MFE Math Bootcamp HW 2

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1) Matrix

a) compute P^2. Interpret results. Are there assumptions attached to the conclusion?

In [105…
```
import numpy as np

P = np.array([[0.84, 0.13, 0.03, 0],[0.10, 0.77, 0.09, 0.04], [0.10, 0.20, 0.65, 0.

P_squared = P @ P

P_squared
```

Out[105…
```
array([[0.7216, 0.2153, 0.0564, 0.0067],
       [0.17  , 0.6239, 0.1308, 0.0753],
       [0.169 , 0.297 , 0.4435, 0.0905],
       [0.    , 0.    , 0.    , 1.    ]])
```

P^2 based on my initial interpretation is the 2 year transition probabilities that all asset classes at the firm (investment grade, speculative, junk and defaulted unsecured debt) transition to a desired asset class after 2 years. For instance the 0.7216 probability is the probability that all investment grade debt assets transition to the 4 assets after a year, then transition back to investment grade debt after the second year. This is hard to articulate for me but it kinda makes sense if matrix multiplication calculations are done by hand. The main assumptions attached to my conclusion is that the economic/financial parameters that created probability of transitions in the first year are conserved in the second year. If the economic/financial environemental parameters arent the same after a year, the second matrix we multiply P by, wouldnt be P.

b) Find Eigenvectors and Eigenvalues of P

In [107…
```
evals, evecs = np.linalg.eig(P)

for i in range(len(evals)):
    print( f'eigenvalue: { evals[i]:.2f}; eignvector: {evecs[:,i]}\n')
```

```
eigenvalue: 0.98; eignvector: [0.64393093 0.54754569 0.53436568 0.        ]

eigenvalue: 0.72; eignvector: [ 0.69673268 -0.52319313 -0.4907469   0.        ]

eigenvalue: 0.56; eignvector: [ 0.11082422 -0.44136358  0.89045841  0.        ]

eigenvalue: 1.00; eignvector: [0.5 0.5 0.5 0.5]
```

c) Use b) to diagnalize P

In [115…

```
# D = (P^-1)AP

# Matrix we want to diagnolize: A
# matrix of eigenvectors: evecs = P
# Inverse of eigenvector matrix: np.linalg.inv(evecs) = P^-1
evecs_inv = np.linalg.inv(evecs)

# Diagonalization

D = evecs_inv @ P @ evecs

D
```

Out[115…

```
array([[ 9.75436747e-01, -1.11022302e-16,  8.32667268e-17,
         2.22044605e-16],
       [-1.38777878e-16,  7.21249270e-01, -2.77555756e-17,
        -9.71445147e-17],
       [-2.77555756e-17,  2.77555756e-17,  5.63313983e-01,
        -7.02563008e-17],
       [ 0.00000000e+00,  0.00000000e+00,  0.00000000e+00,
         1.00000000e+00]])
```

