1.4 Portfolio Selection: Introduction to Multistage Problems

General Multistage Stochastic Optimization Problem

The problem at stage $t = T, \dots, 1$ takes the form:

$$\min_{z_t, y_t, x_t} (c^T z_t + h^T y_t + c^T x_t + Q_{t+1}(x_t, y_t, d_{[t]})) \tag{1.32}$$

subject to:

$$\begin{aligned} y_t &= y_{t-1} + x_{t-1} - A^T z_t \\ 0 &\leq z_t \leq d_t \\ y_t &\geq 0 \end{aligned}$$

Optimal value is denoted by $Q_t(x_{t-1}, y_{t-1}, d_{[t]})$. The function $Q_{t+1}(x_t, y_t, d_{[t]})$ is defined as the expected future cost:

$$Q_{t+1}(x_t, y_t, d_{[t]}) := \mathbb{E}\{Q_{t+1}(x_t, y_t, D_{[t+1]}) | D_{[t]} = d_{[t]}\}$$

Initial Problem (First Stage)

At stage $t = 1$, the problem is to determine the first-order quantities x_0:

$$\min_{x_0 \geq 0} c^T x_0 + \mathbb{E}[Q_1(x_0, D_1)] \tag{1.33}$$

Key Difference: Unlike a simple two-stage formulation, $Q_1(x_0, D_1)$ is not given in a computationally accessible form but is itself a result of recursive optimization.

1.4.1 Static Model

Problem Setup

- **Capital:** Initial capital W_0.
- **Assets:** n assets, invest amount x_i in asset i.
- **Return Rates:** R_i (per one period) are unknown (uncertain).
- **Objective:** Distribute W_0 in an optimal way.
- **Total Wealth after one period:** $W_1 = \sum_{i=1}^n \xi_i x_i$, where $\xi_i := 1 + R_i$.
- **Balance Constraint:** $\sum_{i=1}^n x_i \leq W_0$. If cash is an investment option, it becomes $\sum_{i=1}^n x_i = W_0$.

1. Maximizing Expected Return (Basic Model)

$$\max_{x \geq 0} \mathbb{E}[W_1] \quad \text{s.t.} \quad \sum_{i=1}^n x_i = W_0 \tag{1.34}$$

- **Expected Wealth:** $\mathbb{E}[W_1] = \sum_{i=1}^n \mathbb{E}[\xi_i] x_i = \sum_{i=1}^n \mu_i x_i$, where $\mu_i := \mathbb{E}[\xi_i] = 1 + \mathbb{E}[R_i]$.
- **Optimal Solution:** Invest everything in the asset with the largest expected return rate, i.e., $x_k = W_0$ for $k = \arg\max_i \mu_i$, and $x_j = 0$ for $j \neq k$.
- **Intuitive Critique:** This solution is generally not appealing as it concentrates all wealth in one asset, leading to high risk if that asset performs poorly.

2. Maximizing Expected Utility of Wealth

An alternative approach to address risk is to maximize the expected utility of wealth.

$$\max_{x \geq 0} \mathbb{E}[U(W_1)] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = W_0 \tag{1.35}$$

- **Utility Function $U(W_1)$:** Must be a concave nondecreasing function.
- **Example Utility Function (Piecewise Linear):**

$$U(W) := \begin{cases} (1+q)(W-a) & \text{if } W > a \\ (1+r)(W-a) & \text{if } W \leq a \end{cases} \tag{1.36}$$

  - a: amount to pay after return (threshold).
  - q: interest rate for investing additional wealth ($W > a$).
  - r: interest rate for borrowing if wealth is low ($W \leq a$).
  - Condition: $r > q \geq 0$.

