ST. PAUL SCHOOL

ANSWER SHEET

Examination__________ 20 - 20

Name of Student ______________________ Date ______

Roll No. ______________ Examiner Signature ______

Subject ______________

Parent Signature ______________ Class Teacher Signature ______

Q2) we have two states of win | lose and outcome of having dinner together

∴

$P(\text{win next} \mid \text{win current}) = 0.8$

$P(\text{lose next} \mid \text{win current}) = 0.2$

$P(\text{win next} \mid \text{lose current}) = 0.3$

$P(\text{lose next} \mid \text{lose current}) = 0.7$

$P(\text{dinner} \mid \text{win}) = 0.7$

$P(\text{dinner} \mid \text{lose}) = 0.2$

the transition matrix would be
$$\begin{array}{c c} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

~~let π_W & π_L be ste~~

let π_W & π_L be steady-state prob^s of winning & losing

at steady state $\pi_W = 0.8\pi_W + 0.3\pi_L$

$\pi_W + \pi_L = 1$

solving these we get

$\pi_W = 0.6 \quad \pi_L = 0.4$

∴ team wins 60 % of the games in long run

(2) $P(\text{dinner}) = P(\text{dinner}|\text{win}) \times P(\text{win}) + P(\text{dinner}|\text{lose}) \times P(\text{lose})$

$P(\text{dinner}) = 0.7 \times 0.6 + 0.2 \times 0.4$

$= 0.5$

$\therefore$ 50% of games result in a team dinner in the long run

(3) As prob^y of having dinner after any given game is 0.5 & each game is indp^t trial, expected waiting line follows a Geometric distribution.

$\therefore$ expected games $= \frac{1}{0.5} = 2$

Q6) a) the probability of going from g to h in one single step is 0. It is not 0 when we can obtain h from g by swapping two entries of second row.

In this case it is $\frac{1}{^{26}C_2}$ because there are $^{26}C_2$ swaps which is equally likely.

The stationary ~~permutation~~ distribution is uniform distrib^n over all 26! permutations because we can perform enough swaps to ~~gro~~ go from one permut^n to another

→ Monte Carlo simulation gives probability $\approx$ $\underline{0}$.

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$\because$ The given chain is irreducible, there is not a unique π. but a one-parameter family.

Within each class, we can find invariant measures:

Class $\{1, 2\}$:

$$\pi_1 = 0.5\pi_1 + 0.25\pi_2$$
$$\pi_2 = 0.5\pi_1 + 0.75\pi_2$$

Solving these two eqns,

$$0.5\pi_1 = 0.25\pi_2 \Rightarrow \pi_2 = 2\pi_1$$

One unnormalized solution can be $\pi_1 = 1$ & $\pi_2 = 2$.

Class $\{3, 4\}$:

$$\pi_3 = 0.25\pi_3 + 0.75\pi_4$$
$$\pi_4 = 0.75\pi_3 + 0.25\pi_4$$

Solving,

$$0.75\pi_3 = 0.75\pi_4 \Rightarrow \pi_3 = \pi_4$$

$\pi_3 = \pi_4 = 1$ is one unnormalized soln.

Any convex combination of these two "subchain" distributions is global stationary distr.

$\omega \in [0, 1]$ and $\pi_1 = \frac{\omega}{3}$, $\pi_3 = \frac{1-\omega}{2}$

$\pi_2 = 2\omega/3$, $\pi_4 = \frac{1-\omega}{2}$

$\sum_i \pi_i = 1$ and $\pi Q = \pi$.

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(b) Since, chain is transient, no stationary distribution exists.

→ A time-homogeneous Markov chain on a countable infinite space admits a stationary distr. π only if it is recurrent.

(c) Pr of earning ₹5 payoff by 1 pm

Strike $K = 125$.

~~$S_0 = 10$~~

→ To earn exactly ₹5 you need to hit $S_t \geq K + 5 = 130$

→ Every 5s from 10 am to 1 pm

That is $3h = 10,800\,s$ or 2160 steps.

→ Hitting 130 → Random walk in ticks starting at 0 and reaching +1000 ticks before or at step 2160.

→ Each step has mean drift 0.05 ticks and variance

$$\mathrm{Var}(\Delta) = E[\Delta^2] - (E[\Delta])^2$$

$$= 0.15 - 0.05^2 \approx 0.1475$$

For $N = 2160$ steps,

$$E[\text{total ticks}] = 2160 \times 0.05 = 108.$$

$$SD = \sqrt{2160 \times 0.1475} \approx 18.$$

To cover 1000 ticks, $\frac{1000}{18} \approx 55.5$ upward fluctuation in 2160 steps.