d) Use c) to find lim as n approaches infinity of P^ infinity

- See attachment at end of notebook

2) Pod Shop Quant trading analyst

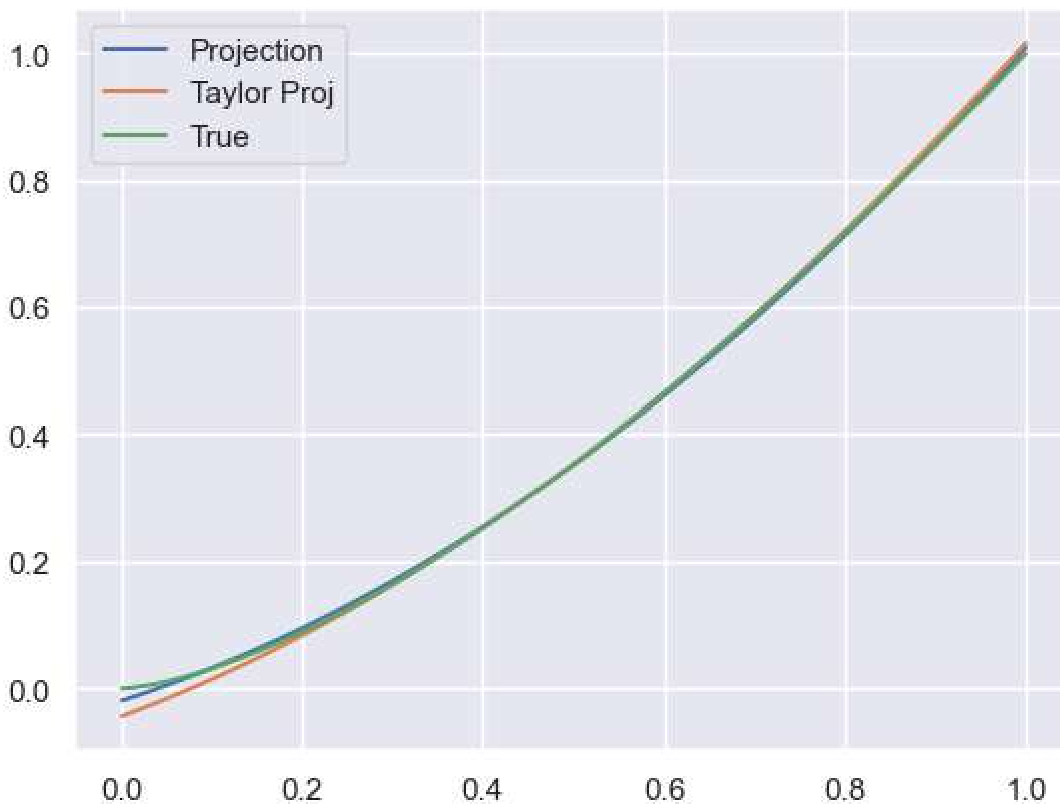
For Parts a) b) c) please see scanned pages

d) Graph Results from b) and c)

In [117…

```
import numpy as np, matplotlib.pyplot as plt, seaborn as sns

# Import numerical integrator
from scipy.integrate import quad

# seaborn
sns.set_theme()


# Define inner product
inner = lambda f, g: quad(lambda x: f(x) * g(x), 0, 1)[0]

# f
f = lambda x: x**(3/2)

# basis
u1, u2, u3 = lambda x: 1, lambda x: x - 1/2, lambda x: (x**2) - x +(1/6)
```

```
# inner products
a_proj, b_proj, c_proj = inner(u1, f)/inner(u1, u1), inner(u2, f)/inner(u2, u2), in

# Define functions
proj = lambda x: a_proj * u1(x) + b_proj * u2(x) + c_proj * u3(x)
taylor_proj = lambda x: 0.53*(x**2) + (0.53*x) - 0.0435

# Get the x-values for plot
x_vals = np.linspace(0, 1, 100)


plt.plot(x_vals, proj(x_vals), label = 'Projection')
plt.plot(x_vals, taylor_proj(x_vals), label = 'Taylor Proj')
plt.plot(x_vals, f(x_vals), label = 'True')


plt.legend()


plt.show()
```



e) Compute norm between x^2/3 and results from b) and c)

In [121…

```
norm = lambda f: np.sqrt(inner(f, f))

# Let's calculate the norms
norm_proj, norm_taylor = norm(lambda x: x**(3/2) - proj(x)), norm(lambda x: x**(3/2
```

```
print(f'Using the projection approximation the norm is {norm_proj:.3f}.')
print(f'Using the taylor projection approximation the norm is {norm_taylor:.3f}.')
```

```
Using the projection approximation the norm is 0.005.
Using the taylor projection approximation the norm is 0.011.
```

3) Minimizing the function L_2

a) Please see scanned pages

b) Use minimize in scipy for the result

In [123…

```
import numpy as np
from scipy.optimize import minimize

def L(params):
    a, b = params
    return ((3.02 - a + b)**2) + ((2.18 - a)**2) + ((4.08 - a - b)**2) + ((-0.03 -

minimize(L,[0, 0] )
```

Out[123…

```
  message: Optimization terminated successfully.
  success: True
   status: 0
      fun: 7.4999100000000025
        x: [ 2.675e+00 -1.041e+00]
      nit: 5
      jac: [ 5.960e-08 -5.960e-08]
 hess_inv: [[ 1.500e-01 -5.000e-02]
            [-5.000e-02  5.000e-02]]
     nfev: 21
     njev: 7
```