- **Two-Stage Stochastic Linear Program:** For the utility (1.36), problem (1.35) can be formulated as:

$$\max_{x \geq 0} \mathbb{E}[Q(x, \xi)] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = W_0 \tag{1.37}$$

  where $Q(x, \xi)$ is the optimal value of the second-stage problem:

$$\max_{y,z \in \mathbb{R}_+} (1+q)y - (1+r)z \quad \text{s.t.} \quad \sum_{i=1}^{n} \xi_i x_i = a + y - z \tag{1.38}$$

  Intuition: The first stage (x) determines initial investment. The second stage (y, z) represents actions (investing/borrowing) taken after random returns ξ are realized.

3. Risk Control via Variance Constraint (Mean-Variance Models)

Another approach is to maximize expected return while controlling the investment risk, measured by variance.

- **Variance of Wealth:** $W_1 = \sum \xi_i x_i$. Let Σ be the covariance matrix of $\xi = (\xi_1, \ldots, \xi_n)^T$. Then, $\text{Var}[W_1] = x^T \Sigma x$.
- **Optimization Problem:**

$$\max_{x \geq 0} \sum_{i=1}^{n} \mu_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = W_0, \quad x^T \Sigma x \leq \nu \tag{1.39}$$

  where $\nu > 0$ is a specified constant (maximum allowable variance).

- **Properties:**
  - Since Σ is positive semidefinite, $x^T \Sigma x$ is convex quadratic. Thus, (1.39) is a convex problem.
  - It has at least one feasible solution (e.g., investing in cash, where $\text{Var}[W_1] = 0$).
  - Its feasible set is compact, so an optimal solution exists.
  - It satisfies the Slater condition, meaning no duality gap between (1.39) and its dual (1.40).

- **Dual Problem (Min-Max Form):**

$$\min_{\lambda \geq 0} \max_{x \geq 0, \sum x_i = W_0} \left\{ \sum_{i=1}^{n} \mu_i x_i - \lambda (x^T \Sigma x - \nu) \right\} \tag{1.40}$$

- **Equivalent Problem (Penalized Form):** There exists a Lagrange multiplier $\bar{\lambda} \geq 0$ such that (1.39) is equivalent to:

$$\max_{x \geq 0} \sum_{i=1}^{n} \mu_i x_i - \bar{\lambda} x^T \Sigma x \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = W_0 \tag{1.41}$$

  Intuition: This objective function represents a compromise between expected return and its variability (risk).

- **Alternative Formulation (Minimizing Variance):**

$$\min_{x \geq 0} x^T \Sigma x \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = W_0, \quad \sum_{i=1}^{n} \mu_i x_i \geq \tau \tag{1.42}$$

  where τ is a specified minimum expected return.

- **Key Insight:** Problems (1.39)-(1.42) are quadratic programming problems that can be efficiently solved. They depend only on the first and second order moments (μ_i and Σ) of the random data ξ, not its complete probability distribution.

4. Risk Control via Chance Constraints

This approach controls risk by imposing a probabilistic constraint on the outcome.

$$\max_{x \geq 0} \sum_{i=1}^{n} \mu_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = W_0, \quad \Pr\left\{ \sum_{i=1}^{n} \xi_i x_i \geq b \right\} \geq 1 - \alpha \tag{1.43}$$

- **Interpretation:** The total wealth $W_1 = \sum \xi_i x_i$ must not fall below a chosen amount b with at least $1 - \alpha$ probability.
- **Normal Distribution Assumption:** If $\xi \sim N(\mu, \Sigma)$, then $W_1 \sim N(\sum \mu_i x_i, x^T \Sigma x)$.
- **Conversion to Deterministic Form:** Using the standard normal CDF $\Phi(z)$:

$$\Pr\{W_1 \geq b\} = \Pr\left\{ Z \geq \frac{b - \sum \mu_i x_i}{\sqrt{x^T \Sigma x}} \right\} = 1 - \Phi\left(\frac{b - \sum \mu_i x_i}{\sqrt{x^T \Sigma x}} \right) \tag{1.44}$$

  The constraint $\Pr\{W_1 \geq b\} \geq 1 - \alpha$ becomes $\Phi\left(\frac{b - \sum \mu_i x_i}{\sqrt{x^T \Sigma x}} \right) \leq \alpha$. This can be rewritten as:

$$b - \sum_{i=1}^{n} \mu_i x_i + z_\alpha \sqrt{x^T \Sigma x} \leq 0 \tag{1.45}$$

  where $z_\alpha := \Phi^{-1}(\alpha)$ is the α-quantile of the standard normal distribution. If $0 < \alpha < 1/2$, then $z_\alpha < 0$, and the constraint is convex.

