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Compare and contrast the Two-Factor Hull-White Model with other interest short rate models

Introduction

This report aims to, first of all, study the **Two factor Hull White model** in contrast to the previously discussed models (Ho Lee, Vasicek, Hull white 1 factor model, CIR models). Secondly, we wish to study and apply Two factor Hull White model on interest rate caplets using both analytical and Monte Carlo method leading to comparison between two methods outcomes. The report follows the given structure:

1. Interest rate models: General Introduction
2. One factor models
3. Two Factor Hull White (2FHW) model
4. Concluding Comparisons
5. Caplet pricing
6. References

Interest Rate Models: A General Introduction

The basic aim of any interest rate model is to be able to describe future evolutions of interest rate while agreeing to the existing term structure (family of interest based on maturity date). Building such models is quite complicated as the interest rates for different maturity across all products are closely interlinked, hence, they evolve simultaneously in time which makes their joint evolution a whole lot more complicated that stock prices.
Here, we will expressly focus on short rate models. In these models, we take instantaneous spot rate as the stochastic state variable. The advantage of these models is their simplicity and the fact that they help us conclude with exact formulas for pricing of bonds and options. The flaw in these models is the fact that the stochastic state variable is a mathematical idealization and cannot be directly observed in real time market.

One Factor models

Introduction

The term ‘one factor’ above indicates represents the single source of uncertainty. We will also focus on the final distribution of the interest rate and make comparisons across models.

1. **Ho Lee model [1986]**
No arbitrage/ exogenous term structure model
Risk neutral Dynamics of ‘r_t’
$$d(r_t) = \theta(t)dt + \sigma dW_t$$
Here, W -> Brownian motion, σ-> constant and θ(t) is deterministic and fitted to the initial term structure for the purpose of simulation.

Basic integration of above equation gives us the following equation for ‘r_r’
$$r_t = r_0 + \int_0^t \theta(s)\, ds + \sigma W_t$$
Here, it is easy to observe that ‘r_r’ is Normally distributed with following mean and variance values.
$$\mathbb{E}(r_t) = r_0 + \int_0^t \theta(s) ds$$
$$Var(r_t) = \sigma^2 t$$
Flaws:
There is positive probability for ‘r_t’ being negative.
This model doesn’t have mean reverting property. Also as time gap increases, the variance keeps on increasing and the mean value could explode depending on the function ‘θ’.

2. **Vasicek Model [1977]**
Equilibrium models which are endogenous in nature
Risk neutral Dynamics of ‘r_t’
$$d(r_t) = (\theta - \alpha r_t)dt + \sigma dW_t$$
Here, W -> Brownian motion, σ, α and θ-> constant
In order to find distribution of ‘r_t’, we consider a function $f(t,r_t)=e^{\alpha t}r_t$
By Ito’s Lemma,
$$d(f(t, r_t)) = \alpha e^{\alpha t} r_t dt + e^{\alpha t} dr_t$$
Inputting the value of dr_t,
$$\begin{aligned} d\big(f(t, r_t)\big) &= \alpha e^{\alpha t} r_t dt + e^{\alpha t}\big((\theta - \alpha r_t)dt + \sigma dW_t\big) \\ &= \theta e^{\alpha t} dt + \sigma e^{\alpha t} dW_t \end{aligned}$$
Integrating over the interval (0, t), we get:
$$r_t = r_0 e^{-\alpha t} + \frac{\theta}{\alpha}(1 - e^{-\alpha t}) + \int_0^t \sigma\, e^{\alpha(s-t)} dW_s$$

Here, it is easy to observe that ‘r_r’ is Normally distributed with following mean and variance values.
$$\mathbb{E}(r_t) = r_0 e^{-\alpha t} + \frac{\theta}{\alpha}(1 - e^{-\alpha t})$$
$$Var(r_t) = \int_0^t \sigma^2 e^{2\alpha(s-t)} ds = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

Flaw:
There is positive probability for ‘r_t’ being negative

3. **Cox-Ingersoll-Ross (CIR) Model [1985]** – deals with the flaw of Vasicek's model
   Equilibrium models which are endogenous in nature
   Risk neutral Dynamics of 'r_t'
   $$d(r_t) = (\theta - \alpha r_t)dt + \sigma\sqrt{r_t}dW_t$$
   Here, W -> Brownian motion, σ, α and θ-> constant
   In order to find distribution of 'r_t', we consider a function $f(t,r_t)=e^{\alpha t}r_t$
   By Ito's Lemma,
   $$d(f(t, r_t)) = \alpha e^{\alpha t}r_t dt + e^{\alpha t}dr_t$$
   Inputting the value of dr_t,
   $$d\big(f(t, r_t)\big) = \alpha e^{\alpha t}r_t dt + e^{\alpha t}\big((\theta - \alpha r_t)dt + \sigma\sqrt{r_t}dW_t\big)$$
   $$= \theta e^{\alpha t}dt + \sigma\sqrt{r_t}e^{\alpha t}dW_t$$
   Integrating over the interval (0, t), we get:
   $$r_t = r_0e^{-\alpha t} + \frac{\theta}{\alpha}(1 - e^{-\alpha t}) + \int_0^t \sigma\sqrt{r_t}e^{\alpha(s-t)}dW_s$$
   Above 'r_t' follows non central chi-square distribution

