General Logical Rules:

$$\frac{}{\Gamma, A \vdash A}\ (\mathrm{id}_A)$$

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash C}{\Gamma \vdash C}\ (\mathrm{cut})$$

Name	Connective	Left Rule	Right Rule
and conjunction	$\wedge$	$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ (\wedge_L)$	$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge_R)$
or disjunction	$\vee$	$\dfrac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ (\vee_L)$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee_{R_1}) \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee_{R_2})$
“top” truth triviality	$\top$	$\dfrac{\Gamma \vdash C}{\Gamma, \top \vdash C}\ (\top_L)$	$\dfrac{}{\Gamma \vdash \top}\ (\top_R)$
“bottom” falsehood absurdity	$\bot$	$\dfrac{}{\Gamma, \bot \vdash C}\ (\bot_L)$	N/A
implication	$\to$	$\dfrac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, (A \to B) \vdash C}\ (\to_L)$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash (A \to B)}\ (\to_R)$

General Linear Rules:

$$\frac{}{\bullet, A \vdash A}\ (\mathrm{id}_A)$$

$$\frac{\Delta \vdash A \qquad \Delta', A \vdash C}{\Delta, \Delta' \vdash C}\ (\mathrm{cut})$$

$$\frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}\ (\text{logical context})$$

Name	Connective	Left Rule	Right Rule	Polarity
Tensor	$\otimes$	$\dfrac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\ (\otimes_L)$	$\dfrac{\Delta \vdash A \qquad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B}\ (\otimes_R)$	+, L always applies
"with"	&	$\dfrac{\Delta, A \vdash C}{\Delta, A \& B \vdash C}\ (\&_{L_1})$ $\quad \dfrac{\Delta, B \vdash C}{\Delta, A \& B \vdash C}\ (\&_{L_2})$	$\dfrac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \& B}\ (\&_R)$	−, R always applies
	1	$\dfrac{\Delta \vdash C}{\Delta, 1 \vdash C}\ (1_L)$	$\dfrac{}{\bullet \vdash 1}\ (1_R)$	+, L always applies
	0	$\dfrac{}{\Delta, 0 \vdash C}\ (0_L)$	N/A	+, L always applies
Implication	$\multimap$	$\dfrac{\Delta \vdash A \qquad \Delta', B \vdash C}{\Delta, \Delta', (A \multimap B) \vdash C}\ (\multimap_L)$	$\dfrac{\Delta, A \vdash B}{\Delta \vdash (A \multimap B)}\ (\multimap_R)$	−, R always applies
"bang" "of course"	!	$\dfrac{\Gamma, A; \Delta \vdash C}{\Gamma; \Delta, !A \vdash C}\ (!_L)$	$\dfrac{\Gamma; \bullet \vdash A}{\Gamma; \bullet \vdash !A}\ (!_R)$	−, R always applies

Further Reading:

- The section titled "Linear Logic and Session-based Concurrency — Frank Pfenning" at `https://www.cs.uoregon.edu/research/summerschool/summer13/curriculum.html`
- `http://logitext.mit.edu/Intuitionistic/main`, and/or the tutorial at `http://logitext.mit.edu/tutorial`
- `https://click-and-collect.linear-logic.org/`
- The Stanford Encyclopedia of Philosophy's entry on Linear Logic: `https://plato.stanford.edu/entries/logic-linear/`
- The Wikipedia page on Linear Logic: `https://en.wikipedia.org/wiki/Linear_logic`

Linear Examples:

- Tensor ($\otimes$)
  $A \otimes B$ is like having both A and B
  - $$\frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C} \; (\otimes_L)$$
    To show that bread (A) tensor cheese (B) makes a meal (C), you must show that bread together with cheese makes a meal
  - $$\frac{\Delta \vdash A \qquad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B} \; (\otimes_R)$$
    To show that from a dollar Δ and another dollar (Δ') you can get bread (A) tensor cheese (B), you must show that a dollar (Δ) gives bread and also that a dollar (Δ') gives cheese.
- "with" (&)
  $A\&B$ says that you can have either of A or B (and the resource consumer gets to pick)
  - $$\frac{\Delta, A \vdash C}{\Delta, A\&B \vdash C} \; (\&_{L_1})$$
    To show that a meal (A) with dreams (B) gives satisfaction, it suffices to show that a meal gives satisfaction (it does not matter whether or not dreams do).
  - $$\frac{\Delta, B \vdash C}{\Delta, A\&B \vdash C} \; (\&_{L_2})$$
    Alternatively, to show that a meal (A) with dreams (B) gives satisfaction, it suffices to show that dreams give satisfaction (it does not matter whether or not a meal does).
  - $$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A\&B} \; (\&_R)$$
    If bread (A) and cheese (B) each cost a dollar (Δ), you can get bread with cheese for a single dollar.
- Triviality, Truth (1)
  - $$\frac{\Delta \vdash C}{\Delta, 1 \vdash C} \; (1_L)$$
    You can throw away trivialities without losing anything of value.
  - $$\frac{}{\bullet \vdash 1} \; (1_R)$$
    Even if you have nothing of value, you can conclude trivialities (such as zero dollars).
- Absurdity, Falsehood (0)
  - $$\frac{}{\Delta, 0 \vdash C} \; (0_L)$$
    From absurdity (e.g., a proof that $1 = 0$), you can get any resource you like (such as infinite money).
- Implication, "lolli" ($\multimap$)
  - $$\frac{\Delta \vdash A \qquad \Delta', B \vdash C}{\Delta, \Delta', (A \multimap B) \vdash C} \; (\multimap_L)$$
    Suppose you have two cups flour and one cup of water (collectively Δ), hunger for bread (Δ'), and a magical item ($A \multimap B$) that turns bread dough (A) into bread (B). To derive satiation (C), it suffices to prove that you can make bread dough from two cups flour and one cup water (this is $\Delta \vdash A$), and also that you can derive satiation from having hunger for bread combined with bread (this is $\Delta', B \vdash C$).
  - $$\frac{\Delta, A \vdash B}{\Delta \vdash (A \multimap B)} \; (\multimap_R)$$
    To derive that flour lolli bread in the context Δ, you need to show that adding flour to the resources in Δ allows you to derive bread.
- "bang", "of course" (!)
  - $$\frac{\Gamma, A; \Delta \vdash C}{\Gamma; \Delta, !A \vdash C} \; (!_L)$$
    If you have an infinite resource (e.g., "of course we'll always have air" or "of course we have a stove") (A), you can assume this resource as part of the logical context, as an axiom or pervasive assumption. (The logical context rule about ; will then let you get as many copies of this resource as you want.)
  - $$\frac{\Gamma; \bullet \vdash A}{\Gamma; \bullet \vdash !A} \; (!_R)$$
    To prove that something is of course present/true, you must prove it without using up any resources.