- **Equivalent Problem:** Similar to (1.41), there exists $\eta \geq 0$ such that (1.43) is equivalent to:

$$\max_{x \geq 0} \sum_{i=1}^{n} \mu_i x_i - \eta \sqrt{x^T \Sigma x} \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i = W_0 \tag{1.46}$$

- **Value-at-Risk (VaR):** In financial engineering, the $(1 - \alpha)$-quantile of a random variable Y (representing losses) is called Value-at-Risk:

$$\text{V@R}_\alpha(Y) := H^{-1}(1 - \alpha) \tag{1.47}$$

  where $H(\cdot)$ is the CDF of Y.

- **Chance Constraint as VaR Constraint:** The chance constraint in (1.43) can be written as:

$$\text{V@R}_{1-\alpha}\left(-\sum_{i=1}^{n} \xi_i x_i \right) \leq -b \quad \text{or equivalently} \quad \text{V@R}_\alpha\left(b - \sum_{i=1}^{n} \xi_i x_i \right) \leq 0 \tag{1.48}$$

- **Critique of Normal Assumption:** A closed-form VaR constraint is possible due to the normal distribution assumption. However, this assumption is not very realistic for return factors ξ_i, which cannot be negative.

1.4.2 Multistage Portfolio Selection

Goal: Rebalance portfolios over time periods $t = 1, \dots, T-1$, without injecting additional cash.

Problem Formulation

- **Decision at t:** $x_t = (x_{1t}, \dots, x_{nt})$ amounts invested in assets.
- **Random Process:** Return rates $R_{1t}, \dots, R_{nt}$ form a random process. Let $\xi_t = (\xi_{1t}, \dots, \xi_{nt})$ where $\xi_{it} := 1 + R_{it}$.
- **Adaptive Decisions:** $x_t = x_t(\xi_{[t]})$, where $\xi_{[t]} = (\xi_1, \dots, \xi_t)$ is the history of random data up to time t.
- **Implementable Policy:** A sequence of functions $x_t(\xi_{[t]})$ for $t = 0, \dots, T-1$ (with x_0 constant).
- **Feasibility Conditions (w.p. 1):**
  - Non-negativity: $x_{it}(\xi_{[t]}) \geq 0$.
  - Balance of Wealth: $\sum_{i=1}^n x_{it}(\xi_{[t]}) = W_t$.
- **Wealth Propagation:** $W_t = \sum_{i=1}^n \xi_{it} x_{i,t-1}(\xi_{[t-1]})$.
- **Overall Objective:** Maximize expected utility of wealth at the last period T.

$$\max \mathbb{E}[U(W_T)] \tag{1.49}$$

This is a multistage stochastic programming problem.

Dynamic Programming Equations (Backward Recursion)

- **Last Stage ($t = T-1$):** Given $\xi_{[T-1]}$ and x_{T-2}, solve:

$$\max_{x_{T-1} \geq 0, W_T} \mathbb{E}\{U[W_T] | \xi_{[T-1]}\} \quad \text{s.t.} \quad W_T = \sum_{i=1}^n \xi_{iT} x_{i,T-1}, \quad \sum_{i=1}^n x_{i,T-1} = W_{T-1} \tag{1.50}$$

  The optimal value is denoted $Q_{T-1}(W_{T-1}, \xi_{[T-1]})$.

