Submission Number: 1

Group Number: 26

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Introduction

In this paper, we will first discuss the history and development of stochastic processes/ integration through the 20^{th} century starting from contributions of Louis Bachelier followed by how his ideas influenced Paul Samuelson research/contributions to the field of Mathematical Finance. We will further discuss the contributions of Japanese mathematicians like Kiyosi ito to the field of mathematical finance. The historical discussion will be concluded by discussing the relevance of Paul-Andre Meyer's 'Strasbourg School' of probability. In the next part, we will try to explain how Brownian motion and Ito's Lemma help us in derivation of Black-Scholes Model. The last part of the paper will discuss Black-Scholes equation's partial differential equation in greek form along with an illustrative example of delta-hedging.

(a)

The study of stochastic process is centered around the concept of Brownian motion. One of the first attempt to model Brownian motion was Louis Bachelier (-) when he tried to apply it on Paris stock market in 1900. Bachelier tried to apply the concept of probability to study the static state stock market at an instant. He tried to exploit the concepts of Central Limit Theorem and inferred that market noise should be a memoryless process. Therefore, according to him the changes in stock prices have to be independent and normally distributed. He further drew from the physics' heat equation and Markov properties. The paper was not well received as economics was a totally unfamiliar field for mathematical application at the time. Even though Bachelier's paper predated Einstein's paper on 'Brownian motion', it only got its deserved recognition decades later in mid-1960's when Paul Samuelson proved the opening statement of Bachelier's dissertation ***'past, present and even discounted future events are reflected in market price, but often show no apparent relation to price changes'***. Paul's paper in conjunction with Fama's work became the foundation for the theory of 'Efficient Market Hypothesis'. This theory revolutionized the world of empirical finance and the associated debated are still ongoing. Two crucial insights which later became fundamentals of 'Options Pricing Theory' are: a) discounted future price -> martingale & b) part (a) is extendable to spot prices. In his next paper, he worked with H P M Kean Jr to develop today's geometric Brownian motion. This model avoided the pitfalls of Bachelier's model as it ensured that stock price always stays positive.
As the time went by Bachelier's work became an inspiration to all upcoming researchers in the field and was also cited by following Leonard Savage, Joseph L. Doob, Andrey Kelmogorov etc.
Let's not jump ahead, Daniell's approach to measure theory (1913) along with N. Wieners's (1923) combining it with Fourier approach lead to mathematically proving Brownian motion (Wiener's process). Two key results that relating to stochastic integration from Wiener's work: a) paths of Brownian motion-> non zero finite quadratic variation & b) the paths of Brownian motion have infinite variation. Wiener also developed 'Multiple Wiener Integral', which was too complicated and was later refined and improved by Kiyoshi Ito.
The next development in the field came from A. N. Kolmogorov when he laid down the foundation of stochastic integration. He established mathematical basis for understanding probability theory using Measure Theory. In his paper, he worked on Continuous Markov processes and showed that their diffusion essentially depends on two components: drift and random component. He then related the probability distribution to the solution of the partial differential equation, Kolmogorov's equations.

(b)

Next, we discuss Kiyosi Ito, Father of stochastic integration, developed concept of stochastic integral and stochastic differential equation. He further noticed that each markovian particle performs a homogenous (w.r.t time) differential process, hence, the foundation of Ito calculus. His first paper on stochastic integration was published in 1944. In the same year, Kakutani found the link between Brownian motion and harmonic motion. Around the same time, Doob found the connection between martingales and harmonic functions, so, he was working on Martingale oriented probability theory. Doob also explained about 'Strong Markov processes' property'. In 1948, E. Hille and K.Yosida, independently clarified the roles of infinitesimal generators in Markov processes. Ito further developed stochastic differential equation of Markov processes and connected it to Kolmogorov's equation leading to derivation of Ito's formula. He further established the isometry property using the concept of independent increment of Brownian motion. Post establishing isometry property, he further extended it to jointly measurable non-anticipating processes. His research was carried forward by Doob, aiming to fully utilize the property of independence of each increment, hence, establishing the Doob-Decomposition theorem for sub-martingales. The interest in Ito's treatment of Markov process only increased further after G. A. Hunt published his papers on Axiomatic Probablity theory (1957 and 1958).

