Note Title

Differential & Difference Equations [MAT 2001] Lecture-1

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Matrix :- A matrix is defined as a rectangular array or arrangement in rows or columns of scalars subject to certain rules of operations.

For example:- $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}_{2\times 2}$ or $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}_{2\times 2}$

If mn number are arranged in the form of a rectangular array having m rows & n columns, then A is called an $m\times n$ matrix & each of the mn numbers is called an element of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$= [a_{ij}]_{m\times n}$, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$

Types of Matrices :-

① **Row Matrix :** A matrix having only one row & any numbers of columns is called a row matrix.

For example:- $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$

② **Column Matrix :** A matrix having only one column & any number of rows is called a column matrix.

For Example:- $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times 1}$

③ Square Matrix : A matrix in which the number of rows is equal to the number of columns is called a square matrix.

Examples: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}_{2\times 2}$, $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}_{3\times 3}$.

④ Null Matrix : A matrix in which each element is zero is called a null matrix or zero matrix.

For example : $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

⑤ Diagonal Matrix :- A square matrix in which all the non-diagonal elements are zero is called a diagonal matrix.

Example :- $A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}$

⑥ Scalar Matrix : A diagonal matrix in which all the diagonal elements are equal to a scalar, say k, is called a scalar matrix.

Ex. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3\, I_{3\times 3}$.

⑦ Identity Matrix :- A scalar matrix in which each diagonal element is unity or 1 is called a unit matrix or identity matrix

For example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} = I_{2\times2}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} = I_{3\times3}$

(8) Upper Triangular Matrix: A square matrix in which all the elements below the principal diagonal are zero is called an upper triangular matrix.

$\Rightarrow$ $A = [a_{ij}]$ is an upper triangular if $a_{ij} = 0$ for $i > j$

For example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times3}$

(9) Lower Triangular Matrix: A square matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix.

$\Rightarrow$ $A = [a_{ij}]_{m\times n}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$

For example: $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(10) Symmetric Matrix: A square matrix $A = [a_{ij}]$ is said to be symmetric if $A = A^t$ or if the transpose of the matrix is equal to the matrix itself.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & h & g \\ h & b & d \\ g & d & c \end{bmatrix}$

(11) Skew-Symmetric Matrix: A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $\boxed{A = -A^t}$ or if the transpose of the matrix is equal to the negative of the matrix.

For example:
$$\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

$A = [a_{ij}]$

$a_{ij} = -a_{ji}$

$a_{ii} = -a_{ii}, \ j = i$

$2a_{ii} = 0$

$a_{ii} = 0$

(12) Orthogonal Matrix: A square matrix A is called an orthogonal matrix if $AA^t = I$

For example: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^t = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$AA^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

(13) Nilpotent Matrix: A square matrix A is said to be nilpotent if $A^k = 0$. where k is the least positive integer.

Ex. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(14) Idempotent Matrix: A square matrix A is said to be idempotent if $A^2 = A$.

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$

(15) Involutary Matrix : A square matrix A is said to be involutary if $A^2 = I$.

Ex.
$$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$