Week 2 Exercises

Help Center

Lecture 2

Note II.11

Balls switching chambers: the Ehrenfest model.
Consider a system of two chambers A and B (also classically called "urns"). There are N distinguishable balls, and, initially, chamber A contains them all. At any instant $\frac{1}{2}$, $\frac{3}{2}$, $\ldots$, one ball is allowed to change from one chamber to the other. Let $E_n^{[\ell]}$ be the number of possible evolutions that lead to chamber A containing ℓ balls at instant n and $E^{[\ell]}(z)$ the corresponding EGF. Then

$$E^{[\ell]}(z) = \binom{N}{\ell}(\cosh z)^{\ell}(\sinh z)^{N-\ell}, \qquad E^{[N]}(z) = (\cosh z)^N \equiv 2^{-N}(e^z + e^{-z})^N.$$

[Hint: the EGF $E^{[N]}$ enumerates mappings where each pre-image has an even cardinality.] In particular, the probability that urn A is again full at time $2n$ is

$$\frac{1}{2^N N^{2n}} \sum_{k=0}^{N} \binom{N}{k} (N-2k)^{2n}.$$

This famous model was introduced by Paul and Tatiana Ehrenfest in1907, as a simplified model of heat transfer. It helped resolve the apparent contradiction between irreversibility in thermodynamics (the case $N \to \infty$) and recurrence of systems undergoing ergodic transformations (the case $N < \infty$).

Note II.31

Combinatorics of trigonometrics. Interpret $\tan \frac{z}{1-z}$, $\tan \tan z$, and $\tan(e^z - 1)$ as EGFs of combinatorial classes.

Lecture 3

Note III.17

Leaves and node-degree profile in Cayley trees. For Cayley trees, the bivariate EGF with u marking the number of leaves is the solution to

$$T(z,u) = uz + z(e^{T(z,u)} - 1)$$

(By Lagrange inversion, the distribution is expressible in terms of Stirling partition numbers.) The mean number of leaves in a random Cayley tree is asymptotic to ne^{-1}. More generally, the mean number of nodes of outdegree k in a random Cayley tree of size~n is asymptotic to

$$n \cdot e^{-1} \, \frac{1}{k!}$$

Degrees are thus approximately described by a Poisson law of rate 1.

Note III.21

After Bhaskara Acharya (circa 1150 AD). Consider all the numbers formed in decimal with digit 1 used once, with digit 2 used twice, ..., with digit 9 used nine times. Such numbers all have 45 digits. Compute their sum S and discover, much to your amazement that S equals `458755596000061532190847692863999999999999999954124440399993846780915230713600000`. This number has a long run of nines (and further nines are hidden!). Is there a simple explanation? This exercise is inspired by the Indian mathematician Bhaskara Acharya who discovered multinomial coefficients near 1150 AD.

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