(c)

P.A. Mayer's first paper focused on establishing Doob-Meyer Decomposition for sub-martingales using the language of potential theory. During the period of 1966-80, Paul and his co-workers worked the general theory of processes (known as Strasbourg School). This theory is focused on establishing the foundation of Continuous time stochastic processes' theory. Later, Ito and S. Watanabe defined locale martingales which ended up supporting Doob's original conjecture. P.A. Meyer's skeletal development stochastic integral was further worked on by Philippe Courrege (1964). He pioneered predictable σ-algebra. One thing which he missed out on was to prove a change of variables formula. This was done in 1967 in an influential paper of H. Kunita and S. Watanabe. This paper worked with the ideas of orthogonality of martingales mentioned by P.A. Meyer. Simultaneously, Motoo and Watanabe worked on theory of stable spaces of martingales, hence, the concept of 'Market Completeness' came into being. They further extended the idea of quadratic variation as a kind of inner product, which helped prove general change to variable formula, hence, extending Ito's formula for martingales
The concise contribution of P.A. Meyer's Strasbourg School are as follows: -

a. Defined pseudo inner product, which is applicable to all martingale processes
b. Importance of predictable α-algebra. Integrand should be a predictable process so as to ensure that stochastic integral agrees with the path be pat construction according to Lebesgue-Stieltjes integration.

After the 4 papers from the lecture series, P.A. Meyer worked with Doleans-Dade on a paper after a gap of three years. In this paper, they remove the assumption of Filtrations being left continuous. The stated that filtration doesn't have a fixed time of discontinuity. Because of the removal of above assumption, they managed to end up with a purely martingale theory. This paper also saw the coining of the term 'semi-martingale'.

d), e) and f)

To answer the question, the best way is to just derive the Black Scholes PDE.
So, let 'S' be the price of the stock and allow it to follow geometric Brownian motion process(a)

Then,

$$dS = \mu S_t dt + \sigma S_t dW_t$$

We assume that interest rate is constant.
Then differential of the bank account is as follows:

$$dB = rB_t dt$$

We assume that volatility(σ) and strike price(K) are constant, then the option price 'V' is as follows:

$$V = V(T - t, S_t)$$

By Ito's lemma, (b)

$$dV = \frac{dV}{dt}dt + \frac{dV}{dS}dS + \frac{1}{2}\frac{d^2V}{dS^2}dS^2$$

Apply the value of dS, we get

$$dV = \frac{dV}{dt}dt + \frac{dV}{dS}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{d^2V}{dS^2}dS^2$$
$$dV = (\frac{dV}{dt} + \mu S_t \frac{dV}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2})dt + \sigma \frac{d\ V}{dS}dW$$

Above is basically a drift term and a stochastic term.

In case of delta hedging, we aim to ensure that portfolio value remains unchanged. We try and do this by investing in a combination of derivative and its underlying.
Let us invest in 'Δ' units of Stock and 'α' be the amount borrowed from the bank.
Then,

$$\pi = \Delta S + \alpha B$$

Taking differential and substituting values of dS and dB
We get,

$$d\pi = (\Delta \mu S + \alpha r B)dt + \ \Delta \sigma S dW$$

Combined dV and dπ represent our hedging strategy.
So, the addition of the two should lead to zero coefficient of dW term,
So,

$$\Delta \sigma S + \sigma S \frac{dV}{dS} = 0$$
$$\Delta = -\frac{dV}{dS}$$

Then our combined portfolio becomes:

$$d(V + \pi) = (\frac{dV}{dt} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + \alpha r B)dt$$

Inputing 'Δ' in bank/stock portfolio, we get

$$\pi = -\frac{dV}{dS}S + \alpha B$$

Now, the total portfolio only has a deterministic term, so, it must grow at the risk free rate.

$$d(V + \pi) = (V + \pi)rdt$$

Substitute for π and simplify the equation to get the Black Scholes PDE,
Which is,

$$\boldsymbol{\frac{dV}{dt} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0}$$

There are several variables known as 'Greeks' to assess risk in optional markets.

a) Δ(Delta)- change of option price w.r.t. asset price
b) θ (Theta)- change of option price w.r.t. time
c) γ (Gamma)- change of Δ w.r.t. asset price

Considering above point, we can re-write the above equation as follows:

$$\boldsymbol{\theta + \frac{1}{2}\sigma^2 S^2 \gamma - rS\Delta - rV = 0}$$

Simplifying, we get

$$\boldsymbol{\theta + \frac{1}{2}\sigma^2 S^2 \gamma = r(S\Delta + V)}$$

Here,
θ represents reduction in option value as time to maturity reduces
Since, the case is of delta hedged strategy, therefore, Δ is approximately 0
Assuming V is greater than or equal to 0
Therefore, to make LHS = RHS, the second term involving γ has to be positive. To ensure that γ has to be positive.

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