Result from a) matches result from b)

- alpha = 2.675, beta = -1.041

c) Use minimize and the new L_1 defined to find the new alpha and beta values

In [125…

```
import numpy as np
from scipy.optimize import minimize

def L_1(params):
    a, b = params
    return ( abs(3.02 - a + b) + abs(2.18 - a) + abs(4.08 - a - b) + abs(-0.03 - a

minimize(L_1,[0, 0] )
```

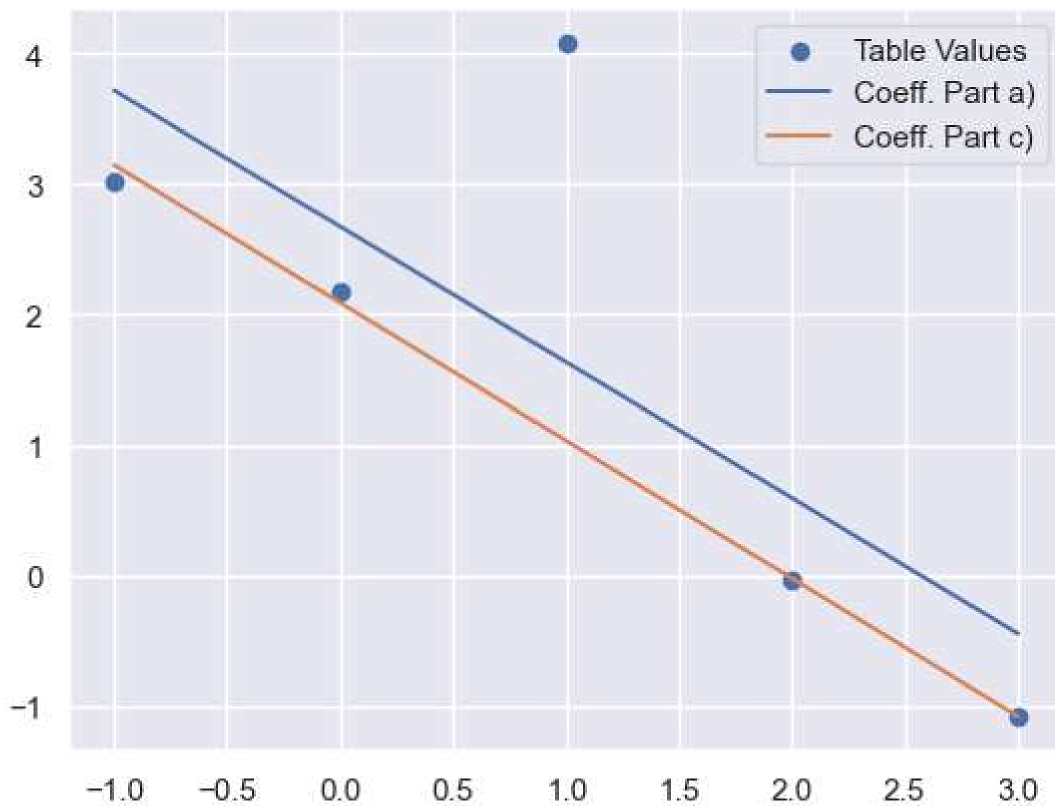
Out[125…

```
  message: Desired error not necessarily achieved due to precision loss.
  success: False
   status: 2
      fun: 3.2700000324377347
        x: [ 2.090e+00 -1.057e+00]
      nit: 10
      jac: [-1.000e+00 -1.354e+00]
 hess_inv: [[ 9.579e-01 -4.498e-01]
            [-4.498e-01  2.115e-01]]
     nfev: 240
     njev: 76
```

d) Use Scatter to graph points on table. Use Plot to graph lines corresponding to the coeff. in a) c)

In [127…

```python
import matplotlib.pyplot as plt, seaborn as sns

# table values
x_t = [-1,0,1,2,3]
y_t = [ 3.02, 2.18, 4.08, -0.03, -1.08]

# define function
f_a = lambda x: 2.675 + (-1.041*x)

# define function
f_c = lambda x: 2.090 + (-1.057*x)

# seaborn
sns.set_theme()

plt.scatter(x_t, y_t, label = 'Table Values')

# Get the x-values for plot
x_vals = np.linspace(-1, 3, 200)


plt.plot(x_vals, f_a(x_vals), label = 'Coeff. Part a)')
plt.plot(x_vals, f_c(x_vals), label = 'Coeff. Part c)')
plt.legend()
plt.show()
```



