Contribution

HIMANSHU : Q 1, 2, 3, 7
HARSHITA : Q 4, 5, 6, 9
SONALI : Q 8, 10, 11; Code

1) N letters
N envelopes

let no letter be in correct envelope
$\therefore$ it is possible for $(N-1)\times(N-1)\ldots$ i.e. $(N-1)^N$

the probability $= \frac{(N-1)^N}{N^N} = \left(\frac{N-1}{N}\right)^N$

as $N-1$ incorrect letters can be filled in each envelope

$\therefore$ for at least one letter correctly
$$P = 1 - \left(\frac{N-1}{N}\right)^N$$

Now $N = 50$ $\quad P = 0.635$

2) here we have three cases
Prize in 1, 2, 3 and host opens 2
$\therefore$ ~~P (Prize in 1 | opens 2) =~~

$P(\text{opens } 2 \mid \text{prize in } 1) = \frac{1}{2}$
$\because$ she can open any of 2 or 3 with equal prob^y

$P(\text{opens } 2 \mid \text{prize in } 2) = 0$ $\quad\because$ from given cond^n

$P(\text{opens } 2 \mid \text{prize in } 3) = 1$ $\quad\because$ he has to open 2 as its only blank option

$\therefore$ by Baye's theorem
$$\frac{1\times\frac{1}{3}}{1\times\frac{1}{3} + 0\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{3}\left(1+\frac{1}{2}\right)} = \frac{2}{3}$$

$\therefore$ expected winnings if switched $= 1000 \times \frac{2}{3}$
$= \$666.6$

8) a) $P(A \cap B | C) = P(A | B \cap C)\, P(B | C)$ $\Rightarrow$ True

$$\frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A | B \cap C)\, P(B \cap C)}{P(C)} = P(A | B \cap C)\, P(B | C)$$

b) $P(A \cap B | C) = P(A | C) P(B | C)$ for indp^{t} A & B $\Rightarrow$ False

$$\frac{P(A \cap B \cap C)}{P(C)}$$

we are given only indp^{ce} of A & B with no information about indpendence with respect to C hence assuming its not true, the above eq^{n} is false

c) given: $P(A | D \cap B^c) > P(A | D \cap B)$

$P(A | D^c \cap B^c) > P(A | D^c \cap B)$

then : $P(A | B) > P(A | B^c)$ $\Rightarrow$ ~~True~~ False

we can write

$$P(A|B) = P(A | D \cap B)\, P(D|B) + P(A | D^c \cap B)\, P(D^c | B)$$
$$P(A|B^c) = P(A | D \cap B^c)\, P(D | B^c) + P(A | D^c \cap B^c)\, P(D^c | B^c)$$

we can see that each component is greater for $P(A|B^c)$ than $P(A|B)$

$\therefore$ $P(A|B^c) > P(A|B)$

(others terms are just probabilities so ~~are~~ they are positive hence won't affect the inequality)

7) a) not reaching originator:

every time the person can tell n-1 people

$\therefore$ $(n-1)^4$ ways to spread

total ways $(n)^4$

$\therefore$ $P = \left(\frac{n-1}{n}\right)^4$

b) not repeated to any person:

we will take permutations to calculate

$\therefore$ ${}^nP_4$ ways

total ways n^4

$\therefore$ $P = \frac{{}^nP_4}{n^4} = \frac{n!}{(n-4)!\,n^4}$

For N people in group

a) at each step: $\frac{{}^{n-1}C_N}{{}^nC_N} = \frac{n-N}{n}$

for 4 times: $\left(\frac{n-N}{n}\right)^4$

b) ~~assuming~~ as we are sharing with grp of N

$\therefore$ every time it goes $(n-N), (n-2N) \ldots$

$$\therefore \frac{\prod_{i=0}^{4-1} \binom{n-iN}{N}}{({}^nC_N)^4}$$

$\longrightarrow$ every time we can choose the N group randomly which can include repitition

Date ①

(Q4) Construct / disprove

(i) A discrete random variable X for which $E(X)$ is finite and $E(X^2)$ is not finite :-

Let us define X as a discrete random variable taking the integer values $n = 1, 2, 3, \ldots$

Define PMF of x as $P(X=n) = \frac{c}{n^2}$ for $n \geq 1$.

computing c such that total probability is 1

$$\sum_{n=1}^{\infty} \frac{c}{n^2} = 1 \Rightarrow c = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

$$P(X=n) = \frac{6}{\pi^2 n^2}$$

now, $E(X) = \sum_{n=1}^{\infty} n \cdot P(X=n) \Rightarrow \sum_{n=1}^{\infty} n \cdot \frac{6}{\pi^2 n^2} \Rightarrow \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n}$.