   $$\mathbb{E}(r_t) = r_0e^{-\alpha t} + \frac{\theta}{\alpha}(1 - e^{-\alpha t})$$
   $$\mathrm{Var}(r_t) = \int_0^t \sigma^2 r_t e^{2\alpha(s-t)}ds$$
   Integrating by parts, we get
   $$\mathrm{Var}(r_t) = r_0\frac{\sigma^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) + \frac{\theta\sigma^2}{2\alpha}(1 - e^{-\alpha t})^2$$
   'r_t' cannot be negative conditional on the fact that $2\theta \geq \sigma^2$ as it would ensure that 'r_t' never touches 0.

   Flaw:
   Highly sensitive to parameter choices.
   Accuracy of the model is highly dependent on the fact whether market is experiencing low/high volatility period. Therefore, this model fails to fit the market at all times.

4. **One factor Hull White (1FHW) model [1990]**
   No arbitrage/ exogenous model
   Risk neutral Dynamics of 'r_t'
   $$d(r_t) = (\theta(t) - \alpha r_t)dt + \sigma dW_t$$
   Here, W -> Brownian motion, σ and α -> constant and θ(t) is a deterministic function fitted to markets term structure. 'α' can be seen as rate of mean reversion.
   In order to find distribution of 'r_t', we consider a function $f(t,r_t)=e^{\alpha t}r_t$
   By Ito's Lemma,
   $$d(f(t, r_t)) = \alpha e^{\alpha t}r_t dt + e^{\alpha t}dr_t$$
   Inputting the value of dr_t,
   $$d\big(f(t, r_t)\big) = \alpha e^{\alpha t}r_t dt + e^{\alpha t}\big((\theta(t) - \alpha r_t)dt + \sigma dW_t\big)$$
   $$= \theta(t)e^{\alpha t}dt + \sigma e^{\alpha t}dW_t$$
   Integrating over the interval (0, t), we get:

$$r_t = r_0 e^{-\alpha t} + \int_0^t \theta(s) e^{-\alpha(s-t)} ds + \int_0^t \sigma\, e^{\alpha(s-t)} dW_s$$

Here, it is easy to observe that 'r_r' is Normally distributed with following mean and variance values.

$$\mathbb{E}(r_t) = r_0 e^{-\alpha t} + \int_0^t \theta(s) e^{-\alpha(s-t)} ds$$

$$\mathrm{Var}(r_t) = \int_0^t \sigma^2 e^{2\alpha(s-t)} ds = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

A better model as it allows for long run mean to change according to the deterministic function '$\theta(t)$'.

Flaw:

There is positive probability for 'r_t' being negative.

Not good for large volatility cases in the market (just like the previously discussed CIR model)

Summary

The No arbitrage/exogenous models are preferable as they start with the current term structure and aims to track its change rather that the endogenous models which aim to fit to market's yield curve of all shapes based on the three parameters. It is infeasible, hence, fitting these models to the market is almost an impossibility. Though, these models could still be effective in case of certain yield curve shapes of the market. One last thing to know is the fact that it has been observed on certain occasions that banks have lent/borrowed on negative interest rates, so, the flaw that we have mentioned regarding 'negative interest rates' might not be that much of an issue.

2FHW models [2006]

Origin

The 2FHW model can be seen as an extension to 1FHW model with an additional stochastic component added to the deterministic term '$\theta(t)$'. This additional term allows our model to move (in a graph) other than parallel as the short/long term rates could fluctuate in variety of ways. An example of such a situation would be an event when the yield curve steepens.

