Locus of Lines and Points; Some Observations

Harshit Chhabra

August 2021

Ever wondered why a fan is only able to trace a 2 Dimensional Circle? You Probably have, as I did. And, the answer is pretty obvious, but what I want to talk about is some interesting and fascinating observations I made, once I decided I wanted to describe the whole business mathematically. Funnily enough, it's intuitive enough that I won't be using any fancy equations (or any equations for that matter). I know this topic is intuitive and interesting enough that it can be understood fairly easily.

1 So What is a Locus?

Simply speaking, it is just the path traced by a point under some arbitrary constraints. Now what does that mean? Exactly what it says. If you take a mathematical point with some Cartesian coordinates *(h,k)* and made it follow some arbitrary rules, like a function perhaps, the path traced by that point is called the Locus of that Point. That's it. So for example if I have a point on a graph, and the constraint is $y = x^2$; then Mathematically, I would say the Locus of a point on the graph of the given constraint is a Parabola. So, I guess in the end, you could even say that the Concept of a Locus is just a fancy way to visualize or even describe various functions. But, as we'll see in just about a few seconds, it is a powerful tool that gives us the ability to describe a lot of things and helps us visualize equations.

2 A circle, you say?

I am sure most of you, regardless of knowledge of formal Math, have at some point wondered the simple question of 'Why does a Fan move only in a circle?' Why not a square or some other fancy shape. Now we'll see how the Concept of a Locus helps us answer this question.

We all know the most basic property of a circle; it's Radius. It is used to find the circle's perimeter, its area. If you know the radius of the circle, you know everything about it, and I mean everything, even the fact why it's a circle. You see, radius has an interesting property; it is constant all over the circle. The Radius of a circle does not change. Another property of a circle is that its Center is fixed. Now combine these two and therein lies our answer.

Coming back to the concept of a Locus, imagine a Line L, with fixed length, l and let's put a pin in it at one point, i.e., let's fix one of it's end points (because it really is a line segment) to a fixed coordinate on a Cartesian Coordinate system. Let's call that point O with the coordinates *(0,0)*. Now let the other end point move freely. Now we want to see what that other free end point traces. We want to describe it's Locus. And you guessed it, it's a circle.

"The Locus of a line with fixed length and a fixed point is a Circle"

Alright let's bring in one equation, the equation of a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

where, r is the Radius of the circle, and *(h,k)* are the variable coordinates of the free point of the line segment.

And that's it, we've explained why a fan only traces a circle: Because it has a fixed center and the length of the blades is fixed and equal to the radius. But wait, we just explained that the Locus of One line is a circle. There are 3 blades in a regular fan, sometimes there are 4. What about 5 blades.

Relax, the equations will form a group all of which will trace the same circle. I am not going into the details, because frankly, I myself don't know the details. I am a high school student, albeit a curious one. But if you're curious, you could look into Family of equations and their Locus.

3 So, that's it?

Well not really. When I was thinking about this topic, I took the liberty to ask myself; Can we somehow, generalize this concept in 3 Dimensions? What can we say about the Locus of a Line in 3D space, fixed at a point, let's say the origin?

Well, at the outset, it is obvious that the projection of the Locus free end point on a 2D surface, can trace more that a circle. In fact, it can trace any figure. And similarly, the line, with its fixed length, can trace any 3D Figure, in that 3D space.

But here's the special thing, "*The Locus of the line with the Maximum Volume, with fixed length and one fixed point, will be a cone and in turn, the Locus of the free point on a 2D surface with the maximum surface area WILL be a circle.*"

Now I will be frank, I do not have a rigorous proof here with me, But I will present a qualitative analysis which should make the topic intuitive enough for us to understand.

Now imagine a line in 3d space, with fixed length, as before and one end point fixed, as before. The Other end point, however, is not entirely free. It has a constraint on it as well (author's note: I realized this while writing this paper). The 'free' end point is constrained to an arbitrary 2 Dimensional surface, inside the 3D space. Only then will the hypothesis be true.

Now, I think the reasoning, as to why the hypothesis is true, is even more simpler than I was intending it to be. Due to the constraints, the Locus of the projection of the line, when that projection is maximized (i.e., fixed) will be a circle. And its 3D counterpart, because the line is fixed at a point, will be a cone. I hope that's simple enough.

The reason I went through all this was mere curiosity, I was playing with one of those cloth hangers when I was thinking of this. It was really fun.

4 What if we Add another Dimension?

Yes, that's right. I am curious what we can say about such a system in 4 dimensions. I should warn you though, I admittedly have little knowledge here and anything I say can be wrong or misinformed. What I'm trying to do here is create enough curiosity for myself and people to not be afraid to take a simple question and keep going with it.

Now, coming back to the question. First, we have reformulate our question itself for 4 Dimensions as we cannot visualize the system anymore. Okay so here it is "*We are talking about a 3D figure with a fixed property, like Volume or height perhaps, and its Locus on a 3d surface, while it is constrained* *to a 2d surface on one end, and a 3d surface on the other*"

Now, I will admit, I have no idea what we can say about such a system. However, I would like to point out an example which we can use to help us understand the problem at least.

5 Light Cones

Light Cones are really interesting objects that come up in Physics while studying Einstein's Theory of Relativity. They are spacetime diagrams that 'trace' a particles trajectory in Minkowski space.I used the word 'trace' to relate to the topic of Locus more naturally.

Here's the neat thing, Minkowski space is 4 Dimensional; it treats time as the 4th Dimension. This means that Light Cones are very interesting 4 dimensional Figures. Here's another neat thing, they have a fixed point; the present: We make Light cones relative to the present. So if we ignore the light cone from the past, and restrict ourselves to a certain arbitrary time in the future, say 10 seconds later, we could have an object with the desired properties. Of course, Light Cones have constraints of their own, you can look Light Cones up if you want.

My main point again, is to strike up enough curiosity about the subject so that people are as interested in it as I am.

6 Conclusion

I did not make any comments about the behaviour of any Locus in 4D space. That is simply because I do not know anything useful. I was merely pointing out the fact that we can, and this is the 3rd time I am thinking about this topic and I will be honest with you, the example of Light Cones just came to me.

So, in the end I hope I made a simple boring topic interesting for some of you, and gave you tiny bit of inspiration to keep wondering about little things, who knows maybe you'll end up at Light Cones too.