4) Convertible Bond Question

a) Write P as a function of c,r, S, T and sigma. Evaluate when c = 0.05, r = 0.08, S = 65, T = 5, sigma = .35

In [129…

```
import numpy as np
from scipy.stats import norm

def P(c,r,S,T,sigma):

    d1 = ((np.log(S/100)) + (r + ((sigma**2)/2)*T)) / (sigma * (np.sqrt(T)))

    d2 = (d1 - (sigma * np.sqrt(T)))

    P = (S*norm.cdf(d1)) - ((100*np.exp(-1*r*T))*(norm.cdf(d2))) + (100*np.exp(-1*r
        ((c/2)*np.exp((-1*r*2)/2)) + \
        ((c/2)*np.exp((-1*r*3)/2)) + \
        ((c/2)*np.exp((-1*r*4)/2)) + \
        ((c/2)*np.exp((-1*r*5)/2)) + \
        ((c/2)*np.exp((-1*r*6)/2)) + \
        ((c/2)*np.exp((-1*r*7)/2)) + \
        ((c/2)*np.exp((-1*r*8)/2)) + \
        ((c/2)*np.exp((-1*r*9)/2)) + \
        ((c/2)*np.exp((-1*r*10)/2))

    return P
```

```
#Define Variables

c = 0.05

r = 0.08

S = 65

T = 5

sigma = 0.35

P_a = P(c,r,S,T,sigma)
P_a
```

Out[129…

```
84.81345701937335
```

b) Compute partial derivative of P with respect to S and R. Evaluate when c = .05, r = .08, T = 5, sigma = .35

In [131…

```
#Define Variables and d1, d2 functions

c = 0.05

r = 0.08

T = 5

S = 65

d1 = ((np.log(S/100)) + (r + ((sigma**2)/2)*T)) / (sigma * (np.sqrt(T)))

d2 = (d1 - (sigma * np.sqrt(T)))

sigma = 0.35



dp_ds = norm.cdf(d1)

dp_dr =  ((T * (100*np.exp(-1*r*T))*(norm.cdf(d2)))) + (-100*T*np.exp(-1*r*T)) + ((
        ((-c/2)*np.exp((-1*r*2)/2)) + \
        ((-3*c/4)*np.exp((-1*r*3)/2)) + \
        ((-c)*np.exp((-1*r*4)/2)) + \
        ((-5*c/4)*np.exp((-1*r*5)/2)) + \
        ((-3*c/2)*np.exp((-1*r*6)/2)) + \
        ((-7*c/4)*np.exp((-1*r*7)/2)) + \
        ((-2*c)*np.exp((-1*r*8)/2)) + \
        ((-9*c/4)*np.exp((-1*r*9)/2)) + \
        ((-5*c/2)*np.exp((-1*r*10)/2))

print(f' partial derivative of P wrt S is {dp_ds}')

print(f' partial derivative of P wrt r is {dp_dr}')
```

```
partial derivative of P wrt S is 0.4773116015175033
partial derivative of P wrt r is -268.45337919932155
```

c) Write python function that returns r1 given S0, S1 and r0

In [82]:
```python
def r_transform(S0,S1,r0):

    r1 = (S0/S1) ** 0.15

    return r1
```

d) Compute partial derivative of function in c) wrt S1. S0 = S1 = 65. r0 = .08

In [133…
```python
S1 = 65
S0 = 65
r0 = 0.08

dr_dS1 = 0.15*r0*((S0/S1)**-0.85)*(-S0*(S1**-2))
dr_dS1
```

Out[133…
```
-0.00018461538461538461846153846153846
```

e) Use results from b) and d) to find total derivative of convertible bond price wrt value of underlying equity

In [135…
```python
# use dp_ds, dp_dr from b and dr_dS1 from d

total_derivative = dp_ds + (dp_dr * dr_dS1)

print(f' Using the answers from b) and d), the total derivative value is: {total_de
```

```
Using the answers from b) and d), the total derivative value is: 0.5268722253696857
```