- **Intermediate Stages ($t = T-2, \dots, 1$):**

$$\max_{x_t \geq 0, W_{t+1}} \mathbb{E}\{Q_{t+1}(W_{t+1}, \xi_{[t+1]}) | \xi_{[t]}\} \quad \text{s.t.} \quad W_{t+1} = \sum_{i=1}^n \xi_{i,t+1} x_{it}, \quad \sum_{i=1}^n x_{it} = W_t \tag{1.51}$$

  The optimal value is denoted $Q_t(W_t, \xi_{[t]})$.

- **First Stage ($t = 0$):**

$$\max_{x_0 \geq 0, W_1} \mathbb{E}[Q_1(W_1, \xi_1)] \quad \text{s.t.} \quad W_1 = \sum_{i=1}^n \xi_{i1} x_{i0}, \quad \sum_{i=1}^n x_{i0} = W_0 \tag{1.52}$$

Simplification under Stagewise Independence

Assumption: The process ξ_t is (stochastically) independent of $\xi_1, \dots, \xi_{t-1}$ for $t = 2, \dots, T$.

- **Consequence:** Conditional expectations become unconditional. The value function $Q_t(W_t)$ does not depend on $\xi_{[t]}$.
- **Logarithmic Utility Function ($U(W) := \ln W$):**
  - **Property (Homogeneity):** $Q_{T-1}(a\omega, \xi_{[T-1]}) = Q_{T-1}(\omega, \xi_{[T-1]}) + \ln a$.

- **Optimal Value for Q_{T-1}:** $Q_{T-1}(W_{T-1}, \xi_{[T-1]}) = v_{T-1}(\xi_{[T-1]}) + \ln W_{T-1}$, where $v_{T-1}(\xi_{[T-1]})$ is the optimal value of (1.54) with $W_{T-1} = 1$:

$$\max_{x_{T-1} \geq 0} \mathbb{E}\left[\ln\left(\sum_{i=1}^{n} \xi_{iT} x_{i,T-1}\right) \middle| \xi_{[T-1]}\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i,T-1} = 1 \tag{1.54}$$

- **Myopic Optimization Problem:** For $t = T-1, \ldots, 1, 0$, the problem becomes:

$$\max_{x_t \geq 0} \mathbb{E}\left[\ln\left(\sum_{i=1}^{n} \xi_{i,t+1} x_{it}\right) \middle| \xi_{[t]}\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{it} = W_t \tag{1.56}$$

  Intuition: The decision at each stage only considers the immediate next period's wealth, given current information.

- **First Stage Problem ($t = 0$):**

$$\max_{x_0 \geq 0} \mathbb{E}\left[\ln\left(\sum_{i=1}^{n} \xi_{i1} x_{i0}\right)\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i0} = W_0 \tag{1.57}$$

- **Optimal Policy:** $x_t(W_t) := W_t x_t^*$, where x_t^* is the optimal solution of (1.56) for $W_t = 1$. This is a myopic policy.

- **Overall Optimal Value ($t = 0$):** $\vartheta^* = \ln W_0 + v_0 + \sum_{t=1}^{T-1} \mathbb{E}[v_t(\xi_{[t]})]$ If stagewise independent, $v_t(\xi_{[t]}) = v_t$ (constant).

• **Power Utility Function ($U(W) := W^\gamma$, with $1 > \gamma > 0$):**

  - **Property (Homogeneity):** $Q_{T-1}(W_{T-1}) = W_{T-1}^\gamma Q_{T-1}(1)$.
  - **Myopic Optimization Problem:** For $t = 0, \ldots, T-1$:

$$\max_{x_t \geq 0} \mathbb{E}\left[\left(\sum_{i=1}^{n} \xi_{i,t+1} x_{it}\right)^\gamma\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{it} = W_t \tag{1.60}$$

  - **First Stage Problem ($t = 0$):**

$$\max_{x_0 \geq 0} \mathbb{E}\left[\left(\sum_{i=1}^{n} \xi_{i1} x_{i0}\right)^\gamma\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i0} = W_0 \tag{1.61}$$

  - **Optimal Policy:** $x_t(W_t) = W_t x_t^*$, where x_t^* is the optimal solution of (1.60) for $W_t = 1$. Again, a myopic policy.

