Solution to Question 2: Pen and Paper Option Pricing

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Option Characteristics

- **Option Type:** European Call
- $S_0 = 100$, $K = 105$, $T = 10$ days

Part A: Discrete Binomial Model

Assume each day the stock moves by ± 1 with equal probability.

(a) Probability of Ending In-the-Money

The stock follows a symmetric random walk. Let $X_i \in \{+1, -1\}$. After 10 days:

$$S_{10} = S_0 + \sum_{i=1}^{10} X_i$$

To end in-the-money: $S_{10} > 105 \Rightarrow \sum X_i > 5$.

Let n_+ be the number of upward moves ($X_i = +1$). We require:

$$2n_+ - 10 > 5 \Rightarrow n_+ > 7.5 \Rightarrow n_+ \in \{8, 9, 10\}$$

$$\begin{aligned} P(\text{ITM}) &= \sum_{k=8}^{10} \binom{10}{k} \left(\frac{1}{2}\right)^{10} \\ &= \frac{1}{1024}(45 + 10 + 1) = \frac{56}{1024} \approx 0.0547 \end{aligned}$$

(b) Expected Payoff

$$\begin{aligned}
\text{Payoff} &= \max(S_{10} - K, 0) \\
\text{Possible values:} & \\
\text{If 8 up: } S = 108 &\Rightarrow \text{Payoff} = 3 \\
\text{If 9 up: } S = 110 &\Rightarrow \text{Payoff} = 5 \\
\text{If 10 up: } S = 112 &\Rightarrow \text{Payoff} = 7 \\
\text{Expected payoff} &= \frac{1}{1024}(45 \times 3 + 10 \times 5 + 1 \times 7) = \frac{202}{1024} \approx 0.197
\end{aligned}$$

(c) Fair Value (No Discounting)

$$\boxed{\text{Fair Value} \approx 0.197}$$

Part B: Continuous Normal Model

(a) Determine Daily σ

$$E[|X|] = 1 = \sigma\sqrt{\frac{2}{\pi}} \Rightarrow \sigma = \sqrt{\frac{\pi}{2}} \approx 1.2533$$

10-day std. dev: $\sigma_{10} = \sigma\sqrt{10} \approx 3.96$

(b) Expected Payoff as Integral

Let $S_T \sim N(100, 3.96^2)$, then

$$\mathbb{E}[\max(S_T - 105, 0)] = \int_{105}^{\infty} (S - 105) f_{S_T}(S)\, dS$$

where f_{S_T} is the PDF of $N(100, 3.96^2)$.

(c) Numerical Evaluation

Python code provided below.

Part C: Uniform Model

(a) Support $[a, b]$

Given $\mathbb{E}[|X|] = 1$ for uniform $[-a, a]$:

$$1 = \mathbb{E}[|X|] = \int_{-a}^{a} \frac{|x|}{2a} dx = \frac{a}{2} \Rightarrow a = 2$$

So support is $[-2, 2]$.

(b) Final Prices

Simulate 10-day walk from $[-2, 2]$, with mean zero.

(c) Estimate Option Value (Simulation)

Python code provided below.

Python Code for Part B and C

```
import numpy as np
from scipy.stats import norm, uniform

# Part B: Normal model
mu, sigma = 100, 3.96
K = 105

# Expected payoff integral
S = np.linspace(105, 130, 1000)
payoff = (S - K) * norm.pdf(S, mu, sigma)
expected_payoff = np.trapz(payoff, S)
print("Expected payoff (Normal):", expected_payoff)

# Part C: Uniform model simulation
np.random.seed(42)
n_sim = 100000
steps = 10
moves = uniform.rvs(loc=-2, scale=4, size=(n_sim, steps))
final_prices = 100 + moves.sum(axis=1)
call_payoffs = np.maximum(final_prices - 105, 0)
print("Estimated call price (Uniform):", call_payoffs.mean())
```