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MFE Math Bootcamp

1)

d) Compute $\lim_{n \to \infty} P^n$ using c

$D = I^{-1}(P) I$

$$D = \begin{bmatrix} 0.98 & 0 & 0 & 0 \\ 0 & 0.72 & 0 & 0 \\ 0 & 0 & 0.56 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* From c

$$\lim_{n \to \infty} P^n \Rightarrow \lim_{n \to \infty} I^{-1} P^n I \Rightarrow \lim_{n \to \infty} D^n$$

* raise diagonals to power of ∞

$$\Rightarrow \boxed{\lim_{n \to \infty} D^{\infty} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

2) $\int_0^1 [x^{3/2} - \hat{f}(x)]^2 dx$

$\langle f,g \rangle = \int_0^1 f(x) g(x) dx$

a) Orthogonalize the basis $(1, x, x^2)$

$\vec{u}_1 = 1$

$\vec{u}_2 = x - \frac{\langle x, 1 \rangle}{\|1\|^2} \cdot 1$

- $\langle x, 1 \rangle = \int_0^1 x(1) dx = \int_0^1 x dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$
- $\|1\|^2 \Rightarrow \langle 1, 1 \rangle = \int_0^1 1 \cdot 1 dx = 1$

$= x - (1/2 / 1) \cdot 1$

$= x - 1/2$

- $\langle x^2, 1 \rangle = \int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1$
  $= \frac{1}{3}$

$\vec{u}_3 = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x^2, x - 1/2 \rangle}{\langle x - 1/2, x - 1/2 \rangle}(x - 1/2)$

- $\langle x^2, x - 1/2 \rangle$
  $= \int_0^1 x^2 (x - 1/2) dx$
  $\int_0^1 x^3 - 1/2 x^2 dx$
  $= \left[\frac{1}{4}x^4 - \frac{1}{6}x^3\right]_0^1$
  $= \left(\frac{1}{4} - \frac{1}{6}\right) = \frac{6}{24} - \frac{4}{24} = \frac{1}{12}$

$= x^2 - \frac{1}{3} - \frac{\frac{1}{12}}{\frac{1}{12}}(x - 1/2)$

$= x^2 - \frac{1}{3} - x + \frac{1}{2}$

$= x^2 - x + \frac{1}{6}$

- $\langle x - 1/2, x - 1/2 \rangle = \int_0^1 (x - 1/2)(x - 1/2) dx$
  $\int_0^1 x^2 - x + 1/4 \, dx$
  $\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right]_0^1$
  $= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \Rightarrow 1/12$
  $\frac{4}{12} - \frac{6}{12} + \frac{3}{12} = 1/12$

- check $\Rightarrow$ $\vec{u}_1$ and $\vec{u}_3$ orthog

$\int_0^1 (x^2 - x + \frac{1}{6}) dx = 0$

$\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x\right]_0^1$

$\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\right)$

$\frac{2}{6} - \frac{3}{6} + \frac{1}{6} = 0$ ✓

b) use result from a) to project $x^{3/2}$ onto subspace of polynomials of order at most 2

$\vec{u}_1 = 1$

$\vec{u}_2 = x - 1/2$

$\vec{u}_3 = x^2 - x + \frac{1}{6}$

- $f(x) = c_1 + c_2(x - \frac{1}{2}) + c_3(x^2 - x + \frac{1}{6})$
- $c_1 = a_{proj} = \frac{\langle 1, x^{3/2} \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^1 x^{3/2} dx}{1} \Rightarrow \left[\frac{2}{5} x^{5/2}\right]_0^1 \Rightarrow \frac{2}{5}$
- $c_2 = b_{proj} = \frac{\langle x - \frac{1}{2}, x^{3/2} \rangle}{\langle x - \frac{1}{2}, x - \frac{1}{2} \rangle} \Rightarrow \frac{\int_0^1 (x - \frac{1}{2})(x^{3/2}) dx}{\int_0^1 (x - \frac{1}{2})(x - \frac{1}{2}) dx}$