  - **Overall Optimal Value:** $\vartheta^* = W_0^\gamma \prod_{t=0}^{T-1} \eta_t$, where η_t is the optimal value of (1.60) for $W_t = 1$.

• **Important Note:** This myopic behavior is rather exceptional and is typically destroyed when transaction costs are introduced.

Chapter 4. Optimization Models with Probabilistic Constraints

Types of Probabilistic Constraints

• **Joint Probabilistic Constraint:** A set of conditions must hold simultaneously with a certain probability.

$$\Pr\{g_j(x, Z) \leq 0, \forall j \in \mathcal{J}\} \geq p$$

• **Individual Probabilistic Constraints:** Each condition must hold with its own specified probability.

$$\Pr\{g_j(x, Z) \leq 0\} \geq p_j, \quad \forall j \in \mathcal{J}$$

Stochastic Orders

A framework for comparing random variables.

- **Definition 4.3: First Order Dominance ($X \geq_{(1)} Y$):** X dominates Y in the first order if $F_X(\eta) \leq F_Y(\eta)$ for all $\eta \in \mathbb{R}$.
  - **Interpretation:** X is "larger" than Y in a probabilistic sense; X has less probability mass on smaller values.
  - **Equivalent form (using inverse CDF):** $F_X^{-1}(p) \geq F_Y^{-1}(p)$ for all $p \in (0,1)$.
  - Can be interpreted as a continuum of probabilistic (chance) constraints.
- **Higher Order Distribution Functions:** Defined recursively:

$$F_X^{(k)}(\eta) = \int_{-\infty}^{\eta} F_X^{(k-1)}(t)dt \quad \text{for } k = 2, 3, 4, \ldots$$

- **Second Order Distribution Function ($F_X^{(2)}(\eta)$):**

$$F_X^{(2)}(\eta) = \int_{-\infty}^{\eta} F_X(\alpha)d\alpha = \mathbb{E}[(\eta - X)^+] \tag{4.7}$$

  Interpretation: This is the **Expected Shortfall** (or mean excess loss) above η.
  - **Properties:** Continuous, nonnegative, nondecreasing, and convex .
- **Second Order Dominance ($X \geq_{(2)} Y$):** X dominates Y in the second order if $\mathbb{E}[(\eta - X)^+] \leq \mathbb{E}[(\eta - Y)^+]$ for all $\eta \in \mathbb{R}$. Interpretation: X has smaller expected shortfalls than Y.
- **Definition 4.4: k-th Order Dominance ($X \geq_{(k)} Y$):** X dominates Y in the k-th order if $F_X^{(k)}(\eta) \leq F_Y^{(k)}(\eta)$ for all $\eta \in \mathbb{R}$.

Stochastic Optimization with Stochastic Ordering Constraints

- **General Form (First Order):**

$$\min_{x \in \mathcal{X}} c(x) \quad \text{s.t.} \quad \Pr\{g(x, Z) \leq \eta\} \leq F_Y(\eta), \quad \forall \eta \in [a, b] \tag{4.9}$$

- **General Form (k-th Order):**

$$\min_{x \in \mathcal{X}} c(x) \quad \text{s.t.} \quad F_{g(x,Z)}^{(k)}(\eta) \leq F_Y^{(k)}(\eta), \quad \forall \eta \in [a, b] \tag{4.10}$$

Example 4.5: Portfolio Selection with Stochastic Ordering Constraints

Objective: Maximize expected return of portfolio $\sum R_i x_i$. Constraint: Portfolio return rate must dominate a benchmark outcome Y.