This is the harmonic series, which diverges, hence X

Let's choose ; $P(X=n) = \frac{c}{n^3}$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3) \Rightarrow c = \frac{1}{\zeta(3)}$$

$$E(X) = \sum_{n=1}^{\infty} n \cdot \frac{1}{\zeta(3) \cdot n^3}$$

$$\Rightarrow \frac{1}{\zeta(3)} \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{6\zeta(3)} < \infty.$$

But $E(X^2) = \sum_{n=1}^{\infty} n^2 \cdot \frac{1}{\zeta(3) \cdot n^3}$

$$\Rightarrow \frac{1}{\zeta(3)} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Therefore $P(X=n) = \frac{1}{\zeta(3) \cdot n^3}$.

b) A continuous random variable X such that ~~E(X)~~ $E(X)$ is finite but $E(X^2)$ is not finite.

We need a PDF such that
$$\int_0^\infty x f(x)\,dx < \infty \quad \text{and} \quad \int_0^\infty x^2 f(x)\,dx = \infty$$

$$f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_1^\infty x \cdot \frac{2}{x^3}\,dx \Rightarrow 2\int_1^\infty \frac{1}{x^2}\,dx = 2$$

$$E(X^2) = \int_1^\infty x^2 \cdot \frac{2}{x^3}\,dx \Rightarrow 2\int_1^\infty \frac{1}{x}\,dx = \infty.$$

Hence the cont. radom variable is $f(x) = \frac{2}{x^3}$; $\underline{x \geq 1}$

(c) A random variable where $E(X) = 1$ but $E(e^{-X}) < \frac{1}{3}$.

Because e^{-x} is convex, and by Jensen's inequality, we have
$$E(e^{-X}) \geq e^{-E(X)} \Rightarrow e^{-1} \approx 0.3679 \geq \frac{1}{3}$$

So for any random var. with $E(X) = 1$
$$E(e^{-X}) \geq e^{-1} \geq \frac{1}{3}$$

Hence it is not possible.

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(Q5) Let $M = \max(x_1, x_2, x_3 \cdots x_N)$ where $x_i \in \{1, 2, \cdots N\}$

Each draw is independent & uniformly distributed.

$$P(X_i \leq k) = \frac{k}{N}.$$

$$P(M \leq k) = P(X_1 \leq k, X_2 \leq k, \cdots X_N < k) = \left(\frac{k}{N}\right)^n$$

Then

$$P(M = k) = P(M \leq k) - P(M \leq k-1)$$

$$= \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n.$$

$$E(M) = \sum_{k=1}^{N} k \cdot P(M = k) = \sum_{k=1}^{N} k \left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n\right]$$

(Q6) Let X and Y be 2 randomly chosen points on $[0, d]$.

$$X, Y \sim \text{Uniform}(0, d)$$

$$P\left(|X - Y| < \frac{d}{3}\right) \longrightarrow \text{To find.}$$

Assuming $d = 3$; $P\left(|X - Y| < \frac{1}{3}\right)$

Let us consider the unit square $[0,1] \times [0,1]$, where all points (x, y) are equally likely (due to uniform independent distribution)

$$|x - y| < \frac{1}{3}.$$

This corresponds to region in b/w $\left[y = x + \frac{1}{3} \text{ and } y = x - \frac{1}{3}.\right]$

Area of region outside the strip $|x - y| < \frac{1}{3}$

$$2\left(1-\frac{1}{3}\right)^2 \Rightarrow \frac{1}{2}\left(\frac{2}{3}\right)^2 \Rightarrow \frac{1}{2}\cdot\frac{4}{9} \Rightarrow \frac{2}{9}.$$

Total area $= \frac{4}{9}$

area inside $\Rightarrow 1-\frac{4}{9} = 5/9$

So $P(|X-Y|) < \frac{d}{3}$ is $\frac{5}{9}$.

(Q9) Let $F(x)$ and $G(x)$ be distribution functions of 2 independent real valued random variables.

$$F(x) = P(X \le x)$$
$$G(x) = P(Y \le x)$$

Both F and G are non decreasing, right continuous with $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

$\lim_{x\to-\infty} G(x) = 0$ & $\lim_{x\to\infty} G(x) = 1$.

The convolution of $(F*G)(x) \Rightarrow \int_{-\infty}^{+\infty} F(x-y)\,dG(y)$.

now, we need to prove that this is too a distribution function.