Risk neutral Dynamics of 'r_t'

$$d(r(t)) = (\theta(t) + u(t) - \alpha r(t))dt + \sigma_1 dW_1(t)$$

and

$$d(u(t)) = -bu(t)dt + \sigma_2 dW_2(t)$$

Here, W_1 and W_2 are different brownian motions; α, b and σ's -> constants, $b<\alpha$ ensures that as 't' tends to infinity our 2FHW model will tend to 1FHW model. $\theta(t)$ is deterministic function which is fitted to the initial term structure. Also $dW_1 dW_2 = \rho dt$, r (0) =r_0 and u (0) = 0

Following a similar approach to 1FHW model, [Consider two functions as $e^{\alpha t}r_t$ and $e^{bt}u_t$ and apply ito's Lemma followed by integrating them over interval of (s,t) for s<t]
We get the following equations

$$r(t) = r(s)e^{-\alpha(t-s)} + \int_s^t \theta(v)\, e^{-\alpha(t-v)} dv + \int_s^t u(v)\, e^{-\alpha(t-v)} dv + \sigma_1 \int_s^t e^{-\alpha(t-v)}\, dW_1(t)$$

$$u(t) = u(s)e^{-b(t-s)} + \sigma_2 \int_s^t e^{-b(t-v)}\, dW_2(t)$$

Solving the 'u' integral in the first equation

$$\int_s^t u(v)\, e^{-\alpha(t-v)} dv = \int_s^t u(s)\, e^{-\alpha(t-v)-b(v-s)} dv + \sigma_2 \int_s^t e^{-\alpha(t-v)} \left[\int_s^v e^{-b(v-x)}\, dW_2(x)\right] dv$$

$$= u(s)\frac{e^{-b(t-s)} - e^{-\alpha(t-s)}}{\alpha - b} + \sigma_2 e^{-\alpha t} \int_s^t e^{(\alpha-b)v}\, [\int_s^v e^{bs} dW_2(x)] dv$$

Now, we apply 'by parts integration method' on the double integral above to get below equation

$$\int_s^t e^{(\alpha-b)v} \left[\int_s^v e^{bs} dW_2(x)\right] dv = \frac{1}{\alpha - b}\int_s^t \left(\int_s^v e^{bx} dW_2(x)\right) d_v\left(e^{(\alpha-b)v}\right)$$

$$= \frac{1}{\alpha - b}\left[e^{(\alpha-b)t}\int_s^t e^{bx} dW_2(x) - \int_s^t e^{(\alpha-b)v} d_v\left(\int_s^v e^{bx} dW_2(x)\right)\right]$$

$$= \frac{1}{\alpha - b}\int_s^t \{e^{-b(t-v)} - e^{-\alpha(t-v)}\} dW_2(v)$$

Combining all above calculation in our initial equation, then we get the following

$$r(t) = r(s)e^{-\alpha(t-s)} + \int_s^t \theta(v)\, e^{-\alpha(t-v)} dv + \sigma_1 \int_s^t e^{-\alpha(t-v)}\, dW_1(t)$$
$$+ u(s)\frac{e^{-b(t-s)} - e^{-\alpha(t-s)}}{\alpha - b} + \frac{\sigma_2}{\alpha - b}\int_s^t \{e^{-b(t-v)} - e^{-\alpha(t-v)}\} dW_2(v)$$

When we take s=0, we get

$$r(t) = r(0)e^{-\alpha t} + \int_0^t \theta(v)\, e^{-\alpha(t-v)} dv + \sigma_1 \int_0^t e^{-\alpha(t-v)}\, dW_1(t)$$
$$+ \frac{\sigma_2}{\alpha - b}\int_0^t \{e^{-b(t-v)} - e^{-\alpha(t-v)}\} dW_2(v)$$

Now we define the following modification

$$\sigma_3 = ({\sigma_1}^2 + \frac{{\sigma_2}^2}{(\alpha-b)^2} + 2\rho\frac{\sigma_1\sigma_2}{b-\alpha})^{\frac{1}{2}}$$

$$dW_3(t) = \frac{\sigma_1 dW_1(t) - \frac{\sigma_2}{\alpha - b} dW_2(t)}{\sigma_3}$$

$$\sigma_4 = \frac{\sigma_2}{\alpha - b}$$

Our final equation becomes:

$$r(t) = r(0)e^{-\alpha t} + \int_0^t \theta(v)\, e^{-\alpha(t-v)} dv + \sigma_3 \int_0^t e^{-\alpha(t-v)}\, dW_3(t) + \sigma_4 \int_0^t e^{-b(t-v)} dW_2(v)$$

2FHW model is mostly studied in conjunction with Two additive Gaussian Factor model as it is easier to calculate. In order to do that we define ther following equation

$$X(t) = r(t) + \delta u(t)$$

Here, $\delta = \frac{1}{b-\alpha}$

Then

$$dX(t) = dr(t) + \delta du(t)$$

Inputting the value of dr(t) and du(t) from initial definition of 2FHW model

We get,

$$dX(t) = \big(\theta(t) + u(t) - \alpha r(t) - \delta b u(t)\big)dt + \sigma_1 dW_1(t) + \delta\sigma_2 dW_2(t)$$
$$= \big(\theta(t) - \alpha X(t)\big)dt + \sigma_3 dW_3(t)$$