- **First Order Dominance Constraint:**

$$\max_{x \geq 0} \sum_{i=1}^{n} \mathbb{E}[R_i]x_i \quad \text{s.t.} \quad \Pr\left\{\sum_{i=1}^{n} R_i x_i \leq \eta\right\} \leq P_\eta, \quad \forall \eta \in \mathbb{R}, \quad \sum_{i=1}^{n} x_i = 1 \tag{4.11}$$

  where $P_\eta = \Pr\{Y \leq \eta\}$ is the CDF of Y.
- **k-th Order Dominance Constraint:**

$$\max_{x \geq 0} \sum_{i=1}^{n} \mathbb{E}[R_i]x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} R_i x_i \geq_{(k)} Y, \quad \sum_{i=1}^{n} x_i = 1 \tag{4.12}$$

- **Second Order Dominance Interpretation:** The constraint $\sum R_i x_i \geq_{(2)} Y$ is equivalent to:

$$\mathbb{E}\left[\left(\eta - \sum_{i=1}^{n} R_i x_i\right)^+\right] \leq \mathbb{E}[(\eta - Y)^+], \quad \forall \eta \in \mathbb{R}$$

  This means the expected shortfall of the portfolio return must be less than or equal to that of the benchmark.

- **Relation to AV@R:** Second order dominance is a continuum of Average Value-at-Risk (AV@R) or Conditional Value-at-Risk (CVaR) constraints.

Convexity in Probabilistic Optimization

- Define $A_1(Y) := \{X \in L_1(\Omega, \mathcal{F}, P) : X \geq_{(1)} Y\}$.
- Define $A_2(Y) := \{X \in L_1(\Omega, \mathcal{F}, P) : X \geq_{(2)} Y\}$.
- **Proposition 4.48:** For any $Y \in L_1(\Omega, \mathcal{F}, P)$, the set $A_2(Y)$ is convex and closed.
  - **Proof Sketch:** Relies on the fact that $X \geq_{(2)} Y$ is equivalent to $\mathbb{E}[(\eta - X)^+] \leq \mathbb{E}[(\eta - Y)^+]$ for all $\eta \in \mathbb{R}$. The functional $X \mapsto \mathbb{E}[(\eta - X)^+]$ is convex in X, which implies the convexity of $A_2(Y)$.
- **Note:** $A_1(Y)$ is closed but not convex in general (Example 4.49).
- **Relationship:** $A_1(Y) \subseteq A_2(Y)$, and thus $\text{conv}(A_1(Y)) \subseteq A_2(Y)$.
- **Theorem 4.50:** If Ω is finite and elementary events have equal probabilities, then $\text{conv}(A_1(Y)) = A_2(Y)$.
  - **Proof Sketch:** Under these conditions, second order stochastic dominance coincides with weak majorization, which can be represented by doubly stochastic matrices (Birkhoff's theorem).

Chapter 5. Statistical Inference: SAA Method for Multistage Problems

Example 5.34: Portfolio Selection using SAA

The Sample Average Approximation (SAA) method replaces expectations with sample averages to solve stochastic programs.

- **SAA for Multistage Problem:**
  - At stage $t = 1$, for N_0 samples ξ_1^i:

$$\hat{Q}_{1,N_1}(W_1, \xi_1^i) = \sup_{x_1 \geq 0} \left\{ \frac{1}{N_1} \sum_{j=1}^{N_1} U((\xi_{t+1}^j)^T x_1) : e^T x_1 = W_1 \right\} \tag{5.248}$$

  - At stage $t = 0$:

$$\max_{x_0 \geq 0, W_1} \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{Q}_{1,N_1}(\xi_1^i) x_0, \xi_1^i) \quad \text{s.t.} \quad e^T x_0 = W_0 \tag{5.249}$$

- **Logarithmic Utility ($U(W) = \ln W$) and Stagewise Independence:**
  - **True Optimal Value:** $\vartheta^* = \ln W_0 + \sum_{t=0}^{T-1} v_t$, where v_t is the optimal value of:

$$\max_{x_t \geq 0} \mathbb{E}\left[\ln\left((\xi_{t+1})^T x_t\right)\right] \quad \text{s.t.} \quad e^T x_t = 1 \tag{5.251}$$