(i) Non decreasing:- Since $F(x-y)$ is non decreasing in x

let $x_1 < x_2$, then

$$F(x_1-y) \le F(x_2-y) \text{ for all } y.$$

$$H(x_1) = \int F(x_1-y)\,dG(y) \le \int F(x_2-y)\,dG(y) = H(x_2)$$

So H is non decreasing.

Right continuity :-

Suppose $x_n \to x$; Then for every y, $F(x_n - y) \to F(x-y)$

By applying the monotone convergence theorem;

$$H(x_n) = \int F(x_n - y)\, dG(y) + \int F(x-y)\, dG(y) = H(x).$$

Therefore

$H(x)$ is right continuous.

Also; as $x \to -\infty$, $F(x-y) \to 0$ for all y.

So $H(x) \to \int 0\, dG(y) = 0$

as $x \to \infty$, $F(x-y) \to 1$ for all y.

$H(x) = \int 1\, dG(y) = 1$

Therefore $H(x) = (F * G)(x)$ satisfies all properties of a distribution function.

8. $A_1, A_2, A_3, \dots A_n$ be n independent events.

To prove: $P(\cap A_i^c) \leq e^{-P(A_1)-P(A_2)\dots-P(A_n)}$

Sol^n.

Since the events $A_1, \dots A_n$ are independent so are their complements.

$$\therefore \quad P\left(\cap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P\left(A_i^c\right)$$

For any A_i,

$$P(A_i^c) = 1 - P(A_i)$$

$$P\left(\cap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} \left(1 - P(A_i)\right)$$

So, we need to prove

$$\prod_{i=1}^{n} \left(1 - P(A_i)\right) \leq e^{-P(A_1)-P(A_2)\dots-P(A_n)}$$

For any event A_i,

$$1 - P(A_i) \leq e^{-P(A_i)}$$

For any real x, consider the inequality

$$1 - x \leq e^{-x}$$

let $f(x) = e^{-x} - (1-x)$

$f(0) = 0$ and $f'(x) = -e^{-x} + 1$

8. $A_1, A_2, A_3, \cdots A_n$ be n independent events.

To prove: $P(\cap A_i^c) \leq e^{-P(A_1)-P(A_2)\cdots\cdots-P(A_n)}$

Sol^n.

Since the events $A_1, \ldots A_n$ are independent so are their complements.

$$\therefore \quad P\left(\cap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P\left(A_i^c\right)$$

For any A_i,

$$P(A_i^c) = 1 - P(A_i)$$

$$P\left(\cap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} \left(1 - P(A_i)\right)$$

So, we need to prove

$$\prod_{i=1}^{n} \left(1 - P(A_i)\right) \leq e^{-P(A_1)-P(A_2)\cdots-P(A_n)}$$

For any event A_i,

$$1 - P(A_i) \leq e^{-P(A_i)}$$

For any real x, consider the inequality

$$1 - x \leq e^{-x}$$

let $f(x) = e^{-x} - (1-x)$

$f(0) = 0$ and $f'(x) = -e^{-x} + 1$

$f'(x) \geq 0$ for $x \geq 0$

So. And since $f(0) = 0$. $f(x) \geq 0$ for all $x \geq 0$.

$\therefore \quad 1 - e^{-x} - (1-x) \geq 0$.

so, $e^{-x} \geq (1-x)$

$\Rightarrow \quad e^{-P(A_i)} \geq 1 - P(A_i) \quad \left[\because P(A_i) \geq 0 \ \& \ P(A_i) \leq 1\right]$

Applying $1 - P(A_i) \leq e^{-P(A_i)}$ to each factor in the product.

$$\prod_{i=1}^{n} (1 - P(A_i)) \leq \prod_{i=1}^{n} \exp(-P(A_i))$$
$$= \exp\left(-\sum_{i=1}^{n} P(A_i)\right)$$

$$\Rightarrow P\left(\cap A_i^c\right) \leq e^{-\sum_{i=1}^{n} P(A_i)}$$

10. X be a non-negative random variable with cumulative distribution function $F(x) = P\{X \leq x\}$.

<u>To show:</u> $E[X] = \int_0^\infty (1 - F(x))\, dx$

by showing $\int_\Omega \int_0^\infty I_{[0, X(\omega)]}(x)\, dx\, dP(\omega)$

is equal to both $E[X]$ and $\int_0^\infty (1-F(x))\,dx$

Sol^n.

$$E[X] = \int_0^\infty (1-F(x))\,dx$$

can be shown using "two-sided Fubini argument.