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Week - 2 Assignment

1. Two - State loop

State space - $\{1, 2, 3, 4\}$

(a) Transition Matrix

$$Q = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{pmatrix} \end{array}$$

(b) The chain divides into two closed communicating classes:

Class $\{1, 2\}$ and $\{3, 4\}$.

Each of these is finite and irreducible, so all 4 states are recurrent.

→ No transient states

(c) A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses.

It is represented as a row vector π, whose entries are probabilities summing to 1, ~~ang~~ and given Q, it satisfies

$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$

$\cancel{\pi = \pi P}$

$\pi = \pi Q$ $\qquad \sum_i^4 \pi_i = 1 \quad \pi_i \geq 0$

If $(\omega = 1)$,

$$\pi = \left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)$$ Class $\{1, 2\}$.

If $(\omega = 0)$

$$\pi = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$$ Class $\{3, 4\}$

Both are stationary dist^r.

5. Stock Price Model

(a) Discrete-time Markov Chain state-space

$$S = \{120.00, 120.01, 120.02, \ldots\} \cup \{\ldots, 119.99, 119.98 \ldots\}.$$

Each 5 sec move:

→ +1 tick (+ ₹0.01) with probab. 0.10
→ 0 tick ₹ with probab 0.85
→ -1 tick with probab 0.05

i.e. $p = P_r\{\Delta = +1\} = 0.10$, $q = P_r\{\Delta = -1\} = 0.05$,

$r = 0.85$

So, $E[\Delta] = (+1)\cdot(0.10) + (0)\cdot(0.85) + (-1)\cdot(0.05)$

$= 0.05 > 0$

→ A 1D $\boxed{\text{random walk}}$ with nonzero drift on an infinite state-space is transient.

→ Specific kind of Markov Chain where the next state depends on the current state by "stepping" up or down acc^r. to some fixed probabilities.

(Q3)

(a) Let the 2 rooms be ROOM1 & ROOM2.
Let state space be

C_1 = cat in ROOM1
C_2 = cat in ROOM2.

Transition matrix $\Rightarrow$ $Q_{cat} = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$

Let stationary state be $\pi_{cat} = (\pi_1, \pi_2)$
such that

$$\pi_{cat} Q_{cat} = \pi_{cat}.$$

$$\left.\begin{aligned} ① \quad \pi_1 &= 0.2\pi_1 + 0.8\pi_2 \\ ② \quad \pi_2 &= 0.8\pi_1 + 0.2\pi_2 \end{aligned}\right\} \text{ where } \pi_1 + \pi_2 = 1$$

On solving eq 1 & 2 simultaneously,

$$\pi_{cat} = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Mouse chain

Let state space be
M_1 : mouse in room 1
M_2 : mouse in room 2

$$Q_{mouse} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

Let stationary distribution be $\pi_{mouse} = (\mu_1, \mu_2)$
where

$$\left.\begin{aligned} \mu_1 &= 0.7\mu_1 + 0.6\mu_2 \\ \mu_2 &= 0.3\mu_1 + 0.4\mu_2 \end{aligned}\right] \quad \mu_1 + \mu_2 = 1$$

On solving : $\pi_{mouse} = \left(\frac{2}{3}, \frac{1}{3}\right)$.

(b) Is $Z_0, Z_1, Z_2 \dots$ a markov chain?
Lets define the combined system $\Rightarrow$
$\because$ There are a total of 4 possible states for (cat, mouse) positions $\Rightarrow \{(1,1), (1,2), (2,1), (2,2)\}$

So state space of $Z_n \Rightarrow$

$\{ 1 \Rightarrow (1,1)$; $2 = (1,2)$, $3 \Rightarrow (2,1)$; $4 = (2,2) \}$

(Z_n) is a Markov Chain.

Transition Probability of $Z_n \Rightarrow (i,j)$ to $Z_{n+1} = (k,l)$ is

$$P(Z_{n+1} = (k,l) \mid Z_n = (i,j)) = P(C_{n+1} = k \mid C_n = i). \; P(M_{n+1} = l \mid M_n = j)$$

Product of Individual Transitions = satisfies MARKOV PROPERTY.

(Q4)

TYPE	location	legal moves	count
① CORNER	(1,1), (1,8), (8,1), (8,8)	3	4
② EDGE	Along borders/ (Not corners)	5	24
③ INNER EDGE	Squares next to borders (not on it)	8	36
④ CENTER	none of above.	8	36

In such chains

$$\pi(i) \propto \deg(i)$$

number of legal moves from squares 'i'

In order to find probabilities:-

Corner Square $\pi = \frac{3}{420} = \frac{1}{140}$

Edge square , $\pi = \frac{5}{420} = \frac{1}{84}$

Interior square , $\pi = \frac{8}{420} = \frac{2}{105}$

Inner Edge Squares $= \frac{8}{420} = \frac{2}{105}$

These are the stationary probabilities for them.