\partial^2 F}{\partial (u')^2} v'(x)^2 \right) dx.$$

This is the expression for the second variation of the functional $J[u]$ solely in terms of the Euler-Lagrange equation.

$$Let\ Q = \frac{\partial^2 F}{\partial u^2} v(x)^2 + 2 \frac{\partial^2 F}{\partial u \partial u'} v(x) v'(x) + \frac{\partial^2 F}{\partial (u')^2} v'(x)^2$$

Now if $Q > 0$ then the function has a minima else if Q¡0 then the function has a maxima.