1) $\int_0^1 \left(x^{5/2} - \frac{1}{2} x^{3/2}\right) dx$

$\left[\frac{2}{7} x^{\frac{7}{2}} - \frac{1}{5} x^{5/2}\right]_0^1$

$\left[\frac{2}{7} - \frac{1}{5}\right] = \frac{3}{35}$

2) $\int_0^1 x^2 - \frac{1}{2}x - \frac{1}{2}x + \frac{1}{4} dx$

$\int_0^1 x^2 - x + \frac{1}{4} dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right]_0^1$

$\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{12}$

$c_2 = \frac{\frac{3}{35}}{\frac{1}{12}} \Rightarrow \frac{3}{35} \cdot \frac{12}{1} = \frac{36}{35}$

- $c_3 = c_{proj} = \frac{\langle x^2 - x + \frac{1}{6}, x^{3/2} \rangle}{\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6} \rangle} \Rightarrow$

1) $\int_0^1 x^{3/2}(x^2 - x + \frac{1}{6}) dx$

$\int_0^1 \left(x^{7/2} - x^{5/2} + \frac{1}{6} x^{3/2}\right) dx$

$\left[\frac{2}{9} x^{9/2} - \frac{2}{7} x^{7/2} + \frac{2}{30} x^{5/2}\right]_0^1$

$= \frac{2}{9} - \frac{2}{7} + \frac{2}{30} = 0.00317$

$\frac{2}{30} \cdot \frac{5}{2}$ $\frac{5}{30}$ $\frac{1}{6}$

2) $\int_0^1 (x^2 - x + \frac{1}{6})(x^2 - x + \frac{1}{6}) dx$

$\cancel{x^4} - \cancel{x^3} + \cancel{\frac{1}{6}x^2} - \cancel{x^3} + \cancel{x^2} - \frac{1}{6}x + \cancel{\frac{1}{6}x^2} - \frac{1}{6}x + \frac{1}{36}$

$\int_0^1 \left(x^4 - 2x^3 + \frac{8}{6}x^2 - \frac{2}{6}x + \frac{1}{36}\right) dx$

$\left[\frac{1}{5}x^5 - \frac{2}{4}x^4 + \frac{8}{18}x^3 - \frac{2}{12}x^2 + \frac{1}{36}x\right]_0^1$

$\left(\frac{1}{5} - \frac{2}{4} + \frac{8}{18} - \frac{2}{12} + \frac{1}{36}\right)$

$\left(\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}\right) \Rightarrow 0.0056$

$\boxed{f(x) = 0.4 + 1.03(x - \frac{1}{2}) + \overset{0.00317}{0.0056}(x^2 - x + \frac{1}{6})}$

c) Second degree taylor polynomial ~~to approximate~~ centered at $x = 1/2$ to approximate $x^{3/2}$

* n^{th} degree taylor polynomial for $f(x)$ near 0:

$$f(x) \approx P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{(i)!}(x-0)^i$$

$$f(x) = \left(\frac{1}{2}\right)^{3/2}(x)^0 + \underline{\frac{3}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}}\left(x-\frac{1}{2}\right) + \frac{\frac{3}{4}\left(\frac{1}{2}\right)^{-1/2}}{2}\left(x-\frac{1}{2}\right)^2$$

$$f(x) = 0.354 + 1.06x - 0.53 + \overset{0.53}{\cancel{1.06}}\left(x^2 - x + \frac{1}{4}\right)$$

$$\cancel{f(x) = 0.354 + 1.06x - 0.53 \quad 1.06x^2 - 1.06x + 0.265}$$

$$\cancel{= 1.06x^2 + 0.089}$$

$$f(x) = 0.354 + 1.06x - 0.53 + 0.53x^2 - 0.53x + 0.1325$$

$$f(x) = 0.53x^2 + 0.53x - 0.0435$$

3)

x_i	-1	0	1	2	3
y_i	3.02	2.18	4.08	-0.03	-1.08

* approx f with function in form $\alpha + \beta x$

a)

f whose x, y coordinates are above. Minimize ~~loss~~ ℓ_2 with respect to α an β to find optimal coefficients

$$\ell_2(\alpha,\beta) = \sum_{i=1}^{5} (y_i - \alpha - \beta x_i)^2$$

$$\ell_2 = ((3.02) - \alpha + \beta)^2 + (2.18 - \alpha)^2 + (4.08 - \alpha - \beta)^2 + (-0.03 - \alpha - 2\beta)^2 + (-1.08 - \alpha - 3\beta)^2$$