Consider the indicator function,

$$I_{[0,X(\omega)]}(x) = \begin{cases} 1, & 0 \le x \le X(\omega) \\ 0, & \text{otherwise} \end{cases}$$

Now,

$$\int_\Omega \int_0^\infty I_{[0,X(\omega)]}(x)\,dx\,dP(\omega)$$

Inner integral

$$\int_0^\infty I_{[0,X(\omega)]}(x)\,dx = \int_0^{X(\omega)} 1\cdot dx = X(\omega)$$

So,

$$\int_\Omega \left[\int_0^\infty I_{[0,X(\omega)]}(x)\,dx\right] dP(\omega) = \int_\Omega X(\omega)\,dP(\omega)$$

$$= E[X]$$

Thus,

$$E[X] = \int_{\Omega} \int_{0}^{\infty} I_{[0, X(\omega)]}(x)\, dx \; dP(\omega)$$

Since $I_{[0, X(\omega)]}(x) \geq 0$ by Fubini's theorem, we can swap the integrals.

$$\int_{\Omega} \int_{0}^{\infty} I_{[0, X(\omega)]}(x)\, dx \; dP(\omega) = \int_{0}^{\infty} \int_{\Omega} I_{[0, X(\omega)]}(x)\, dx \; dP(\omega)$$

So, for,

$$\int_{0}^{\infty} \int_{\Omega} I_{[0, X(\omega)]}(x)\, dP(\omega)\; dx$$

<u>Inner integral</u>

$$\int_{\Omega} I_{[0, X(\omega]}(x)\, dP(\omega) = P\{\omega : 0 \leq x \leq X(\omega)\}$$
$$= P(X \geq x\}$$
$$= 1 - F(x)$$

<u>So</u>,

$$\int_{0}^{\infty} [1 - F(x)]\, dx \leftarrow \text{Outer integral}$$

So,

$$\int_{-\infty}^{\infty} e^{u(z\sigma+\mu)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \sigma \cdot dz$$

$$= \int_{-\infty}^{\infty} e^{u(z\sigma+\mu)} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cancel{\sigma} \cdot dz$$

$$= \int_{-\infty}^{\infty} e^{u\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot e^{uz\sigma} \cdot dz$$

$$= \frac{e^{u\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2uz\sigma + u^2\sigma^2 - u^2\sigma^2)} dz$$

$$= e^{u\mu} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(z-u\sigma)^2}}{\sqrt{2\pi}} e^{\frac{u^2\sigma^2}{2}} dz$$

$$= e^{u\mu} \cdot e^{\frac{u^2\sigma^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-u\sigma)^2}{2}} dz}_{N(z:\mu=u\sigma,\ \sigma^2=1)} = 1$$

[integration of probability density f^n]

$$= e^{\left(u\mu + \frac{u^2\sigma^2}{2}\right)}$$

So, $E[e^{uX}] = e^{u\mu + \frac{u^2\sigma^2}{2}}$

Putting these together,

$$\int_{\Omega}\int_{0}^{\infty} I_{[0,X(\omega)]}(x)\,dx\,dP(\omega) = \int_{0}^{\infty}(1-F(x))\,dx$$

11. u is a fixed number in $\mathbb{R}$.
$\varphi(x) = e^{ux}$ is convex function for all $x \in \mathbb{R}$.
$X \to$ normal random variable with $\mu = E[X]$ and $\sigma = [E(X-\mu)^2]^{1/2}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(i) To verify: $E[e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2}$

Computing the mgf of X,

$$M_X(u) = E[e^{uX}]$$

Given, $X \sim N(\mu, \sigma^2)$

$$M_X(u) = E[e^{uX}] = \int_{-\infty}^{\infty} e^{ux} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{ux}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Let $z = \frac{x-\mu}{\sigma} \Rightarrow x = z\sigma + \mu$

$dx = \sigma\,dz$

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(ii) To verify the Jensen's Inequality

Soln.

$$\varphi(x) = e^{4x}$$

Given, $\varphi(x)$ is convex. So,

$$\varphi''(x) = 4^2 e^{4x} \geq 0 \text{ for all } x.$$

Jensen's inequality for any convex φ says:

$$E[\varphi(X)] \geq \varphi(E[X])$$

$$E[e^{4X}] \geq e^{4E[X]} = e^{4\mu}$$

$$E[e^{4X}] = e^{4\mu + \frac{1}{2}4^2\sigma^2} \geq e^{4\mu}$$

Hence, verified.