- **SAA Optimal Value:** $\hat{\vartheta}_N = \ln W_0 + \sum_{t=0}^{T-1} \hat{v}_{t,N_t}$, where $\hat{v}_{t,N_t}$ is the optimal value of:

$$\max_{x_t \geq 0} \frac{1}{N_t} \sum_{j=1}^{N_t} \ln\left((\xi_{t+1}^j)^T x_t\right) \quad \text{s.t.} \quad e^T x_t = 1 \tag{5.253}$$

- **Bias:** $\mathbb{E}[\hat{\vartheta}_N] \geq \vartheta^*$ (SAA estimator is upward biased for maximization problems).
- **Additive Bias Growth:** $\mathbb{E}[\hat{\vartheta}_N] - \vartheta^* = \sum_{t=0}^{T-1} (\mathbb{E}[\hat{v}_{t,N_t}] - v_t)$. Bias grows additively with the number of stages .
- **Variance:** Due to independent samples, $\text{Var}[\hat{\vartheta}_N] = \sum_{t=0}^{T-1} \text{Var}[\hat{v}_{t,N_t}]$. Variance also grows additively .

- **Power Utility Function ($U(W) = W^\gamma$, with $0 < \gamma \leq 1$) and Stagewise Independence:**
  - **True Optimal Value:** $\vartheta^* = W_0^\gamma \prod_{t=0}^{T-1} \eta_t$, where η_t is the optimal value of:

$$\max_{x_t \geq 0} \mathbb{E}\left[\left((\xi_{t+1})^T x_t\right)^\gamma\right] \quad \text{s.t.} \quad e^T x_t = 1 \tag{5.257}$$

  - **SAA Optimal Value:** $\hat{\vartheta}_N = W_0^\gamma \prod_{t=0}^{T-1} \hat{\eta}_{t,N_t}$, where $\hat{\eta}_{t,N_t}$ is the optimal value of:

$$\max_{x_t \geq 0} \frac{1}{N_t} \sum_{j=1}^{N_t} \left((\xi_{t+1}^j)^T x_t\right)^\gamma \quad \text{s.t.} \quad e^T x_t = 1 \tag{5.259}$$

  - **Multiplicative Bias Growth:** $\mathbb{E}[\hat{\vartheta}_N] = \vartheta^* \prod_{t=0}^{T-1} (1 + \beta_{t,N_t})$, where β_{t,N_t} is the relative bias of $\hat{\eta}_{t,N_t}$. Bias grows multiplicatively , and can grow exponentially fast with the number of stages if relative biases are constant.

- **Statistical Validation Analysis:** While SAA provides a valid lower bound, the fast growth of bias and variance with the number of stages is a significant challenge.

Chapter 6. Risk Averse Optimization

Risk Averse Portfolio Selection

Objective: Minimize the risk of the negative return (maximize risk-adjusted return).

$$\min_{x \in \mathcal{X}} \rho\left(-\sum_{i=1}^n \xi_i x_i\right) \quad \text{s.t.} \quad \sum_{i=1}^n x_i = W_0, \quad x \geq 0 \tag{6.175}$$

where ρ is a chosen risk measure.

Example: Mean-Risk Measure

- **Form:** $\rho(Z) := \mathbb{E}[Z] + cD[Z]$, where $c > 0$ and $D[\cdot]$ is a dispersion measure (e.g., standard deviation $\sqrt{\text{Var}[Z]}$ or variance $\text{Var}[Z]$) .
- **Critique:** Such mean-risk measures are not always monotone . This can lead to counter-intuitive results where a stochastically dominant asset (e.g., ξ_1 dominates ξ_2) is deemed "riskier" by ρ, leading to non-optimal solutions from a practical standpoint.

Coherent Risk Measures

If ρ is a coherent risk measure , the problem (6.175) can be written in a min-max (dual) form.