- $$\frac{\partial \ell}{\partial \alpha} = -2(3.02 - \alpha + \beta) - 2(2.18 - \alpha) - 2(4.08 - \alpha - \beta) - 2(-0.03 - \alpha - 2\beta) - 2(-1.08 - \alpha - 3\beta)$$

$$= -6.04 + 2\alpha - 2\beta - 4.36 + 2\alpha - 8.16 + 2\alpha + 2\beta + 0.06 + 2\alpha + 4\beta + 2.16 + 2\alpha + 6\beta$$

$$= 10\alpha + 10\beta - 16.34 = 0$$

1) $10\alpha + 10\beta = 16.34$

- $$\frac{\partial \ell}{\partial \beta} = 2(3.02 - \alpha + \beta) - 2(4.08 - \alpha - \beta) - 4(-0.03 - \alpha - 2\beta) - 6(-1.08 - \alpha - 3\beta)$$

$$= 6.04 - 2\alpha + 2\beta - 8.16 + 2\alpha + 2\beta + 0.12 + 4\alpha + 8\beta + 6.48 + 6\alpha + 18\beta$$

2) $10\alpha + 30\beta = ~~8.08~~ -4.48$

- Solve for α and β

1) $10\alpha + 10\beta = 16.34$

2) $(10\alpha + 30\beta = ~~8.08~~ -4.48) \times -1$

~~$-20\beta = 7.46$~~

~~$\beta = -0.343$~~

~~$\alpha = 2.027$~~

$$\alpha = 2.675$$

$$\beta = -1.041$$

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial \alpha^2} & \frac{\partial^2 f}{\partial \alpha \partial \beta} \\ \frac{\partial^2 f}{\partial \beta \partial \alpha} & \frac{\partial^2 f}{\partial \beta^2} \end{vmatrix} \Rightarrow \begin{vmatrix} 10 & 10 \\ 10 & 30 \end{vmatrix}$$

$D = 300 - 100 = 200$

$D = 200$, $D > 0$, test works

$D > 0$, $f_{\alpha\alpha} \Rightarrow 10$, $f_{\alpha\alpha}(2.027, -0.393) > 0$

$\boxed{\text{Local min}}$

$\boxed{\alpha = \cancel{2.027}\ 2.675,\ \beta = \cancel{-0.393}\ -1.041,\ \text{Local min}}$

4)

$$P = S\,\Phi(d_1) - 100e^{-rT}\,\Phi(d_2) + 100e^{-rT} + \sum_{k=1}^{2T} \frac{c}{2} e^{-rk/2}$$

b) $\frac{\partial P}{\partial S}$ and $\frac{\partial P}{\partial r}$

- $$\frac{\partial P}{\partial S} = \Phi(d_1)$$

- $$\frac{\partial P}{\partial r} = \left(-100e^{-rT}\right)\cdot(-T) + 100e^{-rT}(-T) + \frac{c}{2}e^{-r(1/2)}\cdot(-1/2)$$
$$+ \frac{c}{2}e^{-r(2/2)}\cdot(-1) + \frac{c}{2}e^{-r(3/2)}\cdot(-3/2) + \frac{c}{2}e^{-r(4/2)}\cdot(-2)$$
$$+ \frac{c}{2}e^{-r(5/2)}\cdot(-5/2) + \frac{c}{2}e^{-r(6/2)}\cdot(-3) + \frac{c}{2}e^{-r(7/2)}\cdot(-7/2) +$$
$$\frac{c}{2}e^{-r(8/2)}\cdot(-4) + \frac{c}{2}e^{-r(9/2)}\cdot(-9/2) + \frac{c}{2}e^{-r(10/2)}\cdot(-5)$$

d) $$r_1 = \left(S_0\cdot(S_1)^{-1}\right)^{0.15} r_0$$

$$\frac{\partial r_1}{\partial S_1} = \left((0.15)(r_0)\left((S_0/S_1)^{-0.85}\right)\right)\cdot -S_0(S_1)^{-2}$$