With σ_3 and dW_3 as follows

$$\sigma_3 = (\sigma_1{}^2 + \frac{\sigma_2{}^2}{(\alpha-b)^2} + 2\rho\frac{\sigma_1\sigma_2}{b-\alpha})^{\frac{1}{2}}$$
$$dW_3(t) = \frac{\sigma_1 dW_1(t) - \frac{\sigma_2}{\alpha-b} dW_2(t)}{\sigma_3}$$

Now we consider another stochastic process,

$$\omega(t) = -\delta u(t)$$

Post differentiation, we get

$$d\omega(t) = -b\omega(t)dt + \frac{\sigma_2}{\alpha-b} dW_2(t)$$

Let $\sigma_4 = \frac{\sigma_2}{\alpha-b}$, the equation becomes following

$$d\omega(t) = -b\omega(t)dt + \sigma_4 dW_2(t)$$

Hence, we can further writhe r(t) as following

$$r(t) = X(t) + \omega(t) + \phi(t)$$

Where

$$dX(t) = -\alpha X(t)dt + \sigma_3 dW_3(t)$$
$$d\omega(t) = -b\omega(t)dt + \sigma_4 dW_2(t)$$

And

$$\phi(t) = r(0)e^{-\alpha t} + \int_0^t \theta(v)\, e^{-\alpha(t-v)} dv$$

If we see the general G2++ model equations and compare, then it is easy to see the similarity between the two models.

$$r(t) = x(t) + y(t) + \phi(t)$$

Where

$$dx(t) = -\alpha x(t)dt + \sigma\, dW_1(t)$$
$$dy(t) = -by(t)dt + \eta\, dW_2(t)$$

And $\phi(t)$ is a deterministic function which equals r_0 at time 0. Then comparing the two system of equations we get required transformation from one to another.

Concluding Comparisons

Overall, we have studied two types of models: Exogenous and Endogenous. Basic difference is the fact that exogenous models are able to capture the exact shape of the yield curve, hence helps with better precision while the endogenous methods are less complex. Therefore, whenever the situation arises such that the exact shape prediction doesn't matter and accurately knowing the directional trend is more important than endogenous models are preferred (e.g. long term prediction about the market or portfolio). Whereas, when it calls for precise calculations, we would prefer exogenous models, more accurate the better (e.g. pricing of derivatives). Then comes the question of complexity of the model which can sometimes be limited by computational power or even mathematical rigor.

Caplet Pricing

What is an interest rate caplet derivative?

Interest Rate Caplet is a derivative which helps in limiting loss due to rising interest rates. It basically involves setting a cap/strike interest rate and if the Interbank rate rises above this strike rate only then will the option buyer be paid. Therefore, payoff is Max {L (T_1, T_2) – X,0}

Here,

$$L(T_1, T_2) = \frac{1 - P(T_1, T_2)}{(T_2 - T_1)P(T_1, T_2)}$$

Here, P (T_1, T_2) is price of a zero coupon bond at time T_1 which is maturing at time T_2 with unit face value ($1 or 1 Rupee).

Target audience

This instrument can be used for various purposes like hedging, Arbitrage, Direction trading, for locking yield and managing portfolio duration. Seeing the wide array of applications, the target

audience includes banks, hedge funds, banks, dealers/traders, Corporates, Mutual funds and NBFCs to name a few.
The inputs required are as follows
Notional amount (N)– Amount under consideration
Cap on interest rate(X)
T_1 -> start date
T_2-> End date
Then ,Price of Caplet = $N(1 + X(T_2 - T_1))\mathbb{E}\left[e^{-\int_t^{T_1} r_s ds}\left(\frac{1}{1+X(T_2-T_1)} - P(T_1, T_2)\right)^+ | \mathcal{F}_t\right]$
(The Analytical method related calculations are done in the attached excel sheet)

In the given case, we are considering the scenario where the Caplet buyer wants to reduce interest rate risk for the period of year 2022. We have taken yield related data US Treasury's par yield curve and USD LIBOR data as a replacement for L (T_1, T_2) for given start dates.

Monte Carlo Method
The basic approach here is to start with fitting the deterministic function to current term structure. Following this step, we set up limiting ranges for various parameters along with corresponding step size for each parameter. Now, we us a grid like approach in order to calculate each unique combination's caplet price. Now, we just select the one which is best fit curve based on minimum sum of square error(SSE). The curve with least SSE is the best fit curve. This curve's corresponding parameters can be used to predict future yields/prices. Once the market situation drastically changes, we will have to start anew.

Comparison between analytical approach and Monte Carlo approach
Analytical approach is computationally very easy and need in depth understanding at the time of each application whereas Monte Carlo approach computationally rigorous but still follows a certain pattern once encoded (that is, even the system itself could do it) while giving a much better fit of the yield curve.

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