- **Min-Max Form:**

$$\min_{x \in \mathcal{X}} \sup_{\zeta \in \mathcal{A}} \mathbb{E}_\zeta\left[\sum_{i=1}^n \xi_i x_i\right] \tag{6.177}$$

  where $\mathcal{A}$ is the dual set associated with ρ.

- **Saddle Point:** The problem has a saddle point $(\bar{x}, \bar{\zeta})$ such that:

$$\bar{\zeta} \in \partial\rho\left(\sum_{i=1}^{n} \xi_i \bar{x}_i\right) \quad \text{and} \quad \bar{x} \in \arg\max_{x\in\mathcal{X}} \sum_{i=1}^{n} \mu_i x_i \tag{6.178}$$

- **Game-Theoretical Interpretation:**
  - Investor's strategy: x (portfolio allocations, e.g., fractions of W_0).
  - Opponent's (market's) strategy: ζ (a measure from the dual set $\mathcal{A}$).
  - The risk-averse solution corresponds to the equilibrium of this game.

Multistage Risk Averse Optimization

A nested formulation using conditional risk mappings.

$$\min \rho_0[\rho_1[\dots\rho_{T-1}[\rho_T[W_T]]\dots]] \tag{6.296}$$

- **Conditional Risk Mappings:** ρ_t represents a conditional risk mapping at stage t. Examples include conditional AV@R.
- **Constraints:**

$$\begin{aligned} W_{t+1} &= \sum_{i=1}^{n} \xi_{i,t+1} x_{it}, \\ \sum_{i=1}^{n} x_{it} &= W_t, \\ x_t &\geq 0 \end{aligned}$$

  for $t = 0, \dots, T-1$.
- **Risk-Neutral Case:** If $\rho_t(\cdot) := \mathbb{E}[\cdot]$, the problem reduces to the risk-neutral formulation.
- **Stagewise Independence and Positive Homogeneity of ρ_T:**
  - **Last Stage ($t = T-1$):**

$$\min_{x_{T-1}\geq 0,\, W_T} \rho_T[W_T] \quad \text{s.t.} \quad W_T = \sum_{i=1}^{n} \xi_{iT} x_{i,T-1}, \quad \sum_{i=1}^{n} x_{i,T-1} = W_{T-1} \tag{6.297}$$

  - Its optimal value is $Q_{T-1}(W_{T-1}) = W_{T-1} v_{T-1}$, where v_{T-1} is the optimal value of:

$$\min_{x_{T-1}\geq 0} \rho_T\left[\sum_{i=1}^{n} \xi_{iT} x_{i,T-1}\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{i,T-1} = 1 \tag{6.298}$$

  - **Myopic Policy:** The optimal policy is $x_t(W_t) = W_t x_t^*$, where x_t^* is the optimal solution of:

$$\min_{x_t\geq 0} \rho_{t+1}\left[\sum_{i=1}^{n} \xi_{i,t+1} x_{it}\right] \quad \text{s.t.} \quad \sum_{i=1}^{n} x_{it} = 1 \tag{6.299}$$

    This implies a myopic behavior similar to that for logarithmic/power utility under stagewise independence.

Alternative Multiperiod Risk Averse Approach

$$\min \rho[W_T] \quad \text{s.t. wealth propagation and budget constraints} \tag{6.300}$$

- **Explicit Risk Measure Example:**

$$\rho(\cdot) := (1-\beta)\mathbb{E}[\cdot] + \beta \text{AV@R}_{\alpha}(\cdot), \quad \beta \in [0,1],\ \alpha \in (0,1) \tag{6.301}$$

- **Problem (6.300) becomes:**

$$\min (1-\beta)\mathbb{E}[W_T] + \beta \mathbb{E}[r - W_T]^+ \quad \text{s.t. wealth propagation and budget constraints} \tag{6.302}$$

- **Key Result:** For this type of composite risk measure, the optimal policy is NOT myopic and the property of time consistency is NOT satisfied. This highlights that more complex risk measures or dependencies lead to more involved and non-myopic decision-making over time.